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Two-dimensional elastic inclusion problems

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Abstract. A system of equations is derived for determining the elastic fields in an inclusion and its surrounding finite matrix when the inclusion suffers a physical change and, if not constrained by the matrix, would undergo a deformation E^0_{ij}. A method for obtaining the exact solution of these equations, when the matrix and inclusion have the same elastic constants, is described and the particular problem of the square inclusion in an infinite matrix solved.

1. In this paper an inclusion will be understood to be a region, within a finite or infinite elastic medium, which suffers a physical change. In the absence of the elastic medium surrounding the inclusion, generally referred to as the matrix, the physical change would cause the inclusion to undergo a deformation E^0_{ij}. The matrix and inclusions may have the same or different elastic constants, but it is assumed that the particular case of the cavity, where the inclusion has zero elastic constants, is excluded.

Eshelby ((2)) determined the elastic field of an ellipsoidal inclusion in an infinite matrix by what has been referred to as the Point Force method. In (3) he derived the elastic field of the infinite matrix surrounding the ellipsoid. More recently, Jaswon and Bhargava ((4)) have applied Eshelby's Point Force method to specifically two-dimensional problems. As Jaswon and Bhargava point out, Eshelby's three-dimensional method of solution involves 'integrals of a formidable nature' and even for the two-dimensional case the integrals still prove difficult. By restricting their attention to only two-dimensional problems they have been able to overcome this difficulty, to some extent, by making some use of the complex variable methods of solution.

In general, the Point Force method can only be applied to problems where the matrix and inclusion have the same elastic constants. However, if the inclusion has a constant elastic field, an example being the elliptic or ellipsoidal inclusion, the inhomogeneous case can be solved.

More recently Bhargava ((1)) has applied the theorem of minimum energy in the classical theory of elasticity to two-dimensional inclusion problems.

The solution of two-dimensional problems by a combination of the Point Force and complex variable methods, as described by Jaswon and Bhargava, is restricted by the following three main difficulties.

(i) The Cauchy integrals are awkward to evaluate, particularly those for $\psi(z)$, for all but the simplest cases.

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(ii) The need to express the equation of the contour of the inclusion in the form $\bar{z} = f(z)$.

(iii) The inability of the method to deal with the problem of an inclusion, with different elastic constants from those of the matrix, except for the particular case of the inclusion with a constant elastic field.

The second difficulty can be overcome by conformally mapping the inclusion onto the unit circle, provided such a mapping is available in closed form. However, not many shapes fall into this class. While approximate mappings in closed form are available for almost any shape, it will be found that the computation of the Cauchy integrals mentioned in (i) is a major obstacle to their use.

2. In (5), section 58·1⁰, Muskhelishvili determines the stresses caused by inserting into a circular hole of radius R, in an infinite plate, a circular disc with an initially slightly larger radius $R+\epsilon$. In section 109 he discusses a generalization of this problem solved by D. I. Sherman as a Hilbert problem. In this generalization a number of discs are inserted into holes, of slightly smaller dimensions, in a finite plate. However, the plate and the discs have the same elastic constants. These two references suggest an alternate approach to two-dimensional elastic inclusion problems.

It will be assumed that initially the matrix and inclusion have a common boundary L which is either a simple smooth contour or a closed union of a finite number of simple finite smooth arcs. Further, it will be assumed that the matrix and inclusion always remain in contact as the inclusion tends to expand.

For the purpose of describing the basis of the method consider first a single inclusion, boundary L, tending to undergo a prescribed deformation in an infinite matrix. The process can be divided into a number of stages as follows. Cut the inclusion from the matrix and let it undergo the known deformation, free from constraint, resulting in the displacement $(u_0^+ + iv_0^+)$, say. In the sequel the + and − superscripts will refer to the inclusion and matrix respectively and the subscript 0 to the initial deformation. The final equilibrium configuration of the inclusion can now be reached by applying the unknown stress R say, the reaction of the matrix to the initial deformation of the inclusion, to the inclusion boundary. On applying the same unknown stress R to the edge of the hole (due to the removal of the inclusion) in the matrix, the matrix will also take up its final equilibrium configuration. The process is completed by replacing the inclusion back into the matrix and rewelding along the common boundary L. During these imaginary steps the displacement $(u^+ + iv^+)$ of the inclusion boundary L, from the completion of the known initial deformation, is equal to $-(u_0^+ + iv_0^+)$ to reduce it to its original shape plus $(u^- + iv^-)$, the displacement of the matrix boundary L, to reach its final equilibrium configuration. Hence

$$(u^+ + iv^+) = -(u_0^+ + iv_0^+) + (u^- + iv^-) \quad \text{on} \quad L. \tag{2·1}$$

It will be noted that no assumption regarding the matrix reaction R has been made.

Consider now a general finite region S_0 containing a finite number of inclusions. Let the boundary of S_0 be denoted by L_0 and the union of the contours of the inclusions by L. Put

$$-(u_0^+ + iv_0^+) = g(z) = g_1(z) + ig_2(z) \quad \text{on} \quad L \tag{2·2}$$

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and using the basic result (the notation of (5) is used throughout this paper) relating the displacements u, v to the analytic stress functions ϕ, ψ and $\Phi(z) = \phi'(z)$

$$2\mu(u+iv) = \kappa\phi(z) - z\overline{\Phi(z)} - \overline{\psi(z)} \tag{2·3}$$

and the general boundary condition of the first fundamental problem

$$\phi(z) + z\overline{\Phi(z)} + \overline{\psi(z)} = f(z), \tag{2·4}$$

where

$$f(z) = i\int^z (X_n + iY_n)\,ds,$$

X_n and Y_n being the components of boundary traction, the equations to be solved are derived as follows.

Substituting (2·3) into (2·1), indicating the different elastic constants of inclusion and matrix by subscripts 1 and 2, respectively, gives on L

$$\mu_2\{\kappa_1\phi^+(z) - z\overline{\Phi^+(z)} - \overline{\psi^+(z)}\} = 2\mu_1\mu_2 g(z) + \mu_1\{\kappa_2\phi^-(z) - z\overline{\Phi^-(z)} - \overline{\psi^-(z)}\}. \tag{2·5}$$

As the stresses are in equilibrium across L (2·4) gives

$$\phi^+(z) + z\overline{\phi^+(z)} + \overline{\psi^+(z)} = \phi^-(z) + z\overline{\Phi^-(z)} + \overline{\psi^-(z)} \quad \text{on} \quad L. \tag{2·6}$$

The boundary condition on L_0 is on the other hand

$$\phi(z) + z\overline{\Phi(z)} + \overline{\psi(z)} = f(z), \tag{2·7}$$

where

$$f(z) = i\int^z (X_n + iY_n)\,ds \quad \text{on} \quad L_0.$$

Since these equations were derived it has been noted that Sherman in (6) states the equations (2·5), (2·6) and (2·7) and describes a general method for solving them.

For the particular case of $\mu_1 = \mu_2$ and $\kappa_1 = \kappa_2$ (i.e. the matrix and inclusion of the same material) adding and subtracting the equations (2·5) and (2·6) gives on L

$$\phi^+(z) - \phi^-(z) = \frac{2\mu}{\kappa+1}\,g(z) \tag{2·8}$$

and, after use of (2·7),

$$\psi^+(z) - \psi^-(z) = -\frac{2\mu}{\kappa+1}\{\bar{z}g'(z) + \overline{g(z)}\} = \frac{2\mu}{\kappa+1}\,h(z), \tag{2·9}$$

where

$$h(z) = -\{\bar{z}g'(z) + \overline{g(z)}\} \quad \text{and} \quad g'(z) = \frac{dg(z)}{dz}.$$

By (5), section 108, the solutions of (2·8) and (2·9) can be written down immediately in the form

$$\left.\begin{aligned} \phi(z) &= \phi_1(z) + \frac{\mu}{\pi i(\kappa+1)}\int_L \frac{g(t)\,dt}{t-z}, \\ \psi(z) &= \psi_1(z) + \frac{\mu}{\pi i(\kappa+1)}\int_L \frac{h(t)\,dt}{t-z}, \end{aligned}\right\} \tag{2·10}$$

where $\phi_1(z)$ and $\psi_1(z)$ are functions holomorphic in S_0. Writing

$$\left.\begin{aligned} \phi_*(z) &= \frac{\mu}{\pi i(\kappa+1)}\int_L \frac{g(t)\,dt}{t-z}, \\ \psi_*(z) &= \frac{\mu}{\pi i(\kappa+1)}\int_L \frac{h(t)\,dt}{t-z} \end{aligned}\right\} \tag{2·11}$$

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(2·10) becomes

$$\left.\begin{aligned}\phi(z) &= \phi_1(z)+\phi_*(z),\\ \psi(z) &= \psi_1(z)+\psi_*(z).\end{aligned}\right\} \tag{2·12}$$

Substituting (2·12) into (2·7) gives

$$\phi_1(z)+z\overline{\Phi_1(z)}+\overline{\psi_1(z)} = f_1(z) \quad \text{on} \quad L_0, \tag{2·13}$$

where $$f_1(z) = f(z)-\phi_*(z)-z\overline{\Phi_*(z)}-\overline{\psi_*(z)}.$$

3. If an inclusion tends to undergo a homogeneous deformation E^0_{11}, E^0_{22}, without any rigid body displacement, $u_0^+ = E^0_{11}x$ and $v_0^+ = E^0_{22}y$. Equation (2·2) gives

$$g(t) = -\tfrac{1}{2}(E^0_{11}+E^0_{22})\,t-\tfrac{1}{2}(E^0_{11}-E^0_{22})\,\bar{t}, \tag{3·1}$$

where t is a general point on the contour L. Substituting equation (3·1) into (2·9) gives

$$h(t) = \tfrac{1}{2}(E^0_{11}-E^0_{22})\,t+(E^0_{11}+E^0_{22})\,\bar{t}+\tfrac{1}{2}(E^0_{11}-E^0_{22})\,\bar{t}\,\frac{d\bar{t}}{dt}. \tag{3·2}$$

With these expressions the integrals in equations (2·10) can be evaluated by integration around the contour and the use of Cauchy's formula.

4. Consider the application of the method, described above, to the square inclusion, with boundary L given by $(x^2-1)(y^2-1) = 0$, enclosed in an infinite matrix and tending to undergo a homogeneous deformation E^0_{11}, E^0_{22}.

Using the results

$$\int_L \frac{\bar{t}\,dt}{t-z} = (2-z)\ln\frac{1+i-z}{1-i-z}-(2i-z)\ln\frac{-1+i-z}{1+i-z}-(z+2)\ln\frac{-1-i-z}{-1+i-z}$$
$$+(z+2i)\ln\frac{1-i-z}{-1-i-z},$$

$$\int_L \frac{\{\bar{t}(d\bar{t}/dt)\}\,dt}{t-z} = -(2-z)\ln\frac{1+i-z}{1-i-z}-(2i-z)\ln\frac{-1+i-z}{1+i-z}+(z+2)\ln\frac{-1-i-z}{-1+i-z}$$
$$+(z+2i)\ln\frac{1-i-z}{-1-i-z},$$

substitution of the expressions for $g(t)$ and $h(t)$ from section 3, into the equations (2·11) gives:

(a) for the inclusion

$$\phi_*^+(z) = -\frac{\mu(E^0_{11}+E^0_{22})}{\kappa+1}z+\frac{\mu(E^0_{11}-E^0_{22})}{\pi i(\kappa+1)}\left\{z\ln\frac{z^2-2i}{z^2+2i}-\ln\frac{(z-i)^2-1}{(z+i)^2-1}-i\ln\frac{(z-1)^2+1}{(z+1)^2+1}\right\}, \tag{4·1}$$

$$\psi_*^+(z) = -\frac{2\mu(E^0_{11}+E^0_{22})}{\pi i(\kappa+1)}\left\{z\ln\frac{z^2-2i}{z^2+2i}-\ln\frac{(z-i)^2-1}{(z+i)^2-1}-i\ln\frac{(z-1)^2+1}{(z+1)^2+1}\right\}+\frac{2\mu(E^0_{11}-E^0_{22})}{\kappa+1}$$
$$\times z-\frac{\mu(E^0_{11}-E^0_{22})}{\pi i(\kappa+1)}\left\{\ln\frac{(z-i)^2-1}{(z+i)^2-1}-i\ln\frac{(z-1)^2+1}{(z+1)^2+1}\right\}; \tag{4·2}$$

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(*b*) for the matrix

$$\phi_*^-(z) = \frac{\mu(E_{11}^0 - E_{22}^0)}{\pi i(\kappa+1)}\left\{z\ln\frac{z^2-2i}{z^2+2i} - \ln\frac{(z-i)^2-1}{(z+i)^2-1} - i\ln\frac{(z-1)^2+1}{(z+1)^2+1}\right\}, \tag{4·3}$$

$$\psi_*^-(z) = \frac{-2\mu(E_{11}^0 - E_{22}^0)}{\pi i(\kappa+1)}\left\{z\ln\frac{z^2-2i}{z^2+2i} - \ln\frac{(z-i)^2-1}{(z+i)^2-1} - i\ln\frac{(z-1)^2+1}{(z+1)^2+1}\right\} - \frac{\mu(E_{11}^0 - E_{22}^0)}{\pi i(\kappa+1)}$$
$$\times\left\{\ln\frac{(z-i)^2-1}{(z+i)^2-1} - \ln\frac{(z-1)^2+1}{(z+1)^2+1}\right\}, \tag{4·4}$$

with the following restrictions of the imaginary parts of the logarithms:

$$\theta_1 = \Im\left(\ln\frac{z^2-2i}{z^2+2i}\right) = \arctan\frac{4(y^2-x^2)}{(x^2+y^2)^2-4},$$

$$\text{for } z \text{ inside } L \begin{cases} 0 < \theta_1 < \frac{1}{2}\pi & (|y| < |x|), \\ -\frac{1}{2}\pi < \theta_1 < 0 & (|y| > |x|), \end{cases}$$

$$\text{for } z \text{ outside } L \begin{cases} -\pi < \theta_1 < 0 & (|y| < |x|), \\ 0 < \theta_1 < \pi & (|y| > |x|), \end{cases}$$

$$\theta_2 = \Im\left(\ln\frac{(z-i)^2-1}{(z+i)^2-1}\right) = \operatorname{artan}\frac{4x\{2-(x^2+y^2)\}}{4-8x^2+(x^2+y^2)^2},$$

$$\text{for } z \text{ inside } L \begin{cases} 0 \leqslant \theta_2 < \pi & (x \geqslant 0), \\ -\pi < \theta_2 \leqslant 0 & (x \leqslant 0), \end{cases}$$

$$\text{for } z \text{ outside } L \begin{cases} x > 1, & \begin{cases} (x-1)^2+y^2 > 1 & (-\pi < \theta_2 < 0), \\ (x-1)^2+y^2 \leqslant 1 & (-\frac{3}{2}\pi < \theta_2 < -\frac{1}{2}\pi), \end{cases} \\ -1 \leqslant x \leqslant 1 & (-\frac{1}{2}\pi < \theta_2 < \frac{1}{2}\pi), \\ x < -1, & \begin{cases} (x+1)^2+y^2 > 1 & (0 < \theta_2 < \pi), \\ (x+1)^2+y^2 \leqslant 1 & (\frac{1}{2}\pi < \theta_2 < \frac{3}{2}\pi), \end{cases} \end{cases}$$

$$\theta_3 = \Im\left(\ln\frac{(z-1)^2+1}{(z+1)^2+1}\right) = \arctan\frac{4y\{(x^2+y^2)-2\}}{4-8y^2+(x^2+y^2)^2},$$

$$\text{for } z \text{ inside } L \begin{cases} 0 \leqslant \theta_3 < \pi & (y \leqslant 0), \\ -\pi < \theta_3 \leqslant 0 & (y \geqslant 0), \end{cases}$$

$$\text{for } z \text{ outside } L \begin{cases} y > 1, & \begin{cases} (y-1)^2+x^2 > 1 & (0 < \theta_3 < \pi), \\ (y-1)^2+x^2 \leqslant 1 & (\frac{1}{2}\pi < \theta_3 \leqslant \frac{3}{2}\pi), \end{cases} \\ -1 \leqslant y \leqslant 1 & (-\frac{1}{2}\pi \leqslant \theta_3 \leqslant \frac{1}{2}\pi), \\ y < -1, & \begin{cases} (y+1)^2+x^2 > 1 & (-\pi < \theta_3 < 0), \\ (y+1)^2+x^2 \leqslant 1 & (-\frac{3}{2}\pi < \theta_3 < -\frac{1}{2}\pi). \end{cases} \end{cases}$$

From (5), the stresses can be obtained from the equations

$$X_x + Y_y = 4\Re\phi(z),$$

$$Y_y - X_x + 2iX_y = 2\{\bar{z}\Phi'(z) + \psi(z)\}.$$

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Hence, by substituting from equations (4·1) to (4·4), the sum of the principal stresses within the inclusion is

$$X_x^+ + Y_y^+ = 4\Re\left\{-\frac{\mu(E_{10}^0+E_{22}^0)}{\kappa+1}+\frac{\mu(E_{10}^0-E_{22}^0)}{\pi i(\kappa+1)}\ln\frac{z^2-2i}{z^2+2i}\right\} \tag{4·5}$$

and inside the matrix

$$X_x^- + Y_y^- = 4\Re\left\{\frac{\mu(E_{11}^0-E_{22}^0)}{\pi i(\kappa+1)}\ln\frac{z^2-2i}{z^2+2i}\right\}. \tag{4·6}$$

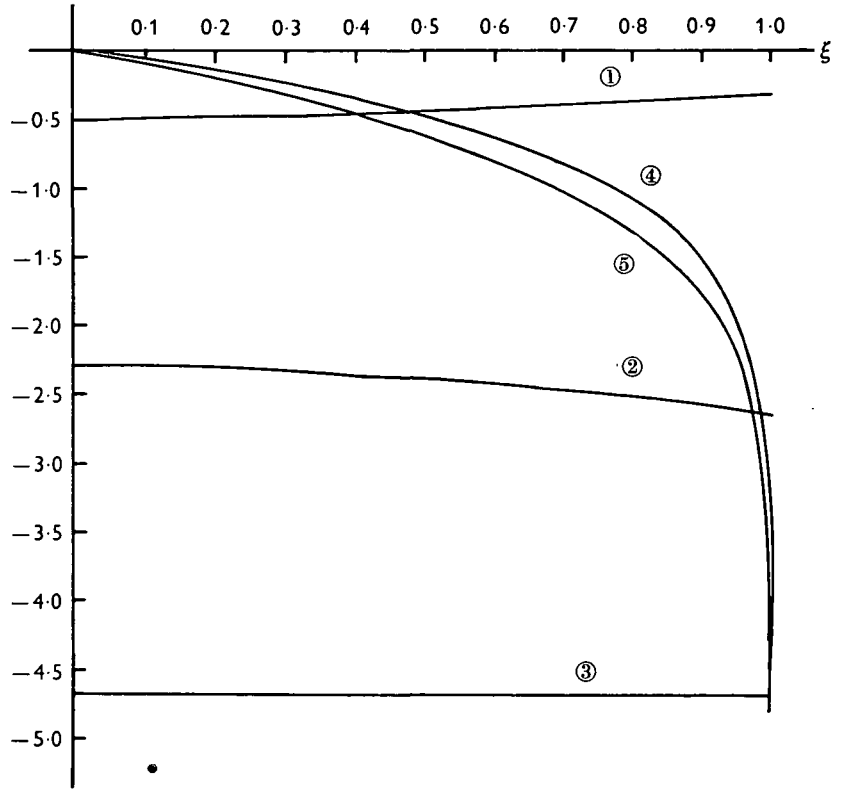


Fig. 1

The analysis can be checked by verifying that the direct stress along the boundary of the square is continuous. For example, as

$$X_x - iX_y = 2\Re\Phi(z) - \bar{z}\Phi'(z) - \Psi(z),$$

$$X_x^+ = \Re\left[\frac{2\mu(E_{11}^0-E_{22}^0)}{\kappa+1}\left\{\frac{1}{\pi i}\ln\frac{z^2-2i}{z^2+2i}-1\right\}+\frac{2\mu(E_{11}^0+E_{22}^0)}{\kappa+1}\left\{\frac{1}{\pi i}\ln\frac{z^2-2i}{z^2+2i}-1\right\}\right.$$
$$\left.+\frac{8\mu(E_{11}^0-E_{22}^0)}{\pi(\kappa+1)}\left\{\frac{2-z\bar{z}}{z^4+4}\right\}\right],$$

$$X_x^- = \Re\left[\frac{2\mu(E_{11}^0-E_{22}^0)}{\kappa+1}\frac{1}{\pi i}\ln\frac{z^2-2i}{z^2+2i}+\frac{2\mu(E_{11}^0+E_{22}^0)}{\kappa+1}\frac{1}{\pi i}\ln\frac{z^2-2i}{z^2+2i}+\frac{8\mu(E_{11}^0-E_{22}^0)}{\pi(\kappa+1)}\left\{\frac{2-z\bar{z}}{z^4+4}\right\}\right]$$

showing $X_x^+ = X_x^-$ on $x = \pm 1$, $-1 \leqslant y \leqslant 1$.

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Let the boundary stresses in the square inclusion be expressed in the form

$$X_x = \alpha_1 \frac{\mu}{\kappa+1} E^0_{11} + \alpha_2 \frac{\mu}{\kappa+1} E^0_{22},$$

$$Y_y = \beta_1 \frac{\mu}{\kappa+1} E^0_{11} + \beta_2 \frac{\mu}{\kappa+1} E^0_{22},$$

$$X_y = \gamma_1 \frac{\mu}{\kappa+1} E^0_{11} + \gamma_2 \frac{\mu}{\kappa+1} E^0_{22},$$

where the α_i, β_i, $\gamma_i (i = 1, 2)$ are real valued functions of position. Their values were computed and their graphs are shown in the diagram above where for $x = 1, y = \xi, \alpha_1$, is given by curve 2, α_2 by 1, β_1 by 1, β_2 by 3, γ_1 by 4 and γ_2 by 5. For $x = \xi, y = 1, \alpha_1$ is given by curve 3, α_2 by 1, β_1 by 1, β_2 by 2, γ_1 by 5 and γ_2 by 4.

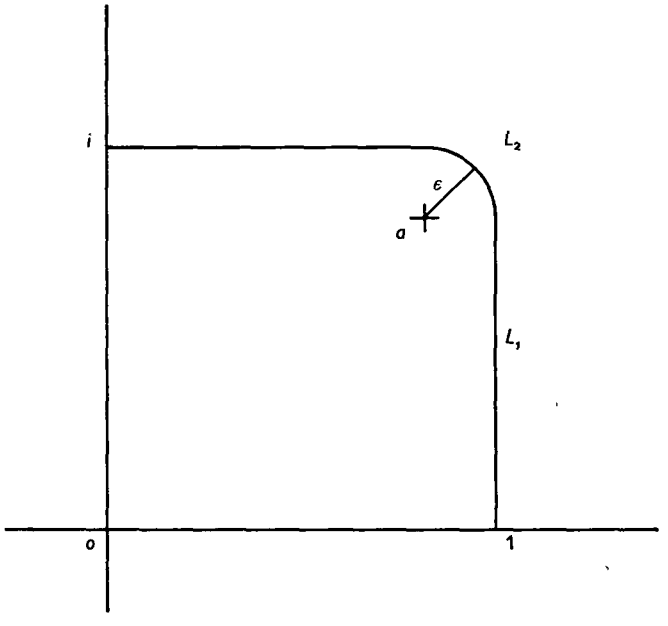


Fig. 2

5. It will be noted that the shear stresses at the corners are infinite. This is, of course, due to the discontinuity in the direct stress. In order to show that our solution is valid we propose to show that it is the limiting case of the solution for a square inclusion with rounded corners; for this solution no discontinuities arise and as a result all stresses remain finite.

We now consider the square inclusion to have its corners rounded to a circular arc of radius ϵ. Applying the analysis of sections 2 and 3 we obtain

$$\tfrac{1}{2}[Y_y - X_x + 2iX_y] - 2\pi i A = A \int_L \frac{(\bar{t}-\bar{z})}{(t-z)^2} \frac{d\bar{t}}{dt} dt + 2B \int_L \frac{\bar{t}}{(t-z)^2} dt, \qquad (5{\cdot}1)$$

where

$$A = \tfrac{1}{2}\mu(E^0_{11} - E^0_{22})/\pi i(\kappa+1) \quad \text{and} \quad B = \tfrac{1}{2}\mu(E^0_{11} + E^0_{22})/\pi i(\kappa+1).$$

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Since X_x and Y_y remain bounded for $\epsilon \geqslant 0$ we shall obtain our aim by establishing continuity of the right-hand side of (5·1). It is clearly sufficient for us to show that the functions defined on the right-hand side of (5·1) are bounded and continuous inside and on L and that as $\epsilon \to 0$ the right-hand side tends to the solution for the square.

It is clear that the integrals in (5·1) define holomorphic functions inside L so that we merely have to consider the situation on L. The symmetry of the region also permits us to concentrate on the region $0 \leqslant \arg z \leqslant \frac{1}{4}\pi$. Let the centre of the bounding circular arc be $a = 1-\epsilon+i\beta$ and let L_1 denote the straight right-hand boundary and L_2 the circular arc. We see immediately that singularities on the boundary in the region considered can only arise from the integrals taken along L_1 and L_2 which we now proceed to evaluate. We start with

$$\int_{L_1+L_2} \frac{\bar{t}-\bar{z}}{(t-z)^2}\frac{d\bar{t}}{dt}\,dt.$$

An easy calculation shows that $d\bar{t}/dt = -1$ along L_1 and $d\bar{t}/dt = -e^{-2i\theta}$ along L_2. Writing $\gamma = a-z$ we obtain

$$\int_{L_1} \frac{\bar{t}-\bar{z}}{(t-z)^2}\frac{d\bar{t}}{dt}\,dt = -i\int_{-2\beta}^{0} \frac{\bar{\gamma}+\epsilon-iy}{(\gamma+\epsilon+iy)^2}\,dy$$

$$= \frac{\bar{\gamma}+\epsilon}{\gamma+\epsilon} - \frac{\bar{\gamma}+\epsilon+2i\beta}{\gamma+\epsilon-2i\beta} + \ln(\gamma+\epsilon) - \ln(\gamma+\epsilon-2i\beta)$$

and

$$\int_{L_2} = -i\epsilon\int_0^{\frac{1}{2}\pi} \frac{\bar{\gamma}+\epsilon e^{-i\theta}}{(\gamma+\epsilon e^{i\theta})^2} e^{-i\theta}\,d\theta = -\frac{\bar{\gamma}+\epsilon}{\gamma+\epsilon} - \frac{\bar{\gamma}i+\epsilon}{i(\gamma+i\epsilon)} - \frac{\epsilon}{\gamma}(i-1) - \frac{3\epsilon^2-2\bar{\gamma}\gamma}{\gamma^2}$$

$$+(i+1)\frac{\epsilon}{\gamma}(3\epsilon^2-2\bar{\gamma}\gamma) + \frac{\epsilon^2}{\gamma^4}(3\epsilon^2-2\bar{\gamma}\gamma)\ln(\epsilon-i\gamma) - \frac{\epsilon^2}{\gamma^4}(3\epsilon^2-2\bar{\gamma}\gamma)\ln(\gamma+\epsilon).$$

When $\epsilon \to 0$ $\gamma \to 1+i-z$ and the above expression is seen to tend to 0 so that the solution given earlier for the square is the limiting case of the solution being discussed here, provided this condition is also satisfied by the second integral on the right-hand side of (5·1). Further on combining the two integrals

$$\int_{L_1+L_2} \frac{\bar{t}-\bar{z}}{t-z}\frac{d\bar{t}}{dt} = \frac{1}{\gamma^4}\ln(\gamma+\epsilon)\{\gamma^4-3\epsilon^4+2\epsilon^2\bar{\gamma}\gamma\} + \frac{\bar{\gamma}+\epsilon+i\beta}{\gamma+\epsilon-i\beta} - \ln(\gamma+\epsilon-i\beta) - \frac{\bar{\gamma}-i\epsilon}{\gamma+i\epsilon}$$

$$+(1-i)\frac{\epsilon}{\gamma} - \frac{3\epsilon^2-2\bar{\gamma}\gamma}{\gamma^2} + (i+1)\frac{\epsilon}{\gamma}(3\epsilon^2-2\bar{\gamma}\gamma) + \frac{\epsilon^2}{\gamma^4}(3\epsilon^2-2\bar{\gamma}\gamma)\ln(\epsilon-i\gamma).$$

When

$$\gamma = -\epsilon+\zeta, \quad \gamma\bar{\gamma} = \epsilon^2-\epsilon(\zeta+\bar{\zeta})+\zeta\bar{\zeta}, \quad \gamma^4 = \epsilon^4-4\epsilon^3\zeta+O|\zeta|^2,$$

hence the first term is

$$\frac{1}{\gamma^4}\ln\zeta\{-4\epsilon^3\zeta-2\epsilon^3(\zeta+\bar{\zeta})+O|\zeta|^2\}$$

which is continuous and tends to 0 as $\zeta \to 0$. All other terms are clearly continuous as $(\gamma+\epsilon)$ tends to 0.

We next consider

$$\int_{L_1+L_2} \frac{\bar{z}\,dt}{(t-z)^2}$$

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and obtain

$$\int_{L_1} \frac{\bar{t}\,dt}{(t-z)^2} = -\frac{\bar{a}+\epsilon}{a+\epsilon-z} + \frac{\bar{a}+\epsilon+2i\beta}{a+\epsilon-2i\beta-z} - \ln\frac{a+\epsilon-z}{a+\epsilon-z-2i\beta}$$

and

$$\int_{L_2} \frac{t^2\,dt}{(t-z)^2} = +\frac{\bar{a}-\epsilon}{a+\epsilon-z} + \frac{-\bar{a}+i\epsilon}{a-z+i\epsilon} + \frac{\epsilon^2}{(z-a)^2}\ln\frac{a-z+\epsilon}{a-z+i\epsilon} + \frac{i\pi\epsilon^2}{2(z-a)^2} - \epsilon\frac{1+i}{a-z}.$$

Again it can be seen by inspection that as $\epsilon \to 0$ the second integral will vanish which establishes our claim that the solution for the square is the limiting case of the solution obtained here as $\epsilon \to 0$.

Adding the integrals, we obtain

$$\int_{L_1+L_2} \frac{\bar{t}}{(t-2)^2}\,dt = \frac{(\epsilon+z-a)[\epsilon-(z-a)]}{(z-a)^2}\ln(a-z+\epsilon) + \frac{\bar{a}+\epsilon+i2\beta}{a+\epsilon-i2\beta-z}$$
$$+\ln(a+\epsilon-z-2i\beta) - \frac{\bar{a}-i\epsilon}{a-z+i\epsilon} + \frac{i\pi\epsilon^2}{2(z-a)^2} - \epsilon\frac{1+i}{a-z} - \frac{\epsilon^2}{(z-a)^2}\ln(a-z+i\epsilon).$$

The only singularities that arise in the region considered are in the first term when z tends to $a+\epsilon$ but again the term tends to 0 so that the singularity is removable and the solution is shown to be everywhere continuous.

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