The fact that we have a new parallelogram is evident from the symmetry in the figure. It is similar to the original as the affixes of its corresponding vertices are simply those of the original multiplied by the same constant. Indeed, multiplication by $\frac{-i}{\sqrt{3}}$ is a scaling by $\frac{1}{\sqrt{3}}$ in tandem with a clockwise rotation of 90° about the origin, which also immediately settles (3), (4).

The proof can be easily modified to show that the theorem also holds for inwardly constructed triangles.

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103.26 A proof of Clarke's conjecture

Introduction

In 2001, the late Robert J. Clarke [1] proposed the problem: Show that, for $n > m \geqslant 1$, the following identity holds:

$$\frac{(-1)^{mn}}{F_{n+m}}\prod_{r=1}^{2m}\frac{F_r}{F_{n+m-r}} = \frac{1}{F_{n+m}} + \sqrt{2}\sum_{r=1}^{2m}\left(\frac{\sin\left[2(r+1)+(-1)^m\right]\frac{\pi}{4}}{F_{n+m-r}}\prod_{s=1}^{r}\frac{F_{2m+1-s}}{F_s}\right), \quad (1)$$

where F_n is the nth Fibonacci number.

In this paper, our main goal is to give a solution for Clarke's conjecture. Our method is to rewrite the conjecture in terms of the Fibonomial coefficients. Further we will find a q-form of the conjecture satisfied by any real number q. Further we will give a proof for this q-form. Moreover, we will also provide a general case of the conjecture by choosing a particular value of q. Finally we will present a Lucas counterpart of the conjecture.

The Gibonacci polynomial sequence

For $n \geqslant 2$, define the second order linear recursive sequence $\{g_n(x)\}$ by

$$g_n(x) = a(x)\,g_{n-1}(x) + b(x)\,g_{n-2}(x),$$

where the initial polynomials are $g_0(x)$ and $g_1(x)$, and, the coefficients $a(x)$ and $b(x)$ are arbitrary polynomials. The sequence $\{g_n(x)\}$ is called the *gibonacci polynomial* sequence.

Suppose $a(x) = x$ and $b(x) = 1$. When $g_0(x) = 0$ and $g_1(x) = 1$, the sequence $\{g_n(x)\}$ is the *Fibonacci polynomial* sequence $\{f_n(x)\}$, and when $g_0(x) = 2$ and $g_1(x) = x$, the sequence $\{g_n(x)\}$ is the *Lucas polynomial* sequence $\{l_n(x)\}$. In particular, for a non-zero integer p, $f_n(p) = U_n(p)$ and $l_n(p) = V_n(p)$, the nth generalised Fibonacci and Lucas numbers, respectively. (For details, we refer to [2, 3].)

For the special case $p = 1$, the sequences $\{U_n(p)\}$ and $\{V_n(p)\}$ are reduced to the usual Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$, respectively.

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The nth *Pell polynomial* $p_n(x)$ and the nth *Pell-Lucas polynomial* $q_n(x)$ are given by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. Correspondingly, for $n \geqslant 0$, the nth Pell number P_n and the nth Pell-Lucas number Q_n are given by $p_n(1) = P_n$ and $q_n(1) = 2Q_n$, respectively. (For more details, see [4].)

The Binet formulas of the sequences $\{f_n(x)\}$ and $\{l_n(x)\}$ are

$$f_n(x) = \frac{\alpha(x)^n - \beta(x)^n}{\alpha(x) - \beta(x)} \text{ and } l_n(x) = \alpha(x)^n + \beta(x)^n,$$

where $\alpha(x) = \frac{1}{2}\left(x + \sqrt{x^2+4}\right)$ and $\beta(x) = \frac{1}{2}\left(x - \sqrt{x^2+4}\right)$.

In particular, when $x = p$, the Binet formulas of the generalised Fibonacci and Lucas sequences are

$$U_n(p) = \frac{\alpha(p)^n - \beta(p)^n}{\alpha(p) - \beta(p)} \text{ and } V_n(p) = \alpha(p)^n + \beta(p)^n,$$

where $\alpha(p)$ and $\beta(p)$ are defined as before. For brevity, we denote $\alpha(1)$ and $\beta(1)$ by α and β, respectively.

The generalised Fibonomial and Gaussian q-binomial coefficients

For integers n and k such that $0 \leqslant k \leqslant n$, the generalised Fibonomial coefficients are defined by

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_U = \frac{U_1(p)\,U_2(p)\ldots U_n(p)}{\left(U_1(p)\,U_2(p)\ldots U_k(p)\right)\left(U_1(p)\,U_2(p)\ldots U_{n-k}(p)\right)}$$

with $\begin{Bmatrix} n \\ n \end{Bmatrix}_U = \begin{Bmatrix} n \\ 0 \end{Bmatrix}_U = 1$ and 0 for $k < 0$ or $k > n$. (For details, see [5, 6]). This definition may also be generalised to one using the Gibonacci polynomial sequence $\{g_n(x)\}$.

If $U_n(p) = F_n$ (that is, the case $p = 1$), then the generalised Fibonomial coefficients $\begin{Bmatrix} n \\ k \end{Bmatrix}_U$ are reduced to the usual Fibonomial coefficients $\begin{Bmatrix} n \\ k \end{Bmatrix}_F$.

The Gaussian q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;\,q)_n}{(q;\,q)_k\,(q;\,q)_{n-k}},$$

where q is any real number and $(x;\,q)_n$ is the q-Pochhammer symbol, $(x;q)_n = (1-x)(1-xq)\ldots(1-xq^{n-1})$ with $(x;\,q)_0 = 1$. (For more details, we refer to [7, 8].)

We rewrite the Binet formulas as

$$U_n(p) = \frac{\alpha(p)^n - \beta(p)^n}{\alpha(p) - \beta(p)} = \alpha(p)^{n-1}\frac{1-q^n}{1-q} \tag{2}$$

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and

$$V_n(p) = \alpha(p)^n + \beta(p)^n = \alpha(p)^n\left(1 + q^n\right), \tag{3}$$

where $q = \dfrac{\beta(p)}{\alpha(p)} = -\alpha^2(p)$ or equivalently $\alpha(p) = iq^{-1/2}$, where $i = \sqrt{-1}$. The equations (2) and (3) are called the q-forms of the Binet formulas of $\{U_n(p), V_n(p)\}$, respectively.

The relationship between the generalised Fibonomial and Gaussian q-binomial coefficients is

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_U = \alpha^{k(n-k)}(p)\begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{4}$$

A proof of Clarke's conjecture

Firstly, in order to rewrite the right-hand side of the conjecture given in (1) with a simple sign function rather than a trigonometric expression, we give the following result.

Lemma 1: For integers r and m,

$$\sqrt{2}\left(\sin\left[2(r+1) + (-1)^m\right]\frac{\pi}{4}\right) = (-1)^{r/2}\begin{cases} i(-1)^m & \text{if } r \text{ is odd,} \\ 1 & \text{if } r \text{ is even.} \end{cases}$$

Proof: Consider

$$\begin{aligned} &\sqrt{2}\left(\sin\left[2(r+1) + (-1)^m\right]\frac{\pi}{4}\right) \\ &= \sqrt{2}\left[\sin\left((r+1)\frac{\pi}{2}\right)\cos\left((-1)^m\frac{\pi}{4}\right) + \cos\left((r+1)\frac{\pi}{2}\right)\sin\left((-1)^m\frac{\pi}{4}\right)\right] \\ &= (-1)^{r/2}\begin{cases} i(-1)^m & \text{if } r \text{ is odd,} \\ 1 & \text{if } r \text{ is even,} \end{cases} \end{aligned}$$

as claimed.

By Lemma 1 and the definition of the Fibonomial coefficients, the conjecture (1) takes the form

$$\frac{(-1)^{mn}}{F_{n+m}}\begin{Bmatrix} n+m-1 \\ 2m \end{Bmatrix}_F^{-1} = \sum_{r=0}^{2m}\begin{Bmatrix} 2m \\ r \end{Bmatrix}_F (-1)^{\binom{r+1}{2}+mr}\frac{1}{F_{n+m-r}}. \tag{5}$$

Firstly, we will state the q-form of the conjecture given in (5). By the relations (2) and (4), the right-hand side of (5) equals

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$$\sum_{r=0}^{2m}(-1)^{\binom{r+1}{2}+mr}\frac{1-q}{\alpha^{n+m-r-1}(1-q^{n+m-r})}\alpha^{r(2m-r)}\begin{bmatrix}2m\\ r\end{bmatrix}_q,$$

which, by $\alpha = iq^{-1/2}$, equals

$$-(1-q)q^{(-m-n-1)/2}i^{1-m-n}\sum_{r=0}^{2m}\begin{bmatrix}2m\\ r\end{bmatrix}_q(-1)^r q^{\binom{r+1}{2}-mr}\frac{1}{(1-q^{r-m-n})}.$$

Similarly, by $\alpha = iq^{-1/2}$, the left-hand side of (5) equals

$$\frac{(-1)^{mn}(1-q)}{\alpha^{n+m-1}(1-q^{n+m})}\alpha^{2m(m+1-n)}\begin{bmatrix}n+m-1\\ 2m\end{bmatrix}_q^{-1}$$

$$=\frac{1-q}{1-q^{n+m}}\begin{bmatrix}n+m-1\\ 2m\end{bmatrix}_q^{-1}i^{1-n-m}q^{(m+n-1)/2}q^{m(n-m-1)}.$$

By equating the q-forms of both sides, the q-form of the conjecture given in (5) is

$$\sum_{r=0}^{2m}\begin{bmatrix}2m\\ r\end{bmatrix}_q(-1)^r q^{\binom{r+1}{2}-mr}\frac{1}{1-q^{r-m-n}}=-q^{mn+n-m^2}\frac{1}{1-q^{n+m}}\begin{bmatrix}n+m-1\\ 2m\end{bmatrix}_q^{-1}. \quad (6)$$

In order to prove Clarke's conjecture, we will prove (6) instead of (5) for any real q. When $q=\dfrac{1-\sqrt{5}}{1+\sqrt{5}}$, the equality (6) is reduced to the conjecture given in (5).

We recall an auxiliary result from [9] before giving the proof of (6).

Lemma 2: For any real number q, non-zero real number a, non-negative integer n and any integer t such that $t \geqslant -n$, the following sum identity holds:

$$\sum_{r=0}^{n}\begin{bmatrix}n\\ r\end{bmatrix}_q(-1)^r q^{\binom{r+1}{2}+tr}\frac{1}{1-aq^r}=a^{-t}\frac{(q;q)_n}{(a;q)_{n+1}}-\sum_{j=0}^{t-1}\begin{bmatrix}n+j\\ n\end{bmatrix}_q a^j. \quad (7)$$

Now we are going to move to the proof of (6) for any real q. For this, it is enough to take $t=-m$, $n=2m$ and $a=q^{-m-n}$ in (7). So the left-hand side of (7) is

$$\sum_{r=0}^{2m}\begin{bmatrix}2m\\ r\end{bmatrix}_q(-1)^r q^{\binom{r+1}{2}-mr}\frac{1}{(1-q^{r-m-n})}, \quad (8)$$

which, by Lemma 2, equals

$$q^{-m^2-mn}\frac{(q;q)_{2m}}{(q^{-m-n};q)_{2m+1}},$$

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which, by the definition of the q-Pochhammer symbol, equals

$$q^{-m^2-mn}\frac{(q;\,q)_{2m}}{(1-q^{-m-n})(1-q^{-m-n+1})\ldots(1-q^{m-n})}.$$

Since $(1-q^{-m-n+k}) = -q^{-m-n+k}(1-q^{m+n-k})$ for $0 \leqslant k \leqslant 2m$, the last statement equals

$$-q^{-m^2+nm+n}\frac{(q;\,q)_{2m}}{(1-q^{m+n})(1-q^{m+n-1})\ldots(1-q^{n-m})},$$

which, by multiplying both the numerator and denominator by $(q;\,q)_{n-m-1}$, equals

$$-q^{-m^2+mn+n}\frac{(q;\,q)_{2m}\,(q;\,q)_{n-m-1}}{(q;\,q)_{n+m}}.$$

In terms of the Gaussian q-binomial coefficients, the last statement equals

$$-\frac{1}{1-q^{n+m}}\,q^{mn+n-m^2}\begin{bmatrix} n+m-1 \\ 2m \end{bmatrix}_q^{-1}.$$

Consequently, we get

$$\sum_{r=0}^{2m}\begin{bmatrix} 2m \\ r \end{bmatrix}_q (-1)^r q^{\binom{r+1}{2}-mr}\frac{1}{1-q^{r-m-n}} = -q^{mn+n-m^2}\frac{1}{1-q^{n+m}}\begin{bmatrix} n+m-1 \\ 2m \end{bmatrix}_q^{-1},$$

which completes the proof.

As mentioned before, when $q = \dfrac{1-\sqrt{5}}{1+\sqrt{5}}$ in (6), we obtain a proof of Clarke's conjecture.

Some further results

One may obtain similar or more general sum identities by choosing particular values of q in (6). For example, by taking $q = \dfrac{\beta(p)}{\alpha(p)}$ for a non-zero integer p, we obtain a generalisation of Clarke's conjecture including the generalised Fibonacci sequence $\{U_n\}$ in the form

$$\frac{(-1)^{mn}}{U_{n+m}(p)}\begin{Bmatrix} n+m-1 \\ 2m \end{Bmatrix}_U^{-1} = \sum_{r=0}^{2m}\begin{Bmatrix} 2m \\ r \end{Bmatrix}_U (-1)^{\binom{r+1}{2}+mr}\frac{1}{U_{n+m-r}(p)}. \quad (9)$$

In particular, if we choose $p = 2$, then (9) gives

$$\frac{(-1)^{mn}}{P_{n+m}}\begin{Bmatrix} n+m-1 \\ 2m \end{Bmatrix}_P^{-1} = \sum_{r=0}^{2m}\begin{Bmatrix} 2m \\ r \end{Bmatrix}_P (-1)^{\binom{r+1}{2}+mr}\frac{1}{P_{n+m-r}}, \quad (10)$$

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where P_n and $\left\{ {n \atop k} \right\}_P$ stand for the nth Pell number and the Pellnomial coefficients, respectively.

A Lucas counterpart of Clarke's Conjecture

Our experiments suggest that a Lucas counterpart of Clarke's conjecture is given for $n > m$ by

$$(-1)^{m(n+1)} 5^m \frac{F_1 F_2 \dots F_{2m}}{L_{n-m} L_{n-m+1} \dots L_{n+m}} = \sum_{r=0}^{2m} \left\{ {2m \atop r} \right\}_F (-1)^{\binom{r+1}{2}+mr} \frac{1}{L_{n+m+r}}, \quad (11)$$

where F_n and L_n stand for the nth Fibonacci and Lucas numbers, respectively.

In order to prove this equality, we use the same approach. By using the relations (2, 3, 4), one could convert the Lucas counterpart of the conjecture into its q-form. Here we omit the details. Now we prove the q-form of (11) with the following result.

Lemma 3: For $n > m$,

$$\sum_{r=0}^{2m} \begin{bmatrix} 2m \\ r \end{bmatrix}_q (-1)^r q^{\binom{r+1}{2}-mr} \frac{1}{1+q^{r-m-n}} \quad (12)$$

$$= (-1)^m q^{-m^2+mn+n} \frac{(q;q)_{2m}}{(-q^{n-m};q)_{2m+1}}.$$

Proof: In order to prove the claim, we take $t = -m$, $n = 2m$ and $a = -q^{-m-n}$ in (7). So the left-hand side of (7) takes the form

$$\sum_{r=0}^{2m} \begin{bmatrix} 2m \\ r \end{bmatrix}_q (-1)^r q^{\binom{r+1}{2}-mr} \frac{1}{1+q^{r-m-n}},$$

which, by Lemma 2, equals

$$(-1)^m q^{-m^2-mn} \frac{(q;q)_{2m}}{(-q^{-n-m};q)_{2m+1}},$$

which, by $(1+q^{-n-m+k}) = q^{-n-m+k}(1+q^{n+m-k})$ for $0 \leqslant k \leqslant 2m$, equals

$$(-1)^m q^{-m^2+mn+n} \frac{(q;q)_{2m}}{(-q^{n-m};q)_{2m+1}}. \quad (13)$$

Thus we have the proof.

When $q = \frac{\beta}{\alpha} = \frac{1-\sqrt{5}}{1+\sqrt{5}}$, equation (12) gives us the Lucas counterpart of Clarke's conjecture (11).

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When $q = \dfrac{\beta(p)}{\alpha(p)}$, the general Lucas counterpart of Clarke's conjecture takes the form

$$(-1)^{m(n+1)}\left(p^2+4\right)^m \frac{U_1(p)\dots U_{2m}(p)}{V_{n-m}(p)V_{n-m+1}(p)\dots V_{n+m}(p)} = \sum_{r=0}^{2m}(-1)^{r(r+1)/2+mr}\begin{Bmatrix}2m\\ r\end{Bmatrix}_U \frac{1}{V_{n+m-r}(p)}.$$

A concluding remark

One could also derive generalisations of Clarke's conjecture for the sequences $\{f_n(x)\}$ and $\{l_n(x)\}$ by taking particular values of q in (6) and (12), respectively, but we omit them here.

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