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Relative reciprocities on Dedekind domains

by

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Abstract

Exact sequences in algebraic K-theory contain a lot of information. Here it is shown that by using K-theory exact sequences one can easily derive Bass' description [1] of the SK_1 of an ideal in a Dedekind domain in terms of relative reciprocities.

Key Words: Dedekind domain, reciprocity, algebraic K-theory.

Mathematics Subject Classification 2010: 19B99, 19C20, 19D06.

Introduction

A classical result in algebraic K-theory is the description of the group $\mathrm{SK}_1(R,\mathfrak{a})$ of an ideal $\mathfrak{a}$ of a Dedekind domain R in terms of Mennicke symbols. A complete account of this is given by Bass in [1], where it is also shown that $\mathrm{SK}_1(R,\mathfrak{a})$ is the target group of the universal $\mathfrak{a}$-reciprocity on R: for each abelian group A there is an isomorphism of the group of all $\mathfrak{a}$-reciprocities with values in A with the group $\mathrm{Hom}(\mathrm{SK}_1(R,\mathfrak{a}),A)$. The proofs of these results are long and technical.

Another classical result in algebraic K-theory is Matsumoto's theorem, which describes the group $\mathrm{K}_2(F)$ of a field F in terms of Steinberg symbols. A good account of the original proof is given in [7]. Its proof is long and technical as well.

In [3] a short proof of Matsumoto's theorem was given using the computation of the group SK_1 of the ideal (t^2-t) of the polynomial ring $F[t]$ over a field F.

Let Q be the field of fractions of the Dedekind domain R. In [7] the sequence

$$\mathrm{K}_2(Q) \to \bigoplus_{\mathfrak{p}} (R/\mathfrak{p})^* \to \mathrm{SK}_1(R) \to 0 \tag{1}$$

is shown to be exact. The direct sum in this sequence is over all maximal ideals $\mathfrak{p}$ of R and each of the maps $\mathrm{K}_2(Q) \to (R/\mathfrak{p})^*$ is given by the corresponding tame symbol. The proof of the exactness rests on the computation of the group $\mathrm{SK}_1(R)$ in [1] and this is the only known proof in the framework of classical algebraic K-theory. It is not hard to show that the description of $\mathrm{SK}_1(R)$ can be retrieved from this exact sequence.

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The exact sequences of classical algebraic K-theory fit into the (infinite) long exact sequences of higher algebraic K-theory as introduced by Quillen [8]. In particular, the exact sequence (1) is part of the exact localization sequence associated to R and Q. Thus a proof of its exactness is obtained which is independent of the original technical proof. In section 1 a relative version of the exact localization sequence is described. It contains the exact sequence (2) in section 2, which is a relative version of sequence (1). Using this exact sequence a proof of Matsumoto's theorem is given in section 3 which is similar to the proof in [3]. Finally, in section 4 the description of the group SK_1 of an ideal in a Dedekind domain as given by Bass [1] is derived from the description in section 2. It is shown that Bass' description can be simplified: one type of relation is not needed.

Summarizing, there is a short route from higher algebraic K-theory to the computation of the group SK_1 of an ideal in a Dedekind domain, and from there to Matsumoto's theorem. The hard part of this approach is identifying classical algebraic K-theory with the lower part of higher algebraic K-theory, something that has to be done anyhow when setting up higher algebraic K-theory. The emphasis of this paper is on the organization of algebraic K-theory, more than on new results in this theory.

1. A relative localization exact sequence

The K-groups of an exact category $\mathcal{C}$ were defined by Quillen [8]. Everything in this section is a direct consequence of results in [8]. The K-groups are defined as the homotopy groups of the space $\mathrm{K}(\mathcal{C}) = BQ(\mathcal{C})$, the classifying space of the category $Q(\mathcal{C})$ (as defined in [8]):

$$\mathrm{K}_i(\mathcal{C}) = \pi_{i+1}(\mathrm{K}(\mathcal{C})) \qquad (i \geq 0).$$

For a ring R we denote the category of finitely generated projective R-modules by $\mathcal{P}_R$. It is an exact category and the K-groups of R are the K-groups of this category:

$$\mathrm{K}_i(R) = \mathrm{K}_i(\mathcal{P}_R) \qquad (i \geq 0).$$

For an exact functor $\mathcal{C} \to \mathcal{C}'$ the homotopy fibre of the map $\mathrm{K}(\mathcal{C}) \to \mathrm{K}(\mathcal{C}')$ will be denoted by $\mathrm{K}(\mathcal{C} \to \mathcal{C}')$. The relative K-groups of an ideal $\mathfrak{a}$ of a ring R are the homotopy groups of the homotopy fibre of $\mathrm{K}(\mathcal{P}_R) \to \mathrm{K}(\mathcal{P}_{R/\mathfrak{a}})$:

$$\mathrm{K}_i(R, \mathfrak{a}) = \pi_{i+1}(\mathrm{K}(\mathcal{P}_R \to \mathcal{P}_{R/\mathfrak{a}})) \qquad (i \geq 0).$$

For a Noetherian ring R the category $\mathcal{M}_R$ of finitely generated R-modules is an exact category. For a regular Noetherian ring R the inclusion functor $\mathcal{P}_R \to \mathcal{M}_R$

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induces a homotopy equivalence $\mathrm{K}(\mathcal{P}_R) \to \mathrm{K}(\mathcal{M}_R)$, since finitely generated R-modules have a finite projective resolution.

For the rest of this article we fix the following notations:

R	a Dedekind domain,
Q	the quotient field of R,
$\mathfrak{a}$	a non-zero ideal of R,
S	a set of maximal ideals $\mathfrak{p}$ of R with $\mathfrak{p} \nmid \mathfrak{a}$,
R_S	the ring of all $u \in Q$ with $v_\mathfrak{p}(u) \geq 0$ for all maximal ideals $\mathfrak{p}$ of R with $\mathfrak{p} \notin S$,
$\mathfrak{a}_S$	the ideal $\mathfrak{a}R_S$ of R_S; it consists of all $u \in Q$ with $v_\mathfrak{p}(u) \geq v_\mathfrak{p}(\mathfrak{a})$ for all maximal ideals $\mathfrak{p}$ of R with $\mathfrak{p} \notin S$.

The commutative triangle of ring homomorphisms

$$\begin{array}{ccc} R & \longrightarrow & R/\mathfrak{a} \\ & \searrow \quad \nearrow & \\ & R_S & \end{array}$$

induces a commutative triangle

$$\begin{array}{ccc} \mathcal{P}_R & \longrightarrow & \mathcal{P}_{R/\mathfrak{a}} \\ & \searrow \quad \nearrow & \\ & \mathcal{P}_{R_S} & \end{array}$$

of functors between the categories of finitely generated projective modules.

Since R and R_S are regular Noetherian domains the inclusion functors $\mathcal{P}_R \to \mathcal{M}_R$ and $\mathcal{P}_{R_S} \to \mathcal{M}_{R_S}$ induce homotopy equivalences $\mathrm{K}(\mathcal{P}_R) \to \mathrm{K}(\mathcal{M}_R)$ and $\mathrm{K}(\mathcal{P}_{R_S}) \to \mathrm{K}(\mathcal{M}_{R_S})$. The homotopy fibre $\mathrm{K}(\mathcal{M}_R \to \mathcal{M}_{R_S})$ is homotopy equivalent to $\mathrm{K}(\mathcal{A})$, where $\mathcal{A}$ is the Serre subcategory of S-torsion modules in $\mathcal{M}_R$. By dévissage one has

$$\mathrm{K}_i(\mathcal{A}) \xrightarrow{\sim} \bigoplus_{\mathfrak{p} \in S} \mathrm{K}_i(R/\mathfrak{p}) \qquad (i \geq 0).$$

As a result we have a long exact sequence

$$\cdots \to \mathrm{K}_{i+1}(R_S, \mathfrak{a}_S) \to \bigoplus_{\mathfrak{p} \in S} \mathrm{K}_i(R/\mathfrak{p}) \to \mathrm{K}_i(R, \mathfrak{a}) \to \mathrm{K}_i(R_S, \mathfrak{a}_S) \to \cdots \to \mathrm{K}_0(R_S, \mathfrak{a}_S)$$

of homotopy groups of the homotopy fibres.

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2. Relative reciprocities

In this section we assume that $\mathfrak{a} \neq R$ and for S we take the set of *all* maximal ideals $\mathfrak{p}$ of R satisfying $\mathfrak{p} \nmid \mathfrak{a}$. Then $\mathfrak{a}_S$ is a radical ideal, i.e. an ideal contained in the Jacobson radical of the semi-local ring R_S. Therefore, for the lower relative K-groups we have ([1], Chapter XI, Proposition (1.3)):

$$\begin{aligned} \mathrm{K}_0(R_S,\mathfrak{a}_S) &= 0, \\ \mathrm{K}_1(R_S,\mathfrak{a}_S) &= 1+\mathfrak{a}_S. \end{aligned}$$

Since $\mathrm{SK}_1(R_S,\mathfrak{a}_S) = 0$, from the relative localization exact sequence above one obtains an exact sequence

$$\mathrm{K}_2(R_S,\mathfrak{a}_S) \to \bigoplus_{\mathfrak{p}\in S}(R/\mathfrak{p})^* \to \mathrm{SK}_1(R,\mathfrak{a}) \to 0. \tag{2}$$

The homomorphisms $(R/\mathfrak{p})^* \to \mathrm{SK}_1(R,\mathfrak{a})$ will be denoted by $\kappa_{\mathfrak{p}}$. In [4] a presentation of the group $\mathrm{K}_2(\Lambda,\mathfrak{b})$, where $\mathfrak{b}$ is a radical ideal of a commutative ring Λ, was given in terms of Dennis-Stein symbols. This presentation was based on the presentation given by Maazen and Stienstra [5] for the split case, i.e. the case in which the homomorphism $\Lambda \to \Lambda/\mathfrak{b}$ has a section which is a ring homomorphism. Here we only need generators of $\mathrm{K}_2(R_S,\mathfrak{a}_S)$ and since the map from $\mathrm{K}_2(R_S,\mathfrak{a}_S)$ to $\bigoplus_{\mathfrak{p}\in S}(R/\mathfrak{p})^*$ factors through $\mathrm{K}_2(R_S)$, it suffices to know their images in $\mathrm{K}_2(R_S)$. The image of $\mathrm{K}_2(R_S,\mathfrak{a}_S)$ in $\mathrm{K}_2(R_S)$ is generated by the Dennis-Stein symbols $\langle f,a\rangle$, where $f \in R_S$, $a \in \mathfrak{a}_S$. The map also factors through $\mathrm{K}_2(Q)$ and the image of $\langle f,a\rangle$ in $\mathrm{K}_2(Q)$ is equal to the Steinberg symbol $\{1-fa, f\}$. Here the description of the K_2 of a ring as a subgroup of the Steinberg group is used. Steinberg symbols, and more generally Dennis-Stein symbols, are special elements of the Steinberg group.

A prime $\mathfrak{p}$ of R induces a group homomorphism

$$\partial_{\mathfrak{p}}\colon \mathrm{K}_2(Q) \to (R/\mathfrak{p})^*$$

which maps a Steinberg symbol $\{x,\pi\}$ with $v_{\mathfrak{p}}(x) = 0$ and $v_{\mathfrak{p}}(\pi) = 1$ to $\overline{x}$ in the residue field. The relations for Steinberg symbols imply that for $u,v \in Q^*$:

$$\partial_{\mathfrak{p}}(\{u,v\}) = (u,v)_{\mathfrak{p}},$$

where

$$(u,v)_{\mathfrak{p}} = \overline{(-1)^{v_{\mathfrak{p}}(u)v_{\mathfrak{p}}(v)}u^{v_{\mathfrak{p}}(v)}v^{-v_{\mathfrak{p}}(u)}},$$

the tame symbol [7]. Here the bar stands for taking the class modulo the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ of the valuation ring $R_{\mathfrak{p}}$ and furthermore the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is

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identified with $R/\mathfrak{p}$. It follows that the image of $\mathrm{K}_2(R_S,\mathfrak{a}_S)$ in $\bigoplus_{\mathfrak{p}\in S}(R/\mathfrak{p})^*$ is generated by the elements

$$((1-fa,f)_{\mathfrak{p}})_{\mathfrak{p}\in S},$$

where $f \in R_S$ and $a \in \mathfrak{a}_S$.

The presentation of $\mathrm{SK}_1(R,\mathfrak{a})$ resulting from the above can conveniently be described by means of reciprocities, the group $\mathrm{SK}_1(R,\mathfrak{a})$ being the target group of the universal reciprocity $(\kappa_{\mathfrak{p}})_{\mathfrak{p}}$.

Definition 1 An $\mathfrak{a}$-*reciprocity* on R with values in an Abelian group A is a collection $(\chi_{\mathfrak{p}})_{\mathfrak{p}\in S}$ of group homomorphisms $\chi_{\mathfrak{p}}\colon (R/\mathfrak{p})^* \to A$, one for each $\mathfrak{p} \in S$, such that

$$\sum_{\mathfrak{p}\in S}\chi_{\mathfrak{p}}((1-fa,f)_{\mathfrak{p}}) = 0,$$

for all $f \in R_S$ and $a \in \mathfrak{a}_S$.

Summarizing we have:

Theorem 1 *Let $(\chi_{\mathfrak{p}})_{\mathfrak{p}}$ be an $\mathfrak{a}$-reciprocity on R with values in A. Then there is a unique group homomorphism $\chi\colon \mathrm{SK}_1(R,\mathfrak{a}) \to A$ such that $\chi\kappa_{\mathfrak{p}} = \chi_{\mathfrak{p}}$ for all $\mathfrak{p} \in S$.* □

The group $\mathrm{SK}_1(R,\mathfrak{a})$ is a factor group of the relative general linear group $\mathrm{GL}(R,\mathfrak{a})$ and, as explained in [1], can be presented in terms of Mennicke symbols, elements which can be represented by 2×2-matrices. In section 4 the relative reciprocities as defined above will be compared to the relative reciprocities as defined by Bass [1].

3. A proof of Matsumoto's theorem

Matsumoto's theorem is about the presentation of the K_2 of a field in terms of Steinberg symbols. In [3] a proof of Matsumoto's theorem was given by constructing a relative reciprocity on the affine line. The proof given here is similar.

Let F be a field. In this section we consider the special case:

$R = F[t]$	the polynomial ring in one indeterminate t over F,
$Q = F(t)$	the quotient field of R,
$\mathfrak{a} = (t^2-t)$	the ideal of R generated by t^2-t,
S	the set of irreducible polynomials π over F with $\pi \neq t, 1-t$ and $\pi(0) = 1$,
R_S	the semi-local ring of all $u \in Q$ with $v_t(u), v_{1-t}(u) \geq 0$,
$\mathfrak{a}_S$	the ideal $\mathfrak{a}R_S$ of R_S; it consists of all $u \in Q$ with $v_t(u), v_{1-t}(u) \geq 1$.

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The ring R_S is the ring of rational functions over F that are defined in 0 and 1. By the Chinese remainder theorem we have isomorphisms

$$R/\mathfrak{a} \xrightarrow{\sim} R_S/\mathfrak{a}_S \xrightarrow{\sim} F \times F,$$

induced by $R \subseteq R_S \to F \times F$, $u \mapsto (u(0),u(1))$. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{K}_2(F[t]_S,t^2-t) & \longrightarrow & \mathrm{K}_2(F[t]_S,t) & \longrightarrow & \mathrm{K}_2(F) & \longrightarrow & 0 \\
\| & & \sim\downarrow\partial & & \sim\downarrow & & \\
\mathrm{K}_2(F[t]_S,t^2-t) & \xrightarrow{\partial} & \bigoplus_{\pi\in S}(F[t]/\pi)^* & \longrightarrow & \mathrm{SK}_1(F[t],t^2-t) & \longrightarrow & 0
\end{array}$$

The two maps ∂ in the diagram both factor through $\mathrm{K}_2(Q)$ as described in section 2. The top row results from the exact sequence of relative groups for the ideals (t) and (t^2-t) of $F[t]_S$. Note that $\mathrm{SK}_1(F[t]_S,t^2-t)=0$ since (t^2-t) is a radical ideal and moreover that $\mathrm{K}_2(F[t]/(t^2-t),(t)/(t^2-t)) \cong \mathrm{K}_2(F\times F,0\times F) \cong \mathrm{K}_2(F)$. The bottom row is the exact sequence (2) of section 2 for the ideal (t^2-t) of $F[t]$. The middle vertical map is an isomorphism. This follows from a relative localization exact sequence, the groups $\mathrm{K}_n(F[t],t)$ in that sequence being trivial since F is regular.

The Milnor ring of a field F will be denoted as $\mathrm{K}^{\mathrm{M}}_*(F)$. It is the graded ring generated by elements $l(a)$ in degree 1, one for each $a \in F^*$. The defining relations are

$$\begin{aligned}
l(ab) &= l(a)+l(b) && \text{for all } a,b \in F^*, \\
l(a)l(1-a) &= 0 && \text{for all } a \in F \text{ with } a \neq 0,1.
\end{aligned}$$

It follows that $\mathrm{K}^{\mathrm{M}}_0(F) = \mathbb{Z}$ and that $F^* \xrightarrow{\sim} \mathrm{K}^{\mathrm{M}}_1(F)$, $a \mapsto l(a)$. The Steinberg relations in $\mathrm{K}_2(F)$ imply that $l(a)l(b) \mapsto \{a,b\}$ induces a homomorphism $\mathrm{K}^{\mathrm{M}}_2(F) \to \mathrm{K}_2(F)$.

Theorem 2 (**Matsumoto**) *The homomorphism*

$$\mathrm{K}^{\mathrm{M}}_2(F) \to \mathrm{K}_2(F), \quad l(a)l(b) \mapsto \{a,b\}$$

is an isomorphism.

Proof: The easy part of the theorem is the surjectivity, i.e. the fact that the Steinberg symbols generate $\mathrm{K}_2(F)$, see [7].

It suffices to construct a left inverse. As in [3] this is done by constructing a (t^2-t)-reciprocity on $F[t]$. For each $\pi \in S$ define

$$\chi_\pi : (F[t]/\pi)^* \to \mathrm{K}^{\mathrm{M}}_2(F), \quad \overline{g} \mapsto N_\pi(l(\overline{g})l(\overline{1-\tfrac{1}{t}})),$$

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where $g \in F[t]$ with $\pi \nmid g$. The transfer N_π is the transfer N_v as defined in [2], the discrete valuation v being the one corresponding to the prime ideal (π). We will show that $(\chi_\pi)_{\pi\in S}$ is a (t^2-t)-reciprocity, i.e.

$$\sum_{\pi\in S} \chi_\pi((1-fa,f)_\pi) = 0$$

for all $a \in \mathfrak{a}_S$ and all $f \in F[t]^*_S$. This will follow from the reciprocity law for the Milnor ring [2]:

$$\sum_v N_v(\partial_v(x)) = 0 \quad (\text{for all } x \in \mathrm{K}^{\mathrm{M}}_*(F)),$$

where the sum is over all discrete valuations on $F(t)$ and ∂_v is the degree -1 map from the Milnor ring of $F(t)$ to the Milnor ring of the residue field of v as defined in [2]. For a and f as above we take $x = l(1-fa)l(1-\frac{1}{t})l(f) \in \mathrm{K}^{\mathrm{M}}_3(F)$ and compute $\partial_v(x)$ for all discrete valuations v on $F(t)$. For $b,c,d \in F(t)^*$ with $v(c) = 0$ it follows directly from the definition of ∂_v in [2] that

$$\partial_v(l(b)l(c)l(d)) = \partial_v(l(b)l(d))l(\overline{c}).$$

This identity will be used in the computations. We distinguish four cases.

(1) $v = v_\pi$ with $\pi \in S$.

$$\partial_v(x) = \partial_v(l(1-fa)l(f))l(\overline{1-\tfrac{1}{t}}) = l((1-fa,f)_\pi)l(\overline{1-\tfrac{1}{t}}).$$

(2) $v = v_t$.
Since $l(t)l(1-\frac{1}{t}) = 0$ and $v(1-fa) = 0$ we have

$$\partial_v(x) = -\partial_v(l(1-fa)l(ft^{-v(f)})l(1-\tfrac{1}{t}) = l(\overline{1-fa})l(*) = l(1)l(*) = 0.$$

(3) $v = v_{1-t}$.
Since $l(1-t)l(1-\frac{1}{t}) = 0$ and $v(1-fa) = 0$ we have

$$\partial_v(x) = -\partial_v(l(1-fa)l(f(1-t)^{-v(f)})l(1-\tfrac{1}{t}) = l(\overline{1-fa})l(*) = l(1)l(*) = 0.$$

(4) $v = v_\infty$.

$$\partial_v(x) = \partial_v(l(ft^{-\deg(f)})l(1-\tfrac{1}{t})l(1-af)) = l(*)l(1)v(1-af) = 0.$$

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It follows that $(\chi_\pi)_\pi$ is a (t^2-t)-reciprocity on $F[t]$:

$$\sum_{\pi\in S}\chi_\pi((1-fa,f)_\pi)=\sum_{\pi\in S}N_\pi(l(1-fa,f)_\pi)l(\overline{1-\tfrac{1}{t}}))$$
$$=\sum_{\pi\in S}N_{v_\pi}(\partial_{v_\pi}(x))=\sum_v N_v(\partial_v(x))=0.$$

The reciprocity induces a homomorphism from $\mathrm{K}_2(F)$ to $\mathrm{K}_2^{\mathrm{M}}(F)$. It remains to show that for $b,c\in F^*$ the image of $\{b,c\}\in\mathrm{K}_2(F)$ in $\mathrm{K}_2^{\mathrm{M}}(F)$ is $l(b)l(c)$. Assume that $c\neq 1$. The element $\{b,c\}\in\mathrm{K}_2(F)$ lifts to $\{b,\gamma\}$ in $K_2(F[t]_S,t)$, where $\gamma=(c-1)t+1$. Under ∂ the image of $\{b,\gamma\}$ in $\bigoplus_{\pi\in S}(F[t]/\pi)^*$ is b in the component with index γ and is trivial in the others. Hence,

$$\chi(\partial(\{b,\gamma\}))=\chi_\gamma(b)=l(b)l(1-(1-c))=l(b)l(c).$$

□

4. Bass' relative reciprocities

In this section a description of relative reciprocity is derived which is essentially the same as the one in [1]. The notations of section 2 will be used. Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_r$ be the prime ideals of R which divide $\mathfrak{a}$. Put $v_{\mathfrak{p}_i}(\mathfrak{a})=k_i$ for $i=1,\ldots,r$.

For a maximal ideal $\mathfrak{p}$ of R we write

$$U_\mathfrak{p}=\mathrm{Ker}((R/\mathfrak{p}^{v_\mathfrak{p}(\mathfrak{a})+1})^*\to(R/\mathfrak{p}^{v_\mathfrak{p}(\mathfrak{a})})^*.$$

For $\mathfrak{p}\in S$ this means that $U_\mathfrak{p}=(R/\mathfrak{p})^*$ and for $\mathfrak{p}_i\mid\mathfrak{a}$ we have

$$U_{\mathfrak{p}_i}=\mathrm{Ker}((R/\mathfrak{p}^{k_i+1})^*\to(R/\mathfrak{p}^{k_i})^*)(\cong R/\mathfrak{p}).$$

For $\mathfrak{p}$ a maximal ideal of R, $g\in 1+\mathfrak{a}_S$ and $f\in Q^*$ we define

$$d_\mathfrak{p}(g,f)=\begin{cases}(g,f)_\mathfrak{p} & \text{if } \mathfrak{p}\in S\\ \overline{g}^{v_\mathfrak{p}(f)} & \text{if } \mathfrak{p}\mid\mathfrak{a}.\end{cases}$$

We put $d(g,f)=(d_\mathfrak{p}(g,f))_\mathfrak{p}\in\bigoplus_\mathfrak{p}U_\mathfrak{p}$, where the direct sum is over all maximal ideals of R. Let B be the subgroup of $\bigoplus_{\mathfrak{p}\in S}(R/\mathfrak{p})^*$ generated by all $d(1-af,f)$ with $a\in\mathfrak{a}_S$ and $f\in R_S\setminus\{0\}$, and B^+ the subgroup of $\bigoplus_\mathfrak{p}U_\mathfrak{p}$ generated by all $d(g,f)$ with $g\in 1+\mathfrak{a}_S$ and $f\in Q^*$.

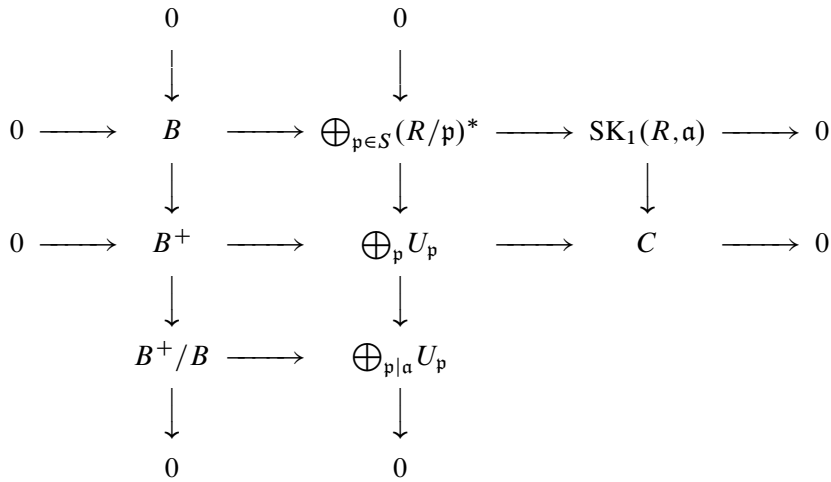
Theorem 3 *Let A be an Abelian group and let $(\chi_\mathfrak{p})_\mathfrak{p}$ be a collection of group homomorphisms $\chi_\mathfrak{p}\colon U_\mathfrak{p}\to A$. Then the following two conditions are equivalent:*

(i) $\sum_\mathfrak{p}\chi_\mathfrak{p}(d_\mathfrak{p}(g,f))=0$ *for all $g\in 1+\mathfrak{a}_S$ and $f\in Q^*$.*

(ii) $(\chi_\mathfrak{p})_{\mathfrak{p}\in S}$ *is an $\mathfrak{a}$-reciprocity on R.*

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Proof: We have a commutative diagram with exact rows and columns:



The theorem is equivalent to $B^+/B \to \bigoplus_{\mathfrak{p}|\mathfrak{a}} U_{\mathfrak{p}}$ being an isomorphism. We will construct an inverse. By the Chinese remainder theorem there exist $\pi_1,\ldots\pi_r \in R$ such that for $i = 1,\ldots,r$ the following is satisfied

$$\begin{cases} v_{\mathfrak{p}_i}(\pi_i) = 1 \\ \pi_i \equiv 1 \pmod{\mathfrak{p}_j} \quad \text{for all } j \neq i. \end{cases}$$

For each j we have to construct a homomorphism $\rho_j : U_{\mathfrak{p}_j} \to B^+/B$. Let $\overline{g} \in U_{\mathfrak{p}_j}$ where $g \in R$. By the Chinese remainder theorem there is a $g' \in R$ with $g' \equiv g \pmod{\mathfrak{p}_j^{k_j+1}}$ and $g' \equiv 1 \pmod{\mathfrak{a}}$. Put

$$\rho_j(\overline{g}) = d(g',\pi_j) + B.$$

This is well-defined: if also $g'' \in R$ with $g'' \equiv g \pmod{\mathfrak{p}_j^{k_j+1}}$ and $g'' \equiv 1 \pmod{\mathfrak{a}}$, then $g'(g'')^{-1} \in 1 + \mathfrak{a}_S$ and $v_{\mathfrak{p}_j}(g'(g'')^{-1}) = 0$, from which it follows that $d(g',\pi_j) - d(g'',\pi_j) \in B$. Since $d(g,f)$ is linear in the first variable, the map ρ_j is a homomorphism. Under $B^+/B \to \bigoplus_{\mathfrak{p}|\mathfrak{a}} U_{\mathfrak{p}}$ the element $d(g',\pi_j) + B$ maps to $((\overline{g'})^{v_{\mathfrak{p}_i}(\pi_j)})_{\mathfrak{p}_i}$. For $i \neq j$ the ith component is trivial and for $i = j$ it is $\overline{g}$.

It remains to check that we constructed a left inverse. Since $d(g,f)$ is linear in the second variable and $d(g,f) \in B$ for $f \in R_S^*$, it suffices to check this on elements $d(g,\pi_j) \in B^+$ with $g \in 1 + \mathfrak{a}_S$. Such a g is a quotient of two elements in $1 + \mathfrak{a}$, so we can assume that $g \in 1 + \mathfrak{a}$. For such $d(g,\pi_j) + B$ it remains to show that $d(g,\pi_j) + B = d(g',\pi_j) + B$, where $g' \in R$ such that $g' \equiv 1 \pmod{\mathfrak{a}}$ and $g' \equiv g \pmod{\mathfrak{p}_j^{k_j+1}}$. This follows from $\frac{1-g(g')^{-1}}{\pi_j} \in R_S$. □

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Finally we show that the notion of relative reciprocity on a Dedekind domain is equivalent to the one defined in [1].

Theorem 4 *Let A be an Abelian group and let $(\chi_\mathfrak{p})_\mathfrak{p}$ be a collection of group homomorphisms $\chi_\mathfrak{p}: U_\mathfrak{p} \to A$. Then the following two conditions are equivalent:*

(i) $\sum_{\mathfrak{p}|f} v_\mathfrak{p}(f)\chi_\mathfrak{p}(\overline{g}) = \sum_{\mathfrak{p}|g} v_\mathfrak{p}(g)\chi_\mathfrak{p}(\overline{f})$ *for all* $f,g \in R$ *with* $g \equiv 1 \pmod{\mathfrak{a}}$, $f \neq 0$ *and* $(f,g) = R$.

(ii) $(\chi_\mathfrak{p})_{\mathfrak{p}\in S}$ *is an* $\mathfrak{a}$*-reciprocity on* R.

Proof: Let D be the subgroup of B^+ generated by all $d(b,c)$ with $b,c \in R$, $b \equiv 1 \pmod{\mathfrak{a}}$, $c \neq 0$ and $(b,c) = R$. We will proof that $D = B^+$. Then the theorem follows from Theorem 3.

Let $g \in 1 + \mathfrak{a}_S$ and $f \in Q^*$. We will prove that $d(g,f) \in D$. This will be done by induction on the number n of maximal ideals $\mathfrak{p}$ with $v_\mathfrak{p}(g)v_\mathfrak{p}(f) \neq 0$.

(1) Let $n = 0$. In the group of fractional ideals of R write Rg and Rf as quotients of ideals of R which are relatively prime: $Rg = \frac{\mathfrak{b}_1}{\mathfrak{c}_1}$ and $Rf = \frac{\mathfrak{b}_2}{\mathfrak{c}_2}$. Since $g \in 1 + \mathfrak{a}_S$ the ideal $\mathfrak{c}_1$ is relatively prime to $\mathfrak{a}$. Choose an ideal $\mathfrak{c}_1'$ in the inverse ideal class of $\mathfrak{c}_1$ and relatively prime to $\mathfrak{a}\mathfrak{b}_2\mathfrak{c}_2$. Choose $c_1 \in R$ with $\mathfrak{c}_1\mathfrak{c}_1' = Rc_1$ and put $b_1 = c_1 g$. Choose $c_1' \in R$ such that $c_1c_1' \equiv 1 \pmod{\mathfrak{a}}$ and c_1' relatively prime to $\mathfrak{b}_2\mathfrak{c}_2$. Then $g = \frac{b_1c_1'}{c_1c_1'}$ and $b_1c_1', c_1c_1' \equiv 1 \pmod{\mathfrak{a}}$. Write $f = \frac{b_2}{c_2}$ with b_2c_2 relatively prime to b_1c_1' and to c_1c_1'. Clearly $d(-,-)$ is bimultiplicative, so we have

$$d(g,f) = d(b_1c_1',b_2) + d(c_1c_1',c_2) - d(bc_1',c_2) - d(c_1c_1',b_2) \in D.$$

(2) Let $n = 1$ and let $\mathfrak{p}_1$ be the unique prime with $v_{\mathfrak{p}_1}(g)v_{\mathfrak{p}_1}(f) \neq 0$. Choose $\mathfrak{r}$ in the ideal class of $\mathfrak{p}_1$ relatively prime to all maximal ideals $\mathfrak{q}$ with $v_\mathfrak{q}(f) \neq 0$ or $v_\mathfrak{q}(\mathfrak{a}) \neq 0$. Choose $h \in Q^*$ such that $Rh = \frac{\mathfrak{r}}{\mathfrak{p}_1}$. Then $h \in R_S^*$. Choose an $h' \in R_S^*$ which is an inverse modulo $\mathfrak{a}_S^2$. Then for $k = hh'$ we have: $k \in 1 + \mathfrak{a}_S^2$, $v_\mathfrak{q}(k) \geq 0$ and $v_\mathfrak{q}(k)v_\mathfrak{q}(f) = 0$ for all maximal ideals $\mathfrak{q} \neq \mathfrak{p}_1$. Put $i = v_{\mathfrak{p}_1}(g)$ and $j = v_{\mathfrak{p}_1}(f)$. Then $d(k^i g, f) \in D$ by (1) and so $d(g,f) \equiv -id(k,f) \pmod{D}$. For all maximal ideals $\mathfrak{q}$ we have $v_\mathfrak{q}(k)v_\mathfrak{q}(f(1-k)^j) = 0$. By (1) we have $d(k, f(1-k)^j) \in D$. Therefore $d(k,f) \equiv -jd(k,1-k) \pmod{D}$. Since $d_\mathfrak{q}(k,1-k) = 1$ for all $\mathfrak{q} \in S$ it remains to show that $d_\mathfrak{q}(k,1-k) = 1$ for all $\mathfrak{q} \mid \mathfrak{a}$. From $1-k \in \mathfrak{a}_S^2$ it follows that for such $\mathfrak{q}$ we have $v_\mathfrak{q}(1-k) \geq 2v_\mathfrak{q}(\mathfrak{a}) \geq v_\mathfrak{q}(\mathfrak{a}) + 1$ and this implies that $d_\mathfrak{q}(k,1-k) = 1$.

(3) Assume that $n > 1$. Let $\mathfrak{p}_1$ be a maximal ideal with $v_{\mathfrak{p}_1}(g)v_{\mathfrak{p}_1}(f) \neq 0$. Choose $h \in Q^*$ with $v_{\mathfrak{p}_1}(h) = -v_{\mathfrak{p}_1}(f)$ and $v_\mathfrak{q}(h)v_\mathfrak{q}(g) = 0$. Then by

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induction hypothesis and (2) it follows that $d(g,f) = d(g,fh) - d(g,h) \in D$. □

Remark In the definition of an $\mathfrak{a}$-reciprocity given in [1] there is the extra condition

$$v_{\mathfrak{q}}(1-g)\chi_{\mathfrak{q}}(\overline{g}) = 0 \qquad \text{for } g \in 1+\mathfrak{a} \text{ and } \mathfrak{q} \mid \mathfrak{a}.$$

This condition is however a consequence of the other condition: choose $g' \in R$ such that

$$\begin{cases} g' \equiv g \pmod{\mathfrak{q}^{v_{\mathfrak{q}}(\mathfrak{a})+1}} \\ g' \equiv 1 \pmod{\mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})+1}} \quad \text{for all } \mathfrak{p} \mid \mathfrak{a} \text{ with } \mathfrak{p} \neq \mathfrak{q}. \end{cases}$$

Assume $\overline{g} \neq \overline{1}$. Then $v_{\mathfrak{q}}(1-g) = v_{\mathfrak{p}}(1-g') = v_{\mathfrak{p}}(\mathfrak{a})$ and we have

$$0 = \sum_{\mathfrak{p}} \chi_{\mathfrak{p}}(d_{\mathfrak{p}}(g',1-g')) = \sum_{\mathfrak{p}\mid\mathfrak{a}} v_{\mathfrak{p}}(1-g')\chi_{\mathfrak{p}}(\overline{g'}) = v_{\mathfrak{q}}(1-g)\chi_{\mathfrak{q}}(\overline{g}).$$

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