The Review of Symbolic Logic
Volume 18, Number 3, September 2025

RNMATRICES FOR MODAL LOGICS

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Abstract. In previous publications, it was shown that finite non-deterministic matrices are quite powerful in providing semantics for a large class of normal and non-normal modal logics. However, some modal logics, such as those whose axiom systems contained the Löb axiom or the McKinsey formula, were not analyzed via non-deterministic semantics. Furthermore, other modal rules than the rule of necessitation were not yet characterized in the framework.

In this paper, we will overcome this shortcoming and present a novel approach for constructing semantics for normal and non-normal modal logics that is based on *restricted* non-deterministic matrices. This approach not only offers a uniform semantical framework for modal logics, while keeping the interpretation of the involved modal operators the same, and thus making different systems of modal logic comparable. It might also lead to a new understanding of the concept of modality.

§1. Introduction. In this paper, we present a novel approach for constructing semantics for normal and non-normal modal logics that is based on *restricted* non-deterministic semantics. This approach proves to be very versatile in the sense that given a finite axiomatic characterization of a modal system, we can construct a semantics, such that the given axiomatic system is sound and complete.

We begin our study with the weakest system of modal logic **M**. This system is an expansion of classical propositional logic (**CPL**) with a unary operator $\ominus$ and is characterized as follows:

- **M** contains all (classical) tautologies.
- **M** is closed under uniform substitution.
- **M** is closed under Modus Ponens.

Received: February 21, 2024.
2020 *Mathematics Subject Classification*: Primary 03B45, 03B50.
Key words and phrases: modal logics, non-deterministic semantics, swap structures, RNmatrices, decidability.

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doi:10.1017/S1755020325100737

https://doi.org/10.1017/S1755020325100737 Published online by Cambridge University Press

This starting point for investigating modal logics is not new. Logicians like Krister Segerberg [34], David Makinson [24], Heinrich Wansing [36] and Lloyd Humberstone [18] started their studies of modal logics with a similar system.[1]

It is evident from the given characterization that there are no axioms that specifically mention the (modal) operator $\ominus$. In particular, there is nothing that characterizes $\ominus$ as a modal operator. The intended interpretation of $\ominus$ as a modal operator will later be established by extending **M** with further axioms and rules, and thus generating the desired modal properties of $\ominus$. As already explained in [18], this provides the broadest possible definition of a modal logic.[2]

In the presentations for the smallest modal system in [24], [34], [36] and [18] the interpretation of the modal operator $\ominus$ is not kept for all extensions of the smallest modal system. In fact, the interpretation of the operator $\ominus$ changes. From no interpretation in **M** to a semantics constituted in possible worlds, where in all extensions, e.g., $\ominus A$ is true in a world iff A is true in all accessible worlds.[3] Similar things can be said about neighborhood frames or some versions of truth-maker semantics. These shifts are made rather abruptly to generate the needed behavior of the modal operator. What then distinguishes our approach is that we keep the interpretation of $\ominus$ the same, thus establishing a uniform theory of modal operators.

The goal of developing a unified theory of modal operators is not new. In recent publications, cf. [10, 28], **M** and some of its normal extensions were investigated as part of a larger discussion concerning non-deterministic semantics for non-normal modal logics[4] and normal modal logics. There, the authors build upon the framework of non-deterministic semantics, which was systematically introduced by Arnon Avron and his collaborators, cf. [3], but already used by Yuri Ivlev and John Kearns in the context of modal logics in [19, 37] and [20, 21]. These results were further developed in [11, 17, 25–27, 29–31].

Those publications by the various authors shared the heuristics of systematically eliminating semantical values or non-determinacy from non-deterministic truth-tables to validate desired axioms. This approach was successful in providing a uniform semantics for a broad class of normal and non-normal modal systems, even for some systems that lack possible worlds semantics, cf. [26]. Hence, the proposed framework is not only more general, but also conceptually conservative since the meaning of modal operators was kept constant.

However, the technique of eliminating values or non-determinacy had limitations. It became apparent that not all modal axioms could be straightforwardly represented in a non-deterministic truth-table format, such as the Gödel–Löb axiom $\ominus(\ominus A \to A) \to \ominus A$ (GL) or the McKinsey formula, as axioms.[5] The possibility of providing non-deterministic truth-tables for such formulas remains uncertain.

[1] They either call it $\mathbf{L}_0$, **PC** or **S**. In the more recent [17] this system is called **0**. The main difference between their and our starting point is that they interpret $\ominus$ from the beginning as a necessity operator.

[2] The system **M** differs from **CPL**, since it cannot be characterized by a finite deterministic matrix, since any such characterization would designate a formula similar to the well-known Dugundji construction, which is of course not derivable in **M**.

[3] If we interpret $\ominus$ as necessity.

[4] Non-normal in the sense that the rule of necessitation is absent.

[5] Coincidentally, those formulas also have no corresponding first-order frame condition, cf. [35].

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Our primary objective is to demonstrate that restricted non-deterministic semantics serves as both a technically viable and conceptually valuable alternative uniform semantics for modal logic.

Hence, we will develop *restricted non-deterministic matrices*, or RNmatrices for short, for modal logics, and we will show how to construct semantics for given axiom systems and prove decidability of all those systems that do not contain modal rules. RNmatrices were introduced in [14], as a generalization of Rmatrices, cf. [32], in order to provide a semantic characterization for some logics of formal inconsistency, such as those found between **mbCcl** and **Cila**, which can not be characterized by finite (non-deterministic) matrices.

As a byproduct, we will generalize Kearns' technique [20] of globally restricting the set of all valuations, in order to validate *any* global modal rule. This will guarantee not only the validity of the rule of necessitation but also other well-known modal rules, like congruentiality, monotonicity or distribution.

This combination of RNmatrices and the generalized technique of globally restricting the set of all valuations, which leads to a uniform semantical framework for modal logics, is novel in the sense that it was nowhere presented in previous publications.

Before we conclude the introduction, we just want to highlight that recently there is a new development under investigation, a combination of Kripke frames and non-deterministic semantics, cf. [15]. The main difference between such hybrid frameworks and our approach is certainly the interpretation of the modal operators, either based on possible worlds or the restricted non-deterministic semantics. Such hybrid frameworks promise yet another generalization of Nmatrices.

Based on this, the aim of this paper is then as follows: in §2, we will introduce RNmatrices for the minimal modal logic **M**. In §3, we will discuss extensions, without any rules for the modal operator, of **M** by making the appropriate restrictions to the set of all valuations. General soundness and completeness results will be presented for all the extensions. This will be followed by §4 where all extensions discussed so far will be enriched with global modal rules, and thus generalizing John Kearns' level technique. In §5, we expand the language with another modal operator, introduce the bimodal minimal logic and its extensions, as well as proofs of soundness and completeness. And in §6, we conclude with results of decidability for modal system axiomatized with a finite set of modal axioms, where no global rules, i.e., rules such that the premises and the conclusion are theorems, are present, that extends the minimal modal logic **M**.

We will show that our semantical approach goes beyond Kripke structures and non-deterministic semantics as considered in previous work. And by focusing on the weakest modal logic **M** and investigating its extensions, we introduce a new hierarchy of (bi-, multi-) modal systems with or without (non-standard, multiple) global rules.

§2. RNmatrices for the minimal modal logic M. Consider a modal propositional signature Σ with unary connectives $\neg$ and $\ominus$ (classical negation and modality, respectively) and a binary connective $\rightarrow$ (material implication). Let $\mathcal{V}$ be a denumerable set of propositional variables $\mathcal{V} = \{p_0, p_1, \ldots\}$ and let $For(\Sigma)$ be the algebra of formulas over Σ freely generated by $\mathcal{V}$. As usual, conjunction $\wedge$, disjunction $\vee$ and bi-implication

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$\leftrightarrow$ are defined from $\neg$ and $\rightarrow$ as follows: $A \wedge B := \neg(A \rightarrow \neg B)$, $A \vee B; = \neg A \rightarrow B$ and $A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$.[6]

In this section, we consider a set of four-valued non-deterministic matrices (Nmatrices, for short) defined from swap structures (see for instance [7, chap. 6] and [13]) in which each truth-value is an ordered pair (or *snapshot*) $z = (z_1, z_2)$ in $\mathbf{2}^2$, for $\mathbf{2} = \{0, 1\}$. Here, z_1 and z_2 represent, respectively, the truth value of A and of $\ominus A$ for a given formula A over Σ. From now on, given a snapshot $z = (z_1, z_2)$, z_1 and z_2 represent the first and second coordinate of the truth-value. This produces four truth-values $\mathsf{t}^- := (1, 0)$, $\mathsf{t}^+ := (1, 1)$, $\mathsf{f}^+ := (0, 1)$ and $\mathsf{f}^- := (0, 0)$. Let V_4 be the set of such truth-values. Accordingly, the set of designated values will be $D_4 = \{z \in V_4 \ : \ z_1 = 1\} = \{\mathsf{t}^-, \mathsf{t}^+\}$. On the other hand, the set of non-designated values is given as $ND_4 = \{\mathsf{f}^-, \mathsf{f}^+\}$.

Because of the intended meaning of the snapshots, negation and implication between snapshots are computed over $\mathbf{2}$ in the first coordinate, while the second one can take an arbitrary value, which will be denoted by $*$. That is:

$$\begin{array}{rcl} \tilde{\neg} z & := & (\sim z_1, *); \\ z \tilde{\rightarrow} w & := & (z_1 \Rightarrow w_1, *). \end{array}$$

Here, $\sim$ and $\Rightarrow$ denote the Boolean negation and the implication in $\mathbf{2}$. Observe that the second coordinate is arbitrary since at this moment $\ominus$ remains uninterpreted, i.e., there are no axioms ruling the value of $\ominus\neg A$ and the value of $\ominus(A \rightarrow B)$.

The interpretation of $\ominus$ is a multioperator[7] which simply 'reads' the second coordinate, while the second coordinate (corresponding to $\ominus\ominus A$) will be arbitrary at this point, as well:

$$\tilde{\ominus} z \ := \ (z_2, *).$$

Let $\mathcal{M} = \langle V_4, D_4, \mathcal{O} \rangle$ be the obtained four-valued Nmatrix, where $\mathcal{O}(\#) = \tilde{\#}$ for every connective $\#$ in Σ. The truth-tables for $\mathcal{M}$ can be displayed as follows:

$\tilde{\rightarrow}$	t^+	t^-	f^+	f^-
t^+	D_4	D_4	ND_4	ND_4
t^-	D_4	D_4	ND_4	ND_4
f^+	D_4	D_4	D_4	D_4
f^-	D_4	D_4	D_4	D_4

A	$\tilde{\neg} A$	$\tilde{\ominus} A$
t^+	ND_4	D_4
t^-	ND_4	ND_4
f^+	D_4	D_4
f^-	D_4	ND_4

DEFINITION 1. *Now, let $\mathcal{F}$ be the set of all the valuations over the Nmatrix $\mathcal{M}$, such that $v \in \mathcal{F}$ iff $v : For(\Sigma) \rightarrow V_4$ is a function satisfying the following properties*:

- $v(\#A) \in \tilde{\#}\, v(A)$ *for* $\# \in \{\neg, \ominus\}$;
- $v(A \rightarrow B) \in v(A) \tilde{\rightarrow} v(B)$.

[6] Note that we could also take $\wedge$, $\vee$ and $\leftrightarrow$ as primitive rather than defined connectives. However, due to the non-truth-functional nature of our semantics, presented below, this would require more care w.r.t. the truth-tables and the formulation of later results. In order to keep our approach accessible to a broader audience, we decided to take smaller set of connectives as primitive.

[7] The term *multioperator* (a.k.a. hyperoperator) is an algebraic term for a non-deterministic function, see [12]. For example, $\tilde{\ominus} : V_4 \rightarrow \mathcal{P}(V_4) \backslash \emptyset$. All connectives in this paper are interpreted as multioperators.

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The logic **M** generated by the Nmatrix $\mathcal{M}$ is then defined as follows: $\Gamma \models_{\mathbf{M}} A$ iff, for every $v \in \mathcal{F}$: if $v(B) \in D_4$ for every $B \in \Gamma$ then $v(A) \in D_4$.

Let $\mathcal{H}$ be the standard Hilbert calculus for **CPL**, presented in the signature Σ (that is, no axioms nor rules for $\ominus$ are given, being *Modus Ponens*, denoted by **MP**, the only inference rule). It is easy to prove the following result (a sketch of the proof will be given in the next subsection).

THEOREM 1 (Soundness and completeness of $\mathcal{H}$ w.r.t. $\mathcal{M}$). *For every* $\Gamma \cup \{A\} \subseteq For(\Sigma)$ *it holds*: $\Gamma \vdash_{\mathcal{H}} A$ *iff* $\Gamma \models_{\mathbf{M}} A$.

REMARK 2. *The Nmatrix semantics* $\mathcal{M}$ *for* ***M*** *was already introduced by H. Omori and D. Skurt in* [28]. *Indeed, the Nmatrix considered in* [28] *uses also four truth-values with the same names, the same multioperators and the same designated values than the ones in* $\mathcal{M}$, *with the only difference that* f^+ *must be changed to* f^- *and vice-versa, due to a different interpretation of those truth-values. Furthermore, the idea of expressing the truth values* (*and so the corresponding multioperations over it*) *as snapshots is unique to this paper.*

Being the minimal modal logic, ***M*** *is nothing else than* ***CPL*** *presented in a language with an uninterpreted modal operator* $\ominus$. *For instance, it does neither satisfy the axiom* $(\mathsf{K}) : \ominus(A \to B) \to (\ominus A \to \ominus B)$ *nor the rule of necessitation. Given that* $\ominus$ *is supposed to represent a given modal operator* (*that it could be, for instance, a possibility operator* $\Diamond$) *this feature should be expected. In particular, we will show that the nature of the modality will strongly depend on our choice of axioms we want to be valid.*

In the next section, the valuations considered over the Nmatrix $\mathcal{M}$ will be restricted to certain conditions, in order to satisfy particular modal axioms which will characterize specific modal logics. It should be observed that a similar investigation was already done in [28], but with a different approach: instead of restricting the set of valuations, there the multioperators of the Nmatrix $\mathcal{M}$ were restricted.

§3. RNmatrices for extensions of M. In what follows, we will introduce and exploit what in [14] was called *restricted Nmatrices* (RNmatrices).

DEFINITION 3 (RNmatrices). *RNmatrices have the form* $\mathcal{RM} = \langle \mathcal{M}, \mathcal{F}' \rangle$, *where* $\mathcal{M}$ *is an Nmatrix and* $\mathcal{F}' \subseteq \mathcal{F}$ *and the set* $\mathcal{F}'$ *is closed under substitutions.*[8]

As proved in [14], any RNmatrix as above generates a Tarskian and structural consequence relation defined as expected: $\Gamma \models_{\mathcal{RM}} A$ iff, for every $v \in \mathcal{F}'$: if $v(B) \in D_4$ for every $B \in \Gamma$ then $v(A) \in D_4$. The aim of the restriction is to satisfy certain modal axiom(s) and, later on, modal rules.

REMARK 4. *From now on, any valuation* $v \in \mathcal{F}$ *over the Nmatrix* $\mathcal{M}$ (*see Definition 1*) *will be written as* $v = (v_1, v_2)$ *such that* $v_1, v_2 : For(\Sigma) \to \mathbf{2}$. *Hence,* $v(A) = (v_1(A), v_2(A))$ *for every formula A. This means that, for all formulas A and B*:

- $v(A) \in D_4$ *iff* $v_1(A) = 1$;
- $v_1(\neg A) = \sim v_1(A)$;

[8] A *substitution* over the signature Σ of $\mathbf{L}^R$ is a function $\sigma : \mathcal{V} \to For(\Sigma)$. Since $For(\Sigma)$ is an absolutely free algebra, each σ can be extended to a unique endomorphism in $For(\Sigma)$ (which will be also denoted by σ). That is, $\sigma : For(\Sigma) \to For(\Sigma)$ is such that $\sigma(\#A) = \#\sigma(A)$ for $\# \in \{\neg, \ominus\}$, and $\sigma(A \to B) = \sigma(A) \to \sigma(B)$. The set of substitutions over σ (seen as endomorphisms in $For(\Sigma)$) will be denoted by $Subs(\Sigma)$.

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- $v_1(\ominus A) = v_2(A)$;
- $v_1(A \to B) = v_1(A) \Rightarrow v_1(B)$.

We start by considering the well-known axiom (K), introduce important notions and prove soundness and completeness of the standard Hilbert-calculus w.r.t. RNmatrices. Then we will discuss other well-known axioms with one modal operator.

Consider

$$(\mathsf{K}) : \ominus(A \to B) \to (\ominus A \to \ominus B).$$

Then, a valuation $v \in \mathcal{F}$ satisfies (K) iff, by Remark 4, $v_1(\ominus(A \to B) \to (\ominus A \to \ominus B)) = 1$ for every A, B iff $v_1(\ominus(A \to B)) \Rightarrow (v_1(\ominus A) \Rightarrow v_1(\ominus B)) = 1$ for every A, B iff $v_1(\ominus(A \to B)) \leq v_1(\ominus A) \Rightarrow v_1(\ominus B)$ for every A, B iff $v_2(A \to B) \leq v_2(A) \Rightarrow v_2(B)$ for every A, B. Hence, the logic $\mathbf{M}_\mathsf{K}$ satisfying axiom (K) is characterized by the RNmatrix $\mathcal{RM}_\mathsf{K} = \langle \mathcal{M}, \mathcal{F}_\mathsf{K} \rangle$ such that

$$\mathcal{F}_\mathsf{K} = \{v \in \mathcal{F} \ : \ v_2(A \to B) \leq v_2(A) \Rightarrow v_2(B) \ \text{ for every } A, B\}.$$

Clearly, $\mathcal{RM}_\mathsf{K}$ is structural. Indeed, let ρ be a substitution and let $v \in \mathcal{F}_\mathsf{K}$. Observe that $v \circ \rho = (v_1 \circ \rho, v_2 \circ \rho)$. Then, for every A, B: $v_2 \circ \rho(A \to B) = v_2(\rho(A \to B)) = v_2(\rho(A) \to \rho(B)) \leq v_2(\rho(A)) \Rightarrow v_2((\rho(B)) = v_2 \circ \rho(A) \Rightarrow v_2 \circ \rho(B)$. Hence $v \circ \rho \in \mathcal{F}_\mathsf{K}$.

Definition 5 (The Hilbert calculus $\mathcal{H}_\mathsf{K}$). *Let $\mathcal{H}_\mathsf{K}$ be the Hilbert calculus over Σ obtained from $\mathcal{H}$ by adding axiom schema* (K).

Definition 6 (Bivaluation semantics for $\mathcal{H}_\mathsf{K}$). *A* bivaluation for $\mathcal{H}_\mathsf{K}$ *is a function $b : For(\Sigma) \to \mathbf{2}$ such that, for every A, B:*

(val 1) $b(\neg A) = {\sim} b(A)$;
(val 2) $b(A \to B) = b(A) \Rightarrow b(B)$;
(val 3) $b(\ominus(A \to B)) \leq b(\ominus A) \Rightarrow b(\ominus B)$.

The consequence relation $\models_2^\mathsf{K}$ w.r.t. bivaluations for $\mathcal{H}_\mathsf{K}$ is defined as follows: $\Gamma \models_2^\mathsf{K} A$ iff, for every b: if $b(B) = 1$ for every $B \in \Gamma$ then $b(A) = 1$.

Remark 7. *Recall that, given a Tarskian and finitary logic $\boldsymbol{L}$, a set of formulas Δ is said to be A-saturated (where A is a formula) if $\Delta \nvdash_\mathbf{L} A$ but $\Delta, B \vdash_\mathbf{L} A$ for every formula B such that $B \notin \Delta$. If Δ is A-saturated then it is a closed theory, that is: $\Delta \vdash_\mathbf{L} B$ iff $B \in \Delta$. It is well-known that, in any Tarskian and finitary logic $\boldsymbol{L}$, if $\Gamma \nvdash_\mathbf{L} A$ then there exists an A-saturated set Δ such that $\Gamma \subseteq \Delta$. Since the logic generated by $\mathcal{H}_\mathsf{K}$ is Tarskian and finitary, it has this property.*

Proposition 1. *Let Δ be an A-saturated set in $\mathcal{H}_\mathsf{K}$. Then, for every formulas A, B:*

(1) $\neg A \in \Delta$ *iff* $A \notin \Delta$;
(2) $A \to B \in \Delta$ *iff either* $A \notin \Delta$ *or* $B \in \Delta$;
(3) *if* $\ominus(A \to B) \in \Delta$ *and* $\ominus A \in \Delta$ *then* $\ominus B \in \Delta$.

Proof. Immediate, by definition of $\mathcal{H}_\mathsf{K}$ and the fact that Δ is a closed theory. □

Corollary 1. *Let Δ be an A-saturated set in $\mathcal{H}_\mathsf{K}$. Then, the characteristic function $b : For(\Sigma) \to \mathbf{2}$ of Δ, given by $b(B) = 1$ iff $B \in \Delta$, is a bivaluation for $\mathcal{H}_\mathsf{K}$.*

Proof. It is immediate from Proposition 1. □

Theorem 2 (Soundness and completeness of $\mathcal{H}_\mathsf{K}$ w.r.t. bivaluation semantics). *Let $\Gamma \cup \{A\} \subseteq For(\Sigma)$. Then: $\Gamma \vdash_{\mathcal{H}_\mathsf{K}} A$ iff $\Gamma \models_2^\mathsf{K} A$.*

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Proof. *(Soundness)*: Clearly every axiom of $\mathcal{H}_{\mathsf{K}}$ is valid w.r.t. bivaluations, and bivaluations preserve trueness through **MP**, that is: if $b(A \to B) = b(A) = 1$ then $b(B) = 1$. Hence, by induction on the length of a derivation in $\mathcal{H}_{\mathsf{K}}$ of A from Γ, it is easy to see the following: $\Gamma \vdash_{\mathcal{H}_{\mathsf{K}}} A$ implies that $\Gamma \models_2^{\mathsf{K}} A$.

(Completeness): Suppose that $\Gamma \nvdash_{\mathcal{H}_{\mathsf{K}}} A$. As observed above, there exists an A-saturated set Δ such that $\Gamma \subseteq \Delta$. By Corollary 1 the characteristic map b of Δ is a bivaluation for $\mathcal{H}_{\mathsf{K}}$ such that $b(B) = 1$ for every $B \in \Gamma$ but $b(A) = 0$. This shows that $\Gamma \not\models_2^{\mathsf{K}} A$. □

LEMMA 1. *Let* b *be a bivaluation for* $\mathcal{H}_{\mathsf{K}}$. *Then, the function* $v : For(\Sigma) \to V_4$ *given by* $v(A) = (b(A), b(\ominus A))$ *for every* A *belongs to* $\mathcal{F}_{\mathsf{K}}$. *In addition*: $b(A) = 1$ *iff* $v(A) \in D_4$, *for every* A.

Proof. Let us first prove that $v \in \mathcal{F}$. By definition of v and by (val 1), $v(\neg A) = (b(\neg A), b(\ominus\neg A)) = (\sim b(A), b(\ominus\neg A)) \in \tilde{\neg}\, v(A)$. By (val 2), $v(A \to B) = (b(A \to B), b(\ominus(A \to B))) = (b(A) \Rightarrow b(B), b(\ominus(A \to B))) \in v(A) \tilde{\to} v(B)$. By definition, $v(\ominus A) = (b(\ominus A), b(\ominus\ominus A)) \in \tilde{\ominus}\, v(A)$. This shows that $v \in \mathcal{F}$. Now, by (val 3), $v_2(A \to B) = b(\ominus(A \to B)) \leq b(\ominus A) \Rightarrow b(\ominus B) = v_2(A) \Rightarrow v_2(B)$. Hence, $v \in \mathcal{F}_{\mathsf{K}}$. Clearly, $b(A) = 1$ iff $v(A) \in D_4$, for every A. □

LEMMA 2. *Let* $v \in \mathcal{F}_{\mathsf{K}}$. *Then, the function* $b : For(\Sigma) \to \mathbf{2}$ *given by* $b(A) = v_1(A)$ *for every* A *is a bivaluation for* $\mathcal{H}_{\mathsf{K}}$. *In addition*: $b(A) = 1$ *iff* $v(A) \in D_4$, *for every* A.

Proof. By Remark 4, b satisfies (val 1) and (val 2). Now, observe that $b(\ominus(A \to B)) = v_1(\ominus(A \to B)) = v_2(A \to B) \leq v_2(A) \Rightarrow v_2(B) = v_1(\ominus A) \Rightarrow v_1(\ominus B) = b(\ominus A) \Rightarrow b(\ominus B)$. Hence, b also satisfies (val 3). It is immediate to see that $b(A) = 1$ iff $v(A) \in D_4$, for every A. □

THEOREM 3 (Soundness and completeness of $\mathcal{H}_{\mathsf{K}}$ w.r.t. the RNmatrix $\mathcal{RM}_{\mathsf{K}}$). *Let* $\Gamma \cup \{A\} \subseteq For(\Sigma)$. *Then*: $\Gamma \vdash_{\mathcal{H}_{\mathsf{K}}} A$ *iff* $\Gamma \models_{\mathcal{RM}_{\mathsf{K}}} A$.

Proof. By Lemmas 1 and 2, it follows that $\Gamma \models_{\mathcal{RM}_{\mathsf{K}}} A$ iff $\Gamma \models_2^{\mathsf{K}} A$. The result follows by Theorem 2. □

REMARK 8. *By simplifying the proof above Theorem 1 can be proven. Indeed, it is enough replacing* $\mathcal{F}_{\mathsf{K}}$ *by* $\mathcal{F}$ *and consider bivaluations just satisfying* (*val* 1) *and* (*val* 2). *Clearly, the calculus* $\mathcal{H}$ *is sound and complete w.r.t. such bivaluations semantics. The proof of this fact is analogous to that of Theorem 2, just by stating a simplified version of Proposition 1 in which item* (3) *is not considered. The rest of the proof follows from here as in the case of* $\mathcal{H}_{\mathsf{K}}$, *with suitable simplifications.*

We will now consider the following well-known axioms with one modal operator and present to corresponding restrictions on the RNmatrices in order to show their versatility. Note that, this list can be easily extended:

- $(\mathsf{Kw}) : A \to (\ominus(A \to B) \to (\ominus A \to \ominus B))$.
- $(\mathsf{T}) : \ominus A \to A$.
- $(\mathsf{Tc}) : A \to \ominus A$.
- $(4) : \ominus A \to \ominus\ominus A$.
- $(4\mathsf{c}) : \ominus\ominus A \to \ominus A$.
- $(\mathsf{GL}) : \ominus(\ominus A \to A) \to \ominus A$.

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By analysis similar to the one given for (K), it is immediate to see that the conditions imposed by these axioms on valuations and bivaluations are, respectively, the following, where $\sqcap$ is the boolean meet:

Axiom	Valuations	Bivaluations
(Kw)	$v_1(A) \sqcap v_2(A \to B) \sqcap v_2(A) \leq v_2(B)$	$b(A) \sqcap b(\ominus(A \to B)) \sqcap b(\ominus A) \leq b(\ominus B)$
(T)	$v_2(A) \leq v_1(A)$	$b(\ominus A) \leq b(A)$
(Tc)	$v_1(A) \leq v_2(A)$	$b(A) \leq b(\ominus A)$
(4)	$v_2(A) \leq v_2(\ominus A)$	$b(\ominus A) \leq b(\ominus\ominus A)$
(4c)	$v_2(\ominus A) \leq v_2(A)$	$b(\ominus\ominus A) \leq b(\ominus A)$
(GL)	$v_2(\ominus A \to A) \leq v_2(A)$	$b(\ominus(\ominus A \to A)) \leq b(\ominus A)$

A general method for obtaining conditions for valuations and bivaluations from a given axiom will be given after Example 14.

Now, let $\mathsf{Ax} \in \{\mathsf{Kw}, \mathsf{T}, \mathsf{Tc}, \mathsf{4}, \mathsf{4c}, \mathsf{GL}\}$, then it is easy to see that all RNmatrices $\mathcal{RM}_{\mathsf{Ax}}$ are structural. Furthermore, the Hilbert calculi $\mathcal{H}_{\mathsf{Ax}}$ for $\mathbf{M}_{\mathsf{Ax}}$ are obtained by adding the axiom (Ax) to $\mathcal{H}$. The consequence relation w.r.t. bivaluations will be denoted by $\models_2^{\mathsf{Ax}}$.

An A-saturated set Δ in $\mathcal{H}_{\mathsf{Ax}}$ satisfies conditions (1) and (2) of Proposition 1 plus the following, respectively:

- if $A \in \Delta$, $\ominus(A \to B) \in \Delta$ and $\ominus A \in \Delta$ then $\ominus B \in \Delta$
- if $\ominus A \in \Delta$, then $A \in \Delta$
- if $A \in \Delta$, then $\ominus A \in \Delta$
- if $\ominus A \in \Delta$, then $\ominus\ominus A \in \Delta$
- if $\ominus\ominus A \in \Delta$, then $\ominus A \in \Delta$
- if $\ominus(\ominus A \to A) \in \Delta$ then $\ominus A \in \Delta$.

From this, it is easy to adapt the corresponding proofs for $\mathcal{H}_{\mathsf{K}}$ in order to obtain the following results.

THEOREM 4 (Soundness and completeness of $\mathcal{H}_{\mathsf{Ax}}$ w.r.t. bivaluation semantics). *Let* $\Gamma \cup \{A\} \subseteq For(\Sigma)$. *Then*: $\Gamma \vdash_{\mathcal{H}_{\mathsf{Ax}}} A$ *iff* $\Gamma \models_2^{\mathsf{Ax}} A$.

THEOREM 5 (Soundness and completeness of $\mathcal{H}_{\mathsf{Ax}}$ w.r.t. the RNmatrix $\mathcal{RM}_{\mathsf{Ax}}$). *Let* $\Gamma \cup \{A\} \subseteq For(\Sigma)$. *Then*: $\Gamma \vdash_{\mathcal{H}_{\mathsf{Ax}}} A$ *iff* $\Gamma \models_{\mathcal{RM}_{\mathsf{Ax}}} A$.

REMARK 9. *We only note that all the results carry over to extensions of* $\boldsymbol{M}$ *by sets of axioms, as well.*

We end this section by showing that it is possible to extend Theorem 5 to *any* axiomatic extension of the basic system $\mathcal{H}$ (which corresponds to the Nmatrix $\mathbf{M}$). That is, any modal axiom (or, in general any finite set Ax of modal axioms) can be added to $\mathcal{H}$, in such a manner that it is possible to obtain conditions on the valuations (and on the bivaluations) faithfully characterizing the resulting Hilbert calculus $\mathcal{H}_{\mathsf{Ax}}$ by the corresponding bivaluation semantics and Nmatrix $\mathcal{RM}_{\mathsf{Ax}}$. In order to do this,

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observe first that, in the given modal signature, any formula A satisfies one and only one of the following conditions:

- A is atomic; or
- $A = \neg B$ for a unique formula B; or
- $A = \ominus B$ for a unique formula B; or
- $A = A_1 \to (A_2 \to (\ldots \to (A_n \to B)\ldots))$ for a unique $n \geq 1$ and unique formulas A_i (for $1 \leq i \leq n$) and B such that either B is atomic, or $B = \neg C$ or $B = \ominus C$ for a unique formula C.

DEFINITION 10. *Given a valuation $v = (v_1, v_2) \in \mathcal{F}$ and a formula A, we define the* immediate unary reduction *of* v *and A as follows*:

- *if A is atomic then $v_i(A)$ reduces to $v_i(A)$, for $i = 1, 2$;*
- *if $A = \neg B$ then $v_1(A)$ reduces to $\sim v_1(B)$;*
- *if $A = \ominus B$ then $v_1(A)$ reduces to $v_2(B)$; and*
- *if $A = A_1 \to (A_2 \to (\ldots \to (A_n \to B)\ldots))$ then $v_1(A)$ reduces to $(v_1(A_1) \sqcap \ldots \sqcap v_1(A_n)) \Rightarrow v_1(B)$; in particular, $v_1(A_1 \to B)$ reduces to $v_1(A_1) \Rightarrow v_1(B)$.*

DEFINITION 11. *Given* v *and A as above, the* unary reduction *of* v *and A is obtained by applying iteratively the immediate unary reductions until no additional immediate reductions can be applied. Since length of the formulas finite, it is always possible to find the unary reduction for any* v *and A.*

DEFINITION 12. *Given* v *and A as above, the* immediate reduction *of* v *and A is defined as follows*:

- *if A is atomic and $j \in \{0, 1\}$ then the equation $v_i(A) = j$ reduces to $v_i(A) = j$, for $i = 1, 2$;*
- *if $A = \neg B$ and $j \in \{0, 1\}$ then the equation $v_1(A) = j$ reduces to $v_1(B) = \sim j$;*
- *if $A = \ominus B$ and $j \in \{0, 1\}$ then the equation $v_1(A) = j$ reduces to $v_2(B) = j$;*
- *if $A = A_1 \to (A_2 \to (\ldots \to (A_n \to B)\ldots))$ then the equation $v_1(A) = 1$ reduces to the condition $v_1(A_1) \sqcap \ldots \sqcap v_1(A_n) \leq v_1(B)$; and*
- *if $A = A_1 \to (A_2 \to (\ldots \to (A_n \to B)\ldots))$ then the equation $v_1(A) = 0$ reduces to the conditions $v_1(A_k) = 1$ (for $1 \leq k \leq n$) and $v_1(B) = 0$.*

DEFINITION 13. *The* reduction *of* v *and A is obtained from an equation $v_i(A) = j$ by applying iteratively the immediate reductions to the obtained equations and the unary reduction to the involved terms inside the conditions.*

Then, starting from the equation $v_1(A) = 1$, it is possible to reduce, in a finite number of steps, any axiom A to a finite set of conditions over valuations in $\mathcal{F}$.

EXAMPLE 14. *Consider the axiom $A = (\ominus B \to B) \wedge (\ominus B \to \ominus\ominus B)$ such that B is a propositional variable. Let $E = (\ominus D \to D) \wedge (\ominus D \to \ominus\ominus D) = \neg((\ominus D \to D) \to \neg(\ominus D \to \ominus\ominus D))$ be a generic instance of A. Hence, the equation $v_1(E) = 1$ immediately reduces to $v_1((\ominus D \to D) \to \neg(\ominus D \to \ominus\ominus D)) = 0$. The latter immediately reduces to the conditions $v_1(\ominus D \to D) = 1, v_1(\neg(\ominus D \to \ominus\ominus D)) = 0$. This immediately reduces to $v_1(\ominus D) \leq v_1(D)$, $v_1(\ominus D \to \ominus\ominus D) = 1$. This immediately reduces to $v_2(D) \leq v_1(D)$, $v_1(\ominus D) \leq v_1(\ominus\ominus D)$. Finally, this reduces to $v_2(D) \leq v_1(D), v_2(D) \leq v_2(\ominus D)$. Observe that, since D is an unspecified formula, it is treated as a propositional variable in the reductions.*

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Clearly, and by using a similar technique, it is possible to obtain in a finite number of steps a finite set of conditions on bivaluations from any axiom or finite set of axioms. Specifically, given a bivaluation b: $b(A)$ and $b(A) = j$ reduce, respectively, to $b(A)$ and $b(A) = j$, if $A \in \mathcal{V}$ or $A = \ominus B$ for some B, for $j = 0, 1$; $b(\neg A)$ and $b(\neg A) = j$ reduce, respectively, to $\sim b(A)$ and $b(A) = \sim j$, for $j = 0, 1$; $b(A_1 \to (A_2 \to (\ldots \to (A_n \to B)\ldots)))$ and $b(A_1 \to (A_2 \to (\ldots \to (A_n \to B)\ldots))) = 1$ reduce, respectively, to $(b(A_1) \sqcap \ldots \sqcap b(A_n)) \Rightarrow b(B)$ and $b(A_1) \sqcap \ldots \sqcap b(A_n) \leq b(B)$; and $b(A_1 \to (A_2 \to (\ldots \to (A_n \to B)\ldots))) = 0$ reduces to the conditions $b(A_k) = 1$ (for $1 \leq k \leq n$) and $b(B) = 0$. Of course these reductions are applied iteratively, arriving in a finite number of steps to a set of conditions over b starting from $b(A) = 1$ for a given axiom A.

Example 15. *Consider the axiom A from Example 14, as well as a generic instance E of it. Then, the following conditions for a bivaluation* b, *starting from the equation $b(E) = 1$, are obtained: $b((\ominus D \to D) \to \neg(\ominus D \to \ominus\ominus D)) = 0$, which reduces to the conditions $b(\ominus D \to D) = 1$, $b(\neg(\ominus D \to \ominus\ominus D)) = 0$. This reduces to $b(\ominus D) \leq b(D)$, $b(\ominus D \to \ominus\ominus D) = 1$, which reduces to $b(\ominus D) \leq b(D)$, $b(\ominus D) \leq b(\ominus\ominus D)$. It should be clear that Lemma 1 can be adapted to the axiomatic extension $\mathcal{H}_A$ of $\mathcal{H}$, taking into consideration Remark 4, showing that the bivaluation semantics obtained for $\mathcal{H}_A$ coincides with the Nmatrix generated in Example 14.*

In general, let Ax be a finite set of modal axioms. By adapting the results above, it follows that the logic generated from the bivaluation semantics obtained from Ax by means of the reductions above coincides with the one generated from the restricted Nmatrix obtained from Ax by the corresponding reductions.

In order to prove completeness of $\mathcal{H}_{\mathsf{Ax}}$ w.r.t. the bivaluation semantics generated from Ax, the following observation is crucial: if Δ is a C-saturated set for $\mathcal{H}_{\mathsf{Ax}}$ then it is a closed theory that contains any instance of any axiom in Ax. This, combined with conditions (1) and (2) of Proposition 1, produces general conditions for Δ (by considering a generic instance of each axiom). Such conditions correspond to the conditions for bivaluations obtained from Ax; that is, Corollary 1 can be adapted to $\mathcal{H}_{\mathsf{Ax}}$. This shows that the proposed method works for *any* set Ax of modal axioms.

Example 16. *Consider once again the axiom A, analyzed in Example 14. Let Δ be a C-saturated set in the axiomatic extension $\mathcal{H}_A$ of $\mathcal{H}$. Then, any instance $E = (\ominus D \to D) \wedge (\ominus D \to \ominus\ominus D)$ of A belongs to Δ. This means that $(\ominus D \to D) \to \neg(\ominus D \to \ominus\ominus D) \notin \Delta$, by Proposition 1(1). By Proposition 1(2), $\ominus D \to D \in \Delta$ and $\neg(\ominus D \to \ominus\ominus D) \notin \Delta$, which implies that $\ominus D \to \ominus\ominus D \in \Delta$. Then, two general properties of Δ are obtained: $(P1)$ if $\ominus D \in \Delta$ then $D \in \Delta$; and $(P2)$ if $\ominus D \in \Delta$ then $\ominus\ominus D \in \Delta$. It is clear that $(P1)$ and $(P2)$ correspond to the conditions for bivaluations obtained from A in Example 15, by identifying '$F \in \Delta$' with '$b(F) = 1$'. This allows us to adapt Corollary 1 to $\mathcal{H}_A$, obtaining so soundness and completeness of $\mathcal{H}_A$ w.r.t. bivaluation semantics, and so w.r.t. its RNmatrix semantics, by Example 15.*

§4. Extensions of M by global inference rules. In this section, the logic **M** and its axiomatic extensions will be enriched with various global inference rules for $\ominus$, and so we will obtain a vast class of modal logics. In all previous works, e.g., [9], [20] or [25], and follow-up articles, the only global inference rule was the rule of necessitation. In this section, however, we will show how to generalize John Kearns'

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original level-valuations technique, in order to capture any global inference rule. We will present furthermore, a generalized soundness and completeness proof and thus showing that any of the Hilbert systems presented before, are sound and complete w.r.t. the corresponding semantics. On top of that, the results below present a general recipe for constructing semantics for various normal and non-normal modal logics.

Throughout this section, **L** will denote any of the modal logics considered in §3. That is, **L** is characterized by a Hilbert calculus $\mathcal{H}_{\mathsf{Ax}}$ obtained by adding the (set of) axiom(s) (Ax) to $\mathcal{H}$. The RNmatrix characterizing **L** will be denoted by $\mathcal{RM}_{\mathsf{Ax}} = \langle \mathcal{M}, \mathcal{F}_{\mathsf{Ax}} \rangle$.

DEFINITION 17. *Let **L** be an extension of **M** characterized by a Hilbert calculus $\mathcal{H}_{\mathsf{Ax}}$. We say that $\mathcal{H}^R_{\mathsf{Ax}}$ is the Hilbert calculus obtained from $\mathcal{H}_{\mathsf{Ax}}$ by adding the (global) inference rule*

$$\frac{E_1 \;\; E_2 \;\; \ldots \;\; E_s}{E} \quad (\mathbf{R})$$

The logic characterized by $\mathcal{H}^R_{\mathsf{Ax}}$ will be denoted by $\mathbf{L}^R$.

REMARK 18. *By a global inference rule we mean any inference rule, where premises and conclusion are theorems.*

Observe that the previous definition can be easily generalized to capture more than one global inference rule (we left the details to the reader). However, to keep matters accessible we restrict ourselves to just one global rule at this point.[9]

The next definition will show the definitions of the notion of derivation in Hilbert calculi that one obtains by adding a global rule. For convenience, some technical notions will be recalled here.

DEFINITION 19. 1. *We say that A is* derivable *or a* theorem *in $\mathcal{H}^R_{\mathsf{Ax}}$, written as $\vdash_{\mathcal{H}^R_{\mathsf{Ax}}} A$, if there exists a finite sequence of formulas $A_1 \ldots A_m$ such that $A_m = A$ and, for every $1 \leq i \leq m$, either $A_i = \sigma(B)$ for some substitution σ and some axiom B in $\mathcal{H}_{\mathsf{Ax}}$, or there exists $j, k < i$ such that $A_j = A_k \to A_i$ (hence A_i is obtained from A_j and A_k by **MP**), or there exist some substitution σ and $j_1, \ldots, j_s < i$ such that $A_{j_k} = \sigma(E_k)$ for every $1 \leq k \leq s$, and $A_i = \sigma(E)$ (hence A_i is obtained from $A_{j_1}, \ldots, A_{j_s}$ by **R**).*

Such a sequence is called a derivation of A in $\mathcal{H}^R_{\mathsf{Ax}}$.

2. *We say that A is derivable in $\mathcal{H}^R_{\mathsf{Ax}}$ from a set Γ of formulas, written as $\Gamma \vdash_{\mathcal{H}^R_{\mathsf{Ax}}} A$, if either $\vdash_{\mathcal{H}^R_{\mathsf{Ax}}} A$, or there exists a nonempty finite set $\{B_1, \ldots, B_k\} \subseteq \Gamma$ such that $B_1 \to (B_2 \to (\ldots \to (B_k \to A) \ldots)$ is derivable in $\mathcal{H}^R_{\mathsf{Ax}}$.*

It is worth noting that $\emptyset \vdash_{\mathcal{H}^R_{\mathsf{Ax}}} A$ iff $\vdash_{\mathcal{H}^R_{\mathsf{Ax}}} A$.

REMARK 20. *It is straightforward to prove that the logic $\mathbf{L}^R$ generated by $\mathcal{H}^R_{\mathsf{Ax}}$ is Tarskian and finitary. Hence, as observed in Remark 7, if $\Gamma \nvdash_{\mathcal{H}^R_{\mathsf{Ax}}} A$ then there exists an A-saturated set Δ in $\mathcal{H}^R_{\mathsf{Ax}}$ such that $\Gamma \subseteq \Delta$.*

Following [9, 20, 28] we define the set of valuations over the RNmatrix $\mathcal{RM}_{\mathsf{Ax}} = \langle \mathcal{M}, \mathcal{F}_{\mathsf{Ax}} \rangle$ as follows.

[9] In Example 27, two global rules for two different modalities are considered.

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DEFINITION 21 (**R**-Level valuations). *We define the set* $\mathcal{F}^R_{\mathsf{Ax}}$ *inductively as follows*:

- $\mathcal{F}^0_{\mathsf{Ax}} = \mathcal{F}_{\mathsf{Ax}}$
- $\mathcal{F}^{m+1}_{\mathsf{Ax}} = \{v \in \mathcal{F}^m_{\mathsf{Ax}} : \forall A \in For(\Sigma), \forall \sigma \in Subs(\Sigma), \text{if } A = \sigma(E)$ *and* $\forall 1 \leq i \leq s, \forall w \in \mathcal{F}^m_{\mathsf{Ax}}(w_1(\sigma(E_i)) = 1),$ *then* $v_1(A) = 1\}$
- $\mathcal{F}^R_{\mathsf{Ax}} = \bigcap_{m=0}^{\infty} \mathcal{F}^m_{\mathsf{Ax}}$.

DEFINITION 22.

(1) *A is valid in* $\mathbf{L}^R$, *denoted by* $\models_{\mathbf{L}^R} A$, *if* $v_1(A) = 1$ *for every* $v \in \mathcal{F}^R_{\mathsf{Ax}}$.
(2) *A is a semantical consequence of* Γ *in* $\mathbf{L}^R$, *denoted by* $\Gamma \models_{\mathbf{L}^R} A$, *if either A is valid in* $\mathbf{L}^R$, *or* $B_1 \to (B_2 \to (\ldots \to (B_k \to A)\ldots)$ *is valid in* $\mathbf{L}^R$ *for some nonempty finite set* $\{B_1, \ldots, B_k\} \subseteq \Gamma$.

LEMMA 3. *If* $\vdash_{\mathcal{H}^R_{\mathsf{Ax}}} A$ *with a derivation of length* $m+1$, *then* $v_1(A) = 1$, *for every* $v \in \mathcal{F}^m_{\mathsf{Ax}}$.

Proof. By induction on $m \geq 0$. If $m = 0$ then $m + 1 = 1$, hence A is an instance of an axioms of $\mathcal{H}^R_{\mathsf{Ax}}$. This means that for any $v \in \mathcal{F}^0_{\mathsf{Ax}} = \mathcal{F}_{\mathsf{Ax}}$, $v_1(A) = 1$, by the soundness of $\mathcal{H}_{\mathsf{Ax}}$ w.r.t. $\mathcal{RM}_{\mathsf{Ax}}$.

Assume the result holds for m, and let A be a theorem of $\mathcal{H}^R_{\mathsf{Ax}}$ with a derivation of the form $A_1 \ldots A_{m+2}$, where $A_{m+2} = A$. If A is an axiom the proof is as above, considering that $\mathcal{F}^{m+1}_{\mathsf{Ax}} \subseteq \mathcal{F}^0_{\mathsf{Ax}}$. We need to check now that the application of any rule preserves the result, that is, $v_1(A) = 1$ for every $v \in \mathcal{F}^{m+1}_{\mathsf{Ax}}$.

MP Let $v \in \mathcal{F}^{m+1}_{\mathsf{Ax}}$. In this case, there is $A_i := (A_j \to A)$, where $i, j \leq m$. By the induction hypothesis (IH), $w_1(A_j) = w_1(A_i) = 1$ for any $w \in \mathcal{F}^m_{\mathsf{Ax}}$. By the definition of the level valuations, $v_1(A_j) = v_1(A_i) = 1$. This implies that $v_1(A) = 1$ simply because of the way $\tilde{\to}$ has been defined.

R Let $v \in \mathcal{F}^{m+1}_{\mathsf{Ax}}$. Assume that there exists a substitution σ such that $A = \sigma(E)$, and that there exists $j_1, \ldots, j_s < i$ such that $A_{j_k} = \sigma(E_k)$ for every $1 \leq k \leq s$. By IH, $w_1(A_{j_k}) = 1$ for every $1 \leq k \leq s$ and every $w \in \mathcal{F}^m_{\mathsf{Ax}}$. Hence, $v_1(A) = 1$, by definition of $\mathcal{F}^{m+1}_{\mathsf{Ax}}$. □

THEOREM 6 (Soundness of $\mathcal{H}^R_{\mathsf{Ax}}$ w.r.t. level valuation semantics). *If* $\Gamma \vdash_{\mathcal{H}^R_{\mathsf{Ax}}} A$ *then* $\Gamma \models_{\mathbf{L}^R} A$.

Proof. Suppose that $\Gamma \vdash_{\mathcal{H}^R_{\mathsf{Ax}}} A$. If $\vdash_{\mathcal{H}^R_{\mathsf{Ax}}} A$, there must be a derivation of length $m + 1 \geq 1$ of A. By Lemma 3, $v'_1(A) = 1$, for every $v' \in \mathcal{F}^m_{\mathsf{Ax}}$ and since $\mathcal{F}^R_{\mathsf{Ax}} \subseteq \mathcal{F}^m_{\mathsf{Ax}}$ we get $\models_{\mathbf{L}^R} A$ and so $\Gamma \models_{\mathbf{L}^R} A$. Now, if there is some nonempty finite set $\{B_1, \ldots, B_k\} \subseteq \Gamma$ such that $\vdash_{\mathcal{H}^R_{\mathsf{Ax}}} C$, where $C := B_1 \to (B_2 \to (\ldots \to (B_k \to A)\ldots)$, it follows as above that $\models_{\mathbf{L}^R} C$ and so $\Gamma \models_{\mathbf{L}^R} A$. □

LEMMA 4. *Let* Δ *be an A-saturated set in* $\mathcal{H}^R_{\mathsf{Ax}}$, *and let* $b^\Delta : For(\Sigma) \to \mathbf{2}$ *given by* $b^\Delta(B) = 1$ *iff* $B \in \Delta$, *for every B. Then* b^Δ *is a bivaluation for* $\mathcal{H}_{\mathsf{Ax}}$ *and so the function* $v^\Delta : For(\Sigma) \to V_4$ *given by* $v^\Delta(B) = (b^\Delta(B), b^\Delta(\ominus B))$, *for every B, belongs to* $\mathcal{F}_{\mathsf{Ax}}$. *In addition, the following holds for every B*: $v^\Delta_1(B) = 1$ *iff* $B \in \Delta$.

Proof. Observe that Proposition 1 describes formal properties of A-saturated sets Δ' in $\mathcal{H}_{\mathsf{K}}$ which can be obtained by analyzing **MP** and the axioms of $\mathcal{H}$ (which give

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conditions (1) and (2)) together with axiom K, which gives condition (3). These conditions imply that the function $b^{\Delta'}$, the characteristic function of Δ', satisfies the clauses defining bivaluation semantics for $\mathcal{H}_{\mathsf{K}}$, proving so Corollary 1. As discussed at the end of §3, the same technique can be applied to Δ'-saturated sets in $\mathcal{H}_{\mathsf{Ax}}$, for any set Ax of modal axioms.

Now, let Δ be an A-saturated set in $\mathcal{H}^R_{\mathsf{Ax}}$, and let b^{Δ} be the characteristic function of Δ. Since $\mathcal{H}_{\mathsf{Ax}} \subseteq \mathcal{H}^R_{\mathsf{Ax}}$, it follows that b^{Δ} satisfies the formal properties for a Δ-saturated set in $\mathcal{H}_{\mathsf{Ax}}$ (collected in the analogous of Proposition 1 for $\mathcal{H}_{\mathsf{Ax}}$) which imply that b^{Δ} is a bivaluation for $\mathcal{H}_{\mathsf{Ax}}$ (recalling that the formal properties of such a bivaluations can be obtained by reductions applied to the axioms in Ax). Hence, by adapting the proof of Lemma 1 to $\mathcal{F}_{\mathsf{Ax}}$ (which is obtained from Ax as described at the end of §3), it follows that $v^{\Delta} \in \mathcal{F}_{\mathsf{Ax}}$. By the very definitions, for every B, it holds: $v^{\Delta}_1(B) = 1$ iff $B \in \Delta$. □

LEMMA 5. *Let A be a formula and let Δ be an A-saturated set in $\mathcal{H}^R_{\mathsf{Ax}}$. Let v^{Δ} be the valuation defined as in Lemma 4. Then, $v^{\Delta} \in \mathcal{F}^n_{\mathsf{Ax}}$ for every $n \geq 0$ and so $v^{\Delta} \in \mathcal{F}^R_{\mathsf{Ax}}$.*

Proof. Let Δ be an A-saturated set in $\mathcal{H}^R_{\mathsf{Ax}}$. The result will be proved by induction over n. If $n = 0$ then the result holds, by Lemma 4 and the fact that $\mathcal{F}^0_{\mathsf{Ax}} = \mathcal{F}_{\mathsf{Ax}}$. Suppose that, for every B and every B-saturated set Δ' in $\mathcal{H}^R_{\mathsf{Ax}}$, $v^{\Delta'} \in \mathcal{F}^k_{\mathsf{Ax}}$ for every $k \leq n$ (IH). From this, $v^{\Delta} \in \mathcal{F}^n_{\mathsf{Ax}}$. Now, let σ be a substitution and $D = \sigma(E)$ such that, for every $w \in \mathcal{F}^n_{\mathsf{Ax}}$, $w_1(\sigma(E_i)) = 1$, for $1 \leq i \leq s$. It will proven by *reductio ad absurdum* that $\vdash_{\mathcal{H}^R_{\mathsf{Ax}}} D$. Thus, suppose that $\nvdash_{\mathcal{H}^R_{\mathsf{Ax}}} D$. By Remark 20, there exists a D-saturated set Δ' in $\mathcal{H}^R_{\mathsf{Ax}}$ and so $D \notin \Delta'$. But, by IH, $v^{\Delta'} \in \mathcal{F}^n_{\mathsf{Ax}}$, hence $v^{\Delta'}_1(\sigma(E_i)) = 1$, for $1 \leq i \leq s$. That is, $\sigma(E_i) \in \Delta'$, for $1 \leq i \leq s$. Since Δ' is a closed theory in $\mathcal{H}^R_{\mathsf{Ax}}$, it follows by rule **R** that $D = \sigma(E) \in \Delta'$, a contradiction. This shows that $\vdash_{\mathcal{H}^R_{\mathsf{Ax}}} D$. But then $D \in \Delta$, since Δ is a closed theory. This means that $v^{\Delta}_1(D) = 1$. From this, it follows that $v^{\Delta} \in \mathcal{F}^{n+1}_{\mathsf{Ax}}$. □

THEOREM 7 (Completeness of $\mathcal{H}^R_{\mathsf{Ax}}$ w.r.t. level valuation semantics). *If $\Gamma \models_{\mathbf{L}^R} A$ then $\Gamma \vdash_{\mathcal{H}^R_{\mathsf{Ax}}} A$.*

Proof. Suppose that $\Gamma \nvdash_{\mathcal{H}^R_{\mathsf{Ax}}} A$. By Remark 20, there exists an A-saturated set Δ in $\mathcal{H}^R_{\mathsf{Ax}}$ such that $\Gamma \subseteq \Delta$. By Lemma 5, $v^{\Delta} \in \mathcal{F}^R_{\mathsf{Ax}}$. Moreover, $v^{\Delta}_1(A) \neq 1$, by Lemma 4. Now, let $\{B_1, \ldots, B_k\}$ be a finite nonempty subset of Γ. Hence, $v^{\Delta}_1(B_i) = 1$ for every i and then, by definition of $\tilde{\to}$, $v^{\Delta}_1(B_1 \to (B_2 \to (\ldots \to (B_k \to A)\ldots)) \neq 1$. This shows that $B_1 \to (B_2 \to (\ldots \to (B_k \to A)\ldots)$ is not valid in $\mathbf{L}^R$ for every nonempty finite set $\{B_1, \ldots, B_k\} \subseteq \Gamma$. From this, we conclude that $\Gamma \not\models_{\mathbf{L}^R} A$. □

From Theorems 6 and 7, we obtain that the modal logic $\mathbf{L}^R$, which is axiomatized by $\mathcal{H}^R_{\mathsf{Ax}}$, is semantically characterized by the RNmatrix $\mathcal{RM}^R_{\mathsf{Ax}} = \langle \mathcal{M}, \mathcal{F}^R_{\mathsf{Ax}} \rangle$.

In the rest of this section, some well-known modal (and so, global) inference rules, see, for example, [6], will be considered in the present framework.

EXAMPLE 23 (Necessitation rule). *Let $\mathcal{H}^N_{\mathsf{Ax}}$ the Hilbert calculus obtained from $\mathcal{H}_{\mathsf{Ax}}$ by adding the necessitation inference rule*:

$$\frac{A}{\ominus A} \quad (\mathbf{N}).$$

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By Definition 21, *the set* $\mathcal{F}^N_{\mathsf{Ax}}$ *of level valuations for* $\mathbf{L}^N$ *can be defined as follows*:

- $\mathcal{F}^0_{\mathsf{Ax}} = \mathcal{F}_{\mathsf{Ax}}$
- $\mathcal{F}^{m+1}_{\mathsf{Ax}} = \{v \in \mathcal{F}^m_{\mathsf{Ax}} : \forall B \in For(\Sigma),\ if\ \forall w \in \mathcal{F}^m_{\mathsf{Ax}}(w_1(B) = 1)\ then\ v_2(B) = 1\}$
- $\mathcal{F}^N_{\mathsf{Ax}} = \bigcap_{m=0}^{\infty} \mathcal{F}^m_{\mathsf{Ax}}$.

Observe that $\mathcal{F}^N_{\mathsf{Ax}}$ *coincides with the original definition of level valuations introduced by Kearns* (*see* [20], [9], [25]).

Example 24 (Distributive rule). *Let* $\mathcal{H}^D_{\mathsf{Ax}}$ *the Hilbert calculus obtained from* $\mathcal{H}_{\mathsf{Ax}}$ *by adding the following distributive inference rule*:

$$\frac{\ominus A \quad \ominus B}{\ominus(A \wedge B)} \quad (\mathbf{D}).$$

By Definition 21, *the set* $\mathcal{F}^D_{\mathsf{Ax}}$ *of level valuations for* $\mathbf{L}^D$ *can be defined as follows*:

- $\mathcal{F}^0_{\mathsf{Ax}} = \mathcal{F}_{\mathsf{Ax}}$
- $\mathcal{F}^{m+1}_{\mathsf{Ax}} = \{v \in \mathcal{F}^m_{\mathsf{Ax}} : \forall A, B \in For(\Sigma)\ if\ \forall w \in \mathcal{F}^m_{\mathsf{Ax}}(w_2(A) = w_2(B) = 1),\ then\ v_2(A \wedge B) = 1\}$
- $\mathcal{F}^D_{\mathsf{Ax}} = \bigcap_{m=0}^{\infty} \mathcal{F}^m_{\mathsf{Ax}}$.

Example 25 (Congruentiality rule). *Let* $\mathcal{H}^C_{\mathsf{Ax}}$ *the Hilbert calculus obtained from* $\mathcal{H}_{\mathsf{Ax}}$ *by adding the congruentiality inference rule*:

$$\frac{A \leftrightarrow B}{\ominus A \leftrightarrow \ominus B} \quad (\mathbf{C}).$$

By Definition 21, *the set* $\mathcal{F}^C_{\mathsf{Ax}}$ *of level valuations for* $\mathbf{L}^C$ *can be defined as follows*:

- $\mathcal{F}^0_{\mathsf{Ax}} = \mathcal{F}_{\mathsf{Ax}}$
- $\mathcal{F}^{m+1}_{\mathsf{Ax}} = \{v \in \mathcal{F}^m_{\mathsf{Ax}} : \forall A, B \in For(\Sigma)\ if\ \forall w \in \mathcal{F}^m_{\mathsf{Ax}}(w_1(A) = w_1(B)),\ then\ v_2(A) = v_2(B)\}$
- $\mathcal{F}^C_{\mathsf{Ax}} = \bigcap_{m=0}^{\infty} \mathcal{F}^m_{\mathsf{Ax}}$.

Example 26 (Monotonicity rule). *Let* $\mathcal{H}^M_{\mathsf{Ax}}$ *the Hilbert calculus obtained from* $\mathcal{H}_{\mathsf{Ax}}$ *by adding the monotonicity inference rule*:

$$\frac{A \to B}{\ominus A \to \ominus B} \quad (\mathbf{M}).$$

By Definition 21, *the set* $\mathcal{F}^M_{\mathsf{Ax}}$ *of level valuations for* $\mathbf{L}^M$ *can be defined as follows*:

- $\mathcal{F}^0_{\mathsf{Ax}} = \mathcal{F}_{\mathsf{Ax}}$
- $\mathcal{F}^{m+1}_{\mathsf{Ax}} = \{v \in \mathcal{F}^m_{\mathsf{Ax}} : \forall A, B \in For(\Sigma)\ if\ \forall w \in \mathcal{F}^m_{\mathsf{Ax}}(w_1(A) \leq w_1(B)),\ then\ v_2(A) \leq v_2(B)\}$
- $\mathcal{F}^M_{\mathsf{Ax}} = \bigcap_{m=0}^{\infty} \mathcal{F}^m_{\mathsf{Ax}}$.

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PROPOSITION 2. *If* $\vdash_{\mathcal{H}^N_{\mathsf{Ax}}} A$ *then, for every* $v \in \mathcal{F}^N_{\mathsf{Ax}}$, $v_2(A) = 1$, *and so* $v(A) = \mathsf{t}^+ = (1,1)$.

Proof. Suppose that $\vdash_{\mathcal{H}^N_{\mathsf{Ax}}} A$. Then $\vdash_{\mathcal{H}^N_{\mathsf{Ax}}} \ominus A$, by **N**-rule. Let $v \in \mathcal{F}^N_{\mathsf{Ax}}$. Hence, $v_2(A) = v_1(\ominus A) = 1$. □

§5. RNmatrices for the minimal bimodal logic Mb. Modal logics usually contain more than one modal operator in their language, e.g., alethic modal operators $\Box$ and $\Diamond$, deontic modal modalities O and P or the multitude of temporal modal operators. While usually those operators are interdefinable, typically $\Diamond A := \neg\Box\neg A$, there are systems where this does not need to be the case, for example, in intuitionistic modal logics. In this section, we introduce bimodal systems, where the different modal operators can be independent of each other or, depending on the modal axioms, are interdefinable.

We will consider a bimodal version of the minimal modal logic **M**, namely, the minimal bimodal logic **Mb**.[10] This logic is defined over a signature Σ_b obtained from Σ by replacing $\ominus$ with two modal operators, which will be denoted by $\ominus$ and $\otimes$.[11] As expected, the snapshots are now triples $z = (z_1, z_2, z_3)$ over **2** in which each coordinate represents a possible truth-value for the formulas A, $\ominus A$ and $\otimes A$, respectively. Hence, eight truth-values are obtained in this way, namely, $\mathsf{t}^{--} := (1,0,0)$, $\mathsf{t}^{-+} := (1,0,1)$, $\mathsf{t}^{+-} := (1,1,0)$, $\mathsf{t}^{++} := (1,1,1)$, $\mathsf{f}^{++} := (0,1,1)$, $\mathsf{f}^{+-} := (0,1,0)$, $\mathsf{f}^{-+} := (0,0,1)$ and $\mathsf{f}^{--} := (0,0,0)$. Let V_8 be the set of such truth-values, and let $D_8 = \{z \in V_8 : z_1 = 1\} = \{\mathsf{t}^{-+}, \mathsf{t}^{++}, \mathsf{t}^{--}, \mathsf{t}^{+-}\}$ be the set of designated values. Hence, $ND_8 = \{\mathsf{f}^{-+}, \mathsf{f}^{++}, \mathsf{f}^{--}, \mathsf{f}^{+-}\}$.

The definition of the multioperators over V_8 interpreting the connectives of Σ_b is a natural generalization of the four-valued case:

$$\begin{array}{rcl} \tilde{\neg}\, z & := & (\sim z_1, *, *); \\ \tilde{\ominus}\, z & := & (z_2, *, *); \\ \tilde{\otimes}\, z & := & (z_3, *, *); \\ z \mathbin{\tilde{\rightarrow}} w & := & (z_1 \Rightarrow w_1, *, *). \end{array}$$

Let $\mathcal{M}_b = \langle V_8, D_8, \mathcal{O}_b \rangle$ be the obtained eight-valued Nmatrix, where $\mathcal{O}_b(\#) = \tilde{\#}$ for every connective $\#$ in Σ_b. Thus, the truth-tables for $\mathcal{M}_b$ are the following:

$\tilde{\rightarrow}$	t^{++}	t^{+-}	t^{-+}	t^{--}	f^{++}	f^{+-}	f^{-+}	f^{--}
t^{++}	D_8	D_8	D_8	D_8	ND_8	ND_8	ND_8	ND_8
t^{+-}	D_8	D_8	D_8	D_8	ND_8	ND_8	ND_8	ND_8
t^{-+}	D_8	D_8	D_8	D_8	ND_8	ND_8	ND_8	ND_8
t^{--}	D_8	D_8	D_8	D_8	ND_8	ND_8	ND_8	ND_8
f^{++}	D_8	D_8	D_8	D_8	D_8	D_8	D_8	D_8
f^{+-}	D_8	D_8	D_8	D_8	D_8	D_8	D_8	D_8
f^{-+}	D_8	D_8	D_8	D_8	D_8	D_8	D_8	D_8
f^{--}	D_8	D_8	D_8	D_8	D_8	D_8	D_8	D_8

[10] Note that in principle we could add finitely many primitive modal operators, as the previous results will carry over.

[11] As in the previous sections, we will leave the interpretation of the modal operators unspecified.

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A	$\tilde{\neg} A$	$\tilde{\ominus} A$	$\tilde{\otimes} A$
t^{++}	ND_8	D_8	D_8
t^{+-}	ND_8	D_8	ND_8
t^{-+}	ND_8	ND_8	D_8
t^{--}	ND_8	ND_8	ND_8
f^{++}	D_8	D_8	D_8
f^{+-}	D_8	D_8	ND_8
f^{-+}	D_8	ND_8	D_8
f^{--}	D_8	ND_8	ND_8

Let $\mathcal{F}_b$ be the set of all the valuations over the Nmatrix $\mathcal{M}_b$. Thus, $v \in \mathcal{F}_b$ iff $v : For(\Sigma_b) \to V_8$ is a function satisfying the following properties:

- $v(\#A) \in \tilde{\#}\, v(A)$ for $\# \in \{\neg, \ominus, \otimes\}$;
- $v(A \to B) \in v(A) \tilde{\to} v(B)$.

Now, any valuation $v \in \mathcal{F}_b$ will be written as $v = (v_1, v_2, v_3)$ such that $v_1, v_2, v_3 : For(\Sigma_b) \to V_8$ and so $v(A) = (v_1(A), v_2(A), v_3(A))$ for every formula A. This means that, for every formulas A and B:

- $v(A) \in D_8$ iff $v_1(A) = 1$;
- $v_1(\neg A) = {\sim} v_1(A)$;
- $v_1(\ominus A) = v_2(A)$;
- $v_1(\otimes A) = v_3(A)$;
- $v_1(A \to B) = v_1(A) \Rightarrow v_1(B)$.

The logic **Mb** generated by the Nmatrix $\mathcal{M}_b$ is defined as follows: $\Gamma \models_{\mathbf{Mb}} A$ iff, for every $v \in \mathcal{F}_b$, $v(A) \in D_8$ whenever $v(B) \in D_8$ for every $B \in \Gamma$.

The proof of the following result can be adapted from the respective one for $\mathcal{M}$. So, let $\mathcal{H}_b$ be the standard Hilbert calculus for **CPL**, presented in the signature Σ_b (that is, no axioms nor rules for $\ominus$ nor for $\otimes$ are given, being **MP** the only inference rule).

THEOREM 8 (Soundness and completeness of $\mathcal{H}_b$ w.r.t. $\mathcal{M}_b$). *For every* $\Gamma \cup \{A\} \subseteq For(\Sigma_b)$, *it holds*: $\Gamma \vdash_{\mathcal{H}_b} A$ *iff* $\Gamma \models_{\mathbf{Mb}} A$.

Axiomatic extensions of $\mathcal{H}_b$ (over the signature Σ_b) can be considered, obtaining in each case an eight-valued and structural characteristic RNmatrix. The definition of the restricted set of valuations, in each extension, is exactly as in the case of $\mathcal{M}$, taking obviously into account that they are subsets of $\mathcal{F}_b$ instead of subsets of $\mathcal{F}$ and that the signature is Σ_b instead of Σ, where $\ominus$ is replaced by $\ominus$ or $\otimes$ in each axiom of the previous examples (of course axioms containing both $\ominus$ and $\otimes$ can also be considered).

Observe that the technique for reducing terms and equations for valuations and bivaluations presented at the end of §3, see Definitions 10–13, can be easily adapted to the bimodal setting.

For instance, consider the following axioms:[12]

(Di) $\otimes A \leftrightarrow \neg\ominus\neg A$
(Bo) $\ominus A \leftrightarrow \neg\otimes\neg A$

[12] Note that if we interpret $\ominus$ and $\otimes$ as $\Box$ and $\Diamond$, respectively, we obtain well-known axioms for modal logics.

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(D) $\ominus A \to \otimes A$
(B) $A \to \ominus\otimes A$
(5) $\otimes A \to \ominus\otimes A$
(C) $\otimes\ominus A \to \ominus\otimes A$.

The conditions imposed by these axioms on valuations and bivaluations are, respectively, the following:

Axiom	Valuations	Bivaluations
(Di)	$v_2(\neg A) = \sim v_3(A)$	$b(\ominus\neg A) = \sim b(\otimes A)$
(Bo)	$v_3(\neg A) = \sim v_2(A)$	$b(\otimes\neg A) = \sim b(\ominus A)$
(D)	$v_2(A) \leq v_3(A)$	$b(\ominus A) \leq b(\otimes A)$
(B)	$v_1(A) \leq v_2(\otimes A)$	$b(A) \leq b(\ominus\otimes A)$
(5)	$v_3(A) \leq v_2(\otimes A)$	$b(\otimes A) \leq b(\ominus\otimes A)$
(C)	$v_3(\ominus A) \leq v_2(\otimes A)$	$b(\otimes\ominus A) \leq b(\ominus\otimes A)$

Let us analyze the case of (Di) in detail: if $B = \otimes A \leftrightarrow \neg\ominus\neg A = \neg((\otimes A \to \neg\ominus\neg A) \to \neg(\neg\ominus\neg A \to \otimes A))$ then $v_1(B) = 1$ reduces to $v_1((\otimes A \to \neg\ominus\neg A) \to \neg(\neg\ominus\neg A \to \otimes A)) = 0$, which reduces to $v_1(\otimes A \to \neg\ominus\neg A) = 1$, $v_1(\neg(\neg\ominus\neg A \to \otimes A)) = 0$, which reduces to $v_1(\otimes A) \leq v_1(\neg\ominus\neg A)$, $v_1(\neg\ominus\neg A \to \otimes A) = 1$. The latter reduces to $v_3(A) \leq \sim v_1(\ominus\neg A)$, $v_1(\neg\ominus\neg A) \leq v_1(\otimes A)$, which reduces to $v_3(A) \leq \sim v_2(\neg A), \sim v_1(\ominus\neg A) \leq v_3(A)$, which reduces to $v_3(A) \leq \sim v_2(\neg A), \sim v_2(\neg A) \leq v_3(A)$. The latter is equivalent to the condition $v_2(\neg A) = \sim v_3(A)$.

Concerning bivaluations, $b(B) = 1$ reduces to $b((\otimes A \to \neg\ominus\neg A) \to \neg(\neg\ominus\neg A \to \otimes A)) = 0$, which reduces to $b(\otimes A \to \neg\ominus\neg A) = 1$, $b(\neg(\neg\ominus\neg A \to \otimes A)) = 0$, which reduces to $b(\otimes A) \leq b(\neg\ominus\neg A)$, $b(\neg\ominus\neg A \to \otimes A) = 1$. The latter reduces to $b(\otimes A) \leq \sim b(\ominus\neg A)$, $b(\neg\ominus\neg A) \leq b(\otimes A)$, which reduces to $b(\otimes A) \leq \sim b(\ominus\neg A)$, $\sim b(\ominus\neg A) \leq b(\otimes A)$. The latter is equivalent to the condition $b(\ominus\neg A) = \sim b(\otimes A)$.

It is worth noting that, if axioms (Di) and (Bo) are taken together then $v(\neg A) = (\sim v_1(A), \sim v_3(A), \sim v_2(A))$ for every A. That is, the restricted valuations are deterministic for negated formulas.

Let Ax be a finite set of modal axioms in the signature Σ_b. Let **Lb** be the bimodal logic given by means of the Hilbert calculus $\mathcal{H}_{b\mathsf{Ax}}$, obtained by $\mathcal{H}_b$ by adding all the axioms in Ax. By adapting the proofs for the monomodal case, it is easy to prove that this logic is semantically characterized by the RNmatrix $\mathcal{RM}_{b\mathsf{Ax}} = \langle \mathcal{M}, \mathcal{F}_{b\mathsf{Ax}} \rangle$. The adaptations to be done with respect to the monomodal case are simple.

LEMMA 6. *Let Δ be an A-saturated set in $\mathcal{H}_{b\mathsf{Ax}}$, and let $b^\Delta : For(\Sigma_b) \to \mathbf{2}$ given by $b^\Delta(B) = 1$ iff $B \in \Delta$, for every B. Then, b^Δ is a bivaluation for* **Lb**.

LEMMA 7. *Let* b *be a bivaluation for* **Lb**. *Then, the function $v : For(\Sigma_b) \to V_8$ given by $v(A) = (b(A), b(\ominus A), b(\otimes A))$ for every A belongs to $\mathcal{F}_{b\mathsf{Ax}}$. In addition: $b(A) = 1$ iff $v_1(A) = 1$, for every A.*

LEMMA 8. *Let* v *be a valuation in $\mathcal{F}_{b\mathsf{Ax}}$. Then, the function $b : For(\Sigma_b) \to \mathbf{2}$ given by $b(A) = v_1(A)$ for every A is a bivaluation for* **Lb** *such that, for every B, $b(B) = 1$ iff $v_1(B) = 1$.*

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THEOREM 9 (Soundness and completeness of $\mathcal{H}_{b\mathsf{Ax}}$ w.r.t. bivaluation semantics and the RNmatrix $\mathcal{RM}_{b\mathsf{Ax}}$). $\Gamma \vdash_{\mathcal{H}_{b\mathsf{Ax}}} A$ *iff* $\Gamma \models_2^{\mathbf{Lb}} A$ *iff* $\Gamma \models_{\mathcal{RM}_{b\mathsf{Ax}}} A$.

As before it is possible, as well, to add global inference rules to any bimodal logic **Lb** as defined above. It is worth noting that the technique for defining level valuation semantics from a given global inference rule **R** presented in §4 does not depend on the modal signature of the underlying logic. This being so, the soundness and completeness results obtained in §4 can be easily adapted to the bimodal (or, in general, the multimodal) case. The only difference will appear in concrete examples of modal inference rules, in which the reduction of the conditions for the level valuations must be done for the bimodal case. By adapting the proofs of Theorems 6 and 7 to the bimodal setting, we obtain the following result

THEOREM 10 (Soundness and completeness of $\mathcal{H}^R_{b\mathsf{Ax}}$ w.r.t. level valuation semantics). $\Gamma \vdash_{\mathcal{H}^R_{b\mathsf{Ax}}} A$ *if and only if* $\Gamma \models_{\mathbf{Lb}^R} A$.

EXAMPLE 27 (Necessitation rules). *Consider the following global inference rules*:

$$\mathbf{N}_\ominus\text{-rule} \quad \frac{A}{\ominus A} \qquad\qquad \mathbf{N}_\otimes\text{-rule} \quad \frac{A}{\otimes A}.$$

Let $\mathcal{H}^{N_\ominus}_{b\mathsf{Ax}}, \mathcal{H}^{N_\otimes}_{b\mathsf{Ax}}, \mathcal{H}^{N_{\ominus\otimes}}_{b\mathsf{Ax}}$ *be the extensions of the Hilbert calculus* $\mathcal{H}_{b\mathsf{Ax}}$ *by adding the* $\mathbf{N}_\ominus$*-rule, the* $\mathbf{N}_\otimes$*-rule, or both. The notion of derivation is then analogous to the one given in Definition* 19. *The addition of one or both of those rules will be denoted by* $\mathbf{Lb}^{N_\ominus}$, $\mathbf{Lb}^{N_\otimes}$ *or* $\mathbf{Lb}^{N_{\ominus\otimes}}$, *respectively. The semantics for those system will be given, as in the previous cases, by considering level valuations over the corresponding RNmatrices. The set of level valuations will be given, respectively, by* $\mathcal{F}^{N_\ominus}_{b\mathsf{Ax}} = \bigcap_{m=0}^{\infty} \mathcal{F}^{\ominus,m}_{b\mathsf{Ax}}$, $\mathcal{F}^{N_\otimes}_{b\mathsf{Ax}} = \bigcap_{m=0}^{\infty} \mathcal{F}^{\otimes,m}_{b\mathsf{Ax}}$ *and* $\mathcal{F}^{N_{\ominus\otimes}}_{b\mathsf{Ax}} = \bigcap_{m=0}^{\infty} \mathcal{F}^{\ominus\otimes,m}_{b\mathsf{Ax}}$. *The sets of level valuations are such that* $\mathcal{F}^{\ominus,0}_{b\mathsf{Ax}} = \mathcal{F}^{\otimes,0}_{b\mathsf{Ax}} = \mathcal{F}_{b\mathsf{Ax}}$, *and*

- $\mathcal{F}^{\ominus,m+1}_{b\mathsf{Ax}} = \{v \in \mathcal{F}^{\ominus,m}_{b\mathsf{Ax}} : \forall B \in For(\Sigma_b),\ \text{if } \forall w \in \mathcal{F}^{\ominus,m}_{b\mathsf{Ax}}(w_1(B) = 1)\ \text{then } v_2(B) = 1\}$
- $\mathcal{F}^{\otimes,m+1}_{b\mathsf{Ax}} = \{v \in \mathcal{F}^{\otimes,m}_{b\mathsf{Ax}} : \forall B \in For(\Sigma_b),\ \text{if } \forall w \in \mathcal{F}^{\otimes,m}_{b\mathsf{Ax}}(w_1(B) = 1)\ \text{then } v_3(B) = 1\}$
- $\mathcal{F}^{\ominus\otimes,m}_{b\mathsf{Ax}} = \mathcal{F}^{\ominus,m}_{b\mathsf{Ax}} \cap \mathcal{F}^{\otimes,m}_{b\mathsf{Ax}}$ *for every* m, *then* $\mathcal{F}^{N_{\ominus\otimes}}_{b\mathsf{Ax}} = \mathcal{F}^{N_\ominus}_{b\mathsf{Ax}} \cap \mathcal{F}^{N_\otimes}_{b\mathsf{Ax}}$.

COROLLARY 2. *All the systems we have considered* (*including those with global rules*) *so far are subsystems of* ***S5***. *This means they are all consistent, i.e., they have A-saturated sets.*

The above corollary does not entail that any possible extension of the logics discussed will be a subsystem of **S5**; in particular, trivial logics are not excluded as possible extensions.

COROLLARY 3. *If* $\vdash_{\mathcal{H}^{N_{\ominus\otimes}}_{b\mathsf{Ax}}} A$ *then* $v_2(A) = v_3(A) = 1$, *and so* $v(A) = \mathsf{t}^{++} = (1, 1, 1)$, *for every* $v \in \mathcal{F}^{N_{\ominus\otimes}}_{b\mathsf{Ax}}$.

Proof. Suppose that $\vdash_{\mathcal{H}^{N_{\ominus\otimes}}_{b\mathsf{Ax}}} A$. Then $\vdash_{\mathcal{H}^{N_{\ominus\otimes}}_{b\mathsf{Ax}}} \ominus A$ and $\vdash_{\mathcal{H}^{N_{\ominus\otimes}}_{b\mathsf{Ax}}} \otimes A$, by the rules for $\ominus$ and $\otimes$. Let $v \in \mathcal{F}^{N_{\ominus\otimes}}_{b\mathsf{Ax}}$. Hence, $v_2(A) = v_1(\ominus A) = 1$ and $v_3(A) = v_1(\otimes A) = 1$. □

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§6. Decidability of non-normal modal logics by means of truth-tables. Despite being defined over finite (four-valued) Nmatrices, the RNmatrices for **S4** and **S5** introduced by Kearns in [20] do not give origin to a decision procedure for these logics by means of truth-tables. The reason is that, in order to check the validity of a formula A, the semantical information provided by A and all of its subfomulas in its truth-table generated by the corresponding Nmatrix is not enough to obtain a full criterion to delete the rows which do not correspond to level valuations. That is, there is no mechanical procedure to determine what rows should be deleted. A solution to this problem, for modal logics **KT** and **S4**, was obtained by L. Grätz in [17], where a mechanical procedure was found to delete the unsound rows from the three-valued non-deterministic truth-tables generated by each formula being tested. Using a related by somewhat different approach, O. Lahav and Y. Zohar proposed in [22] decision procedures for **K** and **KT** based on finite-valued RNmatrices.

It is well-known that not every modal system with global inference rules is decidable. However, even for decidable systems, it is not immediate how to obtain a decision procedure (by means of truth-tables) from the characteristic RNmatrix obtained within our framework. That is, decidability techniques by truth-tables for such logics require a case-by-case analysis, by generalizing the techniques introduced in [17] and [22].

It seems clear that the lack of a decision procedure for a modal logic by means of truth-tables, obtained from its characteristic RNmatrix, is due to the existence of global inference rules, which forces defining level valuations. This infinitary procedure can make it difficult to extract an algorithm from it. This would suggest that any finite axiomatic extension **L** of **M**, without global rules, is decidable by means of truth-tables. That is, the characteristic RNmatrix of **L** would allow to define an algorithm to delete unsound rows from the truth-table of any formula being tested.

In this section, it will be shown how to obtain a truth-table decision procedure, obtained from its RNmatrix, for some of the (non-normal) systems analyzed here, both monomodal and bimodal. The extension of the techniques to be introduced below to the other systems without global rules appears to be immediate.

However, as it will be shown in Example 29, there is a purely axiomatic extension of **M** (called $\mathbf{M}_E$) that is not decidable by the truth-table procedure obtained from the corresponding RNmatrix. This problem will be overcome by means of a detailed analysis of the conditions for the permitted valuations for $\mathbf{M}_E$, which allows to obtain an improved set of conditions to delete unsound rows in the truth-tables (see Theorem 14). This example shows that, even for modal systems without global rules, the definition of a decision procedure by truth-tables must be analyzed case-by-case.

Let us start our analysis. Let $A(p_1, \ldots, p_n)$ be a formula depending on propositional variables $p_1, \ldots, p_n$. From the examples considered in the previous sections, it is immediate to see that in order to check the validity of A in a non-normal system **L** (**Lb**) characterized by an RNmatrix $\mathcal{RM} = \langle \mathcal{M}, \mathcal{F}' \rangle$ (in the monomodal case) or $\mathcal{RM}_b = \langle \mathcal{M}_b, \mathcal{F}'_b \rangle$ (in the bimodal case),[13] a finite row-branching truth table needs to be constructed. This truth-table is obtained by considering the multioperators of the Nmatrix $\mathcal{M}$ (or $\mathcal{M}_b$), starting from the atomic formulas $p_1, \ldots, p_n$.

To be more specific, define the *complexity* $l(B)$ of a formula $B \in For(\Sigma)$ as follows: $l(B) = 0$ if B is a variable; $l(\#B) = l(B) + 1$ if $\# \in \{\neg, \ominus\}$; and $l(B \to C) = l(B) +$

[13] Since each logic satisfies the deduction metatheorem, it is enough to check tautologicity.

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$l(C)+1$. In $For(\Sigma_b)$, the complexity is defined analogously, but now $l(\#B)=l(B)+1$ if $\# \in \{\neg, \ominus, \otimes\}$.

Then, consider the sequence $A_1 \ldots A_k = A$ formed by all the subformulas of A, linearly ordered by complexity (formulas with the same complexity are ordered arbitrarily, with the only restriction of putting $p_1, \ldots, p_n$ in the first n places). From this, $l(A_i) \leq l(A_{i+1})$ for $1 \leq i \leq k$. The table starts by considering all the possible combination of truth-values for the first n-columns $p_1, \ldots, p_n$ (that is, 4^n rows are generated in the monomodal case, and 8^n in the bimodal case). Then, the possible values for the formula $A_{n+1}(p_1, \ldots, p_n)$ are computed in each row, according to the semantics given in $\mathcal{M}$ (or in $\mathcal{M}_b$) to the (unique) conective occurring in A_{n+1}. Consider the four-valued case. By definition of the multioperators of $\mathcal{M}$, each row splits into two ones (since both sets D_4 and ND_4 have elements). In the bimodal case, each row splits into four ones. The procedure continues, by considering the possible values for A_{n+2} on the $2 \cdot 4^n$ (or $4 \cdot 8^n$) rows generated in the previous step, and so on. This procedure will eventually terminate by computing all the possible values for A in all the rows, generating in this way a finite truth-table.

After this, some rows in the obtained finite truth-table need to be be removed, in case they violate the conditions stated by $\mathcal{F}'$ (i.e., they correspond to valuations which do not belong to $\mathcal{F}'$). For instance, if axiom (K) is valid in a monomodal **L**, then any row assigning values x^+, y^- and z^+ to formulas A, B and $A \to B$, respectively (for some $x, y, z \in \{\mathrm{t}, \mathrm{f}\}$), must be removed: indeed in this case $\bar{v}_2(A \to B) = 1 \not\leq 0 = 1 \Rightarrow 0 = \bar{v}_2(A) \Rightarrow \bar{v}_2(B)$, where $\bar{v}$ is the mapping representing the values assigned to the formulas labeling the columns of the given truth-table. That is, $\bar{v}$ *does not* correspond to a valuation in $\mathcal{F}'$, given that it does not satisfy axiom (K), hence it must be removed from the table. It is worth noting that the process of eliminating rows can be done "on the fly", that is, when the table is being constructed. In this way, the first time that a situation is found allowing to delete a row, the row is deleted precisely at that point. This allows to produce smaller truth-tables, turning the process easier to be realized (see, for instance, the case of the bimodal logic characterized by the RNmatrix $\mathcal{RM}_{b\mathsf{K}}$ to be analyzed below).

Let $\mathcal{T}'$ be the obtained (reduced) truth table. If A gets a designated value in every row of $\mathcal{T}'$ then A is declared to be valid; otherwise, it is declared non-valid. By observing the examples of non-normal systems considered in the previous sections, it seems clear that this process can be done algorithmically.

In order to prove that this methodology is sound and complete (that is, in order to verify that this procedure *indeed* represents the RNmatrix semantics given by $\mathcal{RM}'$) the proof proposed in [14, sec. 4.3] for the RNmatrices for the logics C_n will be adapted. Hence, two steps are required:

(i) to prove that any row of $\mathcal{T}'$ represents a valuation in $\mathcal{F}'$;
(ii) to prove that any valuation in $\mathcal{F}'$, restricted to the formulas occurring in the columns of $\mathcal{T}'$, is a row of $\mathcal{T}'$.

From now on, the set of subformulas of a formula A (in $For(\Sigma)$ or $For(\Sigma_b)$) will be denoted by $SF(A)$. A set of formulas Γ is said to be *closed under subformulas* if it satisfies the following: if $A \in \Gamma$ and $B \in SF(A)$ then $B \in \Gamma$.

Let us start with axiom (GL) in the monomodal scenario.

PROPOSITION 3. *Let Γ be a finite set of formulas in $For(\Sigma)$ closed under subformulas. Let $\bar{v} : \Gamma \to V_4$ be a function satisfying the following*: (1) $\bar{v}(\#A) \in \tilde{\#}\,\bar{v}(A)$ *for*

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$\# \in \{\neg, \ominus\}$; (2) $\bar{v}(A \to B) \in \bar{v}(A) \tilde{\to} \bar{v}(B)$; *and* (3) $\bar{v}_2(\ominus A \to A) \leq \bar{v}_2(A)$ (*where* $\bar{v}(A)$ *is written as* $(\bar{v}_1(A), \bar{v}_2(A))$ *for every* A). *Then, there exists a valuation* $v \in \mathcal{F}_{\mathsf{GL}}$ *such that* $v(A) = \bar{v}(A)$ *for every* $A \in \Gamma$.

Proof. Let $F_n = \{A \in For(\Sigma) \ : \ l(A) \leq n\}$, for every $n \geq 0$. Clearly $For(\Sigma) = \bigcup_{n \geq 0} F_n$ and $F_n \subseteq F_{n+1}$ for every $n \geq 0$. The function v will be defined inductively over F_n, for $n \geq 0$, satisfying items (1)–(3) above, such that $v(A) = \bar{v}(A)$ if $A \in \Gamma \cap F_n$. Observe that, if $A \notin \Gamma$ then $\ominus A \notin \Gamma$; analogously, if either $\ominus A \notin \Gamma$ or $A \notin \Gamma$ then $\ominus A \to A \notin \Gamma$. This will allows us to define the values of $v(\ominus A)$ and $v(\ominus A \to A)$ with some degree of freedom in such situations, respectively.

Base $n = 0$. If $A \in \mathcal{V}$ define $v(A) = \bar{v}(A)$ if $A \in \Gamma$, and $v(A)$ arbitrary otherwise. Observe that items (1)–(3) above are vacuously true for v over F_0, and $v(A) = \bar{v}(A)$ if $A \in \Gamma \cap F_0$.

Inductive step. Suppose that v was defined over F_n satisfying items (1)–(3) above, such that $v(A) = \bar{v}(A)$ if $A \in \Gamma \cap F_n$. Let us define v over F_{n+1}. So, let $A \in F_{n+1}$. If $A \in \Gamma$ define $v(A) = \bar{v}(A)$. Otherwise, if $A = \#B$ define $v(A) \in \tilde{\#}\, v(B)$ arbitrarily, for $\# \in \{\neg, \ominus\}$. If $A = B \to C \neq \ominus C \to C$ then define $v(A) \in v(B) \tilde{\to} v(C)$ arbitrarily. Finally, if $A = \ominus C \to C$ define $v(A) \in v(\ominus C) \tilde{\to} v(C)$ (i.e., $v_1(A) = v_1(\ominus C) \Rightarrow v_1(C)$) such that $v_2(A) \leq v_2(C)$. This concludes the definition of $v(A)$ over F_{n+1}. It is immediate to see that v satisfies items (1)–(3) above, and $v(A) = \bar{v}(A)$ if $A \in \Gamma \cap F_{n+1}$. Hence, this procedure produces a function $v \in \mathcal{F}_{\mathsf{GL}}$ such that $v(A) = \bar{v}(A)$ for every $A \in \Gamma$. □

THEOREM 11. *The process for constructing, for any formula* A, *a row-branching truth table based on* $\mathcal{RM}_{\mathsf{GL}}$ *produces a decision procedure for* $\mathbf{M}_{\mathsf{GL}}$. *That is, a formula* A *is valid in* $\mathbf{M}_{\mathsf{GL}}$ *iff the truth-table for* A *constructed by means of the algorithm above assigns to* A *a designated value in any row.*

Proof. Given a formula $A \in For(\Sigma)$, construct the row-branching truth table for it, deleting all the invalid rows, as indicated above, obtaining a reduced table $\mathcal{T}'$. It is worth noting that it is always possible to do this, and the procedure finishes after a finite number of steps. Let $\Gamma = SF(A)$ be the set of subformulas of A. Then, $A \in \Gamma$, and Γ is obviously closed under subformulas. Observe that Γ is the set of formulas labeling the columns of $\mathcal{T}'$. Then, each row of $\mathcal{T}'$ corresponds to a function $\bar{v} : \Gamma \to V_4$ which clearly satisfies the hypothesis of Proposition 3. Hence, there exists $v \in \mathcal{F}_{\mathsf{GL}}$ such that $v(B) = \bar{v}(B)$ for every $B \in \Gamma$, by Proposition 3. Conversely, if v is a valuation in $\mathcal{F}_{\mathsf{GL}}$ then its restriction $\bar{v}$ to the domain Γ coincides, necessarily, with some row of $\mathcal{T}'$. This is a consequence of the form in which $\mathcal{T}'$ was constructed. Hence, v is one of the possible extensions of a row in $\mathcal{T}'$. From this, it follows that A is valid according to $\mathcal{T}'$ (that is, A takes only designated values in the rows of $\mathcal{T}'$) if and only if $v(A)$ is designated for any $v \in \mathcal{F}_{\mathsf{GL}}$ (that is, iff $\models_{\mathcal{RM}_{\mathsf{GL}}} A$). □

Consider now axiom (K). A similar algorithm can be given for constructing a (reduced) truth-table for any formula A obeying the restrictions imposed by $\mathcal{F}_{\mathsf{K}}$, namely: $\bar{v}_2(A \to B) \leq \bar{v}_2(A) \Rightarrow \bar{v}_2(B)$, where $\bar{v}$ represents the values assigned, in an specific row, to the formulas in the columns of the table being constructed, and $\bar{v}(A)$ is written as $(\bar{v}_1(A), \bar{v}_2(A))$ for every A. Hence, the following can be proven

PROPOSITION 4. *Let* Γ *be a finite set of formulas in* $For(\Sigma)$ *closed under subformulas. Let* $\bar{v} : \Gamma \to V_4$ *be a function satisfying the following*: (1) $\bar{v}(\#A) \in \tilde{\#}\, \bar{v}(A)$ *for*

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$\# \in \{\neg, \ominus\}$; (2) $\bar{v}(A \to B) \in \bar{v}(A) \tilde{\to} \bar{v}(B)$; *and* (3) $\bar{v}_2(A \to B) \leq \bar{v}_2(A) \Rightarrow \bar{v}_2(B)$. *Then, there exists a valuation* $v \in \mathcal{F}_{\mathsf{K}}$ *such that* $v(A) = \bar{v}(A)$ *for every* $A \in \Gamma$.

THEOREM 12. *The process for constructing, for any formula* A, *a row-branching truth table based on* $\mathcal{RM}_{\mathsf{K}}$ *produces a decision procedure for* $\mathbf{M}_{\mathsf{K}}$.

Let $\mathbf{GL}^{-}$ be the logic obtained by adding the set of axioms $GLK = \{(\mathsf{GL}), (\mathsf{K})\}$ to $\mathcal{M}$. Then, $\mathbf{GL}^{-}$ is axiomatized by $\mathcal{H}_{GLK}$ and semantically characterized by the RNmatrix $\mathcal{RM}_{GLK} = \langle \mathcal{M}, \mathcal{F}_{GLK} \rangle$ such that

$$\mathcal{F}_{GLK} = \{v \in \mathcal{F} \ : \ v_2(A \to B) \leq v_2(A) \Rightarrow v_2(B) \text{ and } v_2(\ominus A \to A) \leq v_2(A) \text{ for every } A, B\}.$$

In [16, theorem 18] it was shown that axiom (4): $\ominus A \to \ominus\ominus A$ is derivable in **GL**, Gödel–Löb logic of provability, which can be obtained from $\mathbf{GL}^{-}$ by adding the necessitation rule (the $\ominus$-rule in the signature Σ of $\mathbf{GL}^{-}$). By using our framework, we can prove that necessitation rule is essentially required to obtain (4) from **GL**.

PROPOSITION 5. *Axiom* (4) *is not derivable in* $\mathcal{H}_{GLK}$. *Hence, the necessitation rule is required to prove* (4) *in* ***GL***.

Proof. Take a propositional variable p and a valuation $v \in \mathcal{F}_{GLK}$ such that $v(p) = \mathsf{t}^{+}$ and $v(\ominus p) = \mathsf{t}^{-}$. Hence, $v(\ominus\ominus p)$ is not designated, while $v(\ominus p)$ is designated. From this, $v(\ominus p \to \ominus\ominus p)$ is not designated. That is, $\nvdash_{\mathcal{RM}_{GLK}} \ominus p \to \ominus\ominus p$. By soundness, $\nvdash_{\mathcal{H}_{GLK}} \ominus p \to \ominus\ominus p$. □

Consider now an example of bimodal logic: the logic $\mathbf{Mb}_{\mathsf{C}}$ obtained from **Mb** by requiring the validity of axiom (C): $\otimes\ominus A \to \ominus\otimes A$, which is is axiomatized by $\mathcal{H}_{b\mathsf{C}}$ and semantically characterized by the RNmatrix $\mathcal{RM}_{b\mathsf{C}} = \langle \mathcal{M}_b, \mathcal{F}_{b\mathsf{C}} \rangle$ such that

$$\mathcal{F}_{b\mathsf{C}} = \{v \in \mathcal{F}_b \ : \ v_3(\ominus A) \leq v_2(\otimes A) \text{ for every } A\}.$$

As doing before, an algorithm can be defined for constructing a (reduced) truth-table for any formula A by using the restrictions of $\mathcal{F}_{b\mathsf{C}}$. Observe that it is required to check, for each row, the values assigned to pairs of formulas of the form $(\ominus A, \otimes A)$ in order to get rows which are compatible with the definition of $\mathcal{F}_{b\mathsf{C}}$. In order to do this, it may be convenient to exclude the rows that violate the conditions while the truth table is being built, starting from the first pair of the form $(\ominus A, \otimes A)$ found between the columns of the table generated in each step of the process. That is, rows are excluded "on the fly".

PROPOSITION 6. *Let* Γ *be a finite set of formulas in* $For(\Sigma_b)$ *closed under subformulas. Let* $\bar{v} : \Gamma \to V_8$ *be a function satisfying the following*: (1) $\bar{v}(\#A) \in \tilde{\#}\,\bar{v}(A)$ *for* $\# \in \{\neg, \ominus, \otimes\}$; (2) $\bar{v}(A \to B) \in \bar{v}(A) \tilde{\to} \bar{v}(B)$; *and* (3) $\bar{v}_3(\ominus A) \leq \bar{v}_2(\otimes A)$. *Then, there exists a valuation* $v \in \mathcal{F}_{b\mathsf{C}}$ *such that* $v(A) = \bar{v}(A)$ *for every* $A \in \Gamma$.

Proof. The proof will be obtained by adapting the one given for Proposition 3. Thus, let $G_n = \{A \in For(\Sigma_b) \ : \ l(A) \leq n\}$, for every $n \geq 0$. Then $For(\Sigma_b) = \bigcup_{n \geq 0} G_n$ and $G_n \subseteq G_{n+1}$ for every $n \geq 0$. The function v will be defined inductively over G_n, for $n \geq 0$, satisfying items (1)–(3) above, in such a manner that $v(A) = \bar{v}(A)$ if $A \in \Gamma \cap G_n$.

Base $n = 0$. If $A \in \mathcal{V}$ define $v(A) = \bar{v}(A)$ if $A \in \Gamma$, and $v(A)$ arbitrary otherwise. Hence, items (1)–(3) above are vacuously true for v over G_0, and $v(A) = \bar{v}(A)$ if $A \in \Gamma \cap G_0$.

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Inductive step. Suppose that v was defined over G_n satisfying items (1)–(3) above, such that $v(A) = \bar{v}(A)$ if $A \in \Gamma \cap G_n$. Let us now define v over G_{n+1}. So, let $A \in G_{n+1}$. If $A \in \Gamma$ define $v(A) = \bar{v}(A)$. Otherwise, if $A = \neg B$ define $v(A) \in \tilde{\neg} v(B)$ arbitrarily. If $A = B \to C$ define $v(A) \in v(B) \tilde{\to} v(C)$ arbitrarily. Now, the cases $A = \ominus B$ and $A = \otimes B$ will be analyzed simultaneously. That is, the values of $\ominus B$ and $\otimes B$ will be defined simultaneously, for every $B \in G_n$. If both $\ominus B$ and $\otimes B$ belong to Γ then $v(\#B) = \bar{v}(\#B)$ for $\# \in \{\ominus, \otimes\}$. Otherwise, if $\ominus B \notin \Gamma$ but $\otimes B \in \Gamma$, define first $v(\otimes B) = \bar{v}(\otimes B)$ and then define $v(\ominus B) \in \tilde{\ominus} v(B)$ such that $v_3(\ominus B) \leq v_2(\otimes B)$. Now, if $\ominus B \in \Gamma$ but $\otimes B \notin \Gamma$, define first $v(\ominus B) = \bar{v}(\ominus B)$ and then define $v(\otimes B) \in \tilde{\otimes} v(B)$ such that $v_3(\ominus B) \leq v_2(\otimes B)$. Finally, if $\ominus B \notin \Gamma$ and $\otimes B \notin \Gamma$, define first $v(\ominus B) \in \tilde{\ominus} v(B)$ arbitrarily and then define $v(\otimes B) \in \tilde{\otimes} v(B)$ such that $v_3(\ominus B) \leq v_2(\otimes B)$. This concludes the definition of $v(A)$ over G_{n+1}. It is immediate to see that v satisfies items (1)–(3) above, and $v(A) = \bar{v}(A)$ if $A \in \Gamma \cap G_{n+1}$. Clearly, this procedure produces a function $v \in \mathcal{F}_{b\mathsf{C}}$, satisfying that $v(A) = \bar{v}(A)$ for every $A \in \Gamma$. □

THEOREM 13. *The process for constructing, for any formula A, a row-branching truth table based on $\mathcal{RM}_{b\mathsf{C}}$ produces a decision procedure for* $\mathbf{Mb_C}$.

REMARK 28. *It is worth noting that the truth-table method can be extended to inferences from premises. Indeed, by definition of $\mathcal{M}$ and of semantic consequence w.r.t. an Nmatrix, in order to check whether A is a consequence of $\{A_1, \ldots, A_n\}$ a (reduced) table $\mathcal{T}'$ for $\{A_1, \ldots, A_n, A\}$ must be constructed by following this procedure. In this case, it is necessary to consider $\Gamma = \bigcup_{i=1}^{n} SF(A_i) \cup SF(A)$. Finally, it is necessary to check that, in any row of $\mathcal{T}'$, A gets a designated value whenever all the premises A_1,..., A_n get designated values simultaneously. If this is the case, then it is declared that A follows from $\{A_1, \ldots, A_n\}$; otherwise, A does not follow from $\{A_1, \ldots, A_n\}$, and all the counterexamples (restricted to Γ) are obtained from $\mathcal{T}'$. Same observation holds for the bimodal case.*

It seems clear that the decidability method described above can be extended to the other systems considered here. However, it can be seen that the set of conditions for valuations obtained from *any* set of modal axioms does not necessarily produce a decision procedure as it stands. The reason is that certain additional conditions, obtained from the original ones, can be required. Hence, some valid formula can contain in its truth-table some rows which were not deleted by using exclusively the original conditions, and in which the formula gets a non-designated value. Specifically, consider the following example:[14]

EXAMPLE 29. *Consider the following axioms*:

(Ax1) $\ominus(A \to A)$
(Ax2) $\ominus(A \to B) \to \ominus(\ominus A \to B)$.

Let $\mathsf{Ax} = \{(\mathsf{Ax1}), (\mathsf{Ax2})\}$, *and let* $\boldsymbol{M}_E$ *be the axiomatic extension of* $\boldsymbol{M}$ *by adding* Ax. *Clearly,* Ax *imposes the following restrictions on the valuations*: (1) $v_1(\ominus(A \to A)) = 1$; (2) $v_1(\ominus(A \to B)) \leq v_1(\ominus(\ominus A \to B))$. *By Remark* 2, *this is equivalent to the following conditions*: (3) $v_2(A \to A) = 1$; (4) $v_2(A \to B) \leq v_2(\ominus A \to B)$. *However, although this set of conditions is enough for defining an RNmatrix such that* $\boldsymbol{M}_E$ *is sound and complete w.r.t. this RNmatrix, it is not enough for defining sound truth-tables for* $\boldsymbol{M}_E$. *Indeed,*

[14] We would like to thank one of the anonymous referees for pointing out this issue to us and by providing us with Example 29.

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it is immediate to see (*taking* $A = B = p$) *that* $\ominus(\ominus p \to p)$ *is derivable in* $\mathbf{M}_E$, *hence it is valid w.r.t. the obtained RNmatrix. However, conditions* (1)-(2) (*or, equivalently,* (3)-(4)) *are not enough to eliminate in the truth-table all the rows which do not satisfy these conditions*: *indeed, half of the rows of the truth-table for* $\ominus(\ominus p \to p)$ (*the ones producing non-designated values*) *should be deleted.*

p	$\ominus p$	$\ominus p \to p$	$\ominus(\ominus p \to p)$
t^+	t^+	t^+	D_4
		t^-	ND_4
	t^-	t^+	D_4
		t^-	ND_4
t^-	f^+	t^+	D_4
		t^-	ND_4
	f^-	t^+	D_4
		t^-	ND_4
f^+	t^+	f^+	D_4
		f^-	ND_4
	t^-	f^+	D_4
		f^-	ND_4
f^-	f^+	t^+	D_4
		t^-	ND_4
	f^-	t^+	D_4
		t^-	ND_4

The unsound rows in the truth-table of Example 29 were not deleted since none of the formulas involved in conditions (1)-(2) and (3)-(4) can be instantiated with the subformulas of $\ominus(\ominus p \to p)$ in order to apply such conditions.

Another example of a tautology of $\mathbf{M}_E$ which is not recognized by the induced truth-tables method is $A := \ominus(p \to q) \to \ominus(\ominus\ominus p \to q)$ for $p, q \in \mathcal{V}$, $p \neq q$. Indeed, no row can be deleted from its truth-table by means of conditions (3)-(4). Because of this, its truth table contains rows in which A gets a non-designated value. For example, consider a row in which $p, q, p \to q, \ominus p, \ominus\ominus p$ and $\ominus(p \to q)$ get the value t^+, but $\ominus\ominus p \to q$ receives the value t^-. Then, $\ominus(\ominus\ominus p \to q)$ gets a non-designated value, just like A.

However, from the basic conditions (3)-(4), it is possible to *infer* additional ones, that expand the possibility of deleting rows, guaranteeing soundness of $\mathbf{M}_E$ w.r.t. its truth-tables. Indeed, by taking $A = B$ in (4), and taking into account (3), it follows that $v_2(\ominus A \to A) = 1$, for every A. Combining this with (4) once again (by instantiating A with $\ominus A$ and B with A) it follows that $v_2(\ominus\ominus A \to A) = 1$ and, by the same reasoning, that $v_2(\ominus\ominus\ominus A \to A) = 1$, and so on, for every A. Let $+ := \{\mathsf{t}^+, \mathsf{f}^+\}$. By defining $\ominus^0 A := A$ and $\ominus^{n+1} A := \ominus\ominus^n A$, the original conditions can be expanded to the following ones:

(C1) $v(A \to A) = \mathsf{t}^+$;
(C2) $v(\ominus^k A \to A) \in +$, for $k \geq 1$;
(C3) if $v(\ominus^k A \to B) \in +$ for some $0 \leq k < n$ and $A \neq B$ then $v(\ominus^n A \to B) \in +$.

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Let $\mathcal{F}_{\mathbf{M}_E}$ be the set of valuations over the Nmatrix $\mathcal{M}$ satisfying conditions (C1)–(C3). It is clear that a valuation v over the Nmatrix $\mathcal{M}$ satisfies conditions (3)-(4) iff it satisfies conditions (C1)–(C3). Thus, the set $\mathcal{F}_{\mathbf{M}_E}$ is the set of valuations over the Nmatrix $\mathcal{M}$ which characterizes the logic $\mathbf{M}_E$. In Proposition 7, it will be shown that any partial valuation defined over a finite set closed under subformulas which satisfies these conditions can be extended to a valuation for $\mathbf{M}_E$ (i.e., in $\mathcal{F}_{\mathbf{M}_E}$). Observe that half of the rows of the truth-table in Example 29, seen as partial valuations, do not satisfy property (C2), hence, they do not correspond to valuations for $\mathbf{M}_E$.

PROPOSITION 7. *Let Γ be a finite set of formulas in $For(\Sigma)$ closed under subformulas. Let $\bar{v} : \Gamma \to V_4$ be a function satisfying the following*: $(V1)$ $\bar{v}(\#A) \in \tilde{\#}\,\bar{v}(A)$ *for* $\# \in \{\neg, \ominus\}$; $(V2)$ $\bar{v}(A \to B) \in \bar{v}(A) \tilde{\to} \bar{v}(B)$; $(V3)$ $\bar{v}(A \to A) = \mathsf{t}^+$; $(V4)$ $\bar{v}(\ominus^k A \to A) \in +$, *for A and $k \geq 1$ such that $\bar{v}(\ominus^k A \to A)$ is defined*; *and* $(V5)$ *if $\bar{v}(\ominus^k A \to B) \in +$ for some $0 \leq k < n$ then $\bar{v}(\ominus^n A \to B) \in +$.*

Then, there exists a valuation $v \in \mathcal{F}_{\mathbf{M}_E}$ such that $v(A) = \bar{v}(A)$ for every $A \in \Gamma$.

Proof. Let $F_n = \{A \in For(\Sigma) \ : \ l(A) \leq n\}$, for every $n \geq 0$. Then, $For(\Sigma) = \bigcup_{n \geq 0} F_n$ and $F_n \subseteq F_{n+1}$ for every $n \geq 0$. The function v will be defined inductively over F_n, for $n \geq 0$, satisfying items (V1)–(V5), such that $v(A) = \bar{v}(A)$ if $A \in \Gamma \cap F_n$.

Base $n = 0$. If $A \in \mathcal{V}$ define $v(A) = \bar{v}(A)$ if $A \in \Gamma$, and $v(A)$ arbitrary otherwise. Clearly conditions (V1)–(V5) are vacuously true for v over F_0, and $v(A) = \bar{v}(A)$ if $A \in \Gamma \cap F_0$.

Inductive step. Suppose that v was defined over F_n satisfying conditions (V1)–(V5), such that $v(A) = \bar{v}(A)$ if $A \in \Gamma \cap F_n$ (IH). Let us define v over F_{n+1}. So, let $A \in F_{n+1}$. If $A \in \Gamma$ define $v(A) = \bar{v}(A)$. Otherwise, if $A = \#B$ define $v(A) \in \tilde{\#}\, v(B)$ arbitrarily, for $\# \in \{\neg, \ominus\}$. Finally, if $A = B \to C$, consider the following subcases: (i) $B = C$, then define $v(A) = \mathsf{t}^+$ (note that $v(A) \in v(B) \tilde{\to} v(C)$); (ii) $B = \ominus^k C$ for $k \geq 1$, then define $v(A) \in (v(B) \tilde{\to} v(C)) \cap +$ (observe that this set is always nonempty); (iii) $B = \ominus^n D$ for $D \neq C$ such that $v(\ominus^k D \to C) \in +$ for some $0 \leq k < n$, then define $v(A) \in (v(B) \tilde{\to} v(C)) \cap +$ (observe that this set is always nonempty); (iv) none of the situations (i)–(iii) hold, then define $v(A) \in v(B) \tilde{\to} v(C)$ arbitrarily. This concludes the definition of $v(A)$ over F_{n+1}. It is immediate to see that v satisfies (C1)–(C3), and $v(A) = \bar{v}(A)$ if $A \in \Gamma$. That is, this procedure produces a function $v \in \mathcal{F}_{\mathbf{M}_E}$ such that $v(A) = \bar{v}(A)$ for every $A \in \Gamma$. □

It is immediate to see that, given a formula A, the construction of a truth-table for A in $\mathbf{M}_E$ can be done in a finite number of steps by means of a mechanical procedure, such that each row is a function from the set Γ of subformulas of A to V_4 satisfying conditions (V1)–(V5) of Proposition 7. Indeed, when constructing a truth-table for A in $\mathcal{M}$, if a column corresponds to a formula $B \to B$ then every row must assign to it the value t^+. In turn, if the column corresponds to a formula $\ominus^k B \to B$ for some $k \geq 1$ then every row must assign to it a value in $(a \tilde{\to} b) \cap +$ (a nonempty set), where a and b are the values already assigned to $\ominus^k B$ and B in that row. Finally, if the column corresponds to a formula $\ominus^n B \to C$ for some $n \geq 1$ and $B \neq C$ such that there is a previous column corresponding to $\ominus^k B \to C$ for some $0 \leq k < n$ then every row must assign to $\ominus^n B \to C$ a value in $(a \tilde{\to} b) \cap +$ (a nonempty set), where a and b are the values already assigned to $\ominus^n B$ and C in that row, whenever $\ominus^k B \to C$ received a value in $+$ in that row. From this, we arrive to the following result.

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THEOREM 14. *The process for constructing, for any formula A, a row-branching truth table based on $\mathcal{RM}_{\boldsymbol{M}_E}$ produces a decision procedure for $\boldsymbol{M}_E$. That is, a formula A is valid in $\boldsymbol{M}_E$ iff the truth-table for A constructed by means of the algorithm above assigns to A a designated value in any row.*

Proof. Analogous to the proof of Theorem 11. □

The examples discussed in this section show that, given an arbitrary (purely) axiomatic extension of **M**, the definition of a sound and complete decision procedure by truth-tables obtained from the generated RNmatrix is not always guaranteed. As in the case of global rules and level valuations, the decidability must be analyzed case-by-case. It requires a detailed analysis of the set of conditions on the valuations, trying to obtain an equivalent (but improved) set of conditions which guarantees that any partial valuation satisfying the enlarged set of conditions can be extended to a full valuation over the RNmatrix. Finally, it is worth noting that there are in fact undecidable propositional modal logics, cf. [8].[15]

§7. Future work and discussion of the results. In this article, we presented a general semantical framework for modal logics that does not rely on the notion of possible worlds. In particular, we have shown that RNmatrices defined from swap structures in combination with a generalized level-valuations technique, originally developed by Kearns, and presented here for the first time for other modal rules than the rule of necessitation, is expressive enough to incorporate various modal logics, normal or non-normal.

We furthermore, when compared to previous approaches using Nmatrices, cf. [9, 25], have shown that RNmatrices induced by swap structures allow for a more flexible approach to semantics for many valued logics in general, and modal logics in particular, by providing a simple reduction procedure of formulas to restrictions on the set of valuation, that allowed for systems with for example the Löb axiom or the McKinsey formula, as axioms.

More concretely, we showed that our semantical approach goes beyond Kripke structures and non-deterministic semantics as considered in previous work in the following ways:

1. We provided restricted non-deterministic semantics for non-Sahlquist formulas as axioms, such as the Gödel–Löb axiom or the McKinsey formula, which do not have a (first-order) Kripke semantics.
2. Simple extensions of **M** with axioms such as $\Box(A \to A)$ do neither have a Kripke semantics nor a corresponding Nmatrix, cf. [17]. We have shown that there is RNmatrix semantics for such extensions.
3. We presented a decision procedure for selected modal systems and discussed some restrictions to it.
4. By focusing on the weakest modal logic **M** and investigating its extensions, we introduced a new hierarchy of (bi-, multi-) modal systems with or without (non-standard, multiple) global rules.

7.1. Scope and limits of the method. We presented our method for constructing semantics for modal logics as general as possible, with the hope, that the generality of

[15] We thank one of the anonymous referees for pointing out this fact.

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the method is a convincing argument in favor of RNmatrices for modal logics, as an alternative to Kripke semantics. However, we are aware of the fact that our approach is not yet as flexible as Kripke semantics regarding some properties. For example, we have not discussed axiom systems with an infinite number of axioms. While the construction method of RNmatrices for extensions of **M** might give us some arguments that, at least, recursively defined infinite sets of axioms might be expressed in terms of RNmatrices, we might discuss infinite axiom systems, but leave this as a project for future work.

Another limitation of our approach at this moment is a property that is called *analyticity*. In short, if analyticity holds any partial valuation which seems to refute a given formula can be extended to a full valuation (which necessarily refutes that formula too). For example, in [25] it was shown that modal logics defined in terms of Nmatrices with global modal rules do not enjoy this property. It is obvious that the failure of analyticity carries over to RNmatrices with global modal rules. Since the failure of analyticity is related to decidability, it seems our presented semantics for modal logics with global modal rules are not decidable. It should of course be mentioned, that in the absence of such global modal rules it is possible to find decision procedures, but the definition of such procedures has to be analyzed case by case, cf. §6. Needless to say there is gleam of hope. In more recent publications, cf. [17] and [22], it was shown that by a slight adjustment of the level-valuations technique it is possible to regain decidability. The results were proven for the normal modal logics **K** and **KT** expressed in terms of Nmatrices. It is therefore only a matter of time to prove similar results for other modal logics with global modal rules (see, for instance, [23]). However, as already mentioned above, not every modal system with global modal rules is decidable.

7.2. Generalization of the method. There are two straightforward generalizations of our approach to modality that we have not addressed so far. 1) Higher-order versions of RNmatrices for modal logics and 2) n-ary modal operators as defined in [5].

As for 1), this seems to be nice technical exercise by taking inspiration from first-order versions of **FDE**-based logics, cf. [33], where the main feature of the semantics is dividing the extension of predicates into extension/antiextension pairs. Similary, this can be done for first-order versions of extensions of **M**, for example, in the following manner: an *Interpretation* $\mathcal{I}$ is a tuple $\langle U, i_1, i_2\rangle$, where U is a non-empty set and we assign both the *extension* of $i_1(P) \subseteq U^n$ and *modal-extension* $i_2(P) \subseteq U^n$ to each n-ary predicate symbol P. Given any interpretation $\langle U, i_1, i_2\rangle$ we then can define a valuation v from all sentences into $\mathbf{2}^2$, where $\mathbf{2} = \{0, 1\}$. Based on that we would be able to inductively define the value of sentences, similarly to the truth-tables in §2. A first attempt of presenting many-valued non-deterministic first-order semantics for modal logics without global rules can be found in [11]. Of course, one needs to be careful, when adding global rules to those first-order versions. But we leave this endeavor for future research.

As for 2), the situation is slightly more complicated, as for example well-known binary modal operators, strict implication might not be representable just in terms of RNmatrices. This is of course a conjecture and not a proven fact. But in case of strict implication, it seems, the corresponding Kripke semantics implicitly uses a global rule in the definition of the operators, which is something that cannot be expressed in terms of RNmatrices/swap structures alone, at least not in a straightforward manner.

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We could think of defining strict implication $\dashv\!\!\vdash$ in terms of $\rightarrow$, as follows: $v(A \dashv\!\!\vdash B) = v(\Box(A \rightarrow B))$ and depending on our semantics for $\Box$, we could define different notions of strictness. However, without globally restricting the set of all valuations, none of the sentences $A \dashv\!\!\vdash B$ would be a tautology. That is because no sentence $A \rightarrow B$ would be assigned the value t^+ for all valuations. For the moment, we will leave the question of how to define n-ary modal operators open.

7.3. Nmatrices vs. RNmatrices. In this paper, we presented a way to deal with modal systems, with or without global rules, by means of RNmatrix semantics. But the first semantical characterization of the normal modal systems **T**, **S4** and **S5**, as mentioned above, was introduced by J. Kearns by means of four-valued RNmatrices by utilizing the so-called level-valuations technique. There is, however, another semantical approach to non-normal modal logics, i.e., without global modal rules, based on Nmatrices.

Starting in the 1970s, Yuri Ivlev introduced a series of non-normal modal systems which are semantically characterized by (finite) Nmatrices, anticipating in decades the use of Nmatrices for non-classical logics introduced in the 2000s by A. Avron and I. Lev (see [2, 2]). Among other systems, Ivlev introduced a four-valued non-normal version of the system **KT** called S_a+ (see, for instance, [19]). This system was afterward studied and extended in [9] (as Tm) and [25] (as $\mathbf{T}^{-}$). It can be presented in the signature Σ_b, where $\ominus A$ represent $\Box A$ and $\otimes A$ can represent $\Diamond A$ or, alternatively, the modality $\Box\neg A$. The latter was the option chosen in [13], thus defining a swap structure in which the snapshots are triples $z = (z_1, z_2, z_3)$ such that the coordinate z_i represents, respectively, a 0/1 truth-value for the formulas A, $\Box A$ and $\Box\neg A$. However, instead of considering the universe V_8 of eight truth-values of the Nmatrix $\mathcal{M}_b$ and defining an RNmatrix based on the axioms Ax_T of S_a+, the standard approach mentioned in the Introduction was used in [13]. Thus, the axioms in Ax_T imposed restrictions on the universe of snapshots, eliminating some of them, as well as on the multioperations, defining an *Nmatrix* rather than an RNmatrix. Hence, axiom (T) imposes the restrictions $z_2 \leq z_1$ and $z_3 \leq \sim z_1$ (or, equivalently, $z_1 \sqcap z_3 = 0$), and so only four snapshots are allowed, which in turn can be identified with the truth-values of Ivlev's original Nmatrix. This example shows that certain logics (modal, in this case) can be alternatively characterized by a finite-valued Nmatrix or by a finite-valued RNmatrix.

The advantage of our present approach is that the same Nmatrix $\mathcal{M}_b$ can be used for characterizing *any* bimodal logic defined as an axiomatic extension of the minimal bimodal logic **Mb**. In the case of logics other than (non-normal) modal logics, the use of RNmatrices instead of Nmatrices may be more of a necessity than an option.[16]

7.4. Philosophical remarks. Our approach leads to a semantics for which we presented sound and complete axiom systems with global or local rules. In that sense, at least with the addition of modal rules, we are justified to claim that we are actually doing modal logics. However, in the absence of such global modal rules, cf. §2 and §3, it seems, at the very least, questionable what the status of our operator $\ominus$ might be. Surely, we can define, as we did in §3 and §5, restrictions on the set of valuations that validate well-known modal formulas. But this is not yet an argument in favor of the modal

[16] To give a concrete example, it is well-known that da Costa's paraconsistent logic C_1 cannot be characterized a single finite Nmatrix. In turn, the logic C_1 was characterized in [14] by a three-valued decidable RNmatrix.

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nature of $\ominus$. We could furthermore think of concrete well-studied modal systems, such as systems of epistemic or deontic logic, where the rule of necessitation is the source of some paradoxes and therefore not unrestrictedly valid. But even in such systems other global modal rules are present, such as congruentiality. There are logics, called hyperintensional logics, for which even congruentiality fails to hold, cf. [4], and our approach is certainly able to capture such logics, as well, but we should be very clear, that we are not discussing any particular modal operator. Instead, what can be said in favor of our approach, we are able to capture a multitude of different modal concepts under one and the same umbrella – RNmatrices induced by swap structures, with or without global modal rules. Whether this will lead to a new understanding of the concept of modality remains to be open, and needs to be part of a larger investigation and discussion in the future.

Acknowledgements. We would like to thank the two anonymous referees for their valuable comments, suggestions and corrections on the previous version of this paper.

Funding. Marcelo Coniglio thanks the support of the São Paulo Research Foundation (FAPESP, Brazil) through the Thematic Project RatioLog #20/16353-3. Coniglio also acknowledges support from the National Council for Scientific and Technological Development (CNPq, Brazil), through the individual research grant #309830/2023-0. Pawel Pawlowski was supported by fund from FWO (Fonds voor Wetenschappelijk Onderzoek) grant number 1255724N.

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