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ARTICLE

Clique density vs blowups

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Abstract

A well-known theorem of Nikiforov asserts that any graph with a positive K_r-density contains a logarithmic blowup of K_r. In this paper, we explore variants of Nikiforov's result in the following form. Given $r, t \in \mathbb{N}$, when a positive K_r-density implies the existence of a significantly larger (with almost linear size) blowup of K_t? Our results include:

- For an n-vertex ordered graph G with no induced monotone path P_6, if its complement $\overline{G}$ has positive triangle density, then $\overline{G}$ contains a biclique of size $\Omega(\frac{n}{\log n})$. This strengthens a recent result of Pach and Tomon. For general k, let $g(k)$ be the minimum $r \in \mathbb{N}$ such that for any n-vertex ordered graph G with no induced monotone P_{2k}, if $\overline{G}$ has positive K_r-density, then $\overline{G}$ contains a biclique of size $\Omega(\frac{n}{\log n})$. Using concentration of measure and the isodiametric inequality on high dimensional spheres, we provide constructions showing that, surprisingly, $g(k)$ grows quadratically. On the other hand, we relate the problem of upper bounding $g(k)$ to a certain Ramsey problem and determine $g(k)$ up to a factor of 2.
- Any incomparability graph with positive K_r-density contains a blowup of K_r of size $\Omega(\frac{n}{\log n})$. This confirms a conjecture of Tomon in a stronger form. In doing so, we obtain a strong regularity type lemma for incomparability graphs with no large blowups of a clique, which is of independent interest. We also prove that any r-comparability graph with positive $K_{(2h-2)^r+1}$-density contains a blowup of K_h of size $\Omega(n)$, where the constant $(2h-2)^r+1$ is optimal.

The $\frac{n}{\log n}$ size of the blowups in all our results are optimal up to a constant factor.

Keywords: Clique density; large blow-ups; ordered graph; poset graphs

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1. Introduction

1.1. Overview

The extremal number of a graph H, denoted by $\mathrm{ex}(n, H)$, is the maximum number of edges in an n-vertex graph with no copy of H. Its historical roots trace back to Mantel's result from 1907 [25], showing that $\mathrm{ex}(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$. This was generalised by Turán [35] who determined $\mathrm{ex}(n, K_r)$ for all r. Later, the Erdős–Stone–Simonovits theorem [9, 10] asserts that $\mathrm{ex}(n, H) = (1 - \frac{1}{\chi(H)-1} + o(1))\binom{n}{2}$, which asymptotically determines $\mathrm{ex}(n, H)$ for any graph H that is not bipartite. The Erdős–Stone–Simonovits theorem is widely considered as 'the fundamental theorem of extremal graph theory'. This was strengthened by Bollobás, Erdős, and Simonovits [5]

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as follows. Denote by $\varrho_r(G)$ the *K_r-density* of G, which is the number of copies of K_r in G normalised by $\binom{|G|}{r}$. For a graph H, we denote by $H[t]$ the *t-blowup* of H obtained by replacing every vertex of H by an independent set of size t and every edge of H by a copy of $K_{t,t}$. It was shown [5] that every n-vertex graph G with edge density $\varrho_2(G)1 - \frac{1}{r-1} + \varepsilon$ contains a complete r-partite subgraph $K_r[\varepsilon^r \log n]$. By considering random graphs of the same edge density, we see that the logarithmic dependence on n is optimal.

Given that random graphs are highly unstructured, it seems natural to ask whether larger blowups of cliques can be found in graphs from special families. We focus on finding largest possible blowups – those of sizes almost linear in the order of G. Recently, there have been various results of this type under edge density conditions. For instance, Fox, Pach, and Tóth [17] showed that any n-vertex incomparability graph G with $\varrho_2(G) > 0$ contains a $K_2[t]$ with $t = \Omega(\frac{n}{\log n})$. Tomon [34] proved that the complement of an n-vertex string graph G with $\varrho_2(G) > \frac{3}{4}$ contains a $K_2[t]$ with $t = \Omega(n)$. Apart from these recent developments, the Ramsey problem of finding a biclique of linear size in a graph or its complement also receives a great deal of attention [6, 17, 19], as it is closely related to the Erdős–Hajnal conjecture [8].

In a different direction, a beautiful strengthening of the Bollobás–Erdős–Simonovits theorem by Nikiforov [27] replaces the edge density condition by a weaker clique density condition. More precisely, by the so-called supersaturation result of Erdős and Simonovits [11], any graph with $\varrho_2(G) > 1 - \frac{1}{r-1}$ not only contains a single r-clique, but has positive K_r-density $\varrho_r(G) > 0$. Nikiforov [27] showed that a graph having $\varrho_r(G) > 0$ already suffices to imply a logarithmic blowup of K_r.

In this work we combine the above two directions of extensions to find large blowups of cliques in some well-known classes of graphs with clique density conditions instead of edge density. More precisely, we study the following general question.

Question A. *Given a graph class $\mathcal{G}$ and $t \in \mathbb{N}$, find the minimum $r \in \mathbb{N}$ such that if any $G \in \mathcal{G}$ has positive K_r-density $\varrho_r(G) > 0$, then G contains a near-linear-sized blowup of K_t?*

1.2. Our contributions

1.2.1. Ordered graphs forbidding monotone induced path

An *ordered graph* $G_<$ is a graph with a total ordering $<$ on its vertex set $V(G)$. We say $H_<$ is an *induced ordered subgraph* of $G_<$ if there exists an order and adjacency preserving map from $V(H)$ to $V(G)$, i.e. there exists $f : V(H) \to V(G)$ such that for each pair $u, v \in V(H)$, if $u < v$ then $f(u) < f(v)$, and $uv \in E(H)$ if and only if $f(u)f(v) \in E(G)$. For extremal problems concerning ordered graphs, we refer the reader to some of the recent work [2, 7, 23, 26, 28] and the references therein.

Ordered graphs have close connections to geometrically defined graphs. Often, much better quantitative Ramsey results hold for graphs arising from geometric settings. For instance, Fox, Pach, and Tóth [17] proved that for any n-vertex intersection graph G of x-monotone curves in the plane, either G contains a $K_2[\frac{cn}{\log n}]$ or its complement $\overline{G}$ contains a $K_2[cn]$ for some $c > 0$. The original proof in [17] heavily relies on the geometric nature of the x-monotone curves. Recently, Pach and Tomon [29] provided a simpler combinatorial proof using ordered graphs. A *monotone path* P_k is an ordered graph with k vertices $v_1 < v_2 < \cdots < v_k$ in which v_i and v_j are adjacent if and only if $|i - j| = 1$. A key ingredient in their proof is the following result.

Theorem 1.1 (Pach–Tomon [29]). *For every $k \geqslant 3$, there exists $0 \leqslant \beta(k) < 1$ such that the following holds. Let $\varepsilon > 0$ and $G = G_<$ be an n-vertex ordered graph with no induced monotone path P_k and its complement satisfies $\delta(\overline{G}) \geqslant (\beta(k) + \varepsilon)n$. Then $\overline{G}$ contains a copy of $K_2[\frac{cn}{\log n}]$, where $c = c(\varepsilon) > 0$.*

Here $\delta(G)$ stands for the minimum degree of G. This theorem generalises the result of Fox, Pach, and Tóth [17] who proved it for the case $k = 3$ (in this case G is a comparability graph)

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with the optimal constant $\beta(3) = 0$ (see Theorem 1.5). It remains an open problem to determine the optimal value of $\beta(k)$ for larger k. When $k = 5$, Pach and Tomon [29] provided the following example showing that $\beta(5) \geqslant 1/2$. Take the union of two disjoint cliques on $[n/2]$ and $[n]\setminus[n/2]$ respectively and add cross edges randomly and independently with probability $\varepsilon > 0$. The resulting graph G has no induced monotone P_5 and with high probability $\delta(\overline{G}) \geqslant (1/2 - \varepsilon)n$, but the maximum size of a biclique in $\overline{G}$ is $O_\varepsilon(\log n)$.

We confirm that the above construction is optimal, i.e., $\beta(5) = 1/2$. In fact, we prove it in a stronger form, showing that rather than the edge density condition as in Theorem 1.1, large blowups originates from having positive triangle density.

Theorem 1.2. *Let $\varepsilon > 0$ and $G = G_<$ be an n-vertex ordered graph with no induced monotone path P_5 and $\varrho_3(\overline{G}) > \varepsilon$. Then $\overline{G}$ contains a copy of $K_2[\frac{cn}{\log n}]$, where $c = c(\varepsilon) > 0$.*

It is worth pointing out that this theme of '*replacing edge density by clique density*' not only reveals the real cause of the phenomenon occurred (here being the appearance of large biclique), but also implies the stability result for the edge density version. Indeed, if an n-vertex ordered graph with no induced monotone P_5 has edge density close to $1/2$ and no biclique of size $\Omega(\frac{n}{\log n})$, then by Theorem 1.2 it must have zero triangle density. Thus, it follows from the classical Erdős–Simonovits stability result [9] that the graph G must be close to balanced complete bipartite graph.

By considering the union of s disjoint $\frac{n}{s}$-cliques and adding random edges between distinct cliques, we see that $\beta(2s+1) \geqslant 1 - \frac{1}{s}$. After seeing Theorem 1.2, it is tempting to believe that for an induced monotone P_{2s+1}-free G, positive K_{s+1}-density in $\overline{G}$ forces a large biclique in $\overline{G}$. Much to our own surprise, this is already false for the next case when forbidding P_7! It turns out that we need a positive K_5-density.

Theorem 1.3. *For any $\varepsilon > 0$ and an n-vertex ordered graph $G = G_<$ with no induced monotone P_7, if $\varrho_5(\overline{G}) > \varepsilon$, then $\overline{G}$ contains a copy of $K_2[\frac{cn}{\log n}]$, where $c = c(\varepsilon) > 0$.*

On the other hand, there are n-vertex ordered graphs G with no induced monotone path P_7, $\varrho_4(\overline{G}) > 0$ and no biclique of size $n/e^{o(\sqrt{\log n/\log\log n})}$ in $\overline{G}$.

Theorems 1.2 and 1.3 are special cases of our main result. To state it, let $g(k)$ be the minimum $r \in \mathbb{N}$ such that if any n-vertex ordered graph G with no induced monotone P_{2k} satisfies $\varrho_r(\overline{G}) > 0$, then $\overline{G}$ contains a copy of $K_2[\Omega(\frac{n}{\log n})]$. We in fact prove $g(3) = 3$ and $g(4) = 5$, which imply Theorems 1.2 and 1.3 (with the weaker P_6-free or P_8-free condition). For general k, although the construction showing $\beta(2s+1) \geqslant 1 - \frac{1}{s}$ is not optimal, it is not inconceivable that $g(k)$ is linear in k. However, we show that the growth rate of $g(k)$ is rather quadratic, determining $g(k)$ up to a factor of 2.

Our main result reads as follows.

Theorem 1.4. *For $k \geqslant 3$, we have $\lfloor \frac{k^2}{4} \rfloor < g(k) \leqslant \frac{k^2-k+2}{2}$. That is, the followings hold.*

- *For any $\varepsilon > 0$ and an n-vertex ordered graph $G = G_<$ with no induced monotone P_{2k}, if $\varrho_r(\overline{G}) > \varepsilon$, where $r = \frac{k^2-k+2}{2}$, then $\overline{G}$ contains a copy of $K_2[\frac{cn}{\log n}]$, where $c = c(\varepsilon) > 0$.*
- *On the other hand, there are n-vertex ordered graphs G with no induced monotone path P_{2k-1}, $\varrho_{\lfloor \frac{k^2}{4} \rfloor}(\bar{G}) > 0$ and no biclique of size $n/e^{o(\sqrt{\log n/\log\log n})}$ in $\overline{G}$.*

We remark that the $n/\log n$ blowup size in Theorems 1.2–1.4 are optimal up to a constant factor by a construction in [13].

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To prove Theorem 1.4, we reduce the problem of upper bounding $g(k)$ to a Ramsey problem (see Definition 3.1). The lower bound construction is geometric and utilises concentration of measure and isodiametric inequality on high dimensional spheres, motivated by the Bollobás–Erdős graph in Ramsey–Turán theory [3, 4, 14, 24]. It is interesting to see such a connection between these two seemingly unrelated problems.

1.2.2. Graphs on posets

Given a partially ordered set $(P, \prec)$, its *(in)comparability graph* is a graph with vertex set P in which two vertices form an edge if and only if the corresponding elements are (in)comparable in P. Fox, Pach and Tóth [17] proved the following result on existence of large complete bipartite graph in incomparability graphs with positive edge density.

Theorem 1.5 (Fox–Pach–Tóth [17]). *Let $\varepsilon > 0$ and G be an n-vertex incomparability graph with $\varrho_2(G) > \varepsilon$. Then G contains a $K_2[\frac{cn}{\log n}]$, where $c = c(\varepsilon) > 0$.*

Later this result was generalised for $k > 2$ by Tomon [33], who showed that if an incomparability graph G satisfies $\varrho_2(G) \geqslant 1 - \frac{1}{9(k-1)} + \varepsilon$, then G contains a copy of $K_k[\frac{cn}{(\log n)^s}]$, where $c = c(k, \varepsilon) > 0$ and $s = \lceil \log_2 k \rceil$. Tomon proposed the following conjecture.

Conjecture B (Tomon [33]). *Let $\varepsilon > 0$ and G be an n-vertex incomparability graph with $\varrho_2(G) > 1 - \frac{1}{k-1} + \varepsilon$. Then G contains a $K_k[\frac{cn}{(\log n)^s}]$, where $c = c(\varepsilon, k) > 0$ and $s = \lceil \log_2 k \rceil$.*

Our next result resolves Conjecture B in a stronger form using the weaker clique density condition. The size of the blowup $n/\log n$ below is best possible up to a constant factor by [13].

Theorem 1.6. *Let $c \in (0, \frac{1}{2})$, $k \geqslant 2$ and n be sufficiently large. If G is an n-vertex incomparability graph with $\varrho_k(G) \geqslant c$, then G contains a $K_k[\frac{c^4 n}{10^8 k^2 \log n}]$.*

To prove Theorem 1.6, we obtain a strong regularity type lemma for incomparability graphs not containing large blowups of K_k. We prove that such graphs admit regular partitions where almost all pairs of clusters are *homogeneous*, that is, they form either complete or empty bipartite graphs. Such regularity type result is of independent interest. A similar type of partition lemma was recently obtained for intersection graphs of pseudosegments by Fox, Pach, and Suk [16].

Theorem 1.7. *Let $k \geqslant 2$ and $0 < \varepsilon < 1$. Suppose n is sufficiently large and set $q = \frac{\varepsilon^4 n}{10^5 k^2 \log n}$ and $t = \frac{\varepsilon^7 n}{10^{11} k^5}$. Let G be an incomparability graph on n vertices with no copy of $K_k[q]$. Then, there exists a partition $V(G) = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_m$ such that $|V_0| \leqslant \varepsilon n$, $|V_1| = |V_2| = \cdots = |V_m| = t$ and all but at most εm^2 pairs (V_i, V_j), $1 \leqslant i < j \leqslant m$, are homogeneous.*

Next, we consider unions of comparability graphs. Given partial orders $\prec_1, \ldots, \prec_r$ on the same set P, the corresponding *r-comparability graph* $G = G(\prec_1, \ldots, \prec_r)$ is the graph whose vertex set is P and two elements $a, b \in P$ are adjacent if and only if $a \prec_i b$ or $b \prec_i a$ holds for some $1 \leqslant i \leqslant r$. In other words, G is the union of r comparability graphs $G = \bigcup_{i \in [r]} G(\prec_i)$.

Theorem 1.8. *Let $r, h \in \mathbb{N}$ with $h \geqslant 2$, $k = (2h-2)^r + 1$ and $\varepsilon > 0$. Let $G = \bigcup_{i \in [r]} G(\prec_i)$ be an n-vertex r-comparability graph. If $\varrho_k(G) \geqslant \varepsilon$, then some $G(\prec_i)$, $i \in [r]$, contains a copy of $K_h[\frac{\varepsilon n}{r}]$.*

This strengthens a result of Tomon [33] which requires the stronger edge density condition $\varrho_2(G) \geqslant 1 - \frac{1}{(2h-2)^r} + \varepsilon$. The constant $k = (2h-2)^r + 1$ in Theorem 1.8 is best possible by [33, Theorem 4(i)].

Both Theorems 1.6 and 1.8 also fall in the theme of 'replacing edge density by clique density'.

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1.2.3. Hereditary graphs and B_k property

Next we consider Question A for hereditary families. A family of graphs $\mathcal{G}$ is *hereditary*, if for any graph $G \in \mathcal{G}$ and any induced subgraph G' of G, $G' \in \mathcal{G}$. We introduce the following notion of the B_k *property*. A notion similar to the B_2 property has recently been introduced by Fox, Pach, and Suk [16].

Definition. *Let $k \geqslant 2$. A graph class $\mathcal{G}$ has the B_k property with function $f : \mathbb{R}^+ \to \mathbb{R}^+$ if the following holds. For any $\varepsilon > 0$ and n-vertex $G \in \mathcal{G}$, if $\varrho_k(G) > \varepsilon$, then G contains a $K_k[f(\varepsilon)n]$.*

Our next result states that the B_2 property implies the B_k property for all $k \geqslant 2$.

Theorem 1.9. *If a hereditary family of graphs has the B_2 property, then it also has the B_k property for all $k \geqslant 2$ (possibly with a different function f).*

Several classes of graphs are known to have the B_2 property. For instance, Fox, Pach, and Tóth [17] proved it for intersection graphs of planar convex sets as well as for incomparability graphs of posets with bounded dimension. Although the proof in [17] is quite combinatorial, it is not clear how to adapt it to show e.g., that if the intersection graph of planar convex sets has positive triangle density, then it has a linear-size complete tripartite graph, that is, such graphs have the B_3 property. Our Theorem 1.9 circumvents ad-hoc family-specific arguments and show that all families with the B_2 property also have the B_k property.

Here we present another hereditary class with the B_2 property. The *Vapnik-Chervonenkis dimension*, or *VC-dimension* for short, of a graph and, more generally, a set system, is a fundamental parameter that measures its combinatorial complexity. We defer its formal definition to Section 6.

Theorem 1.10. *The family of graphs with VC-dimension d has the B_2 property if and only if $d = 1$.*

Structure of the paper. In Section 2, we provide some useful tools and auxiliary results. The proof of Theorem 1.4 is given in Sections 3 and 4. We prove Theorems 1.6 to 1.8 in Section 5, and Theorems 1.9 and 1.10 in Section 6. Concluding remarks are given in Section 7.

2. Tools and some auxiliary results

All undirected graphs in this paper are simple. For the sake of clarity of presentation, we omit floor and ceiling signs whenever they are not essential.

Let G be a graph. We will denote the set of vertices of G by $V(G)$ and the set of edges by $E(G)$, and define $|G| := |V(G)|$ and $e(G) := |E(G)|$. For any vertex $v \in G$, we define $N(v)$ as the neighbour of G. We say a graph G is (α, β)-*dense* if for every $S \subseteq V(G)$ with $|S| \geqslant \alpha|V(G)|$, $e(G[S]) \geqslant \beta\binom{|S|}{2}$ holds.

For a graph G, let X, Y be disjoint subsets of $V(G)$, and $e(X, Y)$ be the number of edges between X and Y. We define the density of the pair (X, Y) as $d(X, Y) := \frac{e(X,Y)}{|X|\cdot|Y|}$. A pair of vertex sets X and Y is said to be ε-*regular*, if for all subsets $A \subseteq X$, $B \subseteq Y$ satisfying $|A| \geqslant \varepsilon|X|$, $|B| \geqslant \varepsilon|Y|$, we have

$$\left|\frac{e(A, B)}{|A| \cdot |B|} - \frac{e(X, Y)}{|X| \cdot |Y|}\right| = \left|d(A, B) - d(X, Y)\right| \leqslant \varepsilon.$$

A partition of V into $k + 1$ sets $(V_0, V_1, \ldots, V_k)$ is called an ε-*regular partition*, if

- $|V_0| < \varepsilon n$ and for all $1 \leqslant i < j \leqslant k$ we have $|V_i| = |V_j|$, and
- all except εk^2 of the pairs V_i, V_j, $1 \leqslant i < j \leqslant k$, are ε-regular.

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The famous Szemerédi's regularity lemma [31] can be stated as follows.

Lemma 2.1 (Szemerédi regularity lemma, [31]). *For all $m, \varepsilon > 0$, there exists an integer M such that the following holds. If a graph G has $n \geqslant M$ vertices, there exists an integer K satisfying $m \leqslant K \leqslant M$, and an ε-regular partition $\mathcal{P}$ of $V(G)$ with $K+1$ parts.*

We will also use the following simple lemma on the regular pairs.

Lemma 2.2 (Slicing lemma, [22]). *Let (A, B) be an ε-regular pair with density d, and, for $\alpha > \varepsilon$, let $A' \subseteq A$ and $B' \subseteq B$ be subsets with $|A'| \geqslant \alpha|A|$, $|B'| \geqslant \alpha|B|$. Then (A', B') is an ε'-regular pair with $\varepsilon' = \max\{\varepsilon/\alpha, 2\varepsilon\}$. Moreover, let d' be the edge density of (A', B'), we have $|d' - d| < \varepsilon$.*

We will use multiple times the following simple consequence of the regularity lemma.

Lemma 2.3. *For every integer $k \geqslant 2$, and any reals $\eta, \varepsilon > 0$, there is a real $\gamma > 0$ such that the following holds. Let $G = G_<$ be an ordered graph on n vertices with at least ηn^k copies of K_k. Then, there are sets $A_1, \ldots, A_k \subseteq V(G)$ with $A_1 < A_2 < \ldots < A_k$ such that $|A_i| \geqslant \gamma n$ for all $i \in [k]$ and the pair (A_i, A_j) is ε-regular in G with density at least $\frac{\eta}{2}$ for all $1 \leqslant i < j \leqslant k$.*

Proof. Clearly we may assume that $\frac{1}{n} \ll \varepsilon \ll \eta \ll \frac{1}{k}$. Let $\varepsilon' = \frac{\varepsilon}{k}$ and $m = \frac{1}{\varepsilon'}$. We apply Lemma 2.1 to G with parameters ε', m to obtain an ε'-regular partition $V_0, \ldots, V_K$. Note that there are at most $|V_0|n^{k-1} < \varepsilon n^k$ k-cliques in G touching the set V_0. Additionally, there are at most $\varepsilon K^2 |V_1|^2 \cdot n^{k-2} \leqslant \varepsilon n^k$ k-cliques with one of the edges across an irregular pair. Finally, there are at most $K^2|V_1|^2 \cdot \frac{2\eta}{3} \cdot n^{k-2} \leqslant \frac{2\eta}{3} \cdot n^k$ k-cliques with one of the edges across pair with density at most $\frac{2\eta}{3}$. Hence, there exist k sets, without loss of generality the sets $V_1, \ldots, V_k$, such that each of the pairs (V_i, V_j) is ε-regular with density at least $\frac{2\eta}{3}$.

Denote $n_0 = |V_1| = \cdots = |V_k|$. We shall find the desired subsets $A_1, \ldots, A_k$ with $A_i \subseteq V_i$ for $i = 1, 2 \ldots, k$. For a vertex $x \in V(G)$, let $V_{\leqslant x} = \{y \in V(G), \ | \, y \leqslant x\}$.

For $i \in [k]$, set $U_i^1 = V_i$. Then, in steps $t = 1, \ldots, k$, we proceed as follows. Let x be the minimum vertex in $V(G)$ such that for some $i, t \leqslant i \leqslant k$, we have $|U_i^t \cap V_{\leqslant x}| = \frac{n_0}{k}$. Without loss of generality, we may assume that $i = t$. Then, set $A_i = U_i^t \cap V_{\leqslant x}$ and for $j, i+1 \leqslant j \leqslant k$, define $U_j^{t+1} = U_j^t \setminus V_{\leqslant x}$. Observe that by definition of x, we have $|U_j^{t+1}| \geqslant |U_j^t| - \frac{n_0}{k}$. It is easy to see that the described procedure produces sets $A_1, \ldots, A_k$ with $A_1 < A_2 < \ldots < A_k$, $|A_i| = \frac{n_0}{k}$ for all $i \in [k]$ and each of A_i is a subset of a distinct set among $V_1, \ldots, V_k$. Finally, by Lemma 2.2, it follows that each of the pairs (A_i, A_j) is ε-regular with density at least $\frac{2\eta}{3} - \varepsilon' > \frac{\eta}{2}$, as needed. □

Let $0 < \gamma < 1$ be a real number, we say a pair of vertex sets (V_1, V_2) of G is *γ-homogeneous* if the density $d(V_1, V_2)$ is either less than γ or larger than $1 - \gamma$, where in the former case, we call (V_1, V_2) *γ-sparse*, and in the latter case, we call (V_1, V_2) *γ-dense*. We call the pair (V_1, V_2) *homogeneous* if $G[V_1, V_2]$ is empty or complete. A partition of $V(G)$ is called *equitable* if every two parts differ in size by at most one.

We will use the following recent result of Fox and Pham [18].

Lemma 2.4 (Fox–Pham [18]). *Let $\ell \geqslant 2$ and $n \geqslant (100\ell)^5$. Every poset on n elements contains ℓ disjoint sets $A_1, \ldots, A_\ell$ such that*

- *either $A_1 \succ A_2 \succ \cdots \succ A_\ell$ and $|A_i| = \frac{n}{10^4 \ell^5}$ for $1 \leqslant i \leqslant \ell$, or*
- *A_i are pairwise incomparable and $|A_i| = \frac{n}{40\ell^2 \log n}$ for $1 \leqslant i \leqslant \ell$.*

For a given h-dimensional unit sphere $\mathbb{S}^h$, we write λ for the Lebesgue measure, which is normalised such that the unit sphere has Lebesgue measure 1. For two subsets A, B of a unit sphere, we define the Euclidean distance between them to be $d_{\max}(A, B) := \sup\{|a - b| : a \in A, b \in B\}$. We also define the diameter of A as $\operatorname{diam}(A) := d_{\max}(A, A)$.

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We will take advantage of the following lemma in [24].

Lemma 2.5 [24]. *Let $\mu \in (0,1)$ and $A, B \subseteq \mathbb{S}^{k-1}$ with $\lambda(A), \lambda(B) > e^{-\frac{k\mu}{2}}$, then $d_{\max}(A,B) \geqslant 2-\mu$.*

A *spherical cap* is the smaller intersection of the unit sphere with a half-space. Given a spherical cap C bounded by some hyperplane H, we call the point in C with maximum Euclidean distance to H the *centre* of the spherical cap. The distance from the centre to H is the *height* of the spherical cap and the diameter of C is the diameter of the intersection of C and H.

We will use the following lower and upper bounds on the measure of spherical caps.

Lemma 2.6 [24]. *For all $\delta > 0$ and integers $k \geqslant 3$, let $B \subseteq \mathbb{S}^{k-1}$ be the spherical cap consisting of all points with distance at most $\sqrt{2} - \delta/\sqrt{k}$ from a fixed point in $\mathbb{S}^{k-1}$. Then $\lambda(B) \geqslant 1/2 - \sqrt{2}\delta$.*

Lemma 2.7 [32]. *Let $\alpha \in [0,1)$ and $C \subseteq \mathbb{S}^{k-1}$ be a spherical cap with height $1-\alpha$. Then $\lambda(C) \leqslant e^{-k\alpha^2/2}$.*

Recall that a spherical cap with height $1-\alpha$ has diameter $2\sqrt{1-\alpha^2}$.

The following folklore result partitions the sphere into small pieces of equal measure (see e.g., [12]).

Lemma 2.8. *There exists $C > 0$ such that the following holds. Let $0 < \delta < 1$ and $n \geqslant (C/\delta)^k$. Then $\mathbb{S}^{k-1}$ can be partitioned into n pieces of equal measure, each of diameter at most δ.*

We also need the following geometric property, which plays a key role in the original Bollobás–Erdős graph [4].

Theorem 2.9. *For all $k \in \mathbb{N}$ and all $0 < \mu < \frac{1}{4}$, there do not exist four points $p_1, p_2, q_1, q_2 \in \mathbb{S}^k$ such that $|p_1 - p_2| \geqslant 2-\mu$, $|q_1 - q_2| \geqslant 2-\mu$ and $|p_i - q_j| \leqslant \sqrt{2} - \mu$ for all $i, j \in [2]$.*

3. Upper bound for Theorem 1.4

In this section we prove the upper bound on $g(k)$ in Theorem 1.4. To achieve this, we first introduce a framework and a Ramsey variant based on some auxiliary directed graphs.

3.1. A Ramsey problem

Definition 3.1. *Let χ be a red-blue edge-colouring of the ordered clique with vertices $v_1 < v_2 < \ldots < v_k$. Its dependency digraph $D = D(\chi)$ is defined on the vertex set $\{v_1, \ldots, v_k\}$ as follows. For every $i \in [k-1]$ and $j \in [k] \setminus \{i, i+1\}$, if $\chi(v_i v_{i+1}) = \text{red}$ and $\chi(v_i v_j) = \chi(v_{i+1} v_j) = \text{blue}$, then D contains the directed edges (v_j, v_i) and (v_j, v_{i+1}). We say that χ is admissible if $D(\chi)$ is acyclic.*

For $k \geqslant 1$, let $f(k)$ denote the minimum integer N such that in any red-blue edge-colouring of the ordered clique on N vertices, there exist k vertices such that the colouring induced on these k vertices is admissible.

Note that D may contain bidirectional edges, which can be viewed as directed cycles of length two; see Fig. 1 for an example. To clarify, the induced colouring inherits the vertex ordering as well as the colours of the edges.

The main result of this section is the following embedding result which relates the problem of upper bounding $g(k)$ to the Ramsey problem for $f(k)$.

Theorem 3.2. *For any integer $k \geqslant 1$, and real $\eta > 0$, there exists $c > 0$ such that the following holds. Let $G = G_<$ be an ordered graph with no induced monotone P_{2k}. If $\overline{G}$ contains at least $\eta n^{f(k)}$ copies*

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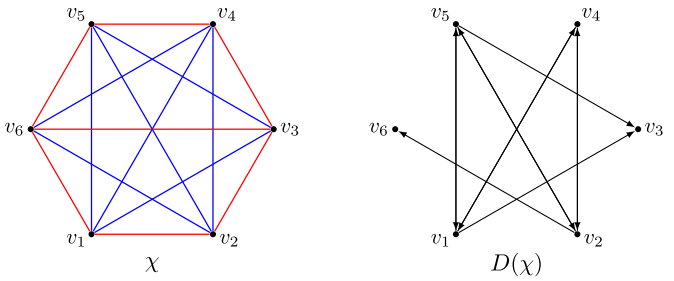


Figure 1. An example of dependency digraph.

of $K_{f(k)}$, then $\overline{G}$ contains a $K_2[\frac{cn}{\log n}]$. In other words,

$$g(k) \leqslant f(k).$$

Let us first derive that $g(3) = f(3) = 3$ and $g(4) = f(4) = 5$, which imply Theorems 1.2 and 1.3. Recall that the optimal construction for $\beta(5) = 1/2$ infers that $g(3) > 2$ and the construction in Theorem 1.4 shows that $g(4) > 4$. Thus, by Theorem 3.2, it suffices to prove $f(3) \leqslant 3$ and $f(4) \leqslant 5$.

Corollary 3.3. *We have $f(3) \leqslant 3$ and $f(4) \leqslant 5$. Consequently, for any $\varepsilon > 0$ and n-vertex ordered graph G with no induced monotone P_6 (or P_8 resp.), if $\varrho_3(\overline{G}) > \varepsilon$ (or $\varrho_5(\overline{G}) > \varepsilon$), then $\overline{G}$ contains a $K_2[\frac{cn}{\log n}]$, where $c = c(\varepsilon) > 0$.*

Proof. To prove $f(3) \leqslant 3$, we need to show that any red-blue colouring χ of $\binom{[3]}{2}$ is admissible. We may assume that $D(\chi)$ has at least one edge and thus by symmetry we may assume that $\chi(1,2) =$ red and $\chi(1,3) = \chi(2,3) =$ blue. But then, $E(D(\chi)) = \{(3,1),(3,2)\}$, which is acyclic. Thus χ is admissible as desired.

To prove $f(4) \leqslant 5$, let χ be a red-blue colouring of $\binom{[5]}{2}$, we need to show that the induced colouring on some set of 4 vertices is admissible. Suppose $D(\chi|_{1234})$ is not acyclic, for otherwise we are done. We claim that $\chi(12) = \chi(34) =$ red. To see this, suppose one of them is blue and by symmetry we may assume that $\chi(34) =$ blue. Now if $\chi(23) =$ red, then $D(\chi|_{1234})$ contains only edges from $\{1,4\}$ to $\{2,3\}$ if $\chi(12) =$ blue or only edges from $\{4\}$ to $\{1,2,3\}$ if $\chi(12) =$ red, both of which are acyclic. Thus we may assume $\chi(23) =$ blue, and then $D(\chi|_{1234})$ contains only edges from $\{3,4\}$ to $\{1,2\}$, which is acyclic.

The above analysis shows that if none of $D(\chi|_{1234}), D(\chi|_{2345}), D(\chi|_{1345})$ is acyclic, then $\chi(12) = \chi(34) = \chi(23) = \chi(13) =$ red and so $D(\chi|_{1234})$ contains only the edge from $\{4\}$ to $\{1,2\}$, which is acyclic, a contradiction. □

We now prove an upper bound for the general case, which together with Theorem 3.2, implies the upper bound for $g(k)$ in Theorem 1.4.

Proposition 3.4. *We have $f(k) \leqslant \frac{k^2-k+2}{2}$.*

Proof. Suppose for a contradiction that there exists a red-blue edge-colouring χ of the ordered clique on vertices $1, 2, \cdots, \frac{k^2-k+2}{2}$ such that the colouring induced on any k vertices is not admissible. If there exists t vertices $i_1 < \cdots < i_t$ such that $\chi(i_r i_{r+1}) =$ blue for any $1 \leqslant r \leqslant t-1$, we say these t vertices form a blue path with order t. For any vertices v_i, let $\sigma(v_i)$ be the order of the longest blue path starting from v_i. If there are no blue edge starting from v_i, set $\sigma(v_i) = 1$. If there exists a blue path with order k, then the colouring induced on these k vertices is admissible as $D(\chi)$ is the empty graph, a contradiction. Let $S_t = \{v_i | \sigma(v_i) = t\}$ for any $t \leqslant k-1$, then these S_t form a partition of the vertices.

Claim 3.5. *For $t \leqslant k-1$, $|S_t| \leqslant k-t$.*

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Proof of claim. Suppose $|S_t| \geqslant k+1-t$, by the definition of S_t, we can pick vertices $i_1 < i_2 < \cdots < i_{k+1-t}$ from S_t and a blue order-t path on vertices $i_{k+1-t} < \cdots < i_k$. Note that $\chi(i_u i_v) = \text{red}$ for any $1 \leqslant u < v \leqslant k+1-t$, as otherwise adding i_u to the blue path of order t starting from i_v yields a blue path of order $t+1$, a contradiction to $i_v \in S_t$. Let D be the dependency digraph induced on the k vertices $i_1, \ldots, i_k$. By the definition of D, for any directed edge $(u, v) \in D$, we must have $u > k+1-t$ and $v \leqslant k+1-t$. Thus, D is acyclic, a contradiction.

Hence, $\frac{k^2-k+2}{2} = \sum_{t\in[k-1]} |S_t| \leqslant \sum_{t\in[k-1]} (k-t) = \frac{k^2-k}{2}$, a contradiction. □

3.2. *Proof of Theorem 3.2*

By taking sufficiently small c, we may assume that n is sufficiently large in terms of k and η. We define additional constants $c, \alpha, \beta, \gamma, \gamma_1, \gamma_2, \varepsilon, \varepsilon_1$, such that

$$0 < 1/n \ll c \ll \alpha \ll \gamma = \gamma_1\gamma_2 < \gamma_1 \ll \varepsilon_1 \ll \gamma_2 \ll \varepsilon \ll \beta \ll \eta < 1/k.$$

Suppose, for the sake of contradiction, that $\overline{G}$ does not contain a copy of $K_2[\frac{cn}{\log n}]$.

Claim 3.6. *G is (α, β)-dense.*

Proof of claim. Suppose otherwise, so there is a set $S \subseteq V(G)$ with $|S| \geqslant \alpha n$ and $e_G(S) < \beta\binom{|S|}{2}$. Removing vertices of degree more than $4\beta|S|$ in $G[S]$, we obtain a set S' of size at least $|S|/2 \geqslant \alpha n/2$ such that $\Delta(G[S']) < 8\beta|S'|$. Then Theorem 1.1 implies the existence of $K_2[\frac{cn}{\log n}]$ in $\overline{G[S']}$, contradicting our assumption. □

By Lemma 2.3, there exist sets $A_1, \ldots, A_{f(k)} \subseteq V(G)$ such that $A_1 < A_2 < \ldots < A_{f(k)}$, for all $i \in [f(k)]$, $|A_i| \geqslant \gamma_1 n$ and for all $1 \leqslant i < j \leqslant f(k)$, (A_i, A_j) is ε_1-regular with density at least $\frac{\eta}{2}$ in $\overline{G}$.

We define an edge-colouring χ_0 of the complete graph on the ordered vertex set $\{1, 2, \ldots, f(k)\}$ as follows. For $1 \leqslant i < j \leqslant f(k)$, let

$$\chi_0(ij) = \begin{cases} \text{blue}, & \text{if } d(A_i, A_j) \geqslant \beta^2 \\ \text{red}, & \text{otherwise.} \end{cases}$$

By definition of $f(k)$, there exist indices $a_1 < \ldots < a_k$ such that $\chi_0[\{a_1, \ldots, a_k\}]$ is admissible. For $i \in [k]$, let $B_i = A_{a_i}$. Moreover, define a colouring $\chi : \binom{[k]}{2} \to \{\text{red}, \text{blue}\}$ by setting $\chi(ij) = \chi_0(a_i a_j)$ and note that χ is admissible. We now show that we can either construct an induced monotone P_{2k} in G by placing two consecutive vertices in each set B_i or there is a biclique of size $\frac{cn}{\log n}$ in $\overline{G}$, thus obtaining a contradiction.

Recall that $|B_i| \geqslant \gamma_1 n \geqslant \alpha n$, so since G is (α, β)-dense, we have $d(G[B_i]) \geqslant \beta$. Applying Lemma 2.3 to $G[B_i]$ for each $i \in [k]$, we obtain two sets $C_{2i-1}, C_{2i} \subseteq B_i$ of size at least $\gamma_2|B_i| \geqslant \gamma_2\gamma_1 n = \gamma n$ with $C_{2i-1} < C_{2i}$ and such that (C_{2i-1}, C_{2i}) is ε-regular in G with density at least $\frac{\beta}{4}$. To summarise, we have that $C_1 < C_2 < \ldots < C_{2k}$, $|C_i| \geqslant \gamma_2|B_i| \geqslant \gamma n$, for all $i \in [2k]$. Moreover, using that $\frac{\varepsilon_1}{\gamma_2} < \varepsilon$, by Lemma 2.2, the pairs (C_i, C_j) are ε-regular. Finally, the following holds for the densities between different pairs. Let $1 \leqslant i < j \leqslant 2k$, then

a) If $\lfloor \frac{i}{2} \rfloor = \lfloor \frac{j}{2} \rfloor$, then $d(C_i, C_j) > \frac{\beta}{4}$.
b) If $\lfloor \frac{i}{2} \rfloor \neq \lfloor \frac{j}{2} \rfloor$, then $d(C_i, C_j) < 1 - \frac{\eta}{4}$.
c) If $\chi(\lfloor \frac{i}{2} \rfloor, \lfloor \frac{j}{2} \rfloor) = \text{red}$, then $d(C_i, C_j) < 2\beta^2$.
d) If $\chi(\lfloor \frac{i}{2} \rfloor, \lfloor \frac{j}{2} \rfloor) = \text{blue}$, then $d(C_i, C_j) > \frac{\beta^2}{2}$.

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Let $D = D(\chi)$ be the dependency digraph of χ as defined in Definition 3.1. By assumption, D is acyclic. Consider a fixed topological ordering of D and for $i \in [k]$, let $\pi(i)$ denote the position of vertex i in this topological ordering. In other words, we have for every edge $ij \in E(D)$, $\pi(i) < \pi(j)$.

Let $(i_1, i_1+1), (i_2, i_2+1), \ldots, (i_{k-1}, i_{k-1}+1)$ be an ordering of the edges $(1,2), \ldots, (k-1,k)$ according to the earliest appearance in π of one of their endpoints. In other words for $j < j'$, we have that $\min\{\pi(i_j), \pi(i_j+1)\} \leqslant \min\{\pi(i_{j'}), \pi(i_{j'}+1)\}$. In particular, when processing (i_t, i_t+1), any j with $\chi(\lfloor \frac{i}{2} \rfloor, \lfloor \frac{j}{2} \rfloor) = \chi(\lfloor \frac{i+1}{2} \rfloor, \lfloor \frac{j}{2} \rfloor) =$ blue must satisfy $j \in I_t$ by the definition of D and the choice of the order.

We shall construct a monotone induced path $x_1, \ldots, x_{2k}$ with $x_j \in C_j$ for all $j \in [2k]$. We first embed all but x_1 and x_{2k} in $k-1$ steps. At step $t \in [k-1]$, we shall define the vertices x_{2i_t}, x_{2i_t+1}. Finally, we embed x_1 and x_{2k}.

Let us now formally describe the embedding procedure. For each $t \in [k]$, just before executing step t we maintain the following invariants:

(I1) The set of already embedded indices is

$$I_t = \bigcup_{j=1}^{t-1} \{\, 2i_j,\ 2i_j+1 \,\} \subseteq [2k].$$

(I2) For every $i \in [2k] \backslash I_t$ there is a candidate set $C_i^t \subseteq C_i$ with the uniform lower bound

$$|C_i^t| \geqslant \beta^{8(t-1)} |C_i|.$$

(I3) For any distinct $i, j \in [2k] \backslash I_t$, the pair (C_i^t, C_j^t) is regular (with parameter at most ε; and by Lemma 2.2 and the parameter hierarchy, whenever invoked below we may in fact use $\varepsilon^{0.9}$). We also repeatedly use the density bounds a)–d) for the base pairs (C_i, C_j).

Formally, we initialise with $C_i^1 = C_i$ for all $i \in [2k]$. After $t-1$ steps, for each $i \in [2k]$ we define

$$C_i^t = \Big(C_i \cap \bigcap_{j \in I_t \cap \{i-1,\, i+1\}} N(x_j) \Big) \backslash \bigcup_{j \in I_t \backslash \{i-1,\, i+1\}} N(x_j).$$

Here we require adjacency only to those of x_{i-1}, x_{i+1} that have already been embedded (i.e., $j \in I_t \cap \{i-1, i+1\}$), and we delete vertices adjacent to any other already embedded x_j (i.e., $j \in I_t \backslash \{i-1, i+1\}$).

Given that we have run the procedure for $t-1 \leqslant k-2$ steps we show we can run step t. Let $i = 2i_t$ and recall that we aim to embed vertices x_i and x_{i+1}. By assumption, we have sets C_i^t and C_{i+1}^t of size at least $\beta^{8(t-1)}|C_i|$ into which we can embed the vertices x_i and x_{i+1}, respectively. Note that

$$\frac{|C_i^t|}{|B_i|} = \frac{|C_i^t|}{|C_i|} \cdot \frac{|C_i|}{|B_i|} \beta^{8(t-1)} \gamma_2 > \varepsilon_1^{0.1},$$

which implies that for all $i, j \in [2k] \backslash I_t$, $i \neq j$, the pair (C_i^t, C_j^t) is $\varepsilon^{0.9}$-regular.

By the $\varepsilon^{0.9}$-regularity of (C_i^t, C_j^t), all but at most $\varepsilon^{0.9}|C_i^t|$ vertices $v \in C_i^t$ satisfy

$$\big| d(v, C_j^t) - d(C_i^t, C_j^t) \big| < \varepsilon^{0.9}.$$

By Lemma 2.2 applied to (C_i, C_j) with subsets (C_i^t, C_j^t) we also have

$$\big| d(C_i^t, C_j^t) - d(C_i, C_j) \big| < \varepsilon^{0.9}.$$

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Hence, for all but at most $\varepsilon^{0.9}|C_i^t|$ vertices $v \in C_i^t$,

$$\left|d\left(v, C_j^t\right) - d\left(C_i, C_j\right)\right| < 2\varepsilon^{0.9} \leqslant \sqrt{\varepsilon},$$

where the last inequality follows from the parameter hierarchy. Discarding the exceptional vertices for each of the fewer than $2k$ choices of j and intersecting over j, we obtain a set

$$C_i' \subseteq C_i^t, \qquad |C_i'| \geqslant (1 - 2k\varepsilon^{0.9})|C_i^t| \geqslant \tfrac{1}{2}|C_i^t|,$$

such that for every $v \in C_i'$ and $j \in [2k]\backslash(I_t \cup \{i, i+1\})$,

$$\left|d\left(v, C_j^t\right) - d(C_i, C_j)\right| < \sqrt{\varepsilon}. \tag{♣}$$

Now we consider two cases.

Case 1. $\chi(i_t, i_t + 1) =$ blue.

Note that $\lfloor \frac{i}{2} \rfloor = i_t$ and $\lfloor \frac{i+1}{2} \rfloor = i_t + 1$. By condition d), we have $d(C_i, C_{i+1}) > \frac{\beta^2}{2}$ and thus by ε-regularity and choices of parameters, $d(C_i', C_{i+1}') > \frac{\beta^2}{2} - \sqrt{\varepsilon} > \beta^3$. In particular, there is a vertex $u \in C_i'$ with $|N(u) \cap C_{i+1}'| \geqslant \frac{\beta^3}{2} \cdot |C_{i+1}'|$. Pick such a vertex u. For $j \in [2k]\backslash(I_t \cup \{i-1, i, i+1\})$, let $C_j' = C_j^t \backslash N(u)$. Furthermore, define $C_{i-1}' = C_{i-1}^t \cap N_G(u)$ if $i - 1 \notin I_t$.

Recall that $d(C_i, C_j) < 1 - \frac{\eta}{4}$ for $j \in [2k]\backslash(I_t \cup \{i-1, i, i+1\})$ by condition b) and $d(C_{i-1}, C_i) > \frac{\beta}{4}$ by condition a). Therefore, for all $j \in [2k]\backslash(I_t \cup \{i, i+1\})$, combining with (♣) we have

$$|C_j'| \geqslant \frac{\beta^3}{100} \cdot |C_j^t| \geqslant \beta^{8(t-1)+4} \cdot \gamma n.$$

By ε-regularity, there is a vertex $v \in N(u) \cap C_{i+1}'$ such that for all $j \in [2k]\backslash(I_t \cup \{i, i+1\})$, we have

$$\left|d\left(v, C_j'\right) - d(C_{i+1}, C_j)\right| < \sqrt{\varepsilon}.$$

Pick such a vertex v and let $x_i = u$ and $x_{i+1} = v$. Recalling that $d(C_{i+1}, C_j) < 1 - \frac{\eta}{4}$ for $j \neq i+2$ and $d(C_{i+1}, C_{i+2}) > \frac{\beta}{4}$, it follows that $|C_j^{t+1}| \geqslant \beta^{8t} \cdot |C_j^t|$ for all $j \in [2k]\backslash I_{t+1}$ as needed.

Case 2. $\chi(i_t, i_t + 1) =$ red.

Since $|C_i'|, |C_{i+1}'| \geqslant \frac{1}{2} \cdot \beta^{8(t-1)}\gamma n > \frac{n}{\log n}$ and $\overline{G}$ is $K_2[\frac{cn}{\log n}]$-free, there exists an edge $uv \in E(G)$ with $u \in C_i'$, $v \in C_{i+1}'$. Set $x_i = u, x_{i+1} = v$. We will show that for all $j \in [2k]\backslash I_{t+1}$, $|C_j^{t+1}| \geqslant \beta^8 |C_j^t|$, which is sufficient by our assumption.

Consider first $j = i - 1$. If $j \notin [2k]\backslash I_t$, there is nothing to prove. Otherwise, $d_G(C_j, C_i) > \frac{\beta}{4}$ by condition a) and $d(C_j, C_{i+1}) < 2\beta^2$ by condition c). Hence, by the definitions of the sets C_i', C_{i+1}', we have

$$|C_j^{t+1}| = |C_j^t \cap N(u)\backslash N(v)| \geqslant |C_j^t| \cdot \left(\frac{\beta}{4} - \sqrt{\varepsilon} - 2\beta^2 - \sqrt{\varepsilon}\right) \geqslant \beta^8 |C_t^j|.$$

Completely analogously, if $i + 2 \in [2k]\backslash I_t$, we have $|C_{i+2}^{t+1}| \geqslant \beta^8 |C_{i+2}^t|$.

Now, consider $j \in [2k]\backslash(I_t \cup \{i-1, i+2\})$. If $\chi(\lfloor \frac{i}{2} \rfloor, \lfloor \frac{j}{2} \rfloor) =$ red or $\chi(\lfloor \frac{i+1}{2} \rfloor, \lfloor \frac{j}{2} \rfloor) =$ red, then $d(C_i, C_j) + d(C_{i+1}, C_j) < 2\beta^2 + 1 - \frac{\eta}{4}$ and so combining with (♣), we have

$$|C_j^{t+1}| \geqslant |C_j^t| \cdot \left(1 - \frac{\eta}{4} - 2\beta^2 - 2\sqrt{\varepsilon}\right) > \beta^8 |C_j^t|,$$

as needed. Then we can assume that $\chi(\lfloor \frac{i}{2} \rfloor, \lfloor \frac{j}{2} \rfloor) = \chi(\lfloor \frac{i+1}{2} \rfloor, \lfloor \frac{j}{2} \rfloor) =$ blue. Then by definition, we can see that $(\lfloor \frac{j}{2} \rfloor, \lfloor \frac{i}{2} \rfloor), (\lfloor \frac{j}{2} \rfloor, \lfloor \frac{i+1}{2} \rfloor) \in E(D)$, which implies that $\pi(\lfloor \frac{j}{2} \rfloor) < \pi(\lfloor \frac{i}{2} \rfloor)$,

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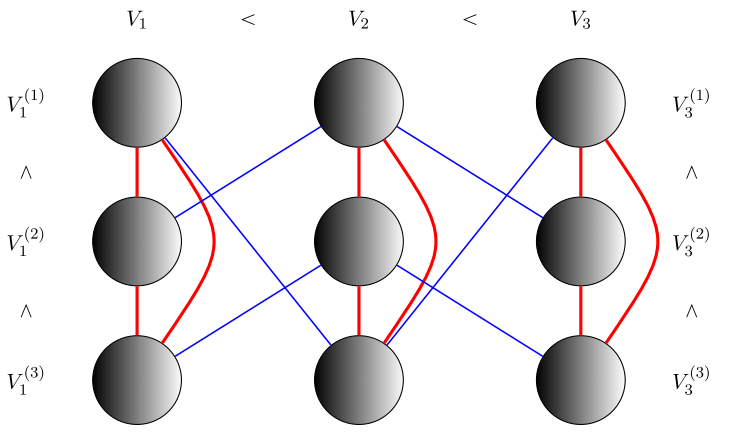


Figure 2. The black shading represents a clique and is of type-1, the red and blue edges are of type-2 and 3 respectively.

$\pi(\lfloor\frac{i+1}{2}\rfloor)$. Therefore, $(\lfloor\frac{j}{2}\rfloor, \lfloor\frac{j}{2}\rfloor+1)$ appears before $\lfloor\frac{i}{2}\rfloor, \lfloor\frac{i}{2}\rfloor+1$ in the sequence $(i_1, i_1+1), \ldots, (i_{k-1}, i_{k-1}+1)$ which in turn implies that $j \in I_t$, contradicting our assumption.

We have shown that we can run all $k-1$ steps of the above procedure. Finally, we have two candidate sets C_1^k, C_{2k}^k of size at least $\beta^{8k}\gamma_2|B_1|$ into which we can embed the vertices x_1 and x_{2k} respectively. Recalling that $d(C_1, C_{2k}) < 1 - \frac{\eta}{4}$, by ε_1-regularity, there are vertices $u \in C_1^k$, $v \in C_{2k}^k$ such that $uv \notin E(G)$. Setting $x_1 = u, x_{2k} = v$, we have that $x_1, x_2, \ldots, x_{2k}$ is a monotone induced path in G, contradicting our assumption.

This finishes the proof of Theorem 3.2.

4. Lower bound construction for Theorem 1.4

In this section, we proceed to construct n-vertex ordered graphs G with no induced monotone path P_{2k-1} such that its complement $\overline{G}$ has positive $K_{\lfloor\frac{k^2}{4}\rfloor}$-density and no biclique of size $n/e^{o(\sqrt{\log n/\log\log n})}$.

Now let $\varepsilon > 0$, n be sufficiently large, $h = \frac{\log n}{\log\log n}$ and $\mu = \frac{\varepsilon}{\sqrt{h}}$. Set $s = \lfloor\frac{k}{2}\rfloor$ and $t = \lfloor\frac{k}{2}\rfloor$. For simplicity, we only provide the proof details for even k so that the floors/ceilings can be dropped; the argument for odd k is identical.

Construction. By Lemma 2.8, as n is sufficiently large, we can partition the unit sphere $\mathbb{S}^{h-1}$ into $\frac{2n}{k}$ many pieces $D_1, D_2, \ldots, D_{\frac{2n}{k}}$ of equal measure such that each piece has diameter at most $\frac{\mu}{4}$. We construct $V(G) = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_s$ as follows. For each $i \in [s]$, we select an arbitrary point from each $D_j, j \in [\frac{2n}{k}]$, as a vertex. So each V_i consists of $\frac{2n}{k}$ vertices. We randomly partition each $V_i = V_i^{(1)} \sqcup V_i^{(2)} \sqcup \cdots \sqcup V_i^{(t)}$ into t parts of equal size. Moreover, we label the vertices so that $V_1 < V_2 < \cdots < V_s$ and for each $i \in [s]$, $V_i^{(1)} < V_i^{(2)} < \cdots < V_i^{(t)}$. For convenience, for a pair of vertices $x, y \in V(G)$, we write $|x-y|$ for the distance between the corresponding points on the sphere.

The edge set of G consists of the following three types of edges; see Fig. 2.

1. For each $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant t$, $G[V_i^{(j)}]$ is a clique.
2. For given $1 \leqslant i \leqslant s$ and $j_1 \neq j_2$, $x \in V_i^{(j_1)}$ and $y \in V_i^{(j_2)}$ are adjacent if $|x-y| \geqslant 2-\mu$.
3. For given $i_1 \neq i_2$, $x \in V_{i_1}$ and $y \in V_{i_2}$ are adjacent if $|x-y| > \sqrt{2}-\mu$.

Claim 4.1. *G does not contain induced monotone P_{2k-1}.*

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Proof of claim. Suppose P is an induced monotone P_{2k-1} on vertices $a_1 < a_2 < \cdots < a_{2k-1}$. Note first that as each $V_i^{(j)}$ induces a clique, it contains at most two vertices of the induced path P. Thus P has at most $2 \cdot t = k$ vertices in each V_i, $i \in [s]$. Consequently, if all but at most one $p_i = |V(P) \cap V_i|$, $i \in [s]$, are at most 2, then $|P| = \sum_{i\in[s]} p_i \leqslant k + 2(s-1) = 2k-2$, a contradiction. We may then assume that at least two p_i are at least 3. Note that as $V_1 < V_2 < \cdots < V_s$, each V_i must contain consecutive vertices in P.

By definitions of type-2 and type-3 edges, there cannot be two type-2 edges $aa' \in V_i$ and $bb' \in V_j$ with $i \neq j$ such that $ab, ab', a'b, a'b' \notin E(G)$ due to Theorem 2.9.

We claim that if any $p_i \geqslant 3$, then $P[V_i]$ contains a type-2 edge. To see this, say $a_r, a_{r+1}, a_{r+2} \in P \cap V_i$. Then these three vertices cannot be in the same $V_i^{(j)}$ for any $j \in [t]$ as $V_i^{(j)}$ induces a clique. Then at least one of the edges $a_r a_{r+1}$ and $a_{r+1}a_{r+2}$ is of type-2.

If there are at least three say $p_i, p_j, p_\ell \geqslant 3$ with $i < j < \ell$. Then by the above discussion, there exist type-2 edges e in $P[V_i]$ and e' in $P[V_\ell]$. Moreover, as $p_j > 0$, there are no edges between these type-2 edges e and e', contradicting Theorem 2.9.

Thus, we may assume that exactly two, say V_1, V_2, containing at least 3 vertices of P, and the remaining parts contain at most 2 vertices each. If $p_1 = p_2 = 3$, then $|P| \leqslant 3 + 3 + 2(s-2) \leqslant k + 2 \leqslant 2k - 2$ as $k \geqslant 4$. Thus, without loss of generality $p_1 \geqslant 4$ and so $a_1, a_2, a_3, a_4 \in P[V_1]$. Then $\{a_1, a_2, a_3\}$ and $P[V_2]$ both contain a type-2 edge and there is no edge between these two edges because of a_4, again contradicting Theorem 2.9. □

Next, we verify that there is no large biclique in $\overline{G}$.

Claim 4.2. *$\overline{G}$ does not contain $K_2[\frac{cn}{\log n}]$ as a subgraph.*

Proof of claim. Suppose there is a copy of $K_2[\frac{cn}{\log n}]$ in $\overline{G}$. By pigeonhole principle, there is a copy of $K_2[\frac{cn}{k\log n}]$ in $\overline{G}$ with parts $S \subseteq V_i$ and $T \subseteq V_j$.

If $i = j$, then by pigeonhole principle again, there exist $a, b \in [t]$ such that $\overline{G}$ contains a copy of $K_2[\frac{2cn}{k^2\log n}]$ with parts $S_a \subseteq V_i^{(a)}$ and $T_b \subseteq V_i^{(b)}$. Note that the Lebesgue measures of the pieces D_j, $j \in [\frac{2n}{k}]$, corresponding to both S_a and T_b are at least $\frac{2cn}{k^2\log n} \cdot \frac{k}{2n} > e^{\frac{-h\mu}{4}}$, therefore, by Lemma 2.5, $d_{\max}(S_a, T_b) \geqslant 2 - \frac{\mu}{2}$. This, together with the fact that the diameter of each piece D_j is at most $\mu/4$, implies that there must be a type-2 edge uv with $u \in S_a$, $v \in T_b$, a contradiction.

If $i \neq j$, then both of the Lebesgue measures of S and T are at least $\frac{cn}{k\log n} \cdot \frac{k}{2n} > e^{\frac{-h\mu}{4}}$. Therefore the diameters of S and T are at least $2 - \frac{\mu}{2}$. We can then find $x, x' \in S \subseteq V_i$ and $y, y' \in T \subseteq V_j$ with $|x - x'|, |y - y'| \geqslant 2 - \mu$. Moreover, by definition of G, all of pairs $xy, xy', x'y, x'y' \notin E(G)$ implies that all of the distances $|x - y|$, $|x - y'|$, $|x' - y|$ and $|x' - y'|$ are at most $\sqrt{2} - \mu$, contradicting Theorem 2.9. □

Claim 4.3. *The expected number of $\frac{k^2}{4}$-cliques in $\overline{G}$ is at least $\eta n^{\frac{k^2}{4}}$ for some $\eta > 0$.*

Proof of claim. The number of $\frac{k^2}{4}$-cliques in $\overline{G}$ is at least the number of $\frac{k^2}{4}$-tuples $(v_i^{(j)} : i \in [s], j \in [t])$ with $v_i^{(j)} \in V_i^{(j)}$ such that the following conditions hold.

- For each $1 \leqslant i \leqslant s$, we have $|v_i^{(j_1)} - v_i^{(j_2)}| < 2 - \mu$ for any $1 \leqslant j_1 < j_2 \leqslant t$.
- For each $1 \leqslant i_1 < i_2 \leqslant s$, we have $|v_{i_1}^{(j_1)} - v_{i_2}^{(j_2)}| \leqslant \sqrt{2} - \mu$ for any $j_1, j_2 \in [t]$.

For each $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant t$, take an arbitrary vertex $a_i^{(j)} \in V_i^{(j)}$. As $V_i^{(j)}$, $j \in [t]$, is a random partition of V_i, we have by Lemma 2.6 that for every $1 \leqslant i_1 < i_2 \leqslant s$ and $j_1, j_2 \in [t]$,

$$\left[\left|a_{i_1}^{(j_1)} - a_{i_2}^{(j_2)}\right| \leqslant \sqrt{2} - \mu\right] \geqslant \frac{1}{2} - \sqrt{2}\varepsilon.$$

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Moreover, for a pair of vertices $a_i^{(j_1)}$ and $a_i^{(j_2)}$ with $j_1 \neq j_2$, $a_i^{(j_1)}a_i^{(j_2)} \in E(G)$ implies $|a_i^{(j_1)} - a_i^{(j_2)}| \geqslant 2-\mu$. Note that for any vertex $v \in V_i$, its neighbours in V_i all lie in a spherical cap C whose points are within distance $d = d_0 \pm \mu/4$ from its centre, where $d_0^2 = 4\mu - \mu^2$. As the height of such a cap is $d^2/2 \leqslant 4\mu$. Thus by Lemma 2.7, the measure of C is at most $e^{-h(1-4\mu)^2/2}$. Consequently,

$$\mathbb{P}\left[\left|a_i^{(j_1)} - a_i^{(j_2)}\right| < 2-\mu\right] \geqslant 1 - e^{-h(1-4\mu)^2/2} > 1-\varepsilon.$$

Therefore, the probability that the selected vertices forming a copy of $\frac{k^2}{4}$-clique is positive, then the claim follows. □

Therefore with positive probability, the random graph G has positive $K_{\frac{k^2}{4}}$-density. This completes the proof of Theorem 1.4.

5. Results in posets

5.1. Incomparability graphs

We first show how Theorem 1.7 implies Theorem 1.6, which we restate for convenience.

Theorem 1.6. *Let $c \in (0, \frac{1}{2})$, $k \geqslant 2$ and n be sufficiently large. If G is an n-vertex incomparability graph with $\varrho_k(G) \geqslant c$, then G contains a $K_k[\frac{c^4 n}{10^8 k^2 \log n}]$.*

Proof of Theorem 1.6. Let G be an incomparability graph on n vertices with at least cn^k k-cliques and for the sake of contradiction, suppose G does not contain any copy of $K_k[q]$, where $q = \frac{c^4 n}{10^8 k^2 \log n}$. Let $\varepsilon = \frac{c}{4}$ and apply Theorem 1.7 to obtain a partition $V(G) = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_m$. Note that $\frac{n}{m} \geqslant t = \frac{\varepsilon^7 n}{10^{11} k^5} \geqslant q$ when n is sufficiently large. We then consider k-cliques in G and list three possible cases as follows.

1. There are at most $|V_0| n^{k-1} \leqslant \varepsilon n^k$ k-cliques containing at least one vertex in V_0.
2. The number of k-cliques in G such that there are at least two vertices in the same part is at most $mt^2 n^{k-2} \leqslant tn^{k-1} \leqslant \varepsilon n^k$.
3. For each k-clique with all vertices belonging to distinct parts $V_{i_1}, V_{i_2}, \ldots, V_{i_k}$, note that at least one pair of these parts is inhomogeneous, for otherwise there is a copy of $K_k[t]$ and thus $K_k[q]$ in G. Since there are at most εm^2 inhomogeneous pairs, the number of k-cliques in G with all vertices in distinct parts among $V_1, \ldots, V_m$ is at most $\varepsilon m^2 t^2 n^{k-2} \leqslant \varepsilon n^k$.

Therefore, G contains at most $3\varepsilon n^k < cn^k$ k-cliques, which is a contradiction to our assumption. □

We proceed to the proof of Theorem 1.7.

Proof of Theorem 1.7. Let $(P, \prec)$ be a partially ordered set on n elements such that G is the incomparability graph of P. Let $<'$ be an arbitrary linear extension of $\prec$, and let $x_1 <' x_2 <' \cdots <' x_n$ be the enumeration of the elements of G by $<'$. Let $s = \frac{10}{\varepsilon}$ and $\ell = \frac{10k}{\varepsilon}$. We equally partition P into s intervals $P_1, \ldots, P_s$, where $P_i = \{x_{\frac{(i-1)n}{s}+1}, \ldots, x_{\frac{in}{s}}\}$ for each $1 \leqslant i \leqslant s$.

Claim 5.1. *Every subset of P of size $\frac{\varepsilon n}{2s}$ contains ℓ disjoint sets $A_1, A_2, \ldots, A_\ell$ satisfying $A_1 \succ A_2 \succ \cdots \succ A_\ell$ and $|A_i| = t$ for $1 \leqslant i \leqslant \ell$.*

Proof of claim. Let S be an arbitrary subset of P of size $\frac{\varepsilon n}{2s}$ and apply Lemma 2.4 to the subposet of P induced on S. If the former conclusion of Lemma 2.4 holds, we obtain ℓ sets $A_1 \succ A_2 \succ \cdots \succ A_\ell$,

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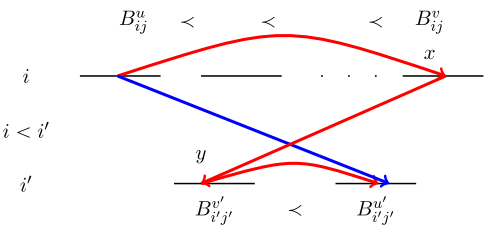


Figure 3. Claim 5.2: arrows in the curves indicate the order in P.

each of size $\frac{|S|}{10^4\ell^5} \geqslant \frac{\varepsilon^7 n}{10^{11}k^5} = t$ as needed. If the latter conclusion of Lemma 2.4 holds, there are ℓ pairwise incomparable sets in P, each of size $\frac{|S|}{40\ell^2 \log |S|} > \frac{\varepsilon^4 n}{10^5 k^2 \log n} = q$. This implies the existence of $K_k[q]$ in G, contradicting our assumption and finishing the proof of the claim. □

Let $m_0 = \lfloor \frac{(1-\varepsilon)n}{st\ell} \rfloor$. Then, for each $i \in [s]$, iteratively using Claim 5.1, we can find ℓm_0 disjoint sets $\{B^u_{ij}\}_{j\in[m_0],u\in[\ell]}$ in P_i, such that $|B^u_{ij}| = t$ and $B^1_{ij} \prec B^2_{ij} \prec \cdots \prec B^\ell_{ij}$ for all $i \in [s], j \in [m_0]$. Let $V_0 = P \setminus \bigcup_{i\in[s],j\in[m_0],u\in[\ell]} B^u_{ij}$ and $m = sm_0\ell$, and we identify $V_1, \ldots, V_m$ with the sets $\{B^u_{ij}\}_{i\in[s],j\in[m_0],u\in[\ell]}$. Note that $|V_0| = n - sm_0\ell t \leqslant \varepsilon n$ as needed. It remains to verify that at most εm^2 pairs (V_i, V_j) are inhomogeneous in G, that is, there are at most εm^2 pairs $((i, j, u), (i', j', u'))$ such that neither $B^u_{ij} \prec B^{u'}_{i'j'}$, nor $B^{u'}_{i'j'} \prec B^u_{ij}$, nor B^u_{ij} and $B^{u'}_{i'j'}$ are incomparable.

Observe that the number of pairs $((i, j, u), (i', j', u'))$ with $i = i'$ is at most $s(m_0\ell)^2 = \frac{m^2}{s} = \frac{\varepsilon m^2}{10}$. To upper bound the number of inhomogeneous pairs coming from different sets $P_i, P_{i'}$, we shall need the following claim.

Claim 5.2. *For fixed pairs $(i, j), (i', j')$ with $i < i'$, the number of pairs $(u, u') \in [\ell]^2$ such that $(B^u_{ij}, B^{u'}_{i'j'})$ is inhomogeneous is at most $2\ell - 1$.*

Before proving the claim, we finish the proof of the lemma. The number of inhomogeneous pairs coming from different sets $P_i, P_{i'}$ is at most $s^2 m_0^2 (2\ell - 1)\frac{2m^2}{\ell} < \frac{\varepsilon m^2}{5}$. Thus the total number of inhomogeneous pairs is at most εm^2, as required.

We are left to prove Claim 5.2. Let Z denote the set of pairs $(u, u') \in [\ell]^2$ such that $(B^u_{ij}, B^{u'}_{i'j'})$ is inhomogeneous. We will show that for any distinct pairs $(u, u'), (v, v') \in Z$, it holds that

$$u + u' \neq v + v'.$$

Indeed, this would prove the claim since trivially $u + u' \in [2, 2\ell]$ for any $(u, u') \in Z$.

Consider two distinct pairs $(u, u'), (v, v') \in Z$ and suppose that $u + u' = v + v'$. Without loss of generality, assume that $u < v$ which then implies $v' < u'$. Since $u < v$, it follows that $B^u_{ij} \prec B^v_{ij}$. Because the pair $(B^v_{ij}, B^{v'}_{i'j'})$ is not incomparable, there exist some $x \in B^v_{ij}, y \in B^{v'}_{i'j'}$ which are comparable in P. Since $i < i', x \in P_i, y \in P_{i'}$ and $x <' y$ by the assumed linear ordering, it follows that $x \prec y$. Finally, since $v' < u'$, we have $y \prec B^{u'}_{i'j'}$ and by transitivity it follows that $B^u_{ij} \prec x \prec y \prec B^{u'}_{i'j'}$ (see Fig. 3), contradicting our assumption that $(u, u') \in Z$. □

Remark 5.3. The dimension of a poset P is the least integer d for which there exist linear extensions $L_1, \ldots, L_d$ whose intersection equals P. If G is the incomparability graph of a poset of dimension at most d, then replacing Lemma 2.4 in the proof of Theorems 1.7 and 1.6 by Theorem 2.3 of Suk–Zeng [30] yields a linear blowup parameter; in particular, we may take $q = \Omega_{k,d}(n)$, and the remainder of the argument goes through without further modification.

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5.2. r-comparability graphs

In this subsection, we prove Theorem 1.8, which we restate here.

Theorem 1.8. *Let $r, h \in \mathbb{N}$ with $h \geqslant 2$, $k = (2h-2)^r + 1$ and $\varepsilon > 0$. Let $G = \bigcup_{i\in[r]} G(\prec_i)$ be an n-vertex r-comparability graph. If $\varrho_k(G) \geqslant \varepsilon$, then some $G(\prec_i)$, $i \in [r]$, contains a copy of $K_h[\frac{\varepsilon n}{r}]$.*

Proof. First we prove the case $r = 1$. For two elements x, y, we write $x \prec y$ if $x \prec_i y$ for every $i \in [r]$. For a copy of K_{2h-1} in G with vertex set $\{x_1, x_2, \ldots, x_{2h-1}\}$, we can assume the order is $x_1 \prec x_2 \prec \cdots \prec x_{2h-1}$. By the pigeonhole principle, there exists an $(h-1)$-tuple $(x_2, x_4, \ldots, x_{2h-2})$ in G such that there are at least εn^h choices of h-tuples $(x_1, x_3, \ldots, x_{2h-1})$ where $x_1 x_2 \cdots x_{2h-1}$ together form a copy of K_{2h-1}. Fix such an $(h-1)$-tuple $(x_2, x_4, \ldots, x_{2h-2})$, let $D_1 := \{x | x \prec x_2\}$, $D_h := \{x | x_{2h-2} \prec x\}$, and $D_i := \{x | x_{2i-2} \prec x \prec x_{2i}\}$ for any $2 \leqslant i \leqslant h-1$. Note that if $x_1 \cdots x_{2h-1}$ form a copy of K_{2h-1}, then $x_{2i-1} \in D_i$ by our above assumption, which also yields that $\prod_{i=1}^h |D_i| \geqslant \varepsilon n^h$. Obviously, $|D_i| \leqslant n$, therefore $|D_i| \geqslant \varepsilon n$ holds for all $1 \leqslant i \leqslant h$. Moreover, for each $1 \leqslant i < j \leqslant h$ and for any pair of vertices $a \in D_i, b \in D_j$, we have $a \prec x_{2i} \prec b$, therefore $G[\bigcup_{i=1}^h D_i]$ contains a copy of $K_h[\varepsilon n]$.

Next we proceed to the proof of the general cases $r \geqslant 2$. Let K be an arbitrary k-clique in G. For each vertex $x \in V(K)$, let $p_i(x)$ be the maximum number t such that there is a chain of the form $x = a_1 \prec_i a_2 \prec_i a_3 \prec_i \cdots \prec_i a_t$, where $a_1, a_2, \ldots, a_t$ are in $V(K)$. Denote $p(x) = (p_1(x), \ldots, p_r(x))$. We first prove the following claim.

Claim 5.4. *For a given k-clique K and any distinct $x, y \in V(K)$, we have $p(x) \neq p(y)$.*

Proof of claim. As x and y are adjacent in G, they are comparable in one of the partial orders. Without loss of generality, assume that $x \prec_1 y$. By definition of $p_i(y)$, we can find a chain of the form $y = a_1 \prec_1 a_2 \prec_1 \cdots \prec_1 a_{p_1(y)}$, such that all the elements $a_2, a_3, \ldots, a_{p_1(y)}$ are in $V(K)$. However, then we have $x \prec_1 y \prec_1 a_2 \cdots \prec_1 a_{p_1(y)}$, which implies that $p_1(x) > p_1(y)$. Therefore, $p(x) \neq p(y)$.

Let $G_i = G(\prec_i)$ be the comparability graph of $\prec_i$. For each k-clique K in G, by definition we have $p_i(x) \geqslant 1$, for all $x \in V(K)$, $i \in [r]$. By Claim 5.4, since $k > (2h-2)^r$, there must exist $x \in V(K)$ and $i \in [r]$ such that $p_i(x) \geqslant 2h-1$. In particular, this implies that G_i contains a $(2h-1)$-clique inside K. By the pigeonhole principle again, there exists some index $j \in [r]$ such that for at least $\frac{\varepsilon n^k}{r}$ many k-cliques K in G, there is a $(2h-1)$-clique in G_j inside K. On the other hand, each $(2h-1)$-clique in G_j can be contained in at most $n^{k-(2h-1)}$ many k-cliques in G, which yields that totally there are at least $\frac{\varepsilon n^{2h-1}}{r}$ many cliques of size $2h-1$ in G_j. Applying the result in the case $r = 1$, we obtain a copy of $K_h[\frac{\varepsilon n}{r}]$ in G_j. □

6. Hereditary graphs and B_k property

6.1. B_2 property implies B_k property in hereditary graphs

Proof of Theorem 1.9. Assume that $\mathcal{G}$ is a hereditary family which has the B_2 property with function f. For any $\varepsilon > 0$, we set a function

$$h(\varepsilon) = \delta \ll \varepsilon_0 \ll \varepsilon_1 \ll \varepsilon_2 \ll \varepsilon_3 \ll \varepsilon_4 \ll \varepsilon.$$

We will show that $\mathcal{G}$ has the B_k property with respect to the function h.

Let G be an n-vertex graph in the family $\mathcal{G}$ such that G has at least εn^k k-cliques. Let $m = \frac{4}{\varepsilon}$. If $n \leqslant m$, then we are done by Nikiforov's result [27]. Otherwise, by Lemma 2.1, there exists $M = M(m, \varepsilon_0)$ such that $V(G)$ admits an ε-regular equitable partition $\mathcal{P}$ with $K+1$ parts $V_0, V_1, \ldots, V_K$, where $m \leqslant K \leqslant M$. Setting $|V_1| = q$, we can see that $(1-\varepsilon_0)\frac{n}{K} \leqslant q \leqslant \frac{n}{m}$.

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We then construct an auxiliary graph G' with vertex set $\{V_1, V_2, \dots, V_K\}$, and a pair of vertices V_i, V_j is adjacent in G' if and only if (V_i, V_j) is ε_0-regular and $d(V_i, V_j) > \frac{\varepsilon}{2}$.

Claim 6.1. *G' contains a k-clique.*

Proof of claim. Suppose that G' is K_k-free, then for any k-clique in G with vertex set $\{v_1, v_2, \cdots, v_k\}$, at least one of the following four types occurs:

- At least one of the vertices $v_1, v_2, \dots, v_k$ belongs to V_0. Note that $|V_0| \leqslant \varepsilon_0 n$, the number of k-tuples of this type is at most $\varepsilon_0 n^k$.
- At least two of the vertices $v_1, v_2, \dots, v_k$ belong to some V_i, where $i \in [K]$. Note that $|V_i| \frac{n}{m}$, the number of the tuples of this type is at most $\frac{n^k}{m} \leqslant \frac{\varepsilon}{4} n^k$.
- There exist some $v_i \in V_a$, $v_j \in V_b$ such that V_a and V_b are not ε_0-regular, where $i, j \in [k]$ and $a, b \in [K]$. Note that there are at most $\varepsilon_0 K^2$ pairs (V_i, V_j) being not ε_0-regular, therefore the number of k-tuples of this type is at most $\varepsilon_0 n^k$.
- There exist some $v_i \in V_a, v_j \in V_b$ such that $d(V_a, V_b) < \frac{\varepsilon}{2}$, where $i, j \in [k]$ and $a, b \in [K]$. Then, for fixed V_a, V_b, there are at most $\frac{\varepsilon q^2}{2}$ choices of v_i, v_j, hence the number of k-tuples of this type is at most $m^2 \cdot \frac{\varepsilon q^2}{2} \cdot n^{k-2} \leqslant \frac{\varepsilon n^k}{2}$.

Hence, the total number of k-cliques in G is less than εn^k, a contradiction.

Without loss of generality, we can assume that $V_1, V_2, \dots, V_k$ form a clique in G'. We consider two cases depending on whether some V_i is locally dense.

Case 1. At least one of the clusters $G[V_1], \dots, G[V_k]$ is $(\varepsilon_1, \varepsilon_2)$-dense. Without loss of generality, we assume that $G[V_1]$ is $(\varepsilon_1, \varepsilon_2)$-dense. We will use induction to show that for $2 \leqslant t \leqslant k$, we can find a copy of $K_t[f(\frac{\varepsilon_2}{3})^{t-1} q]$ in V_1. First, since $G[V_1]$ has density at least ε_2 and B_2 property, we can find a $K_2[f(\frac{\varepsilon_2}{3})q]$ in V_1. Suppose that we have found a copy of $K_t[f(\frac{\varepsilon_2}{3})^{t-1} q]$ in V_1 with parts $P_1, \dots, P_t$. Since $G[V_1]$ is $(\varepsilon_1, \varepsilon_2)$-dense and $\varepsilon_1 \ll \varepsilon_2$, $G[P_1]$ has density at least ε_2, then we can find a copy of $K_2[f(\frac{\varepsilon_2}{3})|P_1|]$ in P_1. Thus, this biclique together with $P_2, \dots, P_t$ contains a copy of $K_{t+1}[f(\frac{\varepsilon_2}{3})^t q]$ in V_1. Taking $t = k$, we get the desired large blowup of K_k.

Case 2. We may then assume that none of $G[V_1], \dots, G[V_k]$ is $(\varepsilon_1, \varepsilon_2)$-dense, then for each $i \in [k]$, we can pick a subset $U_i' \subseteq V_i$ such that $|U_i'| \geqslant \varepsilon_1 q$ and $G[U_i']$ has density at most ε_2. Let $q_0 = \lceil \varepsilon_1 q \rceil$. Pick a subset U_i of U_i' among all possible subgraphs with size q_0 which achieve the minimal density, then $G[U_i]$ has density at most ε_2. We will show that there is a copy of $K_k[\delta n]$ such that the ith-part is contained in U_i.

Claim 6.2. *For any distinct subsets $A_i \subseteq U_i$, $A_j \subseteq U_j$, with $i, j \in [k]$ and $|A_i| = |A_j| \geqslant \varepsilon_3 q_0$, we can find subsets $B_i \subseteq A_i$, $B_j \subseteq A_j$ with $|B_i| = |B_j| \geqslant \varepsilon_4 |A_i|$, such that (B_i, B_j) forms a biclique.*

Proof of claim. Since U_i and U_j are ε_0-regular with density at least $\varepsilon/2$ and $|A_i|, |A_j| \geqslant \varepsilon_3 q_0$, we have $d(A_i, A_j) \geqslant \frac{\varepsilon}{4}$. Hence, $G[A_i \cup A_j]$ has edge density at least $\frac{\varepsilon}{16}$ since $|A_i| = |A_j|$. Thus by the B_2 property, we can find a copy of $K_2[f(\frac{\varepsilon}{16})|A_i|]$ with parts X, Y in $G[A_i \cup A_j]$.

By symmetry, assume $|X \cap A_i| \frac{|X|}{2}$. Let $X_0 = X \cap A_i$. If $|Y \cap A_i| \geqslant \frac{|Y|}{2}$, letting $Y_0 = Y \cap A_i$, we have

$$\varepsilon_2 \geqslant \frac{e(G[U_i])}{\binom{|U_i|}{2}} \geqslant \frac{e(X_0, Y_0)}{|U_i|^2} \geqslant \frac{|X||Y|}{4|U_i|^2} \geqslant \frac{1}{4}\varepsilon_3^2 f\Big(\frac{\varepsilon}{16}\Big)^2,$$

which is a contradiction. Therefore, we have $|Y \cap A_j| \geqslant \frac{|Y|}{2}$. Let $Y_1 = Y \cap A_j$, then X_0, Y_1 form a biclique. Finally, just pick $B_i \subseteq X_0$, $B_j \subseteq Y_1$ with $|B_i| = |B_j| \geqslant \varepsilon_4 |A_i|$.

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Claim 6.3. *For any subgraph H of K_k on vertex set $[k]$, there are sets $W_i \subseteq U_i$ such that*

- $|W_1| = \cdots = |W_k| \geqslant \varepsilon_4^{e(H)} q_0$; *and*
- *if i and j are adjacent in H, then (W_i, W_j) forms a biclique.*

Proof of claim. We use induction on the number of edges in H. When H is a empty graph, taking $W_i = U_i$ suffices.

Assume we have proven the claim for $e(H) \leqslant s$. When $e(H) = s+1$, let H' be the graph obtained by removing one edge e in H. By symmetry, we may assume that $e = \{1, 2\}$. By the induction hypothesis, we can find sets $W_i' \subseteq U_i$ such that

- $|W_1'| = \cdots = |W_k'| \geqslant \varepsilon_4^{|H|-1} q_0$; and
- if i and j are adjacent in H', then (W_i', W_j') forms a biclique.

By Claim 6.2, we can find a subset W_1 of W_1', and subset W_2 of W_2' with $|W_1| = |W_2| \geqslant \varepsilon_4 |W_1'|$ such that (W_1, W_2) is a biclique. Finally, for $3 \leqslant i \leqslant k$, choose a subset W_i of W_i' with size $|W_1|$. It is obvious that $W_1, \ldots, W_k$ satisfy the condition.

Applying Claim 6.3 with $H = K_k$ shows that we can find a $K_k[\delta n]$ such that the ith-part is contained in U_i, finishing the proof. □

6.2. Graphs with bounded VC-dimension

Given a set system $\mathcal{F} \subseteq 2^X$, a set $S \subseteq X$, is said to be *shattered* by $\mathcal{F}$ if for every $B \subseteq S$, there exists a set $F \in \mathcal{F}$ such that $F \cap S = B$. The VC-dimension of $\mathcal{F}$ is the size of the largest set shattered by $\mathcal{F}$. Given a graph $G = (V, E)$, for any vertex $v \in V$, let $N_G(v)$ be the set of its neighbours. The VC-dimension of G is defined to be the VC-dimension of the set system $\{N_G(v) \subseteq V(G) : v \in V(G)\}$.

We start with a simple proposition about graphs with VC-dimension 1.

Proposition 6.4. *Let G be an n-vertex graph with VC-dimension* 1, *then the following holds.*

(1) *If G contains a copy of a triangle with vertices x_1, x_2, x_3, then any vertex $v \in V(G) \backslash \{x_1, x_2, x_3\}$ is adjacent to at least* 2 *vertices of x_1, x_2, x_3.*

(2) *If there is a copy of $P_5 := y_1y_2y_3y_4y_5$, then at least one of the pairs y_1y_4, y_2y_4, y_2y_5 is adjacent.*

Proof. To see (1), if a vertex $v \in V(G) \backslash \{x_1, x_2, x_3\}$ is not adjacent to two vertices, say x_1 and x_2, then the set $\{x_1, x_2\}$ is shattered. To see (2), note that if none of the edges y_1y_4, y_2y_4, y_2y_5 are present, then the set $\{y_2, y_4\}$ is shattered. □

We need the following regularity lemma for graphs of bounded VC-dimension.

Theorem 6.5 [15]. *Let $\gamma \in (0, \frac{1}{4})$ and $G = (V, E)$ be an n-vertex graph with VC-dimension d. Then $V(G)$ has an equitable partition $V(G) = V_1 \cup \cdots \cup V_K$ with $\frac{8}{\gamma} \leqslant K \leqslant c(\frac{1}{\gamma})^{2d+1}$ parts such that all but a γ-fraction of the pairs of parts are γ-homogeneous, where $c = c(d)$ is a constant depending only on d.*

Proof of Theorem 1.10. We first argue that there are graphs with VC-dimension 2 but have no B_2 property. Let H_0 be the point–line incidence bipartite graph of the projective plane PG(2, q). Then H_0 is balanced bipartite with $|A| = |B| = q^2 + q + 1$, is $(q+1)$-regular, and C_4-free. Write $n := |A| + |B| = 2(q^2 + q + 1)$. Obtain H by keeping each edge of H_0 independently with probability $p = c\, q^{-1/3}$ (fixed $c > 0$). Since $H \subseteq H_0$, the graph H is C_4-free and, by Chernoff bounds,

$$\mathbb{E}[e(H)] = p\, e(H_0) = \Theta(q^{-1/3} \cdot q^3) = \Theta(q^{8/3}) = \Theta(n^{4/3}) \quad \text{and} \quad e(H) = \Theta(n^{4/3}) \text{ w.h.p.}$$

In any C_4-free graph a 3-set cannot be shattered, because shattering Y forces two vertices to share two neighbours from Y, creating a C_4. Thus H has VC-dimension at most 2.

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Finally, let $\overline{H}$ be the bipartite complement in $K_{A,B}$. For fixed $A' \subseteq A$ and $B' \subseteq B$ with $|A'| = |B'| = t$,

$$\Pr\big(H[A', B'] = \emptyset\big) = (1-p)^{e(H_0[A',B'])} \leqslant \exp\big(-\Omega(p\,t^2/q)\big).$$

Therefore, by the first-moment method,

$$\mathbb{E}\big[\#K_{t,t} \subseteq \overline{H}\big] \leqslant \binom{n/2}{t}^2 \exp\big(-\Omega(n^{-2/3}t^2)\big) = o(1)$$

for any fixed $\varepsilon > 0$ when $t \geqslant n^{2/3+\varepsilon}$. Consequently, w.h.p. $\overline{H}$ contains no $K_{t,t}$ of size $n^{2/3+o(1)}$, and we obtain graphs of VC-dimension at most 2 without the B_2 property.

Let G be an n-vertex graph with VC-dimension 1 and $e(G) = c\binom{n}{2}$ with $c > 0$. It is left to prove that G contains a $K_2[t]$ with $t = \Omega_c(n)$. We consider two cases.

Case 1. $c \geqslant \frac{99}{100}$.

Claim 6.6. *G contains a subgraph G' with $|G'| \geqslant \frac{9n}{10}$ and $\delta(G') \geqslant \frac{2n}{3}$.*

Proof of claim. We iteratively define a sequence of graphs $G_0 = G \supseteq G_1 \supseteq G_2 \supseteq \cdots$, where for each $i \geqslant 1$, we obtain G_i by removing one vertex with degree less than $\frac{2n}{3}$ from G_{i-1} if it exists, otherwise we stop. Note that the number of non-adjacent pair of vertices in G is at most $\frac{1}{100}\binom{n}{2}$ and each removal decreases the number of non-edges by at least $\frac{n}{3}$. Hence, the process will stop after fewer than $\frac{3}{100}n$ steps. Let G' be the resulting subgraph. It follows that $\delta(G') \geqslant \frac{2n}{3}$ and $|V(G')| \geqslant \frac{9}{10}n$.

The following claim implies that the graph G' is a complete multipartite graph.

Claim 6.7. *For any pair of non-adjacent vertices $u, v \in V(G')$, $N_{G'}(u) = N_{G'}(v)$.*

Proof of claim. Let $u, v \in V(G')$ be a pair of vertices such that $uv \notin E(G')$. Suppose that there exists a vertex w such that $uw \in E(G')$ and $vw \notin E(G')$, note that u and w have a common neighbour $z \in V(G')$ as $\delta(G') \geqslant \frac{2n}{3}$. That means v can be adjacent to at most one vertex of the triangle uwz, a contradiction to Proposition 6.4.

Since G' is a complete multipartite graph and $\delta(G') \geqslant \frac{2n}{3}$, it follows that each of the parts has size at most $\frac{n}{3}$. Since $|V(G')| \geqslant \frac{9n}{10}$, the parts of G' can be split into two sides such that each of the sides has size at least $\frac{n}{5}$, implying that G contains $K_2[\frac{n}{5}]$ as desired.

Case 2. $c < \frac{99}{100}$.

Let $\gamma = 10^{-5}c$. By Theorem 6.5, we can partition $V(G)$ into $V_1 \cup V_2 \cup \cdots \cup V_K$ with $K \leqslant c'(\frac{1}{\gamma})^3$ for some $c' > 0$, such that all but a γ-fraction of the pairs of parts are γ-homogeneous. Since $c\binom{n}{2} \leqslant e(G) \leqslant \frac{99}{100}\binom{n}{2}$, it is easy to see that there are three parts V_i, V_j, V_k such that (V_i, V_j) is γ-dense, (V_j, V_k) is γ-sparse and $|V_i| = |V_j| = |V_k| = m \geqslant \frac{n}{2K}$. Similarly as in Claim 6.6, we remove any vertex from V_j with at most $\frac{4m}{5}$ neighbours in V_i or at least $\frac{m}{5}$ neighbours in V_k. Let $U_j \subseteq V_j$ be the remaining set and observe that $|U_j| \geqslant (1-10\gamma)m$ and for every $u \in U_j$, we have $d_{V_i}(u) \geqslant \frac{4m}{5}$ and $d_{V_k}(u) \leqslant \frac{m}{5}$.

Note that there are at least $\frac{4m}{5}|U_j|\frac{3m^2}{4}$ edges between U_j and V_i so we can pick a vertex $v \in V_i$ with $|N_{U_j}(v)| \geqslant \frac{3m}{4}$. We first claim that $N_{U_j}(v)$ is an independent set. Indeed, if there is an edge u_1u_2 in $N_{U_j}(v)$, then every vertex in V_k should be adjacent to at least one of u_1, u_2 by Proposition 6.4 (1). However, then one of u_1, u_2 has at least $\frac{m}{2}$ neighbours in V_k, a contradiction.

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Moreover, we have the following claim.

Claim 6.8. *For any distinct vertices* $u_1, u_2 \in N_{U_j}(v)$, *either* $N_{V_i}(u_1) \subseteq N_{V_i}(u_2)$, *or* $N_{V_i}(u_2) \subseteq N_{V_i}(u_1)$.

Proof of claim. Suppose this is not the case so there is a vertex $v_1 \in N_{V_i}(u_1) \backslash N_{V_i}(u_2)$ and a vertex $v_2 \in N_{V_i}(u_2) \backslash N_{V_i}(u_1)$. Then $v_1u_1vu_2v_2$ forms a copy of P_5. However, none of the edges v_1u_2, u_1u_2, u_1v_2 are present, contradicting Proposition 6.4 (2).

Now, consider a vertex $u \in N_{U_j}(v)$ minimising $|N_{V_i}(u)|$. Then, for any $u' \in N_{U_j}(v)$, we have $N_{V_i}(u') \supseteq N_{V_i}(u)$ implying the existence of $K_2[\frac{3m}{4}]$ in G.

This completes the proof of Theorem 1.10. □

7. Concluding remarks

Recall that $g(k)$ is the minimum $r \in \mathbb{N}$ such that if any n-vertex ordered graph G with no induced monotone P_{2k} satisfies $\varrho_r(\overline{G}) > 0$, then $\overline{G}$ contains a copy of $K_2[\Omega(\frac{n}{\log n})]$. We determined $g(k)$ up to a factor of 2. It would be interesting to close this gap. To this end, we make the following conjecture on the Ramsey problem for $f(k)$ defined in Definition 3.1.

Conjecture C. $f(k) = \lfloor \frac{k^2}{4} \rfloor + 1$.

By our construction in Section 4, the conjecture above, if true, would be optimal, implying that $g(k) = f(k) = \lfloor \frac{k^2}{4} \rfloor + 1$. With the assistance of computers, we checked that Conjecture C is true for $k \leqslant 6$.

An important theme we systematically investigate in this paper is that of 'replacing edge density by clique density'. We believe that this theme merits further study. A recent result of Holmsen [21] is of this flavour. It is known that for any intersection graph G of axis-aligned boxes in $\mathbb{R}^d$, if G has edge density larger than $1 - \frac{1}{d}$, then it contains a linear-size clique. A corollary of Holmsen's result shows that it suffices to require positive K_{d+1}-density.

Tomon [34] proved that if G is the complement of a string graph and has edge density larger than $\frac{3}{4}$, then G contains a linear-size biclique. Following our theme here, we propose the following conjecture.

Conjecture D. *Let* $\varepsilon > 0$ *and* G *be the complement of a string graph. If* $\varrho_{4s+1}(G) > 0$, *then* G *contains a* $K_{s+1}[\delta n]$, *where* $\delta = \delta(\varepsilon) > 0$.

We remark that taking disjoint union of tight examples in [34] shows that if the conjecture is true, it is optimal in the sense that positive K_{4s}-density does not suffices to guarantee linear-size blowup of K_{s+1}.

In Theorem 1.10, we show that linear-size biclique is guaranteed only in dense graphs with VC-dimension 1. How large a biclique do dense graphs with larger VC-dimension contain? Anthony, Brightwell and Cooper [1] proved that the random graph $G(n,p)$ with edge probability $p = 1 - n^{-\frac{1}{d} + O(\frac{1}{d^2})}$, with high probability, has VC-dimension at most d and is $K_2[t]$-free, where $t = n^{\frac{1}{d} + O(\frac{1}{d^2})}$. Hence, there exist graphs with $(1 - o(1))\binom{n}{2}$ edges, VC-dimension at most d and no biclique of size $n^{\frac{1}{d} + O(\frac{1}{d^2})}$. On the other hand, we can show that any dense graph with VC-dimension d contains a biclique of size $\Omega(n^{\frac{1}{d+1}})$.

Proposition 7.1. *Let* $d \geqslant 2$ *be positive integers and* G *be an* n*-vertex graph with VC-dimension* d. *If* $e(G) \geqslant c\binom{n}{2}$ *for some* $c > 0$, *then* G *contains a copy of* $K_2[c' n^{\frac{1}{d+1}}]$ *for some* c'.

A system $\mathcal{X} \subseteq 2^V$ is *s-separated* if for any $F, F' \in \mathcal{X}$, $|F \triangle F'| \geqslant s$.

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Lemma 7.2 (Haussler [20]). *Let $\mathcal{F} \subseteq 2^V$ be a set system such that $\mathcal{F}$ has VC-dimension d. If $\mathcal{X} \subseteq \mathcal{F}$ is s-separated, then $|\mathcal{X}| \leqslant c_1(\frac{|V|}{s})^d$, where $c_1 = c_1(d)$.*

Proof of Proposition 7.1. Let G be an n-vertex graph with VC-dimension d and $e(G) \geqslant c\binom{n}{2}$. Deleting low degree vertices, we get from G a subgraph G' with $\delta(G') \geqslant cn/2$. Let $\mathcal{F}' := \{N_{G'}(v) \subseteq V(G') : v \in V(G')\}$, the VC-dimension of $\mathcal{F}'$ is also at most d.

Set $s = |G'|^{1-\frac{1}{d+1}}$ and take a maximal s-separated subfamily $\mathcal{X} = \{N_{G'}(v_1), N_{G'}(v_2), \dots, N_{G'}(v_m)\} \subseteq \mathcal{F}'$. We have $m \leqslant c_1(\frac{|G'|}{s})^d$ by Lemma 7.2. Then, partition $V(G')$ into $V_1 \sqcup V_2 \sqcup \cdots \sqcup V_m$ with $v_i \in V_i$ as follows. We put $N_{G'}(v) \in \mathcal{F}'\backslash\mathcal{X}$ into V_i if i is the smallest index such that $|N_{G'}(v) \triangle N_{G'}(v_i)| \leqslant s$. Note that after partitioning, each set in $\mathcal{F}'\backslash\mathcal{X}$ belongs to some part V_i. Moreover, every pair of vertices u, v with their neighbourhoods in the same part satisfies $|N_{G'}(u) \triangle N_{G'}(v)| \leqslant 2s$ by the triangle inequality.

Now pick arbitrary vertices $v_1, v_2, \dots, v_q$ from some part V_i with $|V_i| = \Omega(\frac{n}{m})$, where $q = \Omega(n^{\frac{1}{d+1}})$. As $\delta(G') > (2q+1)s + q$, we see that $\left|\bigcap_{h\in[q]} N_{G'}(v_h)\right| \geqslant |N_{G'}(v_1)| - (2q+1)s \geqslant q$. Note that $|V_i| \Omega(n^{1-\frac{1}{d+1}}) > q$, thus there is a biclique of size q. □

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