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APPROXIMATELY LOCAL DERIVATIONS

EBRAHIM SAMEI

Abstract

Certain linear operators from a Banach algebra A into a Banach A-bimodule X, which are called approximately local derivations, are studied. It is shown that when A is a C*-algebra, a Banach algebra generated by idempotents, a semisimple annihilator Banach algebra, or the group algebra of a SIN or a totally disconnected group, bounded approximately local derivations from A into X are derivations. This, in particular, extends a result of B. E. Johnson that 'local derivations on C*-algebras are derivations' and provides an alternative proof of it.

We were motivated to write this paper by the discovery that *local derivations on* C*-*algebras are derivations*. Let A be a Banach algebra and let X be a Banach A-bimodule. An operator $D\colon A \longrightarrow X$ is a local derivation if, for each $a \in A$, there is a derivation $D_a\colon A \longrightarrow X$ such that $D(a) = D_a(a)$. This concept was introduced by Kadison in [**15**], and it was inspired by his and Ringrose's earlier investigation of Hochschild cohomology of various operator algebras. These maps arise naturally when one seeks conditions to ensure that a given map is a derivation. He showed that if A is a von Neumann algebra and X is a dual Banach A-bimodule, then bounded local derivations from A into X are derivations. Johnson extended Kadison's result and showed that if A is a C*-algebra, then local derivations from A into any Banach A-bimodule are derivations [**12**]. Recently, we have also considered local derivations from certain families of commutative Banach algebras which include the Fourier algebra of a locally compact group [**19**].

Local derivations have also appeared in other contexts. They arose from the study of 'algebraic reflexivity' of the linear space of derivations. In [**16**], Larson studied reflexivity and algebraic reflexivity of certain subspaces of operators, and he asked which algebras have an algebraically reflexive derivation space, or equivalently, for which algebras every local derivation on the algebra is a derivation. He, together with Sourour, showed that the answer to that question is affirmative for $B(\mathcal{X})$, where $\mathcal{X}$ is a Banach space [**17**]. An algebra of operators on a (separable) Hilbert space is called a CSL algebra if it consists of all operators leaving invariant the elements of a given commutative subspace lattice. Crist proved that every bounded local derivation on the direct limit of finite-dimensional CSL algebras via *-extendable embeddings (for example a triangular approximately finite-dimensional (AF) algebra) is a derivation [**5**]. One can also ask for which algebras the linear space of bounded derivations is reflexive.

In order to address these questions, we introduce the concept of an *approximately local derivation*. An operator D from A into X is an approximately local derivation if, for each $a \in A$, there is a sequence of derivations $\{D_{a,n}\}$ from A into X such that $D(a) = \lim_{n\to\infty} D_{a,n}(a)$. Our goal here is to see whether (mainly bounded) approximately local derivations are derivations. The advantage of considering these

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maps rather than local derivations is that they are more general and they allow us to study both reflexivity and algebraic reflexivity at the same time. On the other hand, in the previous studies of local derivations, it has always been helpful to study first the local multipliers (see [**12**, **19**]). However, as shown in Sections 2 and 3, we find it more useful to consider approximately local multipliers. One of our main results (Theorem 3.2) states that, for unital Banach algebras, if bounded approximately local multipliers are multipliers, then bounded approximately local derivations are derivations.

Although reflexivity of the linear space of bounded multipliers is useful, it does not guarantee that bounded approximately local derivations are derivations. However, in Section 4, we introduce a certain class of Banach algebras for which that can happen. We call these algebras *approximately locally unital*. In Theorem 4.4, we show that if A is approximately locally unital, then reflexivity of the linear space of bounded multipliers from A into A^* is sufficient for bounded approximately local derivations to be derivations. We also extend the result to the direct sum and projective tensor product of these Banach algebras. In Sections 5–7, we apply these results to show that bounded approximately local derivations are derivations if they are defined from a C*-algebra, a Banach algebra generated by idempotents, or a semisimple annihilator Banach algebra. Moreover, for C*-algebras, we show that the assumption of 'boundedness' is not necessary. This extends Johnson's result and, in particular, gives an alternative proof of it.

Let $\mathrm{L}^1(G)$ be the group algebra of a locally compact group G. In Section 8, first we show that bounded approximately local multipliers from $\mathrm{L}^1(G)$ are multipliers (Theorem 8.1), and, as a corollary, we conclude that ideals with finite codimension in $\mathrm{L}^1(G)$ have no non-zero bounded point derivations. Then, in Theorem 8.4, we give a type of characterization for bounded approximately local derivations from $\mathrm{L}^1(G)$ which is useful for determining when these maps are derivations. Later in the paper, by using our theory of approximately locally unital Banach algebras, we show that when G is a SIN (that is, a group with small invariant neighbourhoods of the identity) or a totally disconnected group, bounded approximately local derivations from $\mathrm{L}^1(G)$ are derivations. Finally, for a non-discrete group G, we provide an example of an essential Banach $M(G)$-bimodule X and a bounded local derivation T from $M(G)$ into X which is not a derivation.

1. *Preliminaries*

Let X and Y be Banach spaces, and let $L(X,Y)$ and $B(X,Y)$ be the spaces of (linear) operators and bounded operators from X into Y, respectively. Let $\mathcal{S}$ be a linear subspace of $L(X,Y)$, and for each $x \in X$, let $\mathcal{S}x = \{S(x) \mid S \in \mathcal{S}\}$ and $[\mathcal{S}x]$ be the norm-closure of $\mathcal{S}x$. Put

$$\mathrm{ref}_a(\mathcal{S}) = \{T \in L(X,Y) \mid T(x) \in \mathcal{S}x,\ x \in X\},$$

and if $\mathcal{S} \subseteq B(X,Y)$, then put

$$\mathrm{ref}(\mathcal{S}) = \{T \in B(X,Y) \mid T(x) \in [\mathcal{S}x],\ x \in X\}.$$

Suppose that $\mathcal{S} \subseteq L(X,Y)$. Then $\mathcal{S}$ is *algebraically reflexive* if $\mathcal{S} = \mathrm{ref}_a(\mathcal{S})$ and when $\mathcal{S} \subseteq B(X,Y)$, it is *reflexive* if $\mathcal{S} = \mathrm{ref}(S)$. For more on these notions see [**16**] and the references therein.

Let A be a Banach algebra and let X and Y be Banach left A-modules. A map $T \in L(X,Y)$ is a left A-module morphism if $T(ax) = aT(x)$ for all $a \in A$ and $x \in X$.

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In the case when $X = A$, we say that T is a right multiplier. Now let X and Y be Banach right A-modules. A map $T \in L(X, Y)$ is a right A-module morphism if $T(xa) = T(x)a$ for all $a \in A$ and $x \in X$. In the case when $X = A$, we say that T is a left multiplier. For Banach left (right) A-modules X and Y, let ${}_AL(X, Y)$ ($L_A(X, Y)$) and ${}_AB(X, Y)$ ($B_A(X, Y)$) be the linear spaces of left (right) A-module morphisms and bounded left (right) A-module morphisms from X into Y, respectively. In the case when A has a bounded approximate identity, it is known that right (left) multipliers from A into Y are bounded so that ${}_AL(A, Y) = {}_AB(A, Y)$ ($L_A(A, Y) = B_A(A, Y)$).

Let X be a Banach A-bimodule. An operator $D \in L(A, X)$ is a derivation if, for all $a, b \in A$, $D(ab) = aD(b) + D(a)b$. For each $x \in X$, the operator $ad_x \in B(A, X)$ defined by $ad_x(a) = ax - xa$ is a bounded derivation, called an inner derivation. Let $Z^1(A, X)$ and $\mathcal{Z}^1(A, X)$ be the linear spaces of derivations and bounded derivations from A into X, respectively.

Let G be a locally compact group with a fixed left Haar measure λ. The measure algebra $M(G)$ is the Banach space of complex-valued regular Borel measures on G. The space $M(G)$ is identified with the (dual) space of all continuous linear functionals on the Banach space $C_0(G)$, with the duality specified by setting

$$\langle \mu, f \rangle = \int_G f(t)\, d\mu(t), \qquad f \in C_0(G),\ \mu \in M(G).$$

The convolution multiplication $*$ on $M(G)$ is defined by setting

$$\langle \mu * \nu, f \rangle = \int_G \int_G f(st)\, d\mu(s)\, d\nu(t), \qquad f \in C_0(G),\ \mu, \nu \in M(G).$$

We write δ_s for the point mass at $s \in G$; the element δ_e is the identity of $M(G)$, and $\mathrm{l}^1(G)$ is the subalgebra of $M(G)$ generated by the point masses. Then $M(G)$ is a unital Banach algebra, and $\mathrm{L}^1(G)$, the group algebra on G, is a closed ideal in $M(G)$ [**7**, Theorem 3.3.36]. Moreover, $M(G) = \mathrm{L}^1(G) = \mathrm{l}^1(G)$ if and only if G is discrete. The *strong operator topology* on $M(G)$ is defined as follows: a net $\{\mu_\alpha\}$ converges to μ in the strong operator topology if and only if $\mu_\alpha * f \longrightarrow \mu * f$ and $f * \mu_\alpha \longrightarrow f * \mu$ in norm, for every $f \in \mathrm{L}^1(G)$. From [**7**, Theorem 2.9.49, Theorem 3.3.41], both $\mathrm{L}^1(G)$ and $\mathrm{l}^1(G)$ are dense in $M(G)$ in the strong operator topology.

2. *Approximately local multipliers*

Definition 2.1. Let X be a Banach right A-module. An operator T from A into X is a *local* (respectively *approximately local*) *left multiplier* if, for each $a \in A$, there is a left multiplier T_a (respectively a sequence of left multipliers $\{T_{a,n}\}$) from A into X such that $T(a) = T_a(a)$ (respectively $T(a) = \lim_{n\to\infty} T_{a,n}(a)$). If, in addition, T is bounded, then we say that T is a *bounded local left multiplier* (respectively *bounded approximately local left multiplier*). Similarly, for a Banach left A-module X, we can define *local right multipliers*, *bounded local right multipliers*, *approximately local right multipliers* and *bounded approximately local right multipliers* from A into X.

It is clear that local multipliers (respectively bounded local multipliers) are approximately local multipliers (respectively bounded approximately local multipliers). We are interested in determining when bounded approximately local multipliers are multipliers. This seems to be more general than determining the reflexivity of the linear space of bounded multipliers (since, in Definition 2.1, we

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are not assuming that each $T_{a,n}$ is necessary bounded), but the following theorem shows that these two notions are closely related.

We recall that a Banach left (right) A-module X is *essential* if it is the closure of $AX = \text{span}\{ax \,|\, a \in A,\, x \in X\}$ ($XA = \text{span}\{xa \,|\, a \in A,\, x \in X\}$). A Banach A-bimodule X is *essential* if it is essential as both a Banach left and Banach right A-module.

THEOREM 2.2. *Let A be a Banach algebra with $A = \overline{A^2}$, and let Y be a Banach left (respectively right) A-module. Then the following statements are equivalent.*

(i) *Every bounded approximately local right (respectively left) multiplier from A into Y is a right (respectively left) multiplier.*

(ii) *For each essential Banach left (respectively right) A-module X, ${}_AB(X,Y)$ (respectively $B_A(X,Y)$) is reflexive.*

(iii) ${}_AB(A,Y)$ *(respectively, $B_A(A,Y)$) is reflexive.*

Proof. We prove the theorem for the case of a left module. The other case can be proved similarly.

(i) $\Longrightarrow$ (ii): Let $T \in \text{ref}[{}_AB(X,Y)]$. For each $x \in X$, define

$$T_x \colon A \longrightarrow Y, \quad T_x(a) = T(ax), \qquad a \in A. \tag{1}$$

It is easy to check that T_x is a bounded approximately local right multiplier and so it is a right multiplier. Thus $T(acx) = aT(cx)$ for all $a, c \in A$. Now the result follows by essentiality of X.

(ii) $\Longrightarrow$ (iii): This is clear.

(iii) $\Longrightarrow$ (i): Let $T \colon A \longrightarrow Y$ be a bounded approximately local right multiplier. For $b \in A$, we define T_b as in (1). Then $T_b \in \text{ref}[{}_AB(A,Y)] = {}_AB(A,Y)$. Therefore $T(acb) = aT(cb)$ for all $a, c \in A$. The result now follows from the fact that $\overline{A^2} = A$. □

The following theorem shows that if a Banach algebra A has a bounded approximate identity, then the reflexivity of the linear space of bounded multipliers from A into A^* implies that every bounded approximately local multiplier from A is a multiplier.

PROPOSITION 2.3. *Let A be a Banach algebra with a bounded approximate identity such that ${}_AB(A, A^*)$ (respectivly $B_A(A, A^*)$) is reflexive. Then for any Banach left (respectively right) A-module X, every bounded approximately local right (respectively left) multiplier from A into X is a right (respectively left) multiplier.*

Proof. We prove the theorem for the case of a left module. The other case can be treated similarly. In view of Theorem 2.2, it suffices to prove that ${}_AB(A,X)$ is reflexive. Let $T \in \text{ref}[{}_AB(A,X)]$ and $F \in X^*$. Consider

$$K_F \colon X \longrightarrow A^*, \quad \langle K_F(x),\, a\rangle = F(ax), \qquad a \in A,\, x \in X.$$

Then K_F is a bounded left A-module morphism, and so

$$K_F \circ T \in \text{ref}[{}_AB(A, A^*)] = {}_AB(A, A^*).$$

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Thus, for all $a, b \in A$, $K_F(T(ab) - aT(b)) = 0$. Hence, if we put $u = T(ab) - aT(b)$, then $F(cu) = 0$ for all $c \in A$ and $F \in A^*$ so that $cu = 0$. On the other hand, there is a sequence $\{T_n\}$ of bounded right multipliers from A into X such that $T(ab) = \lim_{n\to\infty} T_n(ab)$. Therefore

$$u = \lim_{n\to\infty} T_n(ab) - aT(b) = \lim_{n\to\infty} a[T_n(b) - T(b)].$$

Therefore $u \in \overline{AX}$. Thus, by Cohen's factorization theorem, there are $e \in A$ and $x \in X$ such that $u = ex$. This, together with the fact that $Au = 0$, implies that $u = 0$. □

3. *Approximately local derivations*

DEFINITION 3.1. Let X be a Banach A-bimodule. An operator D from A into X is a *local* (respectively *approximately local*) *derivation* if, for each $a \in A$, there is a derivation D_a (respectively a sequence of derivations $\{D_{a,n}\}$) from A into X such that $D(a) = D_a(a)$ (respectively $D(a) = \lim_{n\to\infty} D_{a,n}(a)$). If, in addition, D is bounded, then we say that D is a *bounded local derivation* (respectively *bounded approximately local derivation*).

Let A be a Banach algebra, and let X be a Banach A-bimodule. It is clear that local derivations from A into X are exactly members of $\mathrm{ref}_a[Z^1(A, X)]$. Also, any element of $\mathrm{ref}_a[Z^1(A, X)]$ or $\mathrm{ref}[\mathcal{Z}^1(A, X)]$ is an approximately local derivation (bounded, in the latter case). However, the converse may not be true, since the derivations $D_{a,n}$ considered in Definition 3.1 need not be bounded. In this section, we primarily study bounded approximately local derivations and address the question of when they are derivations. We start with the following theorem, which determines the relationship between bounded approximately local multipliers and bounded approximately local derivations.

THEOREM 3.2. *Let A be a Banach algebra such that every bounded approximately local multiplier from A into any Banach A-module is a multiplier. Let X be a Banach A-bimodule, and let D be a bounded approximately local derivation from A into X. Then, for all $a, b, c, d \in A$, we have*

$$D(acdb) - D(acd)b - aD(cdb) + aD(cd)b = 0.$$

If, in addition, A is unital, then D is a derivation.

Proof. Let $a \in A$, and let Y be the norm-closure of aX. Then Y is a closed right A-submodule of X. Therefore X/Y is a Banach right A-module. Define the operator $\tilde{D}$ by

$$\tilde{D}\colon A \longrightarrow X/Y, \quad \tilde{D}(b) = D(ab) + Y, \qquad b \in A.$$

Since D is a bounded approximately local derivation, for each $b \in A$, there is a sequence of derivations $\{D_n\}$ from A into X such that $D(ab) = \lim_{n\to\infty} D_n(ab)$. Hence

$$\tilde{D}(b) = \lim_{n\to\infty} D_n(ab) + Y = \lim_{n\to\infty} [aD_n(b) + D_n(a)b] + Y = \lim_{n\to\infty} [D_n(a) + Y]b.$$

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Hence $\tilde{D}$ is a bounded approximately local left multiplier from A into X/Y. Thus, by the hypothesis, $\tilde{D}$ is a left multiplier. Therefore, for all $b, d \in A$, $\tilde{D}(db) = \tilde{D}(d)b$. Hence $D(adb) - D(ad)b \in Y$. Thus there is a sequence $\{x_n\}$ in X (possibly depending upon a, b and d) such that

$$D(adb) - D(ad)b = \lim_{n\to\infty} ax_n. \tag{2}$$

Fix $b, d \in A$, and define

$$T\colon A \longrightarrow X, \quad T(a) = D(adb) - D(ad)b, \qquad a \in A.$$

By (2), T is a bounded approximately local right multiplier and so it is a right multiplier. Therefore, for all $a, c \in A$, $T(ac) = aT(c)$. Thus $D(acdb) - D(acd)b = aD(cdb) - aD(cd)b$.

Let A be unital with identity e. Put $c = d = e$. Then $D(ab) - D(a)b - aD(b) = -aD(e)b$. On the other hand, if $\Delta\colon A \longrightarrow X$ is a derivation, then $a\Delta(e)b = \Delta(ae)b - \Delta(a)eb = 0$. Therefore $aD(e)b = 0$, since $aD(e)b = \lim_{n\to\infty} aD_n(e)b$ for a sequence $\{D_n\}$ of derivations from A into X. Therefore $D(ab) - D(a)b - aD(b) = 0$. Thus D is a derivation. □

The next theorem provides useful criteria for determining whether a bounded approximately local derivation from a Banach algebra with a bounded approximate identity is a derivation.

THEOREM 3.3. *Let A be a Banach algebra with a bounded approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$ such that ${}_AB(A, A^*)$ and $B_A(A, A^*)$ are reflexive. Let X be a Banach A-bimodule, and let D be a bounded approximately local derivation from A into X^*. Then there is a bounded derivation Δ and a right multiplier T from A into X^* such that $D = \Delta + T$. Moreover, if X is essential, then Δ and T are uniquely determined by this property and the following statements are equivalent.*

(i) *D is a derivation.*

(ii) *T is zero.*

(iii) *weak* $-\lim_{\alpha\to\infty} D(e_\alpha) = 0$.*

(iv) *For each $\alpha \in \Lambda$, there is a sequence of bounded derivations $\{D_{\alpha,n}\}$ such that $D(e_\alpha) = \lim_{n\to\infty} D_{\alpha,n}(e_\alpha)$ and $\sup\{||D_{\alpha,n}|| \mid \alpha \in \Lambda, n \in \mathbb{N}\}$ is finite.*

Proof. By Proposition 2.3 and Theorem 3.2 (considering the fact that $A = A^2$), for all $a, b, c \in A$, we have

$$D(acb) - D(ac)b - aD(cb) + aD(c)b = 0.$$

By putting $c = e_\alpha$ and letting $\alpha \to \infty$, we obtain

$$D(ab) - D(a)b - aD(b) + \lim_{\alpha\to\infty} aD(e_\alpha)b = 0. \tag{3}$$

Since $\{D(e_\alpha)\}$ is bounded, there is $x^* \in X^*$ and a subnet $\{D(e_{\alpha_i})\}$ such that $D(e_{\alpha_i}) \longrightarrow x^*$ in the weak* topology. Therefore $D(ab) - D(a)b - aD(b) + ax^*b = 0$. Define $T\colon A \longrightarrow X^*$ by $T(a) = ax^*$ and put $\Delta = D - T$. It is straightforward to check that T is a right multiplier and Δ is a bounded derivation.

(i) $\Longrightarrow$ (iv) and (ii) $\Longrightarrow$ (i) are obvious, and (iii) $\Longrightarrow$ (ii) and the uniqueness follow from the essentiality of X. For (iv) $\Longrightarrow$ (iii), put $K = \sup\{||D_{\alpha,n}|| \mid \alpha \in \Lambda, n \in \mathbb{N}\}$.

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Then for each $a, b \in A,\ x \in X,\ \alpha \in \Lambda$ and $n \in \mathbb{N}$,

$$\begin{aligned}\langle D_{\alpha,n}(e_\alpha), bxa\rangle &= \langle aD_{\alpha,n}(e_\alpha), bx\rangle \\ &= \langle D_{\alpha,n}(ae_\alpha), bx\rangle - \langle D_{\alpha,n}(a), e_\alpha bx\rangle \\ &= \langle D_{\alpha,n}(ae_\alpha - a), bx\rangle - \langle D_{\alpha,n}(a), e_\alpha bx - bx\rangle.\end{aligned}$$

Therefore

$$|\langle D_{\alpha,n}(e_\alpha), bxa\rangle| \leqslant K||ae_\alpha - a||\,||bx|| + K||a||\,||x||\,||e_\alpha b - b||.$$

Therefore, by first letting $n \to \infty$ and then $\alpha \to \infty$, we obtain

$$\langle D(e_\alpha), bxa\rangle \to 0.$$

This completes the proof. □

We note that the statements of the preceding theorem hold if we replace 'right multiplier' with 'left multiplier'. In fact, since A has a bounded approximate identity, we can write every right multiplier from A into X^* as a sum of a left multiplier and an inner derivation from A into X^*.

We recall that a Banach algebra is approximately weakly amenable if every bounded derivation D from A into A^* is approximately inner [**10**].

COROLLARY 3.4. *Let A be an approximately weakly amenable Banach algebra with a bounded approximate identity. If ${}_AB(A, A^*)$ and $B_A(A, A^*)$ are reflexive, then $\mathcal{Z}^1(A, A^*)$ is reflexive.*

Proof. Let $D \in \mathrm{ref}[\mathcal{Z}^1(A, A^*)]$. By Theorem 3.3, there are $\Delta \in \mathcal{Z}^1(A, A^*)$ and $T \in {}_AB(A, A^*)$ such that $D = \Delta + T$. Therefore $T \in \mathrm{ref}[\mathcal{Z}^1(A, A^*)]$. Hence, since A is approximately weakly amenable, for every $a \in A$, there is a sequence $\{x_{a,n}\} \in A^*$ such that

$$T(a) = \lim_{n\to\infty} ax_{a,n} - x_{a,n}a. \tag{4}$$

Now, let $\{e_\alpha \,|\, \alpha \in \Lambda\}$ be a bounded approximate identity for A, and let E be a weak*-cluster point of $\{e_\alpha \,|\, \alpha \in \Lambda\}$ in A^{**}. Then E is a two-sided identity for A in A^{**}. In particular, for all $a \in A$ and $x \in A^*$, $\langle E, ax - xa\rangle = \langle Ea - aE, x\rangle = 0$. Therefore, from (4),

$$\langle E, T(a)\rangle = 0, \qquad a \in A. \tag{5}$$

However, T is a (bounded) right multiplier, and so, by a weak*-cluster point of $T(e_\alpha)$, being taken, there is $y \in A^*$ such that $T(a) = ay$. Thus

$$\langle E, T(a)\rangle = \langle Ea, y\rangle = \langle y, a\rangle.$$

Therefore, from (5), $y = 0$, and hence $T = 0$. Therefore D is a derivation. □

Let φ be a character on the Banach algebra A (that is, φ is a non-zero multiplicative linear functional on A). A functional d on A is called a point derivation at φ if

$$d(ab) = d(a)\varphi(b) + \varphi(a)d(b), \qquad a, b \in A.$$

It is shown in [**10**, Proposition 2.1] that if a Banach algebra has a non-zero bounded point derivation, then it is not approximately weakly amenable. As we see in

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the next proposition and the following corollary, non-zero point derivations also generate local multipliers which are not multipliers. This is noted for a particular example in [**12**, p. 319]. We modify it for the general case.

PROPOSITION 3.5. *Let A be a Banach algebra, let φ be a character on A, and let d be a point derivation at φ. Then the operator $D\colon A \longrightarrow A^*$ defined by $D(a) = d(a)\varphi$ is a derivation which is also both a local left and a local right multiplier. Moreover, if d is bounded, then so is D.*

Proof. It is straightforward to check that D is a derivation, and, if d is bounded, then D is bounded. Therefore it just remains to show that D is both a local left and a local right multiplier. For that, it suffices to show that for every $a \in A$, there is $F_a \in A^*$ such that, for every $b \in A$,

$$\langle D(a), b\rangle = \langle F_a, ba\rangle = \langle F_a, ab\rangle. \tag{6}$$

Take $a \in A$. We consider two cases.

Case I: $\varphi(a) \neq 0$. Put $F_a = D(a)/\varphi(a)$. Then

$$\begin{aligned}\langle F_a, ab\rangle &= 1/\varphi(a)\langle D(a), ab\rangle = [1/\varphi(a)]d(a)\varphi(ab)\\ &= [1/\varphi(a)]d(a)\varphi(a)\varphi(b) = \langle D(a), b\rangle.\end{aligned}$$

Similarly, we have $\langle F_a, ba\rangle = \langle D(a), b\rangle$.

Case II: $\varphi(a) = 0$. Put $F_a = d$. Then it is easy to see that (6) holds. □

COROLLARY 3.6. *Let A be a Banach algebra such that $A = \overline{A^2}$. Suppose that every bounded local left multiplier from A into A^* is a left multiplier. Then A does not have a non-zero bounded point derivation. The same result is true if we replace 'left' with 'right' in the above statement.*

Proof. Let d, φ and D be as considered in Proposition 3.5. Then, by the hypothesis and the same proposition, D is both a bounded derivation and a bounded multiplier. Therefore, a simple calculation shows that D vanishes on $\{ab : a, b \in A\}$, and so does d. Hence $d = 0$. □

Let A be a Banach algebra, let X and Y be left (right) Banach A-modules, and let $T\colon X \longrightarrow Y$ be an operator. Then T is *left-intertwining* (*right-intertwining*) over A if the operator $x \longmapsto T(ax) - aT(x)$; $X \longrightarrow Y$ ($x \longmapsto T(xa) - T(x)a$; $X \longrightarrow Y$) is bounded for all $a \in A$. Suppose that X and Y are Banach A-bimodules. Then T is *intertwining* over A if it is both left-intertwining and right-intertwining. It is easy to see that each derivation is an intertwining map. A classical approach for determining whether an operator $T\colon X \longrightarrow Y$ is bounded is to see first whether it is intertwining. As we see in the next two results, this approach will help us to determine whether an approximately local derivation is bounded.

THEOREM 3.7. *Let A be a Banach algebra with a bounded approximate identity such that every approximately local multiplier from A into any Banach A-module is a multiplier. Then every approximately local derivation from A into any Banach A-bimodule is an intertwining map over A.*

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Proof. Let X be a Banach A-bimodule, let $D\colon A \longrightarrow X$ be an approximately local derivation, and let $a \in A$. Let $\{b_n\}$ be a sequence in A such that $b_n \to 0$. By [**2**, Corollary 11.11, Corollary 11.12], there are $c, d \in A$ and a sequence $\{t_n\} \subseteq A$ such that for each $n \in \mathbb{N}$, $b_n = cdt_n$ and $t_n \to 0$. An argument similar to the one found in Theorem 3.2 shows that the operator $T\colon A \longrightarrow X$ defined by

$$T(b) = D(acb) - aD(cb), \qquad b \in A,$$

is an approximately local left multiplier, and so, by the hypothesis, T is a left multiplier. Thus

$$D(ab_n) - aD(b_n) = T(dt_n) = T(d)t_n \to 0.$$

Therefore D is left-intertwining over A. Similarly, we can show that D is right-intertwining over A. □

COROLLARY 3.8. *Let A be a Banach algebra with a bounded approximate identity such that every approximately local multiplier from A into any Banach A-module is a multiplier. If every derivation from A into any Banach A-bimodule is bounded, then every approximately local derivation from A into any Banach A-bimodule is bounded.*

Proof. The result follows from Theorem 3.7 and [**7**, Corollary 2.7.7]. □

4. *Approximately locally unital Banach algebras*

In Theorem 3.2 we showed that for unital Banach algebras, the reflexivity of the linear space of bounded multipliers is sufficient for bounded approximately local derivations to be derivations. In this section, we extend that result to a considerably larger family of Banach algebras that we call approximately locally unital.

DEFINITION 4.1. Suppose that A is a Banach algebra and $a \in A$. We say that a has a *left* (*right*) *identity* in A if, for some $b \in A$, $ba = a$ ($ab = a$). We say that A is *approximately locally unital* if there are subsets A_{l} and A_r of A such that A is the closed linear span of both A_{l} and A_{r} and each element of A_{l} and A_{r} has a left identity and a right identity in A, respectively. From the definition, it is clear that if A is approximately locally unital, then $A = \overline{A^2}$.

EXAMPLE 4.2. Let A be a commutative regular semisimple Banach algebra, and let A_{c} be the linear subspace of A consisting of those elements with compact support. From [**7**, Proposition 4.1.18], for each $a \in A_{\mathrm{c}}$, there is an element $b \in A$ such that $b = 1$ on $\operatorname{supp} a$, so $ab = ba = a$. Thus A is approximately locally unital if A_{c} is dense in A. Some known examples of this kind of algebra are $C_0(\Omega)$ for a locally compact topological space Ω equipped with the norm $||\cdot||_\infty$ and the Fourier algebra $A(G)$ for a locally compact group G.

PROPOSITION 4.3. *Let A be an approximately locally unital Banach algebra such that ${}_AB(A, A^*)$ (respectively $B_A(A, A^*)$) is reflexive. Then every bounded approximately local right (respectively left) multiplier from A into any Banach left (respectively right) A-module is a right (respectively left) multiplier.*

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Proof. Let X be a Banach left A-module. By Theorem 2.2, it suffices to show that ${}_AB(A,X)$ is reflexive. Therefore let $T \in \mathrm{ref}[{}_AB(A,X)]$. By a similar argument to the one made in the proof of Proposition 2.3, we can show that for all $a,b,c \in A$, $aT(bc) = abT(c)$. Now let A_l be as in Definition 4.1 and take $b \in A_\mathrm{l}$. There is an element $a \in A$ such that $ab = b$. Since $T \in \mathrm{ref}[{}_AB(A,X)]$, for each $c \in A$, There is a sequence $\{T_n\}$ in ${}_AB(A,X)$ such that $T(bc) = \lim_{n\to\infty} T_n(bc)$. Therefore

$$\begin{aligned} T(bc) - bT(c) &= \lim_{n\to\infty} T_n(bc) - bT(c) = \lim_{n\to\infty} T_n(abc) - abT(c) \\ &= a \lim_{n\to\infty} T_n(bc) - abT(c) = aT(bc) - abT(c) = 0. \end{aligned}$$

The final result follows because A is the closed linear span of A_l. The proof in the case of right modules follows along similar lines. □

THEOREM 4.4. *Let A be an approximately locally unital Banach algebra such that ${}_AB(A,A^*)$ and $B_A(A,A^*)$ are reflexive. Then every bounded approximately local derivation D from A into a Banach A-bimodule X is a derivation. In particular, $\mathcal{Z}^1(A,X)$ is reflexive and $\mathrm{ref}_a[Z^1(A,X)] \cap B(A,X) = \mathcal{Z}^1(A,X)$.*

Proof. From Proposition 4.3 and Theorem 3.2, for all $a,b,c,d \in A$,

$$D(acdb) - D(acd)b - aD(cdb) + aD(cd)b = 0. \tag{7}$$

Let A_l and A_r be as in Definition 4.1. Take $a \in A_\mathrm{r}$ and $b \in A_\mathrm{l}$. There are $e,f \in A$ such that $ae = a$ and $fb = b$. Let $g = e + f - ef$. Then it is easy to see that $ag = a$ and $gb = b$. Now, if, in (7), we put $c = d = g$, then

$$D(ab) - D(a)b - aD(b) = -aD(g^2)b.$$

On the other hand, if Δ is any derivation from A into X, then

$$a\Delta(g^2)b = \Delta(ag^2)b - \Delta(a)g^2b = \Delta(a)b - \Delta(a)b = 0.$$

Therefore $aD(g^2)b = 0$, since $aD(g^2)b = \lim_{n\to\infty} aD_n(g^2)b$ for a sequence $\{D_n\}$ of derivations from A into X. Therefore $D(ab) = D(a)b + aD(b)$. The final result follows because A is the closed linear span of both A_l and A_r. □

The following is an application of Theorem 4.4 which will be used later.

THEOREM 4.5. *Let A be a Banach algebra, and let $\{A_i\}_{i\in I}$ be a family of Banach subalgebras of A such that $(\bigoplus_{i\in I} A_i)^- = A$. If, for each $i \in I$, A_i is approximately locally unital and ${}_{A_i}B(A_i,A_i^*)$ and $B_{A_i}(A_i,A_i^*)$ are reflexive, then every bounded approximately local derivation from A into any Banach A-bimodule is a derivation.*

Proof. By the hypothesis, for each $i \in I$, there are subsets A_i^l and A_i^r of A_i such that A_i is the closed linear span of both A_i^l and A_i^r, and each element in A_i^l has a left identity and each element in A_i^r has a right identity in A_i. Therefore, if we put $A_\mathrm{l} = \bigcup_{i\in I} A_i^\mathrm{l}$ and $A_\mathrm{r} = \bigcup_{i\in I} A_i^\mathrm{r}$, then A_l and A_r satisfy the assumption of Definition 4.1, and it follows that A is an approximately locally unital Banach algebra. Therefore, by Theorem 4.4, it suffices to show that ${}_AB(A,A^*)$ and $B_A(A,A^*)$ are reflexive. Let $T \in \mathrm{ref}[{}_AB(A,A^*)]$ and let T_i be the restriction of T to A_i. By the hypothesis and Proposition 4.3, each T_i is a right multiplier. On the other hand, if $a \in A_i$ and $b \in A_j$, where $i \neq j$, then $aT(b) = 0$. To see this, let $\{T_n\}$ be a sequence in

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${}_A B(A, A^*)$ such that $T(b) = \lim_{n\to\infty} T_n(b)$. Then

$$aT(b) = \lim_{n\to\infty} aT_n(b) = \lim_{n\to\infty} T_n(ab) = 0.$$

Thus, for each $a_i, b_i \in A_i$ and $a_j, b_j \in A_j$ with $i \neq j$,

$$T[(a_i + a_j)(b_i + b_j)] = T(a_i a_j + b_i b_j) = a_i T(a_j) + b_i T(b_j) = (a_i + a_j)T(b_i + b_j).$$

Therefore T acts as a right multiplier on the direct sum of $\{A_i\}_{i\in I}$ which is dense in A. Since T is bounded, it follows that $T \in {}_A B(A, A^*)$. Similarly, we can show that $B_A(A, A^*)$ is reflexive. □

Let A and B be Banach algebras, and let $A \widehat{\otimes} B$ be their projective tensor product. There are Banach A-bimodule actions on $A \widehat{\otimes} B$ specified by

$$a \cdot (c \otimes b) = ac \otimes b, \quad (c \otimes b) \cdot a = ca \otimes b, \qquad a, c \in A,\ b \in B.$$

Similarly, we can define Banach B-bimodule actions on $A \widehat{\otimes} B$.

THEOREM 4.6. *Let $\{A_i\}_{i=1}^n$ be a finite set of Banach algebras, and let*

$$\mathcal{A} = \widehat{\bigotimes_{i=1}^{n}} A_i.$$

If, for each i, A_i is approximately locally unital and ${}_{A_i}B(A_i, A_i^*)$ and $B_{A_i}(A_i, A_i^*)$ are reflexive, then every bounded approximately local derivation from $\mathcal{A}$ into any Banach $\mathcal{A}$-bimodule is a derivation.

Proof. First consider the case $n = 2$. For $i = 1, 2$, there are subsets A_i^{l} and A_i^{r} of A_i such that A_i is the closed linear span of both A_i^{l} and A_i^{r}, each element in A_i^{l} has a left identity and each element in A_i^{r} has a right identity in A_i. Put

$$\mathcal{A}_{\mathrm{l}} = \{a_1 \otimes a_2 \mid a_i \in A_i^{\mathrm{l}}\}, \quad \mathcal{A}_{\mathrm{r}} = \{b_1 \otimes b_2 \mid b_i \in A_i^{\mathrm{r}}\}.$$

Then $\mathcal{A}_{\mathrm{l}}$ and $\mathcal{A}_{\mathrm{r}}$ satisfy the assumption of Definition 4.1 and so $\mathcal{A}$ is an approximately locally unital Banach algebra. Therefore, by Theorem 4.4, it suffices to show that ${}_{\mathcal{A}}B(\mathcal{A}, \mathcal{A}^*)$ and $B_{\mathcal{A}}(\mathcal{A}, \mathcal{A}^*)$ are reflexive. We show that ${}_{\mathcal{A}}B(\mathcal{A}, \mathcal{A}^*)$ is reflexive. The other case can be treated similarly. Let $T \in {}_{\mathcal{A}}B(\mathcal{A}, \mathcal{A}^*)$. We claim that $T(c \cdot u) = c \cdot T(u)$ for all $c \in A_1$ and $u \in \mathcal{A}$. To this end, it suffices to show that

$$\langle T(ca_1 \otimes a_2), b_1 \otimes b_2\rangle = \langle T(a_1 \otimes a_2), b_1 c \otimes b_2\rangle, \qquad c \in A_1,\ a_i, b_i \in A_i. \tag{8}$$

Let $a_2 \in A_2^{\mathrm{l}}$ and $b_2 \in A_2^{\mathrm{r}}$. By a similar argument to the one made in Theorem 4.4, there is $g \in A_2$ such that $ga_2 = a_2$ and $b_2 g = b_2$. Hence

$$\begin{aligned}\langle T(ca_1 \otimes a_2), b_1 \otimes b_2\rangle &= \langle T(ca_1 \otimes ga_2), b_1 \otimes b_2\rangle \\ &= \langle (c \otimes g)T(a_1 \otimes a_2), b_1 \otimes b_2\rangle = \langle T(a_1 \otimes a_2), b_1 c \otimes b_2 g\rangle \\ &= \langle T(a_1 \otimes a_2), b_1 c \otimes b_2\rangle.\end{aligned}$$

Thus equation (8) holds since A_2 is the closed linear span of both A_2^{l} and A_2^{r}. Therefore, ${}_{\mathcal{A}}B(\mathcal{A}, \mathcal{A}^*) \subseteq {}_{A_1}B(\mathcal{A}, \mathcal{A}^*)$. On the other hand, $\mathcal{A}$ is an essential Banach A_1-bimodule, and so from Proposition 4.3 and Theorem 2.2, ${}_{A_1}B(\mathcal{A}, \mathcal{A}^*)$ is reflexive. Thus

$$\mathrm{ref}[{}_{\mathcal{A}}B(\mathcal{A}, \mathcal{A}^*)] \subseteq {}_{A_1}B(\mathcal{A}, \mathcal{A}^*). \tag{9}$$

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Similarly, we can show that

$$\mathrm{ref}[{}_{\mathcal{A}}B(\mathcal{A},\mathcal{A}^*)] \subseteq {}_{A_2}B(\mathcal{A},\mathcal{A}^*). \tag{10}$$

Now let $T \in \mathrm{ref}[{}_{\mathcal{A}}B(\mathcal{A},\mathcal{A}^*)]$. Then, for every $a_1 \in A_1$, $a_2 \in A_2$ and $u \in \mathcal{A}$, from (9) and (10) we have

$$T[(a_1 \otimes a_2)u] = T[a_1 \cdot (a_2 \cdot u)] = a_1 \cdot T(a_2 \cdot u) = (a_1 \otimes a_2)T(u).$$

Hence $T \in {}_{\mathcal{A}}B(\mathcal{A},\mathcal{A}^*)$. For $n > 2$, put

$$\mathcal{B} = \bigotimes_{i=1}^{n-1}\!\widehat{\ } A_i.$$

Then $\mathcal{A} = \mathcal{B} \,\widehat{\otimes}\, A_n$. Therefore the result follows by induction. □

5. C*-algebras

Let A be a C*-algebra, and let X be a Banach A-bimodule. It is known that derivations from A into X are bounded [**7**, Theorem 5.3.7], so $\mathcal{Z}^1(A,X) = Z^1(A,X)$. In [**12**], Johnson showed that local derivations from A into X are derivations, that is, $Z^1(A,X)$ is algebraically reflexive. In this section we prove the stronger statement that approximately local derivations from A into X are derivations. In particular, $Z^1(A,X)$ is also reflexive. First, we need the following lemma.

LEMMA 5.1. *Let A be a unital* C*-*algebra. Then every bounded approximately local multiplier from A into any Banach A-module is a multiplier.*

Proof. By Proposition 4.3, it suffices to show that ${}_AB(A,A^*)$ and $B_A(A,A^*)$ are reflexive. First assume that $A = C(K)$ for a compact subset K of $\mathbb{R}$, and R is the restriction map from $C_0(\mathbb{R})$ onto $C(K)$. Then $C(K)$ becomes an essential Banach $C_0(\mathbb{R})$-bimodule for the actions defined by

$$ba = R(b)a, \quad ab = aR(b), \qquad a \in C(K),\ b \in C_0(\mathbb{R}).$$

On the other hand, by [**12**, Proposition 3.1], bounded approximately local multipliers from $C_0(\mathbb{R})$ into $C_0(\mathbb{R})^*$ are multipliers, so, by Proposition 2.3 and Theorem 2.2, $B_{C_0(\mathbb{R})}(C(K),C(K)^*)$ and ${}_{C_0(\mathbb{R})}B(C(K),C(K)^*)$ are reflexive. The final result follows from the fact that

$$B_{C(K)}(C(K),C(K)^*) = B_{C_0(\mathbb{R})}(C(K),C(K)^*),$$
$${}_{C(K)}B(C(K),C(K)^*) = {}_{C_0(\mathbb{R})}B(C(K),C(K)^*).$$

We now consider the general case. Let $T \in \mathrm{ref}[{}_AB(A,A^*)]$, let $a \in A$ be a self-adjoint element, and let $A(a)$ be the C^*-subalgebra of A generated by a and 1. It is well known that there is a compact subset K of $\mathbb{R}$ such that $A(a)$ is isometrically isomorphic to $C(K)$. On the other hand, A is an essential Banach left $A(a)$-module, so, by the preceding case and Theorem 2.2, ${}_{A(a)}B(A,A^*)$ is reflexive. However, ${}_AB(A,A^*) \subseteq {}_{A(a)}B(A,A^*)$, so $T \in {}_{A(a)}B(A,A^*)$, that is, $T(ab) = aT(b)$ for all $b \in A$. The final result follows, since A is the linear span of its self-adjoint elements. The reflexivity of $B_A(A,A^*)$ can be proved similarly. □

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THEOREM 5.2. *Let A be a C*-algebra, and let X be a Banach A-bimodule. Then every approximately local derivation D from A into X is a derivation.*

Proof. Without loss of generality, we can assume that A is unital, for otherwise we may consider the unitalization $A^\sharp$ of A (see [**7**, Definition 3.2.1]) and extend X to a Banach $A^\sharp$-bimodule by defining $1x = x1 = x$ and D to an approximately local derivation from $A^\sharp$ into X by defining $D(1) = 0$. By Theorem 3.2 and Lemma 5.1, in order to show that D is a derivation, it suffices to show that D is bounded. To this end, from [**4**, Corollary 1.2], it suffices to show that the restriction of D to any commutative unital C*-subalgebra $C(\Omega)$ of A is bounded. By a similar argument to the one made in [**12**, Proposition 7.2], every approximately local multiplier from $C(\Omega)$ into any Banach $C(\Omega)$-module is bounded, so, by Lemma 5.1, it is a multiplier. Therefore, since $C(\Omega)$ is unital, by Corollary 3.8, the restriction of D to $C(\Omega)$ is bounded. This completes the proof. □

6. *Banach algebras generated by idempotents*

Let A be a Banach algebra, let E be the set of idempotents in A, and let $A(E)$ be the subalgebra of A generated by E. We say that A is *generated by idempotents* if $A(E)$ is dense in A. It is easy to see that $A(E)$ is the linear span of $\Sigma = \{e_1 \dots e_n \mid n \in N, e_i \in E\}$. Moreover, for each element $u = e_1 \dots e_n$ in Σ, e_1 and e_n are left and right identities for u, respectively.

THEOREM 6.1. *Let A be a Banach algebra generated by idempotents. Then every bounded approximately local derivation from A into any Banach A-bimodule is a derivation.*

Proof. By the remark before the theorem, A is approximately locally unital. Thus, from Theorem 4.4, it suffices to show that ${}_AB(A, A^*)$ and $B_A(A, A^*)$ are reflexive. Let $T \in \mathrm{ref}[{}_AB(A, A^*)]$, and $a \in A$. Let e be an idempotent in A. By the assumption, there is a sequence $\{T_n\} \subseteq {}_AB(A, A^*)$ such that $T(ea) = \lim_{n\to\infty} T_n(ea)$. Hence

$$eT(ea) = \lim_{n\to\infty} eT_n(ea) = \lim_{n\to\infty} T_n(e^2a) = \lim_{n\to\infty} T_n(ea) = T(ea). \tag{11}$$

On the other hand, there is a sequence $\{S_n\} \subseteq {}_AB(A, A^*)$ such that $T(a - ea) = \lim_{n\to\infty} S_n(a - ea)$. Therefore

$$eT(a - ea) = \lim_{n\to\infty} eS_n(a - ea) = \lim_{n\to\infty} S_n(ea - e^2a) = 0. \tag{12}$$

Therefore, from (11) and (12), $T(ea) = eT(a)$. Thus $T \in {}_AB(A, A^*)$ because A is generated by idempotents. The reflexivity of $B_A(A, A^*)$ can be proved similarly. □

EXAMPLE 6.2. (i) Let I be a non-empty set, and let $p \in [1, \infty)$. Then $l^p(I)$ and $c_0(I)$, with pointwise addition and multiplication, are Banach algebras generated by idempotents. Therefore, bounded approximately local derivations from $l^p(I)$ or $c_0(I)$ are derivations.

(ii) Let G be a locally compact totally disconnected group, and let $A(G)$ be the Fourier algebra of G. It is shown in [**9**, p. 188] that $A(G)$ is generated

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by idempotents, so bounded approximately local derivations from $A(G)$ are derivations.

Let X be a Banach space, let $F(X)$ be the algebra of all finite rank operators in $B(X)$, and let Y be an (algebraic) $F(X)$-bimodule. Brešar and Šemrl showed in [**3**, Theorem 3.6] that every linear mapping $D\colon F(X) \longrightarrow Y$ satisfying $D(P) = PD(P) + D(P)P$ for every projection P in $F(X)$ is a derivation. Now assume that Y is, in addition, a normed space and $D\colon F(X) \longrightarrow Y$ is an approximately local derivation (the definition would be similar to Definition 3.1). Then it is straightforward to check that D satisfies the above condition of Brešar and Šemrl so that D is a derivation. Hence we can state the following.

THEOREM 6.3. *Let X be a Banach space, let $F(X)$ be the algebra of all finite rank operators in $B(X)$, and let Y be an $F(X)$-bimodule which is also a normed space. Then every approximately local derivation D from $F(X)$ into Y is a derivation.*

We recall that a *Banach operator algebra* on a Banach space X is a subalgebra of $B(X)$ containing $F(X)$ such that it is a Banach algebra with respect to some norm.

The following corollary follows immediately from the preceding theorem and Corollary 3.6.

COROLLARY 6.4. *Let X be a Banach space, and let A be a Banach operator algebra on X. If $F(X)$ is dense in A, then every bounded approximately local derivation from A into any Banach A-bimodule is a derivation. In particular, A does not have a non-zero bounded point derivation.*

EXAMPLE 6.5. Let X be a Banach space. Then $\mathcal{A}(X)$, the space of approximable operators, $\mathcal{N}(X)$, the space of nuclear operators, and $\mathcal{C}_p(X)$ for $1 \leqslant p < \infty$ when X is a Hilbert space are Banach operator algebras having $F(X)$ as a dense subalgebra. See [**7**, Definition A.3.55, Definition A.3.57; **18**, Chapter 2] for the details.

7. *Semisimple annihilator Banach algebras*

Let A be a Banach algebra and $E \subseteq A$. The left and right annihilator of E are the sets $\operatorname{lan}(E)$ and $\operatorname{ran}(E)$ given by

$$\operatorname{lan}(E) = \{a \in A \mid aE = \{0\}\}, \quad \operatorname{ran}(E) = \{a \in A \mid Ea = \{0\}\}.$$

A is an *annihilator Banach algebra* if, for every closed left ideal L and closed right ideal R,

$$\operatorname{ran}(L) = \{0\} \quad \text{if and only if } L = A,$$
$$\operatorname{lan}(R) = \{0\} \quad \text{if and only if } R = A.$$

EXAMPLE 7.1. (i) Let G be a compact group. Then $L^p(G)$ for $1 \leqslant p < \infty$ and $C(G)$ with the convolution product are semisimple annihilator Banach algebras [**1**, Section 4].

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(ii) Let G be a locally compact group. Then $AP(G)$, the algebra of almost periodic functions on G with convolution product and uniform norm, is a semisimple annihilator Banach algebra [**1**, Section 4].

LEMMA 7.2. *Let A be a semisimple annihilator Banach algebra. Then A is approximately locally unital.*

Proof. Let

$$A_{\mathrm{l}} = \bigcup \{I \mid I \text{ is a minimal right ideal}\}$$

and

$$A_{\mathrm{r}} = \bigcup \{I \mid I \text{ is a minimal left ideal}\}.$$

From [**2**, Proposition 32.17], there is a two-sided ideal of A, denoted by $\mathrm{soc}(A)$, such that the linear span of both A_{l} and A_{r} is dense in $\mathrm{soc}(A)$ and $A = \overline{\mathrm{soc}(A)}$. On the other hand, by [**2**, Proposition 30.6], every minimal right ideal and minimal left ideal has the form eA and Af for some minimal idempotents e and f in A, respectively. In particular, e is a left identity for elements of eA and f is a right identity for the elements of Af. Therefore, A_{l} and A_{r} satisfy the condition considered in Definition 4.1 and so A is approximately locally unital. □

THEOREM 7.3. *Let A be a semisimple annihilator Banach algebra, and let X be a Banach A-bimodule. Then every bounded approximately local derivation from A into X is a derivation.*

Proof. By [**2**, Proposition 32.17], A is the closure of the direct sum of the minimal closed bi-ideals of A. Also, by [**2**, Corollary 32.12], each minimal closed bi-ideal of A is a semisimple annihilator Banach algebra. Therefore, from Lemma 7.2 and Theorem 4.5, it suffices to show that for each minimal closed bi-ideal I, ${}_IB(I, I^*)$ and $B_I(I, I^*)$ are reflexive. Let $T \in \mathrm{ref}[{}_IB(I, I^*)]$. By [**2**, Theorem 32.20], there is a Banach space X such that $\mathrm{soc}(I)$ and $F(X)$ are algebraically isomorphic. Therefore, from Theorem 6.3, the restriction of T to $\mathrm{soc}(I)$ is a right multiplier. Therefore $T \in {}_IB(I, I^*)$ because $I = \overline{\mathrm{soc}(I)}$ and T is bounded. The reflexivity of $B_I(I, I^*)$ can be proved similarly. □

COROLLARY 7.4. *Let A be a semisimple annihilator Banach algebra. Then there is no non-zero bounded point derivation on A.*

Proof. This is a result of the preceding theorem and Corollary 3.6. □

8. *Group algebras*

In this final section, we investigate bounded approximately local derivations from the group algebras. We start with the following essential theorem which states that bounded approximately local multipliers from the group algebras are multipliers.

THEOREM 8.1. *Let G be a locally compact group. Then every bounded approximately local multiplier from $\mathrm{L}^1(G)$ into any Banach $\mathrm{L}^1(G)$-module is a multiplier.*

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Proof. Since $\mathrm{L}^1(G)$ has a bounded approximate identity, from Proposition 2.3, it is enough to show that ${}_{\mathrm{L}^1(G)}B(\mathrm{L}^1(G),\mathrm{L}^1(G)^*)$ and $B_{\mathrm{L}^1(G)}(\mathrm{L}^1(G),\mathrm{L}^1(G)^*)$ are reflexive. First we consider the case when G is discrete. Let $T \in {}_{\mathrm{l}^1(G}B(\mathrm{l}^1(G),\mathrm{l}^1(G)^*)$. Take $x \in G$ and let H be the subgroup generated by x. Then H is an abelian subgroup of G, $\mathrm{l}^1(H)$ is a closed subalgebra of $\mathrm{l}^1(G)$ and $\mathrm{l}^1(G)$ is an essential Banach $\mathrm{l}^1(H)$-bimodule. Also, it is easy to see that the restriction of T to $\mathrm{l}^1(H)$ is a bounded approximately local right multiplier from $\mathrm{l}^1(H)$ into $\mathrm{l}^1(G)^*$. However, the argument in the beginning of [**19**, Section 3], can be employed to show that every approximately local multiplier from $\mathrm{l}^1(H)$ is a local operator in a sense of [**12**, Section 3] or [**19**, Section 2]. Thus, by [**19**, Theorem 2.4], $T|_{\mathrm{l}^1(H)}\colon \mathrm{l}^1(H) \longrightarrow \mathrm{l}^1(G)^*$ is a right multiplier (see also [**19**, Corollary 4.5]). In particular, if e is the identity in G, then $T(\delta_x) = \delta_x T(\delta_e)$. Therefore $T(f) = fT(\delta_e)$, where f is in the linear span of $\Sigma = \{\delta_x \mid x \in G\}$. The final result follows from the fact that T is bounded and $\mathrm{l}^1(G)$ is the closed linear span of Σ.

We now consider the general case. First we note that

$$ {}_{\mathrm{L}^1(G)}B(\mathrm{L}^1(G),\mathrm{L}^1(G)^*)] = {}_{\mathrm{l}^1(G)}B(\mathrm{L}^1(G),\mathrm{L}^1(G)^*) = {}_{M(G)}B(\mathrm{L}^1(G),\mathrm{L}^1(G)^*). \quad (13) $$

To see this, let $T \in {}_{\mathrm{L}^1(G)}B(\mathrm{L}^1(G),\mathrm{L}^1(G)^*)$ and $\mu \in M(G)$. Since $\mathrm{L}^1(G)$ is dense in $M(G)$ in the strong operator topology [**7**, Theorem 2.9.49], there is a net $\{f_i\}_{i\in I}$ in $\mathrm{L}^1(G)$ such that, for all $g \in \mathrm{L}^1(G)$, $f_i * g \longrightarrow \mu * g$ and $g * f_i \longrightarrow g * \mu$ in the L^1-norm. Hence

$$ \begin{aligned} T(\mu * g) &= \text{norm} - \lim_{i\to\infty} T(f_i * g) = w^* - \lim_{i\to\infty} T(f_i * g) \\ &= w^* - \lim_{i\to\infty} f_i * T(g) = \mu * T(g). \end{aligned} $$

Therefore $T \in {}_{M(G)}B(\mathrm{L}^1(G),\mathrm{L}^1(G)^*)$. Since $\mathrm{l}^1(G)$ is also dense in $M(G)$ in the strong operator topology, the other equality in (13) follows by the same argument. On the other hand, from Proposition 2.3, Theorem 2.2 and the result we obtained in the first part,

$$ {}_{\mathrm{l}^1(G)}B(\mathrm{L}^1(G),\mathrm{L}^1(G)^*) $$

is reflexive. Therefore, from (13), ${}_{\mathrm{L}^1(G)}B(\mathrm{L}^1(G),\mathrm{L}^1(G)^*)$ is reflexive. The reflexivity of $B_{\mathrm{L}^1(G)}(\mathrm{L}^1(G),\mathrm{L}^1(G)^*)$ can be proved similarly. □

Corollary 8.2. *Let G be a locally compact group, and let J be an ideal in $\mathrm{L}^1(G)$ with finite codimension. Then every bounded approximately local multiplier from J into any Banach $\mathrm{L}^1(G)$-module is a multiplier. If, in addition, G is amenable, then the result is true for all Banach J-modules.*

Proof. We give the proof for a left module case. The other case can be proved similarly. From [**20**], $J = J^2$, so J is closed and it is an essential Banach $\mathrm{L}^1(G)$-bimodule. Also, for every Banach left $\mathrm{L}^1(G)$-module Y, $T \in {}_JB(J,Y)$, $a \in A$ and $b, c \in J$, we have

$$ T(abc) = abT(c) = aT(bc). $$

Now, since $J = J^2$,

$$ {}_JB(J,Y) = {}_{\mathrm{L}^1(G)}B(J,Y). $$

Hence we have the result, from Theorem 8.1 and Theorem 2.2. If G is amenable, then by Johnson's theorem, $\mathrm{L}^1(G)$ is amenable and so, by [**6**, Corollary 3.8], J has a

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bounded approximate identity. On the other hand, by the preceding part, ${}_JB(J, J^*)$ is reflexive. Therefore, the result follows from Proposition 2.3. □

REMARK 8.3. Let G be a locally compact group, and let J be an ideal in $\mathrm{L}^1(G)$ with finite codimension. Since J^* is a Banach $\mathrm{L}^1(G)$-bimodule, from Corollary 8.2, bounded local multipliers from J into J^* are multipliers. Hence, from Corollary 3.6, there is no non-zero bounded point derivation on J. However, for certain groups, even ideals of codimension 1 need not be weakly amenable [**14**].

The next theorem follows immediately from the preceding theorem and Theorem 3.3.

THEOREM 8.4. *Let G be a locally compact group, let X be a Banach $\mathrm{L}^1(G)$-bimodule, and let D be a bounded approximately local derivation from $\mathrm{L}^1(G)$ into X^*. Then there is a bounded derivation Δ and a right multiplier T from $\mathrm{L}^1(G)$ into X^* such that $D = \Delta + T$. Moreover, if X is essential, then Δ and T are uniquely determined by this property, and the following statements are equivalent for each bounded approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$ for $\mathrm{L}^1(G)$.*

(i) *D is a derivation.*
(ii) *T is zero.*
(iii) $\mathrm{weak}^* - \lim_{\alpha\to\infty} D(e_\alpha) = 0$.

COROLLARY 8.5. *Let G be a locally compact group. Then $\mathcal{Z}^1(\mathrm{L}^1(G), \mathrm{L}^1(G)^*)$ is reflexive.*

Proof. By [**7**, Theorem 5.6.48], $\mathrm{L}^1(G)$ is weakly amenable. Therefore, the result follows from Theorem 8.1 and Corollary 3.4. □

REMARK 8.6. Since $\mathrm{L}^1(G)$ is semisimple, derivations on $\mathrm{L}^1(G)$ are bounded [**13**], so bounded approximately local derivations on $\mathrm{L}^1(G)$ are exactly the members of $\mathrm{ref}[Z^1(\mathrm{L}^1(G), \mathrm{L}^1(G))]$. On the other hand, $\mathrm{L}^1(G)$ is a submodule of $M(G) = C_0(G)^*$ and $C_0(G)$ is an essential Banach $\mathrm{L}^1(G)$-bimodule. Hence, by the preceding theorem, a bounded approximately local derivation D on $\mathrm{L}^1(G)$ is a derivation if and only if $\mathrm{weak}^* - \lim_{\alpha\to\infty} D(e_\alpha) = 0$ for a bounded approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$ for $\mathrm{L}^1(G)$. This provides a useful criterion for determining whether D is a derivation. For example, let G be a SIN group. Then $\mathrm{L}^1(G)$ has a central bounded approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$. However, if Δ is a derivation on $\mathrm{L}^1(G)$, then, by [**7**, Theorem 5.6.53], there is a measure $\mu \in M(G)$ such that $\Delta(f) = f * \mu - \mu * f$ for all $f \in \mathrm{L}^1(G)$. Thus, for each $\alpha \in \Lambda$, $\Delta(e_\alpha) = 0$, and so $D(e_\alpha) = 0$ since $D(e_\alpha) = \lim_{n\to\infty} D_{\alpha,n}(e_\alpha)$ for a sequence of derivations $\{D_{\alpha,n}\}$ on $\mathrm{L}^1(G)$. Hence D is a derivation.

We can extend the result we have obtained in the preceding remark. In order to do that, we will use the theory of approximately locally unital Banach algebras that we developed in Section 4. First we need the following lemma.

LEMMA 8.7. *Let G be a locally compact group, and let H be an open subgroup of G. If $\mathrm{L}^1(H)$ is approximately locally unital, then $\mathrm{L}^1(G)$ is approximately locally unital.*

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Proof. Since H is an open subgroup of G, we can consider $\mathrm{L}^1(H)$ as a closed subalgebra of $\mathrm{L}^1(G)$ where elements in $\mathrm{L}^1(H)$ are exactly those in $\mathrm{L}^1(G)$ which vanishes off H. By the hypothesis, $\mathrm{L}^1(H)$ is approximately locally unital, and so it has subsets B_{l} and B_{r} satisfying the conditions in Definition 4.1. Put

$$A_{\mathrm{l}} = \{f * \delta_x \mid f \in B_{\mathrm{l}}, x \in G\}, \qquad A_{\mathrm{r}} = \{\delta_x * f \mid f \in B_{\mathrm{r}}, x \in G\}.$$

We show that A_{l} and A_{r} satisfy the assumption in Definition 4.1 for $\mathrm{L}^1(G)$. Let $f \in B_{\mathrm{l}}$ and $x \in G$. By the assumption, there is $g \in \mathrm{L}^1(H)$ such that $g * f = f$. Therefore $g * (f * \delta_x) = f * \delta_x$. Hence each element in A_{l} has a left identity. Since $C_{\mathrm{c}}(G)$ is dense in $\mathrm{L}^1(G)$, in order to show that $\mathrm{L}^1(G)$ is the closed linear span of A_{l}, it suffices to show that $C_{\mathrm{c}}(G) \subseteq \overline{\operatorname{span}}\, A_{\mathrm{l}}$. Let $\varphi \in C_{\mathrm{c}}(G)$. Since $\operatorname{supp}\varphi$ is compact, H is open and $G = \bigcup_{x\in G} Hx$, there is a finite subset $\{x_1, \ldots, x_n\}$ of G such that $\operatorname{supp}\varphi \subseteq \bigcup_{i=1}^n Hx_i$, where the union is disjoint. Thus $\varphi = \Sigma_{i=1}^n \varphi_i$, where $\varphi_i = \varphi\chi_{Hx_i}$. It is easy to see that $\varphi_i \in \mathrm{L}^1(G)$ and $\operatorname{supp}\varphi_i \subseteq Hx_i$. Therefore, $\varphi_i * \delta_{x_i^{-1}} \in \mathrm{L}^1(H) = \overline{\operatorname{span}}\, B_{\mathrm{l}}$. However, $\varphi_i = (\varphi_i * \delta_{x_i^{-1}}) * \delta_{x_i}$. Therefore $\varphi_i \in \overline{\operatorname{span}}\, A_{\mathrm{l}}$ for each $i = 1, \ldots, n$. Therefore $\varphi \in \overline{\operatorname{span}}\, A_{\mathrm{l}}$. A similar result can be obtained for A_{r}. □

THEOREM 8.8. *Let G be a locally compact group, and let X be a Banach $\mathrm{L}^1(G)$-bimodule. Then, in either of the following cases, every bounded approximately local derivation from $\mathrm{L}^1(G)$ into X is a derivation.*

(i) *G is a SIN-group.*
(ii) *G is a totally disconnected group.*

Proof. From Theorem 8.1 and Theorem 4.4, it suffices to show that $\mathrm{L}^1(G)$ is approximately locally unital whenever G is a SIN or a totally disconnected group. To this end, we first consider the following two cases.

Case I: G is abelian. In this case $\mathrm{L}^1(G)$ is isometrically isomorphic to the Fourier algebra $A(\widehat{G})$, where $\widehat{G}$ is the dual group of G. Thus, by Example 4.2, $\mathrm{L}^1(G)$ is approximately locally unital.

Case II: G is compact. Then $\mathrm{L}^1(G)$ is a semisimple annihilator Banach algebra by Example 7.1(i).Therefore it is approximately locally unital by Lemma 7.2.

Now let G be a SIN-group. Then, by [**11**, Theorem 2.13], G has an open normal subgroup $V \times K$, where $V \cong \mathbb{R}^n$ for some integer n and K is compact. Since $\mathrm{L}^1(V \times K) \cong \mathrm{L}^1(V) \mathbin{\widehat{\otimes}} \mathrm{L}^1(K)$, from the above two cases and the proof of Theorem 4.6, $\mathrm{L}^1(V \times K)$ is approximately locally unital. Therefore $\mathrm{L}^1(G)$ is approximately locally unital by Lemma 8.7. A similar argument applies when G is a totally disconnected group, because in this case the identity in G has a basis consisting of open compact subgroups. □

THEOREM 8.9. *Let G be a locally compact group. Then every bounded approximately local derivation from $M(G)$ into any Banach $M(G)$-bimodule is a derivation if and only if G is discrete. Moreover, if G is non-discrete, then there is an essential Banach $M(G)$-bimodule X and a bounded local derivation from $M(G)$ into X which is not a derivation.*

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Proof. If G is discrete, then $M(G) = \mathrm{l}^1(G)$, so the result follows from Theorem 8.8. Now assume that G is non-discrete. By [**8**, Theorem 3.2], there is a character φ on $M(G)$ and a non-zero bounded point derivation d on $M(G)$ at φ. Thus $d \in M(G)^*$ and

$$d(ab) = \varphi(a)d(b) + d(a)\varphi(b), \qquad a, b \in A.$$

In particular, if we put $I = \ker d$ and $J = \ker \varphi$, then J is a maximal ideal and I is a (proper) linear subspace of $M(G)$ such that $J^2 \subseteq I$. Moreover, by Proposition 3.5, the bounded operator $D\colon M(G) \longrightarrow M(G)^*$ defined by $D(a) = d(a)\varphi$ is both a derivation and a local left multiplier. Let ψ be the augmentation character on $M(G)$ defined by

$$\psi(\mu) = \mu(G), \qquad \mu \in M(G).$$

Put $M = \ker \psi$. Then M is a maximal ideal in $M(G)$. If T is the restriction of D to M, then T is a bounded local left multiplier from M. We claim that T is not a left multiplier. Otherwise, a simple calculation shows that $aD(b) = 0$ and so $d(b)\varphi(a) = 0$ for all $a, b \in M$. Therefore $M \subset I$ or $M \subset J$. On the other hand, by [**7**, Theorem 3.3.30], $M = M^2$, and so there is no non-zero point derivation on $M(G)$ at ψ and so $\psi \neq \varphi$. Hence $M \nsubseteq J$. Also, $M \nsubseteq I$, for otherwise, since $M(G) = J + M$, we would have

$$M(G) = M(G)^2 \subseteq J^2 + M \subseteq I + I = I,$$

which is impossible. Hence T is not a left multiplier. Now, if we let $X = M(G)^*$ become a Banach M-bimodule by defining the left action to be 0 and the right action to be the one induced from $M(G)^*$, when it is viewed as the dual module of $M(G)$, then $T\colon M \longrightarrow X$ is a bounded local derivation which is not a derivation. Finally, because $M(G) = M \oplus \mathbb{C}\delta_e$, if we define $\delta_e x = x\delta_e = x$ $(x \in X)$ and $T(\delta_e) = 0$, X becomes an essential Banach $M(G)$-bimodule and T extends to a bounded local derivation from $M(G)$ into X, which is not a derivation. □

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References

1. B. A. Barnes, 'Examples of modular annihilator algebras', *Rocky Mountain J. Math.* 1 (1971) 657–665.
2. F. F. Bonsall and J. Duncan, *Complete normed algebras* (Springer, New York, 1973).
3. M. Brešar and P. P. Šemrl, 'Mappings which preserve idempotents, local automorphisms, and local derivations', *Canad. J. Math.* 45 (1993) 483–496.
4. J. Cuntz, 'On the continuity of seminorms on operator algebras', *Math. Ann.* 220 (1976) 171–183.
5. R. L. Crist, 'Local derivations on operator algebras', *J. Funct. Anal.* 135 (1996) 76–92.
6. P. C. Curtis and R. J. Loy, 'The structure of amenable Banach algebras', *J. London Math. Soc.* (2) 40 (1989) 89–104.
7. H. G. Dales, *Banach algebras and automatic continuity* (Oxford University Press, New York, 2000).
8. H. G. Dales, F. Ghahramani and A. Ya. Helemskii, 'The amenability of measure algebras', *J. London Math. Soc.* (2) 66 (2002) 213–226.
9. B. Forrest, 'Fourier analysis on coset spaces', *Rocky Mountain J. Math.* 28 (1998) 173–190.
10. F. Ghahramani and R. J. Loy, 'Generalized notions of amenability', *J. Funct. Anal.* 208 (2004) 229–260.
11. S. Grosser and M. Moskowitz, 'Compactness condition in topological groups', *J. Reine Angew. Math.* 246 (1971) 1–40.

https://doi.org/10.1112/S0024610705006496 Published online by Cambridge University Press

12. B. E. Johnson, 'Local derivations on C*-algebras are derivations', *Trans. Amer. Math. Soc.* 353 (2000) 313–325.
13. B. E. Johnson and A. M. Sinclair, 'Continuity of derivations and a problem of Kaplansky', *Amer. J. Math.* 90 (1968) 1067–1073.
14. B. E. Johnson and M. C. White, 'A non-weakly amenable augmentation ideal', Preprint.
15. R. V. Kadison, 'Local derivations', *J. Algebra* 130 (1990) 494–509.
16. D. R. Larson, 'Reflexivity, algebraic reflexivity and linear interpolation', *Amer. J. Math.* 110 (1988) 283–299.
17. D. R. Larson and A. Sourour, 'Local derivations and local automorphisms of $B(X)$', Proceedings of Symposia in Pure Mathematics 51 (American Mathematical Society, Providence, RI, 1990) 187–194.
18. J. R. Ringrose, *Compact non-self-adjoint operators* (Van Nostrand, London, 1971).
19. E. Samei, 'Bounded and completely bounded local derivations from certain commutative semisimple Banach algebras', *Proc. Amer. Math. Soc.* 133 (2005) 229–238.
20. G. A. Willis, 'Factorization in finite-codimensional ideals of group algebras', *Proc. London Math. Soc.* (3) 82 (2001) 676–700.

Ebrahim Samei
Department of Mathematics
University of Manitoba
Winnipeg
Manitoba
Canada R3T 2N2

umsameie@cc.umanitoba.ca

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