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QUANTILE-BASED STUDY OF (DYNAMIC) INACCURACY MEASURES

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In the present communication, we introduce quantile-based (dynamic) inaccuracy measures and study their properties. Such measures provide an alternative approach to evaluate inaccuracy contained in the assumed statistical models. There are several models for which quantile functions are available in tractable form, though their distribution functions are not available in explicit form. In such cases, the traditional distribution function approach fails to compute inaccuracy between two random variables. Various examples are provided for illustration purpose. Some bounds are obtained. Effect of monotone transformations and characterizations are provided.

Keywords: bounds, characterization, Kerridge's inaccuracy measure, likelihood ratio quantile order, quantile function

Jel Classification: 94A17, 60N05

1. INTRODUCTION

It is well-known that the distribution function (DF) or quantile function (QF) is used to specify a probability model. There are various models which do not have closed-form DFs, though they have explicitly known QFs. For example, Govindarajulu, power-Pareto and various forms of lambda distributions do not carry tractable DFs. Most of the existing tools developed for the investigation of reliability and information properties of several statistical models are based on the DF. Unfortunately, DF-based tools are not suitable to study uncertainty measures of the models having untractable DFs. To investigate the information properties of such models, an alternative methodology based on QF has been introduced and studied in the literature. In modeling and analysis of statistical data, QF-based approach has been shown to be an efficient and equivalent alternative to DF (see Gilchrist [5] and Nair *et al.* [14]). Throughout the paper, we assume the random variables are nonnegative and absolutely continuous. Let X_1 be a random variable with DF F_1 and QF Q_1. Then, the QF of X_1 is defined by

$$Q_1(p) = F_1^{-1}(p) = \inf\{t|F_1(t) \geq p\}, \quad 0 \leq p \leq 1. \tag{1.1}$$

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It means when $0 \leq p \leq 1$ and $0 < t < \infty$, one get $F_1(t) \geq p$ if $Q_1(p) \leq t$, and vice-versa. Similarly, the QF for X_2 with DF F_2 can be defined. We denote the QF of X_2 by Q_2. The study of information measures based on QF is of recent interest. For some development in this direction, we refer to Baratpour and Khammar [1], Kayal [7], Sunoj *et al.* [23], Khammar and Jahanshahi [10], Kayal and Tripathy [8], Sunoj, Sankaran and Nair [24], Sunoj, Krishnan and Sankaran [25], and the references therein.

To measure uncertainty associated with a probability distribution, Shannon [20] introduced the concept of entropy. For n mutually exclusive and exhaustive events $E_1, \ldots, E_n$ in a sample space with respective probabilities $p_1, \ldots, p_n$, the entropy is defined as $\mathcal{H}(\mathbf{p}) = -\sum_{i=1}^{n} p_i \ln p_i$, where $\mathbf{p} = (p_1, \ldots, p_n)$. Various generalizations of entropy have been proposed. One of them was introduced by Kerridge [9]. Suppose p_i is the true probability of the ith event and q_i is the probability of the event assigned by an experimenter. Then, the inaccuracy measure for two probability distributions $\mathbf{p}$ and $\mathbf{q} = (q_1, \ldots, q_n)$ is given by $\mathcal{I}(\mathbf{p}, \mathbf{q}) = -\sum_{i=1}^{n} p_i \ln q_i$. Later, this measure was extended by Nath [16] for continuous random variables. Let X_1 and X_2 have density functions f_1, f_2, DFs F_1, F_2, and survival functions $\bar{F}_1 = 1 - F_1, \bar{F}_2 = 1 - F_2$, respectively. The inaccuracy measure of X_1 and X_2 is

$$\mathcal{I}(X_1, X_2) = -\int_0^\infty f_1(x) \ln f_2(x) dx = -E_{f_1}(\ln f_2(X)), \tag{1.2}$$

where f_1 is the true density function and f_2 is the assigned density function. In other words, (1.2) measures the average uncertainty contained in the proposed model f_2 with respect to the true model f_1. It takes two types of errors into account: one is because of the lack of ample information in experimental results and other is for the wrong specification of the statistical model. If $f_1 = f_2$, then $\mathcal{I}(X_1, X_2)$ reduces to the well-known differential entropy. For X_1, it is given by $\mathcal{J}(X_1) = -\int_0^\infty f_1(x) \ln f_1(x) dx$. We refer to Nath [16] for various properties of (1.2).

In reliability and life testing, the data is often truncated for used items, where (1.2) is not suitable to measure inaccuracy. Here, dynamic versions of (1.2) are of importance. Accordingly, Nair and Gupta [13] extended (1.2) for the residual lifetime distributions. The inaccuracy of two residual random variables $X_{1t} = (X_1 - t | X_1 > t)$ and $X_{2t} = (X_2 - t | X_2 > t)$ is given by

$$\mathcal{I}(X_1, X_2; t) = -\int_t^\infty \frac{f_1(x)}{\bar{F}_1(t)} \ln \left(\frac{f_2(x)}{\bar{F}_2(t)} \right) dx, \tag{1.3}$$

where $t > 0$. Eq. (1.3) is known as the residual inaccuracy measure of X_1 and X_2. Note that if t tends to 0, (1.3) reduces to (1.2). Further, (1.3) becomes residual entropy proposed by Ebrahimi [4] when $f_{2t}(x) = f_{1t}(x)$, where $f_{2t}(x) = f_2(x)/\bar{F}_2(t)$, $x > t$ and $f_{1t}(x) = f_1(x)/\bar{F}_1(t)$, $x > t$. Taneja *et al.* [26] obtained various properties of (1.3). Analogous to the entropy for the past lifetime due to Di Crescenzo and Longobardi [3], recently, Kumar *et al.* [11] introduced inaccuracy measure for two past lifetime random variables as

$$\widetilde{\mathcal{I}}(X_1, X_2; t) = -\int_0^t \frac{f_1(x)}{F_1(t)} \ln \left(\frac{f_2(x)}{F_2(t)} \right) dx, \tag{1.4}$$

where $t > 0$. Eq. (1.4) quantifies inaccuracy carried out by past random lifetimes $X_{1(t)} = (t - X_1 | X_1 < t)$ and $X_{2(t)} = (t - X_2 | X_2 < t)$. Denote the density functions of $X_{1(t)}$ and $X_{2(t)}$ as $f_{1(t)}(x) = f_1(x)/F_1(t)$ and $f_{2(t)}(x) = f_2(x)/F_2(t)$, respectively, where $0 < x < t$. For $f_{1(t)}(x) = f_{2(t)}(x)$, (1.4) reduces to the past entropy. Moreover, when $t \to \infty$, $\widetilde{\mathcal{I}}(X_1, X_2; t)$ equals to (1.2). Kundu and Nanda [12] derived characterizations for various lifetime distributions based on inaccuracy measure when random variables are truncated.

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It has been shown by many authors that the quantile-based approach is an alternative to the traditional DF approach. Motivated with this, Sankaran *et al.* [19] introduced a quantile version of the Kullback-Leibler divergence and studied different properties in the context of lifetime data analysis. More recently, Sunoj *et al.* [24] proposed a quantile-based cumulative Kullback-Leibler divergence and obtained some useful bounds and characterizations. Also, Sunoj *et al.* [25] studied a quantile-based Tsallis entropy and its order statistics. However, Kayal and Tripathy [8] introduced Tsallis divergence of order α and proved some properties including bounds and effect of transformations. Motivated by these, here, we consider quantile version of inaccuracy and its dynamic forms. The proposed measures have several advantages. Firstly, unlike (1.2), (1.3) or (1.4), the computation of the proposed measures are quite simple in cases where the DFs are not tractable while QF's have simpler forms. Secondly, our approach provides an alternative methodology in the study of the inaccuracy measure. There are some properties of the QF that are not shared by DF. Some of them are: (i) sum of two QFs is a QF; (ii) the product of two positive QFs is also a QF; (iii) if $\zeta(.)$ is a nondecreasing function, then $\zeta(Q_1(p))$ is a QF. Applications of these properties provide some new results and better insight into the properties of the measure that are difficult to obtain in the conventional approach. Thus, a formulation of the definition and properties of inaccuracy measure and its dynamic versions based on quantile function is called for.

The rest of the paper proceeds as follows. In Section 2, we aim to study quantile version of the inaccuracy measure. Some important properties are investigated. Effect of monotone transformations on the measure under study is discussed. Examples are provided to illustrate the results. Section 3 explores the quantile-based inaccuracy for residual lifetime distributions. Characterization results and bounds are established. In Section 4, we develop various results based on the past form of the quantile version of inaccuracy measure. Finally, Section 5 concludes the paper.

Throughout the paper, we use "increasing" to mean "nondecreasing", and similarly "decreasing" to mean "nonincreasing". Moreover, "ln" means natural logarithm, and prime denotes derivative. The differentiation and integration wherever used are assumed to exist.

2. QUANTILE-BASED INACCURACY MEASURE

Let X_1 and X_2 be two random variables with QFs Q_1, Q_2 and DFs F_1, F_2, respectively. Further, for a continuous function F_1, from (1.1), we have $F_1Q_1(p) = p$, where $F_1Q_1(.) = F_1(Q_1(.))$. On differentiating $F_1Q_1(.) = p$ with respect to p, we obtain

$$q_1(p)f_1Q_1(p) = 1. \tag{2.1}$$

Here, $f_1Q_1(.) = f_1(Q_1(.))$ is known as the density QF (see Parzen [17]) and $q_1(.)$ is the quantile density function for the QF $Q_1(.)$. Note that $q_1(p) = Q_1'(p)$. Denote $Q_3(.) = Q_2^{-1}(Q_1(.))$ the QF of $F_1(F_2^{-1}(.))$. It is known as the relative inverse between two functions F_2 and F_1. Further, $q_3(p) = Q_3'(p) = f_2(Q_1(p))q_1(p)$. In the following definition, we consider quantile version of the inaccuracy measure given in (1.2).

DEFINITION 2.1: *Let X_1 and X_2 be two nonnegative and absolutely continuous random variables with QFs Q_1 and Q_2, respectively. Then, the quantile-based inaccuracy measure is defined as*

$$\mathcal{I}_Q(X_1, X_2) = -\int_0^1 \ln f_2(Q_1(p))dp = -\int_0^1 \ln\left(\frac{q_3(p)}{q_1(p)}\right)dp. \tag{2.2}$$

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It represents the expected inaccuracy when QF of the true distribution is Q_1 and that of assumed distribution is Q_2. Eq. (2.2) is the sum of the quantile versions of the Kullback-Leibler divergence (Sankaran *et al.* [19]) and the entropy function (Sunoj and Sankaran [21]). Also, (2.2) is a simple mathematical tool to compute inaccuracy when there is no tractable form for either of the DFs. Further, for $Q_1 = Q_2$, $\mathcal{I}_{\mathcal{Q}}(X_1, X_2)$ reduces to the quantile-based entropy $\mathcal{J}_{\mathcal{Q}}(X_1) = \int_0^1 \ln q_1(p)dp$.

Let h_1 and $\widetilde{h}_1$ be the hazard rate and the reversed hazard rate of X_1, respectively. The hazard quantile and reversed hazard quantile functions of X_1 are given by $H_1(p) = h_1(Q_1(p)) = f_1(Q_1(p))/\bar{F}_1(Q_1(p)) = [q_1(p)(1-p)]^{-1}$ and $\widetilde{H}_1(p) = \widetilde{h}_1(Q_1(p)) = f_1(Q_1(p))/F_1(Q_1(p)) = [q_1(p)p]^{-1}$, respectively, where $0 < p < 1$. These are equivalent to the hazard and reversed hazard rate functions in terms of the QF. Denote $H_3(p)$ and $\widetilde{H}_3(p)$ the hazard quantile and reversed hazard quantile functions, respectively corresponding to the distribution with QF Q_3. Then, $H_3(p) = [q_3(p)(1-p)]^{-1}$ and $\widetilde{H}_3(p) = [pq_3(p)]^{-1}$, where $0 < p < 1$. Now, in terms of the hazard quantile and reversed hazard quantile functions, Eq. (2.2) can be written further as

$$\mathcal{I}_{\mathcal{Q}}(X_1, X_2) = -\int_0^1 \ln\left(\frac{H_1(p)}{H_3(p)}\right) dp = -\int_0^1 \ln\left(\frac{\widetilde{H}_1(p)}{\widetilde{H}_3(p)}\right) dp. \tag{2.3}$$

It provides an easy method for the evaluation of $\mathcal{I}_{\mathcal{Q}}(X_1, X_2)$ when the respective hazard quantile and reversed hazard quantile functions of Q_1 and Q_3 are available.

Remark 2.1: For two independent exponential distributions with QFs $Q_1(p) = -(1/\theta_1)\ln(1-p)$ and $Q_2(p) = -(1/\theta_2)\ln(1-p)$, where $\theta_1, \theta_2 > 0$, we obtain $\mathcal{I}_{\mathcal{Q}}(X_1, X_2) = (\theta_2/\theta_1) - \ln\theta_2$ and $\mathcal{I}_{\mathcal{Q}}(X_2, X_1) = (\theta_1/\theta_2) - \ln\theta_1$. This clearly shows that the quantile-based inaccuracy measure is antisymmetric (see Figure 1a), that is, in general, $\mathcal{I}_{\mathcal{Q}}(X_1, X_2) \neq \mathcal{I}_{\mathcal{Q}}(X_2, X_1)$.

Let us now consider examples to illustrate the usefulness of the quantile-based tool given by (2.2) for the computation of inaccuracy measure of X_1 and X_2, where X_1 does not have explicitly known DF, though it has closed-form QF.

Example 2.1: Let X_1 follow Davies distribution with QF $Q_1(p) = c_1 p^{\lambda_1}(1-p)^{-\lambda_2}$, $c_1, \lambda_1, \lambda_2 > 0$ and X_2 follow power distribution with QF $Q_2(p) = c_2 p^{1/c_3}$, $c_2, c_3 > 0$, where $0 < p < 1$. In this case, it is easy to obtain $f_2(Q_1(p)) = (c_1/c_2)^{c_3}(c_3/c_1)p^{\lambda_1(c_3-1)}(1-p)^{(1-c_3)\lambda_2}$. Thus, from (2.2), we get $\mathcal{I}_{\mathcal{Q}}(X_1, X_2) = c_3\ln(c_2/c_1) + \ln(c_1/c_3) + (1-c_3)(\lambda_2 - \lambda_1)$.

Example 2.2: Consider two QFs $Q_1(p) = 2p - p^2$ and $Q_2(p) = -\lambda/\ln p$, where $\lambda > 0$ and $0 < p < 1$. It is noted that $Q_1(p)$ is the QF of a special case of Govindarajulu distribution (see Govindarajulu [6]) and $Q_2(p)$ is the QF of reciprocal exponential distribution. Here, $q_3(p)/q_1(p) = (\lambda/2)(2p-p^2)^{-2}e^{-\lambda/(2p-p^2)}$ which provides

$$\mathcal{I}_{\mathcal{Q}}(X_1, X_2) = \ln\left(\frac{2}{\lambda}\right) - \int_0^1 \ln((2p-p^2)^{-2}e^{-\lambda/(2p-p^2)})dp.$$

This integration can be evaluated numerically for particular values of λ. For example, when $\lambda = 1$, $\mathcal{I}_{\mathcal{Q}}(X_1, X_2) \approx 3.27937$.

Example 2.3: Let X_1 have power-Pareto distribution with QF $Q_1(p) = cp^{\lambda}/(1-p)$, $c, \lambda > 0$ and X_2 have QF $Q_2(p) = ap^{\beta}$, $a, \beta > 0$. Then, $q_3(p)/q_1(p) = (c/a)^{\beta}\beta p^{\lambda(\beta-1)}(1-p)^{1-\beta}$. From (2.2), $\mathcal{I}_{\mathcal{Q}}(X_1, X_2) = (1-\beta)(\lambda+1) - \ln\beta - \beta\ln(c/a)$.

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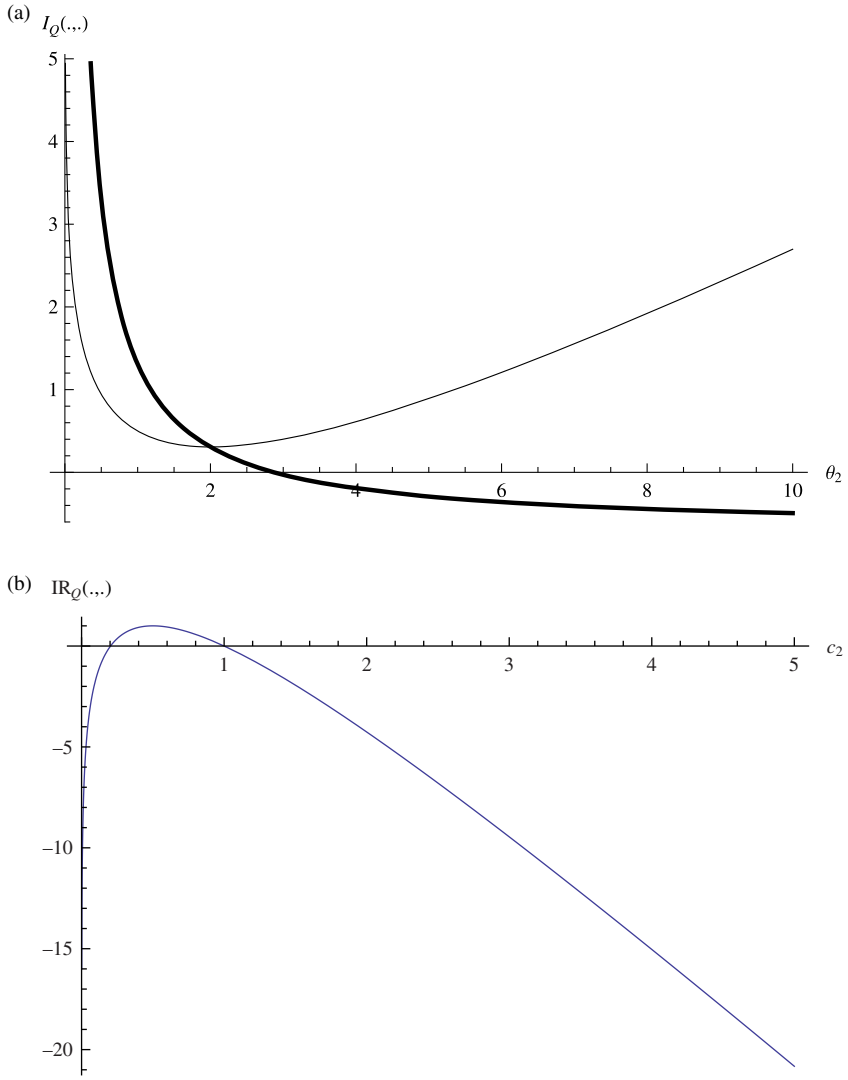


FIGURE **1.** (a) Assuming $\theta_1 = 2$, the inaccuracy measures obtained in Remark 2.1 are plotted with respect to θ_2. The curve with bold (thick) line represents the plot of $\mathcal{I}_{\mathcal{Q}}(X_1, X_2)(\mathcal{I}_{\mathcal{Q}}(X_2, X_1))$. From the graphic, we notice that for the case considered in Remark 2.1, inaccuracy measures cross at the point when $\theta_1 = \theta_2 = 2$. Further, $\mathcal{I}_{\mathcal{Q}}(X_2, X_1) < \mathcal{I}_{\mathcal{Q}}(X_1, X_2)$ for $\theta_2 < 2$ and $\mathcal{I}_{\mathcal{Q}}(X_2, X_1) > \mathcal{I}_{\mathcal{Q}}(X_1, X_2)$ for $\theta_2 > 2$. (b) Plot of the quantile-based inaccuracy ratio as in Example 2.5.

Next, we apply a property of QF (not shared by DF) to study the effect of transformations to quantile-based inaccuracy measure. Let $\psi(x)$ be a continuous and nondecreasing function. Then, $\psi(Q_1(p))$ is the QF of $\psi(X)$. We denote $Q_{\psi(X)}(p) = \psi(Q_1(p))$. Usually, new distributions can be obtained by applying transformations. For example, one may use various transformations such as power transformation, reciprocal transformation and logarithmic transformation to the QFs of basic distributions to get some flexible statistical models. Below, we provide a result which evaluates the inaccuracy when transformations are applied to the QFs of X_1 and X_2.

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THEOREM 2.1: *Let X_1 and X_2 be two random variables with respective QFs Q_1 and Q_2. Then, for two nondecreasing and continuous transformations $\psi_1(.)$ and $\psi_2(.)$, we have*

$$\mathcal{I}_{\mathcal{Q}}(\psi_1(X_1), \psi_2(X_2)) = -\int_0^1 \ln\left(\frac{\frac{d}{dp}[(\psi_2(Q_2(p)))^{-1}(\psi_1(Q_1(p)))]}{\frac{d}{dp}[\psi_1(Q_1(p))]}\right) dp. \tag{2.4}$$

PROOF: The functional in (2.2) can be expressed as

$$\mathcal{I}_{\mathcal{Q}}(X_1, X_2) = -\int_0^1 \ln\left(\frac{\frac{d}{dp}[Q_2^{-1}(Q_1(p))]}{\frac{d}{dp}[Q_1(p)]}\right) dp. \tag{2.5}$$

Further, we have $Q_{\psi_1(X_1)}(p) = \psi_1(Q_1(p))$ and $Q_{\psi_2(X_2)}(p) = \psi_2(Q_2(p))$. Using these in (2.5), the desired result follows. ■

To get a better understanding of Theorem 2.1, we consider the following example. It is known that if X follows Pareto-I with QF $Q(p) = (1-p)^{-1/a}$, $a > 0$, then $\ln X$ has exponential distribution with QF $Q(p) = -(1/a)\ln(1-p)$, $a > 0$, $0 < p < 1$.

Example 2.4: Let X_1 and X_2 be two independent random variables with QFs $Q_1(p) = (1-p)^{-1/a_1}$ and $Q_2(p) = (1-p)^{-1/a_2}$, where $a_1, a_2 > 0$. Moreover, assume $\psi_1(X_1) = \ln X_1$ and $\psi_2(X_2) = \ln X_2$. Then, using (2.4), the quantile-based inaccuracy of two exponential random variables can be computed. Here, $\psi_1(Q_1(p)) = -(1/a_1)\ln(1-p)$ and $\psi_2(Q_2(p)) = -(1/a_2)\ln(1-p)$. Then, $(\psi_2(Q_2(p)))^{-1}(\psi_1(Q_1(p))) = 1-(1-p)^{a_2/a_1}$ implies $\frac{d}{dp}[(\psi_2(Q_2(p)))^{-1}(\psi_1(Q_1(p)))] = (a_2/a_1)(1-p)^{(a_2/a_1)-1}$. Further, $d/dp[\psi_1(Q_1(p))] = 1/a_1(1-p)$. Substituting these expressions of the derivatives in (2.4), we obtain $\mathcal{I}_{\mathcal{Q}}(X_1, X_2) = (a_2/a_1) - \ln a_2$.

It is worth to point out that quantile-based inaccuracy may be positive and negative. In Example 2.3, when $\beta = \lambda = c = 1$ and $\alpha < 1/e$, $\mathcal{I}_{\mathcal{Q}}(X_1, X_2)$ takes negative values. In the following proposition, we present sufficient condition under which $\mathcal{I}_{\mathcal{Q}}(X_1, X_2)$ is always positive. First, let us recall the hazard quantile order.

DEFINITION 2.2: *Let X_1 and X_2 be two nonnegative and absolutely continuous random variables with hazard quantile functions $H_1(p) = 1/(q_1(p)(1-p))$ and $H_2(p) = 1/(q_2(p)(1-p))$, respectively, $0 < p < 1$. Then, X_1 is said to be smaller than X_2 in quantile-based hazard rate order (denoted by $X_1 \leq_{hq} X_2$), if and only if $H_1(p) \geq H_2(p)$, or $q_1(p) \leq q_2(p)$ for all $0 < p < 1$.*

PROPOSITION 2.1: *Suppose that the random variables X_1, X_2 and X_3 have QFs $Q_1(p)$, $Q_2(p)$ and $Q_3(p) = Q_2^{-1}(Q_1(p))$, respectively. Further, assume $X_3 \leq_{hq} X_1$. Then, $\mathcal{I}_{\mathcal{Q}}(X_1, X_2) \geq 0$.*

PROOF: The proof is trivial by (2.2) and Definition 2.2. ■

In this part of the section, we introduce the concept of quantile-based inaccuracy ratio. For X_1 and X_2, it is given by

$$\mathcal{IR}_{\mathcal{Q}}(X_1, X_2) = \frac{\mathcal{I}_{\mathcal{Q}}(X_1, X_2)}{\mathcal{J}_{\mathcal{Q}}(X_1)}, \tag{2.6}$$

where $\mathcal{J}_{\mathcal{Q}}(X_1)$ is the quantile-based Shannon entropy. Eq. (2.6) provides adimensional measure of closeness between X_1 and X_2. If $Q_2 = Q_1$, then (2.6) equals to one. Further, analog to quantile-based inaccuracy measure, (2.6) is also not symmetric, that is,

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$\mathcal{IR}_{\mathcal{Q}}(X_1, X_2) \neq \mathcal{IR}_{\mathcal{Q}}(X_2, X_1)$. Note that the measure $\mathcal{IR}_{\mathcal{Q}}(X_1, X_2)$ given by (2.6) can be interpreted as the amount of dissimilarity carried by the quantile-based Shannon entropy when the QF of the true distribution Q_1 is replaced by another QF Q_2. Further, use of the distribution with QF Q_2 instead of Q_1 provides less information in the sense of quantile-based inaccuracy ratio rather than that carried by Shannon entropy of the distribution with QF Q_1 if $\mathcal{IR}_{\mathcal{Q}}(X_1, X_2)$ is strictly smaller that 1.

The following example computes quantile-based inaccuracy ratio of two re-scaled beta distributions.

Example 2.5: Consider two random variables X_1 and X_2 with respective QFs $Q_1(p) = 1 - (1-p)^{1/c_1}$ and $Q_2(p) = 1 - (1-p)^{1/c_2}$, where $c_1, c_2 > 0$. Then, $\mathcal{I}_{\mathcal{Q}}(X_1, X_2) = ((c_2 - 1)/c_1) - \ln c_2$ and $\mathcal{J}_{\mathcal{Q}}(X_1) = (c_1 - 1)/c_1 - \ln c_1$. Thus, we obtain $\mathcal{IR}_{\mathcal{Q}}(X_1, X_2) = [((c_2 - 1)/c_1) - \ln c_2]/[((c_1 - 1)/c_1) - \ln c_1]$, which is plotted in Figure 1b for $c_1 = 1/2$.

3. QUANTILE-BASED RESIDUAL INACCURACY MEASURE

This section introduces quantile version of the residual inaccuracy of X_1 and X_2 and study its different properties. Let us first set out the following definition. In the sequel, we denote $Q_3(p) = Q_2^{-1}(Q_1(p))$, where $0 < p < 1$.

DEFINITION 3.1: *Let X_1 and X_2 be two nonnegative absolutely continuous random variables with QFs Q_1 and Q_2, respectively. Then, the quantile-based residual inaccuracy for X_{1t} and X_{2t} is defined as*

$$\begin{aligned}\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u) &= -\int_u^1 \frac{f_1(Q_1(p))}{\bar{F}_1(Q_1(u))} \ln\left(\frac{f_2(Q_1(p))}{\bar{F}_2(Q_1(u))}\right) d(Q_1(p)) \\ &= \ln(1 - Q_3(u)) - \left(\frac{1}{1-u}\right)\int_u^1 \ln\left(\frac{q_3(p)}{q_1(p)}\right) dp \\ &= -\left(\frac{1}{1-u}\right)\int_u^1 \ln\left(\frac{q_3(p)}{q_1(p)(1 - Q_3(u))}\right) dp. \end{aligned} \tag{3.1}$$

The quantile-based residual inaccuracy can be expressed in terms of the quantile-based residual Kullback-Leibler divergence (see Sankaran *et al.* [19]) and the quantile-based residual Shannon entropy (see Sunoj and Sankaran [21]). That is, $\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u) = \mathcal{D}_{\mathcal{Q}}(X_1, X_2; u) + \mathcal{J}_{\mathcal{Q}}(X_1; u)$, where $\mathcal{D}_{\mathcal{Q}}(X_1, X_2; u) = \ln((1 - Q_3(u))/1 - u) - (1-u)^{-1} \int_u^1 \ln q_3(p) dp$ and $\mathcal{J}_{\mathcal{Q}}(X_1; u) = \ln(1-u) + (1-u)^{-1} \int_u^1 \ln(q_1(p)) dp$ denote respectively, the quantile-based residual Kullback-Leibler divergence and quantile entropy. Further, if $Q_1 = Q_2$, then $\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u) = \mathcal{J}_{\mathcal{Q}}(X_1; u)$. Analogous to (2.6), the quantile-based residual inaccuracy ratio of X_1 and X_2 can be defined as

$$\mathcal{IR}_{\mathcal{Q}}(X_1, X_2; u) = \frac{\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u)}{\mathcal{J}_{\mathcal{Q}}(X_1; u)}. \tag{3.2}$$

An interpretation of (3.2) similar to (2.6) is as follows. Let $\mathcal{IR}_{\mathcal{Q}}(X_1, X_2; u)$ be strictly <1. This implies that using the conditional distribution $f_2(Q_1(p))/\bar{F}_2(Q_1(u))$ instead of $f_1(Q_1(p))/\bar{F}_1(Q_1(u))$ provides less information in the sense of quantile-based residual inaccuracy ratio rather than that carried out by the Shannon's entropy of the conditional distribution $f_1(Q_1(p))/\bar{F}_1(Q_1(u))$. Further, if $Q_1 = Q_2$, then $\mathcal{IR}_{\mathcal{Q}}(X_1, X_2; u) = 1$. Below, we consider examples to compute quantile-based residual inaccuracy measure.

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Example 3.1: Let a generalized lambda random variable X_1 have QF $Q_1(p) = \lambda_1 + (p^{\lambda_3} - (1-p)^{\lambda_4})/\lambda_2$, where $\lambda_1, \lambda_2, \lambda_4 > 0$, λ_3 is a positive integer and $\lambda_1 - \lambda_2^{-1} \geq 0$. This distribution is known in the literature for its high flexibility. It produces distributions with a wide range of shapes. Further, consider another random variable X_2 with QF $Q_2(p) = p$, $0 < p < 1$. Set $\lambda_1 = \lambda_2 = 1$. Then, $Q_3(p) = 1 + p^{\lambda_3} - (1-p)^{\lambda_4}$ and $q_3(p)/q_1(p) = 1$. Using these in (3.1), we obtain $\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u) = \ln[(1-u)^{\lambda_4} - u^{\lambda_3}]$.

The above example shows the importance of the tool given in (3.1). It can be noted that (1.3) fails to compute inaccuracy between the distributions considered in Example 3.1 since generalized lambda distribution does not have an explicit form of DF. Let X_1 and X_2 be two random variables with QFs Q_1 and Q_2, respectively, which satisfy the proportional hazards model (PHM). Then, its QF is

$$Q_2(p) = Q_1(1 - (1-p)^{1/\theta}),\ 0 < p < 1, \tag{3.3}$$

where $\theta > 0$ is constant. For more properties of PHM, we refer to Nair *et al.* [14] and Nair *et al.* [15]. Our next example is based on (3.3).

Example 3.2: Let two QFs Q_1 and Q_2 satisfy PHM. Then, $Q_3(p) = 1 - (1-p)^{\theta}$. Utilizing this in (3.1), we obtain $\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u) = \theta \ln(1-u) - (1/1-u) \int_u^1 \ln(\theta(1-p)^{\theta-1}/q_1(p))dp$. This can be evaluated numerically for some values of u and θ, provided q_1 is known.

Below we consider an example which shows that $\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u)$ is not monotone.

Example 3.3: Consider two inverted exponential random variables X_1 and X_2 with QFs $Q_1(p) = -\lambda_1/\ln p$ and $Q_2(p) = -\lambda_2/\ln p$, where $\lambda_1, \lambda_2 > 0$ and $0 < p < 1$. Here, $Q_3(p) = p^{\lambda_2/\lambda_1}$. Thus,

$$\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u) = \ln(1 - u^{\lambda_2/\lambda_1}) - \ln\left(\frac{\lambda_2}{{\lambda_1}^2}\right) - \left(\frac{1}{1-u}\right) \int_u^1 \ln\left(p^{\lambda_2/\lambda_1} (\ln p)^2\right) dp. \tag{3.4}$$

For better intuition about the quantile-based residual inaccuracy measure, see Figure 2.

In general, the quantile-based residual inaccuracy measure given in (3.1) is not monotone with respect to u (see the graph as plotted in Figure 2). In the following theorem, we obtain an upper (lower) for $\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u)$ based on the monotonic property of it. Bounds are of also useful when the true measure is difficult to compute.

THEOREM 3.1: *The quantile-based residual inaccuracy measure $\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u)$ is increasing (decreasing) in u, if and only if*

$$\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u) \geq (\leq) \frac{(1-u)q_3(u)}{1 - Q_3(u)} - \ln\left(\frac{q_3(u)}{q_1(u)(1 - Q_3(u))}\right). \tag{3.5}$$

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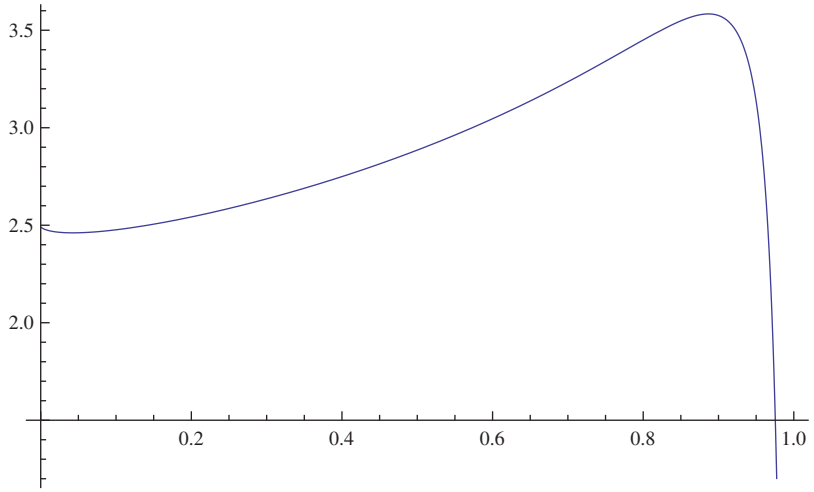


FIGURE **2.** The graph represents the plot of $\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u)$ given by (3.4).

PROOF: From (3.1), we have

$$(1-u)\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u) = (1-u)\ln(1-Q_3(u)) - \int_u^1 \ln\left(\frac{q_3(p)}{q_1(p)}\right) dp. \tag{3.6}$$

Differentiating both sides of (3.6) with respect to u, we obtain

$$\mathcal{I}_{\mathcal{Q}}{}'(X_1, X_2; u) = \frac{1}{1-u}\left[\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u) + \ln\left(\frac{H_1(u)(1-u)q_3(u)}{1-Q_3(u)}\right) - \frac{(1-u)q_3(u)}{1-Q_3(u)}\right], \tag{3.7}$$

where $H_1(u)$ is the hazard QF of X_1. Thus, under the assumption made, the desired result readily follows. The reverse part can be easily gained retracing the steps, and hence omitted. ■

Remark 3.1: Let $Q_2 = Q_1$. From Theorem 3.1, $\mathcal{J}_{\mathcal{Q}}(X_1; u) \geq (\leq)\ 1 + \ln[(1-u)q_1(u)]$.

Remark 3.2: Let Q_1 and Q_2 satisfy PHM. Then, from Theorem 3.1, $\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u) \geq (\leq)$ $\theta - \ln(\theta H_1(u))$.

THEOREM 3.2: *Let the QF* $Q_3(.) = Q_2^{-1}Q_1(.)$ *be concave. Then, we have*

$$\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u) \geq \mathcal{J}_{\mathcal{Q}}(X_1; u) + \ln((1-Q_3(u))H_3(u)), \tag{3.8}$$

where $\mathcal{J}_{\mathcal{Q}}(X_1; u)$ *is the quantile-based residual entropy and* $H_3(u) = [(1-u)q_3(u)]^{-1}$.

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PROOF: From the assumption made, it is evident that $q_3(p) \leq q_3(u)$ for $p \geq u$. Making use of this inequality we have from (3.1) as

$$\begin{aligned}\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u) &\geq \ln(1 - Q_3(u)) - \left(\frac{1}{1-u}\right) \int_u^1 \ln\left(\frac{q_3(u)}{q_1(p)}\right) dp \\ &= \ln(1-u) + \left(\frac{1}{1-u}\right) \int_u^1 \ln(q_1(p)) dp + \ln((1 - Q_3(u))H_3(u)) \\ &= \mathcal{J}_{\mathcal{Q}}(X_1; u) + \ln((1 - Q_3(u))H_3(u)).\end{aligned}$$

This completes the proof. ■

THEOREM 3.3: *Let $F_2(.)$ be concave. Then, we have*

$$\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u) \geq \ln\left(\frac{H_3(u)(1 - Q_3(u))}{H_1(u)}\right). \tag{3.9}$$

PROOF: Under the given assumption, we have $f_2(Q_1(p))$ is decreasing in p. Thus, for $p \geq u$, $f_2(Q_1(p)) \leq f_2(Q_1(u))$. Hence, from (3.1), we get

$$\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u) \geq \ln\left(\frac{\bar{F}_2(Q_1(u))}{f_2(Q_1(u))}\right) = \ln\left(\frac{q_1(u)(1 - Q_3(u))}{q_3(u)}\right),$$

by which, the stated result trivially follows. ■

Next, we obtain bound of $\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u)$ in terms of the quantile-based residual entropy and the hazard quantile functions. In this direction the following definition is useful.

DEFINITION 3.2: *We say X_1 is smaller than X_2 in the likelihood ratio quantile order, denoted by $X_1 \leq_{lrq} X_2$, if and only if $f_2(Q_1(p))/f_1(Q_1(p))$ is increasing in p for all $0 < p < 1$.*

THEOREM 3.4: *Let X_1 and X_2 be two random variables such that $X_1 \leq_{lrq} X_2$. Then,*

$$\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u) \leq \mathcal{J}_{\mathcal{Q}}(X_1; u) + \ln\left(\frac{H_1(u)}{H_2(u)}\right). \tag{3.10}$$

The equality sign holds if and only if $Q_1 = Q_2$.

PROOF: The quantile-based inaccuracy measure given by (3.1) can be re-written as

$$\mathcal{I}_{\mathcal{Q}}(X_1, X_2; u) = -\int_u^1 \frac{f_1(Q_1(p))}{\bar{F}_1(Q_1(u))} \ln\left(\frac{f_2(Q_1(p)) f_1(Q_1(p)) \bar{F}_1(Q_1(u))}{f_1(Q_1(p)) \bar{F}_1(Q_1(u)) \bar{F}_2(Q_1(u))}\right) d(Q_1(p)). \tag{3.11}$$

Further, under the condition made, for $u \leq p$, we have $f_2(Q_1(u))/f_1(Q_1(u)) \leq f_2(Q_1(p))/f_1(Q_1(p))$. Using this inequality in (3.11), the stated inequality readily follows. ■

We consider the following example to illustrate Theorem 3.4.

Example 3.4: Consider two random variables X_1 and X_2 with QFs $Q_1(p) = (1-p)^{-1/\alpha_1}$ and $Q_2(p) = (1-p)^{-1/\alpha_2}$, respectively, where $0 < \alpha_1 < \alpha_2$ and $0 < p < 1$. It is easy to show

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that $X_1 \leq_{lrq} X_2$. Further, $\mathcal{J}_{\mathcal{Q}}(X_1;u) = ((\alpha_1+1)/\alpha_1) - (1/\alpha_1)\ln(1-u) - \ln\alpha_1$, $H_1(u) = \alpha_1(1-p)^{1/\alpha_1}$ and $H_2(u) = \alpha_2(1-p)^{1/\alpha_2}$. Thus, from Theorem 3.4, we obtain an upper bound of $\mathcal{I}_{\mathcal{Q}}(X_1,X_2;u)$ as $((\alpha_1+1)/\alpha_1) - (1/\alpha_2)\ln(1-u) - \ln\alpha_2$.

Suppose X_1 represents the random variable corresponding to a true model with DF F_1, and X_2 represents a random variable corresponding to a candidate model F_2. Then, an important modeling problem is to identify the true model F_1 when the assumed model F_2 is known.

THEOREM 3.5: *Suppose X_1 and X_2 are two nonnegative and absolutely continuous random variables with QFs Q_1 and Q_2, respectively. Further, let X_2 be exponentially distributed with parameter $\lambda_2 > 0$. Then, X_1 is exponential, if and only if $\mathcal{I}_{\mathcal{Q}}(X_1,X_2;u)$ is constant, say κ, provided $\kappa + \ln\lambda_2 > 0$.*

PROOF: Assume that $\mathcal{I}_{\mathcal{Q}}(X_1,X_2;u) = \kappa$. Then, from (3.7), we obtain

$$\kappa + \ln\left(\frac{H_1(u)(1-u)q_3(u)}{1-Q_3(u)}\right) - \frac{(1-u)q_3(u)}{1-Q_3(u)} = 0. \tag{3.12}$$

Further, X_2 follows exponential distribution with parameter λ_2. Thus, $q_3(u)/1-Q_3(u) = \lambda_2 q_1(u)$. Substituting this expression in (3.12), we obtain

$$q_1(u) = \frac{\theta}{1-u}, \tag{3.13}$$

where $\theta = (\kappa + \ln\lambda_2)/\lambda_2 > 0$. From (3.13), it is easy to see that X_1 has exponential distribution with parameter θ. The converse part is straightforward. ■

Below, we provide sufficient condition for $\mathcal{I}_{\mathcal{Q}}(X_1,X_2;u) = c + du$ a linear form under the assumption that X_2 has exponential distribution.

THEOREM 3.6: *Let X_1 and X_2 be two nonnegative and absolutely continuous random variables with QFs Q_1 and Q_2, respectively. Let X_2 be exponentially distributed with parameter λ_2. Then, $\mathcal{I}_{\mathcal{Q}}(X_1,X_2;u) = c + du$, if X_1 has the quantile density function*

$$q_1(u) = \frac{\ln\lambda_2 + c - d + 2du}{\lambda_2(1-u)}. \tag{3.14}$$

PROOF: Under the assumption made, the proof follows from (3.7). Thus, it is omitted. ■

THEOREM 3.7: *Suppose X_1 and X_2 are two nonnegative and absolutely continuous random variables with QFs Q_1 and Q_2, respectively. Further, let X_2 be exponentially distributed with parameter $\lambda_2 > 0$. Then $\mathcal{I}_{\mathcal{Q}}(X_1,X_2;u) = ((a+bu)/(c+du))$ a bilinear form if the quantile density function of X_1 takes the form*

$$q_1(u) = \frac{(a+bu)(c+du) + \ln\lambda_2(c+du)^2 - (1-u)[(c+du)b - (a+bu)d]}{\lambda_2(1-u)(c+du)^2}. \tag{3.15}$$

PROOF: The proof is similar to Theorem 3.6 and hence omitted. ■

Remark 3.3: When $d = 0$, Theorem 3.7 reduces to Theorem 3.6 and when $a = 1, b = 0$, (3.15) provides a new quantile model to the inverse linear characterization for $\mathcal{I}_{\mathcal{Q}}(X_1,X_2;u)$.

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Similar to Theorem 2.1, below we discuss the effect of the transformations on (3.1).

THEOREM 3.8: *For two nondecreasing and continuous transformations $\psi_1(.)$ and $\psi_2(.)$, we have*

$$\mathcal{I}_{\mathcal{Q}}(\psi_1(X_1),\psi_2(X_2);u) = \ln\left(1-(\psi_2(Q_2(u)))^{-1}(\psi_1(Q_1(u)))\right) - \left(\frac{1}{1-u}\right)\int_u^1 \ln\left(\frac{\frac{d}{dp}[(\psi_2(Q_2(p)))^{-1}(\psi_1(Q_1(p)))]}{\frac{d}{dp}[\psi_1(Q_1(p))]}\right)dp. \quad \textbf{(3.16)}$$

PROOF: The proof is similar to Theorem 2.1. Thus, it is omitted for the sake of conciseness. ■

To illustrate Theorem 3.8, we consider the following example.

Example 3.5: Let X_1 and X_2 be two random variables with respective QFs $Q_1(p) = -(1/\theta_1)\ln(1-p)$ and $Q_2(p) = -(1/\theta_2)\ln(1-p)$, where $\theta_1,\theta_2>0$ and $0<p<1$. Consider the transformations $\psi_1(X_1) = X_1^{1/\alpha}$ and $\psi_2(X_2) = X_2^{1/\alpha}$, $\alpha>0$. Then, $\psi_1(Q_1(p)) = (-(1/\theta_1)\ln(1-p))^{1/\alpha}$, $\psi_2(Q_2(p)) = (-(1/\theta_2)\ln(1-p))^{1/\alpha}$ and $(\psi_2(Q_2(p)))^{-1}(\psi_1(Q_1(p))) = 1-(1-p)^{\theta_2/\theta_1}$. Further, $\frac{d}{dp}[(\psi_2(Q_2(p)))^{-1}(\psi_1(Q_1(p)))] = (\theta_2/\theta_1)(1-p)^{(\theta_2/\theta_1)-1}$ and $\frac{d}{dp}[\psi_1(Q_1(p))] = (1/\alpha)(1/\theta_1)^{1/\alpha}(-\ln(1-p))^{(1/\alpha)-1}/(1-p)$. Thus, from (3.16), we obtain

$$\mathcal{I}_{\mathcal{Q}}(X_1^{1/\alpha},X_2^{1/\alpha};u) = \frac{\theta_2}{\theta_1} + \frac{1}{\alpha}\ln\theta_1 - \ln\left(\frac{\theta_2\alpha}{\theta_1}\right) + \frac{1}{1-u}\int_u^1 \ln(-\ln(1-p))^{1/\alpha-1}dp, \quad \textbf{(3.17)}$$

which can be evaluated numerically for some specific values of θ_1,θ_2,α and u. Note that if X follows exponential with QF $Q(p) = -(1/\theta)\ln(1-p)$, $\theta>0$, then $X^{1/\alpha}$, $\alpha>0$ follows Weibull with QF $Q(p) = (-(1/\theta)\ln(1-p))^{1/\alpha}$. Thus, (3.17) is the quantile-based inaccuracy measure for two independent Weibull random variables which has been obtained using appropriate transformations.

Next, we consider three random variables and obtain bounds of the difference of quantile-based inaccuracy measures.

THEOREM 3.9: *Let X_1,Y_1 and Y_2 be three random variables with density functions f_1,g_1 and g_2, survival functions $\bar{F}_1,\bar{G}_1$ and $\bar{G}_2$, QFs Q_1,Q_1^* and Q_2^*, hazard quantile functions H_1,J_1^* and J_2^*, respectively. When $g_1(Q_1(x))/g_2(Q_1(x))$ is decreasing in x, we have*

$$\mathcal{I}_{\mathcal{Q}}(X_1,Y_1;u) - \mathcal{I}_{\mathcal{Q}}(X_1,Y_2;u) \geq \ln\left(\frac{J_2^*(u)}{J_1^*(u)}\right).$$

PROOF: Following (3.1), we have

$$\begin{aligned}\mathcal{I}_{\mathcal{Q}}(X_1,Y_1;u) &= -\int_u^1 \frac{f_1(Q_1(p))}{\bar{F}_1(Q_1(u))}\ln\left(\frac{g_1(Q_1(p))}{\bar{G}_1(Q_1(u))}\right)d(Q_1(p)) \\ &= -\int_u^1 \frac{f_1(Q_1(p))}{\bar{F}_1(Q_1(u))}\ln\left(\frac{g_1(Q_1(p))}{g_2(Q_1(p))}\frac{g_2(Q_1(p))}{\bar{G}_2(Q_1(u))}\frac{\bar{G}_2(Q_1(u))}{\bar{G}_1(Q_1(u))}\right)d(Q_1(p)) \\ &\geq \mathcal{I}_{\mathcal{Q}}(X_1,Y_2;u) - \ln\left(\frac{J_1^*(u)}{J_2^*(u)}\right),\end{aligned}$$

where the inequality is due to the assumption made. This completes the proof. ■

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THEOREM 3.10: *Consider X_1, Y_1 and Y_2 as in Theorem 3.9. Assume that $X_1 \leq_{lrq} Y_1$. Then,*

$$\mathcal{I}_{\mathcal{Q}}(Y_2, X_1; u) - \mathcal{I}_{\mathcal{Q}}(Y_2, Y_1; u) \geq \ln\left(\frac{J_1^*(u)}{H_1(u)}\right).$$

PROOF: From (3.1), we have

$$\mathcal{I}_{\mathcal{Q}}(Y_2, X_1; u) = -\int_u^1 \frac{g_2(Q_1(p))}{\bar{G}_2(Q_1(u))} \ln\left(\frac{f_1(Q_1(p))}{g_1(Q_1(p))} \frac{g_1(Q_1(p))}{\bar{G}_1(Q_1(u))} \frac{\bar{G}_1(Q_1(u))}{\bar{F}_1(Q_1(u))}\right) d(Q_1(p)).$$

Thus, the stated inequality follows from the given assumption after some calculation. So, it is omitted for brevity. ■

4. QUANTILE-BASED PAST INACCURACY MEASURE

Similar to the quantile-based residual inaccuracy, in this section, we study various properties of the quantile-based inaccuracy for past lifetime random variables. Most of the results follow using similar arguments as in the previous section, and hence the details have been omitted for the sake of conciseness. First, consider the following definition.

DEFINITION 4.1: *Let X_1 and X_2 be two nonnegative absolutely continuous random variables with QFs Q_1 and Q_2, respectively. Then, the quantile-based past inaccuracy for $X_{1(t)}$ and $X_{2(t)}$ is given by*

$$\begin{aligned}\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1, X_2; v) &= -\int_0^v \frac{f_1(Q_1(p))}{F_1(Q_1(v))} \ln\left(\frac{f_2(Q_1(p))}{F_2(Q_1(v))}\right) d(Q_1(p)) \\ &= -\left(\frac{1}{v}\right)\int_0^v \ln\left(\frac{q_3(p)}{q_1(p)(Q_3(v))}\right) dp. \end{aligned} \tag{4.1}$$

The quantile-based past inaccuracy measure given by (4.1) can be expressed as the sum of quantile-based past Kullback-Leibler divergence (see Sankaran *et al.* [19]) and quantile-based past entropy (see Sunoj *et al.* [22]). Let $\widetilde{\mathcal{D}_{\mathcal{Q}}}(X_1, X_2; v)(= \ln(Q_3(v)/v) - v^{-1}\int_0^v \ln q_3(p)dp)$ and $\widetilde{\mathcal{J}_{\mathcal{Q}}}(X_1; v)(= \ln v + v^{-1}\int_0^v \ln(q_1(p))dp)$, respectively, denote the quantile-based past Kullback-Leibler divergence and quantile-based past entropy. Then, $\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1, X_2; v) = \widetilde{\mathcal{D}_{\mathcal{Q}}}(X_1, X_2; v) + \widetilde{\mathcal{J}_{\mathcal{Q}}}(X_1; v)$. Similar to (3.2), the quantile-based past inaccuracy ratio of X_1 and X_2 is defined as

$$\widetilde{\mathcal{IR}_{\mathcal{Q}}}(X_1, X_2; v) = \frac{\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1, X_2; v)}{\widetilde{\mathcal{J}_{\mathcal{Q}}}(X_1; v)}. \tag{4.2}$$

When $Q_1 = Q_2$, then $\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1, X_2; v) = \widetilde{\mathcal{J}_{\mathcal{Q}}}(X_1; v)$, and hence $\widetilde{\mathcal{IR}_{\mathcal{Q}}}(X_1, X_2; v) = 1$. Again, $q_1(p) = (p\widetilde{H}_1(p))^{-1}$ and $q_3(p) = (p\widetilde{H}_3(p))^{-1}$, where $\widetilde{H}_1(p)$ and $\widetilde{H}_3(p)$ denote the reversed hazard quantile functions of the distributions with QFs Q_1 and Q_3, respectively. Using these, (4.1) can be written as

$$\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1, X_2; v) = \ln Q_3(v) - \left(\frac{1}{v}\right)\int_0^v \ln\left(\frac{\widetilde{H}_1(p)}{\widetilde{H}_3(p)}\right) dp. \tag{4.3}$$

Eq. (4.3) provides an easy tool to evaluate quantile-based past inaccuracy measure provided $\widetilde{H}_1(p), \widetilde{H}_3(p)$ and $Q_3(p)$ are available. Next, applying (4.1), we compute $\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1, X_2; v)$ for

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X_1 and X_2 obeying proportional reversed hazards model (PRHM). It is known that if X_1 and X_2 have PRHM, then their QFs must satisfy

$$Q_2(p) = Q_1(p^{1/\theta}), \tag{4.4}$$

where $\theta > 0$ is an integer. For detail, we refer to Nair *et al.* [14].

Example 4.1: Let X_1 and X_2 satisfy (4.4). Then, $Q_3(p) = p^\theta$ and hence $q_3(p) = \theta p^{\theta-1}$. From (4.1), $\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1, X_2; v) = \theta \ln v - (1/v)\int_0^v \ln(\theta p^{\theta-1}/q_1(p))dp$. In particular, we take $Q_1(p) = p/(1-p)$, that is, $q_1(p) = (1-p)^{-2}$. Then, for $v = 0.8$ and $\theta = 1$, we get $\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1, X_2; v) \approx 0.972136$.

Example 4.2: Consider $Q_1(p) = 1/\theta_1 + (1/\theta_2)(p^{\theta_3} - (1-p)^{\theta_4})$ and $Q_2(p) = p^\alpha$, where $\theta_1, \theta_2, \theta_3, \theta_4 > 0$ and $\alpha > 0$. Note that $Q_1(p)$ represents life distribution for $\theta_1 - \theta_2^{-1} > 0$. Q_1 is the QF of generalized lambda distribution which does not have tractable DF and Q_2 is the QF of power distribution. In this case, $Q_3(p) = [Q_1(p)]^{1/\alpha} = [1/\theta_1 + (1/\theta_2)(p^{\theta_3} - (1-p)^{\theta_4})]^{1/\alpha}$. After some simple calculation, from (4.1), we obtain

$$\begin{aligned}\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1, X_2; v) = \ln\left[\frac{1}{\theta_1} + \frac{1}{\theta_2}(v^{\theta_3} - (1-v)^{\theta_4})\right]^{1/\alpha} \\ - \frac{1}{v}\int_0^v \ln\left(\frac{1}{\alpha}\left[\frac{1}{\theta_1} + \frac{1}{\theta_2}(p^{\theta_3} - (1-p)^{\theta_4})\right]^{1/\alpha - 1}\right) dp.\end{aligned} \tag{4.5}$$

The above example shows the usefulness of the proposed tool given in (4.1). The following theorem establishes the effect of transformations on the quantile-based past inaccuracy measure given by (4.1).

THEOREM 4.1: *For two continuous and nondecreasing transformations* $\psi_1(.)$ *and* $\psi_2(.)$,

$$\begin{aligned}\widetilde{\mathcal{I}_{\mathcal{Q}}}(\psi_1(X_1), \psi_2(X_2); v) = \ln(\psi_2(Q_2(v))^{-1}(\psi_1(Q_1(v))) \\ - \left(\frac{1}{v}\right)\int_0^v \ln\left(\frac{\frac{d}{dp}[(\psi_2(Q_2(p)))^{-1}(\psi_1(Q_1(p)))]}{\frac{d}{dp}[\psi_1(Q_1(p))]}\right) dp.\end{aligned} \tag{4.6}$$

PROOF: Proof is similar to Theorem 2.1, and thus omitted. ■

To illustrate Theorem 4.1, we consider the following example.

Example 4.3: Consider X_1 and X_2 as in Example 3.5. Further, let $\psi_1(X_1) = X_1^{1/\alpha}$ and $\psi_2(X_2) = X_2^{1/\alpha}$, where $\alpha > 0$. Then, as an application of Theorem 4.1, from (4.6), we obtain quantile-based past inaccuracy of two Weibull random variables as

$$\begin{aligned}\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1^{1/\alpha}, X_2^{1/\alpha}; v) = \ln\left[1 - (1-v)^{\theta_2/\theta_1}\right] - \frac{1}{\alpha}\ln\theta_1 - \ln(\alpha\theta_2/\theta_1) - \frac{\theta_2}{\theta_1 v}\int_0^v \ln(1-p)dp \\ + \frac{1}{v}\int_0^v \ln(-\ln(1-p))^{1/\alpha - 1}dp.\end{aligned} \tag{4.7}$$

Note that this integration can be evaluated numerically for some specific values of the parameters $\theta_1, \theta_2, \alpha$ and v.

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Differentiating (4.1) with respect to v, we obtain

$$\frac{d}{dv}\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1,X_2;v)=\frac{1}{v}\left[v\frac{q_3(v)}{Q_3(v)}-\ln\left(\frac{q_3(v)}{q_1(v)Q_3(v)}\right)-\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1,X_2;v)\right]. \tag{4.8}$$

Now, we have the following result which provides conditions to establish monotonicity of $\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1,X_2;v)$. Proof is omitted as it readily follows from (4.8). Note that bounds of probability measures are useful when either the measure does not have a closed form or is difficult to compute. Various bounds on quantile-based entropy in the past lifetime for different aging classes are available in Sunoj *et al.* [22]. Qiu [18] further derived some sufficient conditions based on certain aging classes for quantile-based entropy in past lifetime. Also, Sankaran *et al.* [19] and Sunoj *et al.* [23], respectively, obtained bounds on quantile-based Kullback-Leibler and Renyi's divergence measures in past lifetime. Motivated with these, we now obtain certain results to get some bounds for quantile-based inaccuracy in past lifetime.

THEOREM 4.2: *The quantile-based past inaccuracy measure is increasing (decreasing) in v, if and only if*

$$\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1,X_2;v)\leq(\geq)\frac{vq_3(v)}{Q_3(v)}-\ln\left(\frac{q_3(v)}{Q_3(v)q_1(v)}\right). \tag{4.9}$$

Remark 4.1: When $Q_2=Q_1$, then from Theorem 4.2, we have $\widetilde{\mathcal{J}_{\mathcal{Q}}}(X_1;v)\leq(\geq)1+\ln(vq_1(v))$.

Remark 4.2: Let X_1 and X_2 satisfy PRHM. That is, their QFs satisfy (4.4). Then, under the assumption as in Theorem 4.2, we have $\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1,X_2;v)\leq(\geq)\ \theta-\ln(\theta\widetilde{H}_1(v))$.

Below, we obtain bounds of $\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1,X_2;v)$ in terms of the quantile-based past entropy and the quantile-based reversed hazard functions.

THEOREM 4.3:

(*i*) *Let $Q_3(p)=Q_2^{-1}(Q_1(p))$ be concave. Then,*

$$\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1,X_2;v)\leq\widetilde{\mathcal{J}_{\mathcal{Q}}}(X_1;v)+\ln(Q_3(v)\widetilde{H}_3(v)), \tag{4.10}$$

where $\widetilde{\mathcal{J}_{\mathcal{Q}}}(X_1;u)$ is the quantile-based past entropy and $\widetilde{H}_3(v)=[vq_3(v)]^{-1}$.

(*ii*) *Let $F_2(.)$ be concave. Then,*

$$\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1,X_2;v)\leq\ln\left(\frac{\widetilde{H}_3(u)Q_3(v)}{\widetilde{H}_1(v)}\right), \tag{4.11}$$

where $\widetilde{H}_1(v)=[vq_1(v)]^{-1}$.

PROOF: The proof of the first part is analogous to that of Theorem 3.2 and the second part is similar to Theorem 3.3. Thus, it is omitted. ■

THEOREM 4.4: *Consider two random variables X_1 and X_2 such that $X_1\leq_{lrq}X_2$. Then,*

$$\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1,X_2;v)\geq\widetilde{\mathcal{J}_{\mathcal{Q}}}(X_1;v)+\ln\left(\frac{\widetilde{H_1}(v)}{\widetilde{H_2}(v)}\right). \tag{4.12}$$

The equality sign holds if and only if $Q_1=Q_2$.

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PROOF: Proof is straightforward and hence omitted. ■

In analogy to Theorems 3.9 and 3.10, we establish the following results. The proofs are omitted for the sake of conciseness.

THEOREM 4.5: *Let X_1, Y_1 and Y_2 be three random variables with density functions f_1, g_1 and g_2, survival functions $\bar{F}_1, \bar{G}_1$ and $\bar{G}_2$, QFs Q_1, Q_1^* and Q_2^*, reversed hazard quantile functions $\widetilde{H}_1, \widetilde{J}_1^*$ and $\widetilde{J}_2^*$, respectively. If $g_1(Q_1(x))/g_2(Q_1(x))$ is increasing in x, then*

$$\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1, Y_1; v) - \widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1, Y_2; v) \geq \ln\left(\frac{\widetilde{J}_2^*(v)}{\widetilde{J}_1^*(v)}\right).$$

THEOREM 4.6: *Consider X_1, Y_1 and Y_2 as in Theorem 4.5. Assume that $X_1 \leq_{lrq} Y_1$. Then,*

$$\widetilde{\mathcal{I}_{\mathcal{Q}}}(Y_2, X_1; v) - \widetilde{\mathcal{I}_{\mathcal{Q}}}(Y_2, Y_1; v) \geq \ln\left(\frac{\widetilde{J}_1^*(v)}{\widetilde{H}_1(v)}\right).$$

Several authors proved that the Lehman alternative family (or PRHM) is characterized by a constant value of various quantile-based past information measures. For instance, see Sankaran *et al.* [19], Sunoj *et al.* [23] and Kayal and Tripathy [8]. Let X_1 and X_2 satisfy PRHM. Then, $q_3(p)/Q_3(p) = \theta/p$. Using this in (4.8), we have

$$v\widetilde{\mathcal{I}_{\mathcal{Q}}}'(X_1, X_2; v) = \theta - \widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1, X_2; v) - \ln(\theta \widetilde{H}_1(v)). \tag{4.13}$$

Let $\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1, X_2; v)$ be free from v. Then, from (4.13), the reversed hazard QF of X_1 must be constant (free from v). However, Block *et al.* [2] showed that there does not exist nonnegative random variable having constant reversed hazard rate function. Using this argument, we have the following remark.

Remark 4.3: There do not exist two nonnegative random variables satisfying PRHM such that $\widetilde{\mathcal{I}_{\mathcal{Q}}}(X_1, X_2; v)$ is independent of v.

5. CONCLUDING REMARKS

The investigation of properties of inaccuracy measure of two lifetime distributions is an important issue in various areas of science and technology. This measure has been developed to quantify the error in the assigned distribution when the true distribution is known. In this paper, we have proposed a quantile-based inaccuracy measure and its dynamic versions. The proposed measure provides an easy tool to compute the inaccuracy between the two models when there is no tractable true DFs. Several examples and properties have been considered to illustrate the usefulness of the measure and its dynamic forms. Characterizations and bounds are also obtained.

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