J. Plasma Physics (2006), *vol.* 72, *part* 6, *pp.* 1209–1213. © 2006 Cambridge University Press
doi:10.1017/S0022377806005952 Printed in the United Kingdom

On the computation of tokamak equilibria with negative core toroidal current

PAULO RODRIGUES and JOÃO P. S. BIZARRO

Centro de Fusão Nuclear, Associação Euratom—IST, Instituto Superior Técnico, 1049-001 Lisboa, Portugal

(Received 8 July 2005 and accepted 16 December 2005)

Abstract. A numerical scheme is presented which is able to solve the Grad–Shafranov (GS) equation in tokamak plasmas with toroidal current reversal for input profiles that are not restricted to any particularly chosen models. Its flexibility is subsequently tested, computing a GS equilibrium with off-axis negative toroidal current density strong enough to generate two nested current-reversal surfaces.

1. Introduction

The observation of confinement regimes displaying nearly zero toroidal current density around the magnetic axis [1, 2], started a discussion about possible Grad–Shafranov (GS) equilibria with negative core toroidal current [3, 4], readily followed by specific solutions displaying an axisymmetric island system [5, 6]. Still, these assumed either flat pressure and toroidal current-density distributions or a rather stiff Solovev-like model, hindering the extrapolation to reactor-like conditions and obscuring the underlying physics.

Adapting a perturbative GS solver [7], it has become possible to handle more general input profiles while retaining some useful insight into the problem posed by the existence of a poloidal-field reversal (PFR) layer inside the plasma, on which the tangential magnetic field and the enclosed toroidal current do vanish [8]. When seeking solutions in the form of a power series in the inverse aspect ratio ε,

$$\psi(r,\theta;\varepsilon) = \psi_0(r) + \sum_{n=1}^{+\infty}\sum_{k=0}^{n} \frac{\varepsilon^n}{n!}\widehat{\psi}_{nk}(r)\cos k\theta, \tag{1.1}$$

such an approach produces a hierarchy of closed integral solutions $\widehat{\psi}_{nk}(r)$ to the linear, non-homogeneous second-order ordinary differential equation (ODE)

$$r^2\widehat{\psi}''_{nk}(r) + r\widehat{\psi}'_{nk}(r) + \left[s(r) - k^2\right]\widehat{\psi}_{nk}(r) = \widehat{b}_{nk}(r), \tag{1.2}$$

where $s(r) = r^2[\ddot{p}_0(r) + \ddot{Y}_0(r)]$, while ψ, $p(\psi)$, and $Y(\psi)$ are, respectively, the normalized poloidal-field flux, plasma pressure, and squared poloidal current, with the dot denoting flux derivatives $d/d\psi$, (r,θ,ϕ) the usual toroidal coordinates, and the non-homogeneous source $\widehat{b}_{nk}(r)$ combining the lower mth-order terms (with $m < n$) and their derivatives, as described elsewhere [7, 8]. The purpose of this

https://doi.org/10.1017/S0022377806005952 Published online by Cambridge University Press

paper is to present a numerical scheme able to efficiently solve (1.2), which needs special handling of the boundary conditions for $k = 1$, as well as the zeroth-order condition

$$-(1/r)(d/dr)[r\psi_0'(r)] = \dot{p}_0(r) + \dot{Y}_0(r) = -J_0^{\phi}(r) \tag{1.3}$$

relating the input profiles $p_0(r) = p[\psi_0(r)]$ and $J_0^{\phi}(r)$ with $Y_0(r) = Y[\psi_0(r)]$ and $\psi_0(r)$. Next, the scheme is applied to a current-density profile with negative values off-axis, of a type which is found in nonlinear resistive magnetohydrodynamic (MHD) simulations [9].

2. Numerical scheme

It is known that the zeroth-order distribution $\psi_0(r)$ cannot be arbitrarily chosen [8], because every flux derivative $\dot{p}_0, \ddot{p}_0, \ldots$ and $\dot{Y}_0, \ddot{Y}_0, \ldots$ is needed when writing the several $\widehat{b}_{nk}(r)$ and, consequently, (1.3) demands $J_0^{\phi}(r)$ to be a function of $\psi_0(r)$ with well-defined derivatives

$$(d/d\psi_0)^n J_0^{\phi}(\psi_0) = [1/\psi_0'(r)](d/dr)(d/d\psi_0)^{n-1} J_0^{\phi}(r), \tag{2.1}$$

even at the inversion radius r_{L} where $\psi_0'(r_{\mathrm{L}}) = 0$. To prevent $(d/dr)(d/d\psi_0)^{n-1} J_0^{\phi}(r)$ from approaching zero at a slower rate than $\psi_0'(r)$ on r_{L}, it suffices to prescribe a model $J_0^{\phi}(\psi_0)$ with a proper Taylor expansion in powers of $\psi_0 - \psi_L$ (with $\psi_{\mathrm{L}} = \psi_0(r_L)$), whose coefficients equate to the derivatives (2.1), and then solve the nonlinear condition

$$r^2\psi_0''(r) + r\psi_0'(r) = r^2 J_0^{\phi}[\psi_0(r)] \tag{2.2}$$

for $\psi_0(r)$, with boundary conditions $\psi_0(0) = 0$ and $\psi_0'(r_{\mathrm{L}}) = 0$. Once the input profiles $p(\psi)$ and $J_0^{\phi}(\psi_0)$ are provided, and $\psi_0(r)$ is extracted from (2.2) through any conventional nonlinear ODE solver, the flux derivatives should be computed as $(d/d\psi_0)^n p_0(r) = (d/d\psi_0)^n p[\psi_0(r)]$ and $(d/d\psi_0)^n J_0^{\phi}(r) = (d/d\psi_0)^n J_0^{\phi}[\psi_0(r)]$, rather than using radial derivatives only as in (2.1), thus avoiding the numerical problems of dividing infinitesimal quantities. Subsequently, after differentiating (1.3), one finds

$$-(d/d\psi_0)^n Y_0(r) = (d/d\psi_0)^n p_0(r) + (d/d\psi_0)^{n-1} J_0^{\phi}(r). \tag{2.3}$$

Despite their usefulness in other contexts, closed-integral forms for $\widehat{\psi}_{nk}(r)$ depend on the homogeneous solutions $g_k(r)$ of (1.2) (see [7]), which may not always be available. Therefore, it is preferable to define an N-point uniform grid $r_0 = 0, \ldots, r_{N-1} = r_{\max}$, in the domain $0 \leqslant r \leqslant r_{\max}$, and solve (1.2) via a finite-difference scheme. If $k \geqslant 2$, boundary conditions $\widehat{\psi}_{nk}^* = \widehat{\psi}_{nk}(r_{nk}^*)$ must be provided in addition to $\widehat{\psi}_{nk}(0) = 0$ (see [7]), yielding a trivial tridiagonal linear system for the $N_{nk}^* - 1$ unknowns $\widehat{\psi}_{nk1}, \ldots, \widehat{\psi}_{nkN_{nk}^*-1}$, where $\widehat{\psi}_{nk0} = 0$ and $\widehat{\psi}_{nkN_{nk}^*} = \widehat{\psi}_{nk}^*$, with $r_{N_{nk}^*} = r_{nk}^*$. The remaining $\widehat{\psi}_{nkN_{nk}^*+1}, \ldots, \widehat{\psi}_{nkN-1}$ are then computed recursively as

$$(i-1)\left(i-\tfrac{1}{2}\right)\widehat{\psi}_{nki} = \widehat{b}_{nki-1} - \left[s_{i-1} - k^2 - 2(i-1)^2\right]\widehat{\psi}_{nki-1} - (i-1)\left(i-\tfrac{3}{2}\right)\widehat{\psi}_{nki-2}. \tag{2.4}$$

https://doi.org/10.1017/S0022377806005952 Published online by Cambridge University Press

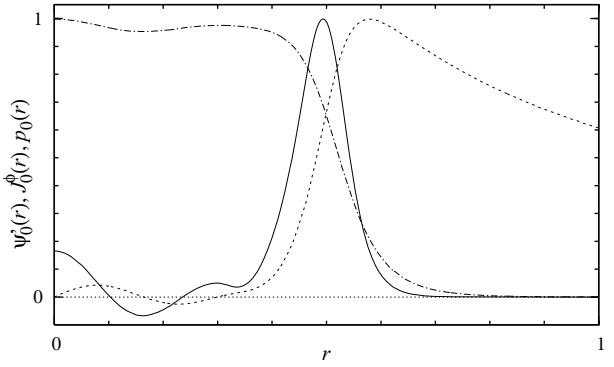


Figure 1. Zeroth-order profiles $J_0^\phi(r)$ (solid curve), $\psi_0'(r)$ (dashed curve), and $p_0(r)$ for $\alpha = 5.55$ and $\gamma = 10$ (dot-dashed curve), normalized to their maximum values.

However, off-axis boundary conditions cannot be provided for $k \leqslant 1$ (see [7]), and a value must be assigned to $\widehat{\psi}_{nk1}$ before using (2.4) for $i = 2, \ldots, N-1$. Near $r = 0$, it is known that $g_k(r) \propto r^k$ (see [7]) and, assuming also $\widehat{b}_{nk}(r) \propto r^l$ there, the closed-integral form for $\widehat{\psi}_{nk}(r)$ (see [7, 8]) may be easily integrated from 0 to r_1, leading to $\widehat{\psi}_{nk1} \approx \widehat{b}_{nk1}(l^2 - k^2)^{-1}$.

3. Results

Letting the current density in the core ($r \leqslant r_\mathrm{L}$) be modelled by

$$J_0^\phi(\psi_0) \propto a_1 + a_2\psi_0, \tag{3.1}$$

one may set $\psi_0(r) \propto 1 - J_0(j_{1,2}r/r_L)$, with $j_{1,2}$ the second zero of $J_1(x)$, and $J_\nu(x)$ a Bessel function of the first kind, which results in two PFR layers at r_L and $r_\mathrm{L}j_{1,1}/j_{1,2}$. For $r > r_\mathrm{L}$, the model is set to

$$J_0^\phi(\psi_0) \propto \frac{b_1 + (\psi_0 - \psi_\mathrm{L})\left[b_2 - b_3 e^{-b_4(\psi_0 - \psi_\mathrm{L})}\right]}{1 - c_1\left[1 - e^{c_2(\psi_0 - \psi_\mathrm{L})}\right]} \tag{3.2}$$

with $b_2 = 800$, $b_3 = 2000$, $b_4 = 1500$, $c_1 = \frac{1}{2}$, and $c_2 = 45$. The parameters a_2 and b_1 are chosen in order to ensure $\psi_0'(r_\mathrm{L}) = 0$ and the continuity of $J_0^\phi(\psi_0)$ over $r_\mathrm{L} = 0.3$, whilst a_1 is adjusted to set the on-axis toroidal current density to 0.5 MA m^{-2}. Figure 1 shows $J_0^\phi[\psi_0(r)]$ after solving (2.2), along with $\psi_0'(r)$ and the pressure profile

$$p(\psi) = \varepsilon^2 A(1 + \alpha\psi)e^{-\gamma\psi} \tag{3.3}$$

for $A = 1.01$, following from typical Joint European Torus parameters: $a = 0.95$ m, $R_0 = 2.96$ m, $B_0 = 2.5$ T, $n_\mathrm{e} = 1.5 \times 10^{19}$ m^{-3}, $T_\mathrm{e} = 6$ keV, $T_\mathrm{i} = 1.8$ keV, and $I_\mathrm{p} = 1$ MA (see [9]). The equilibrium, for boundary conditions $\widehat{\psi}_{22}(1) = 1.5$, $\widehat{\psi}_{32}(1) = 5$, and $\widehat{\psi}_{33}(1) = 4$, is depicted in Fig. 2, where two PFR surfaces may be clearly identified.

https://doi.org/10.1017/S0022377806005952 Published online by Cambridge University Press

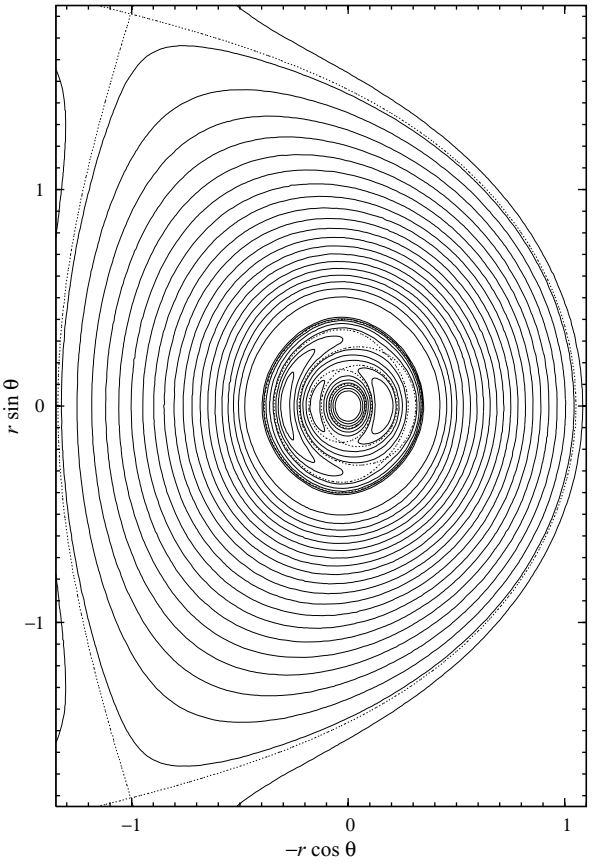


Figure 2. Contours of $\psi(r,\theta) = 5(5+2i)10^{-4}$ for $i = 1,\ldots,9$ (core zone), $\psi(r,\theta) = i/15$ for $i = 1,\ldots,22$ (outer zone), and the separatrices (dotted lines) for $\alpha = 5.55$ and $\gamma = 10$.

4. Conclusions

The robustness of the proposed scheme to solve the GS equation for equilibria with negative toroidal current flowing in the core was demonstrated. In particular, an equilibrium with off-axis negative current density, strong enough to develop two PFR layers, was successfully computed. Equilibria calculations in such conditions are expected to help in understanding the physics behind the current-hole problem.

Acknowledgements

This work has been carried out within the framework of the Contract of Association between the European Atomic Energy Community and the Instituto Superior Técnico (IST), and has also received financial support from the Fundação para a Ciência e a Tecnologia (FCT). The content of this publication is the sole responsibility of the authors and it does not necessarily represent the views of the European Commission, of FCT, of IST, or of their services.

https://doi.org/10.1017/S0022377806005952 Published online by Cambridge University Press

References

[1] Hawkes, N. C. et al. 2001 *Phys. Rev. Lett.* **87**, 115001.
[2] Fujita, T. et al. 2001 *Phys. Rev. Lett.* **87**, 245001.
[3] Chu, M. S. and Parks, P. B. 2002 *Phys. Plasmas* **9**, 5036.
[4] Hammett, G. W. et al. 2003 *Phys. Plasmas* **10**, 4048.
[5] Martynov, A. A. et al. 2003 *Phys. Rev. Lett.* **91**, 085004.
[6] Wang, S. 2004 *Phys. Rev. Lett.* **93**, 155007.
[7] Rodrigues, P. and Bizarro, J. P. S. 2004 *Phys. Plasmas* **11**, 186.
[8] Rodrigues, P. and Bizarro, J. P. S. 2005 *Phys. Rev. Lett.* **95**, 015001.
[9] Stratton, B. C. et al. 2002 *Plasma Phys. Control. Fusion* **44**, 1127.

https://doi.org/10.1017/S0022377806005952 Published online by Cambridge University Press