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On the irregular points for systems with the shadowing property

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Abstract. We prove that when f is a continuous self-map acting on a compact metric space (X, d) that satisfies the shadowing property, then the set of irregular points (i.e., points with divergent Birkhoff averages) has full entropy. Using this fact, we prove that, in the class of C^0-generic maps on manifolds, we can only observe (in the sense of Lebesgue measure) points with convergent Birkhoff averages. In particular, the time average of atomic measures along orbits of such points converges to some Sinai–Ruelle–Bowen-like measure in the weak* topology. Moreover, such points carry zero entropy. In contrast, irregular points are non-observable but carry infinite entropy.

1. *Introduction*

Consider a continuous self-map f acting on a compact metric space (X, d). For a continuous function $\varphi : X \to \mathbb{R}$, every point $x \in X$ and every positive integer n, we can calculate *the Birkhoff average* $(1/n) \sum_{i=0}^{n-1} \varphi(f^i(x))$. According to the classical Birkhoff ergodic theorem, such an average converges for all x from a set carrying full measure for every f-invariant measure. We call such points *φ-regular points*. The complementary set $I_\varphi(f)$ of points where the Birkhoff average diverges is called the *φ-irregular set* and the union $I(f) := \bigcup_{\varphi \in C(X,\mathbb{R})} I_\varphi(f)$ is called the *irregular set*.

From the viewpoint of ergodic theory, the irregular sets $I_\varphi(f)$ and $I(f)$ are too small to be considered. However, if we change the measurement method, they may be quite large. To be concrete, let us recall some related results motivated by topological dynamics.

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Fan *et al* showed, in [**18**], that, for mixing subshifts of finite type (i.e., subshifts with the shadowing property [**43**]), the set $I_\varphi(f)$ is either empty or has full entropy. Later on, Barreira and Schmeling [**4**] studied a broad class of systems including conformal repellers and horseshoes and showed that the irregular set carries full entropy and full Hausdorff dimension. In addition, Olsen [**32–34**] introduced and developed a multifractal framework via deformations of empirical measures. Based on these results, Chen *et al* [**14**] considered irregular points for systems with the specification property. They used a method inspired by the results of Takens and Verbitskiy from [**40**] and showed that $I(f)$ has full topological entropy. Apart from dimensional theory, Li and Wu [**28**] proved that, for systems with the specification property, $I_\varphi(f)$ is either empty or residual, which shows that $I_\varphi(f)$ and $I(f)$ are large from a topological point of view (cf. results in [**15**] on points with maximal oscillation). In [**17**], the authors provide examples with topological mixing, dense periodic points, countably many ergodic measures and an empty irregular set (every point is generic for some ergodic measure).

All dynamical systems in the above mentioned results satisfy the specification property which was first introduced by Bowen to study Axiom A diffeomorphisms [**6**]. It is natural to ask what the situation is beyond the case of specification. More precisely, for which kinds of systems and to what extent can similar conclusions hold? Recently, Thompson achieved some results in this direction. He considered systems with the almost specification property, which is a natural weakening of the specification property (see [**37**]), and he proved, in [**41**], that $I_\varphi(f)$ is either empty or carries full entropy.

In this paper, we focus on another important property which motivated Bowen to define the specification property. Strictly speaking, we consider systems with the shadowing property, providing topological and measure-theoretic characterizations of $I(f)$ in their context.

Before proceeding, we should mention that neither of the properties of almost specification nor shadowing implies the other. This can be seen from the following two examples.

Example 1.1. Consider a homeomorphism $f : [0, 1] \to [0, 1]$ such that 0, 1 are the only fixed points of f. Then it is not hard to check that f has the shadowing property (e.g., see [**13**]). It is clear that the only invariant measures of f are supported on fixed points 0, 1 and one of them is an attractor. This immediately implies that $I(f) = \emptyset$ and one easily sees that $((\delta_0 + \delta_1)/2) \in \mathcal{M}_f([0, 1])$ cannot be approximated by ergodic measures. Thus $([0, 1], f)$ does not satisfy the almost specification property since, for such systems, the ergodic measures are entropy dense [**36**].

Example 1.2. It is well known that every β-shift has the almost specification property [**36**], while the set of β for which the β-shift has the specification property has zero Lebesgue measure [**10**, **39**]. So let us consider some β-shift $\sigma : X_\beta \to X_\beta$ that satisfies the almost specification property but is without the specification property. According to [**24**, Theorem 3.8], when $f : X \to X$ is a continuous map with the shadowing property, the following two statement are equivalent.

- f is surjective and has the almost specification property.
- f is surjective and has the specification property.

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The β-shift is obviously surjective by definition [**42**, p. 179]. Hence (X_β, σ) may not have the shadowing property because then it would have the specification property, which is in contradiction to its selection.

Remark 1.3. It is proved in [**24**, Theorem 3.8] that totally transitive (i.e., (X, f^n) transitive for every $n \geq 1$) maps with the shadowing property have the specification property. In particular, Thompson's results from [**41**] apply to this case.

Some new obstacles appear compared with Thompson's result. For example, there is no hope for transitivity in general, as was the case in Example 1.1. But even the transitive but not mixing case is problematic, while it seems that the relative almost specification property, as defined in [**26**], may still be sufficient to obtain Thompson's results.

Example 1.4. Let (X, f) be the dyadic adding machine (e.g., see [**25**] or [**16**] for definition and basic properties) and let (Y, g) be any mixing map with the shadowing property (e.g., full shift on 2-symbols). Note that every equicontinuous dynamical system on the Cantor set has the shadowing property [**25**], so (X, f) has the shadowing property. Then the dynamical system $(X \times Y, f \times g)$ is transitive and has the shadowing property. But it does not have almost specification, since it is not mixing [**24**].

To deal with cases beyond the specification property, in our approach, we will focus on some ergodic measure μ which captures enough information of the entropy (by the variational principle) and we will manipulate its points by the shadowing property. The main result of the paper is the following.

THEOREM 1.5. *If (X, f) has the shadowing property, then exactly one of the following conditions holds.*

(1) $h_{\text{top}}(f) > 0$, $I(f) \neq \emptyset$ *and has full entropy.*
(2) $h_{\text{top}}(f) = 0$ *and* $I(f) = \emptyset$.

The above result will be the main tool in further studies on generic dynamical systems on some compact manifolds. Before presenting our second result, we first need some notions and notation.

Throughout this paper, M denotes a compact connected manifold admitting a decomposition in the following sense.

Definition 1.6. A finite family $\mathcal{S}$ of pairwise disjoint open subsets of M is called a *decomposition* of M if $M = \bigcup_{U \in \mathcal{S}} \overline{U}$ and each $\overline{U}$ is homeomorphic to a closed ball in $\mathbb{R}^k$. If there exists a decomposition of M, then we say that M *admits a decomposition*.

The class of manifolds admitting a decomposition includes all triangulable manifolds and manifolds with a handlebody. Therefore, the considered class contains all compact manifolds of dimension at most three (see [**5**, **30**]), of dimension at least six (see [**20**]) and all smooth manifolds (see [**11**, **44**]).

Let $C(M)$ be the space of continuous maps on M and let $H(M)$ be the space of homeomorphisms on M. We consider these spaces with C^0-topology: that is, we endow $C(M)$ with the metric

$$d_C(f, g) = \sup_{x \in M} d(f(x), g(x))$$

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and $H(M)$ with the metric

$$d_H(f, g) = d_C(f, g) + d_C(f^{-1}, g^{-1}).$$

It is well known that both spaces $(C(M), d_C)$ and $(H(M), d_H)$ are complete.

A subset $\mathcal{R}$ of a metric space X is *residual* if it contains a countable intersection of dense open sets. A property $\mathcal{P}$ is called *generic* in the space X if there is a residual subset $\mathcal{R}$ of X such that any $f \in \mathcal{R}$ satisfies $\mathcal{P}$. Recent results show that in the class of manifolds with a decomposition, the shadowing property is generic (e.g., see [**23**, **38**]).

Let us denote by

$$\mathcal{E}_n(x) := \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}$$

the nth empirical measure of $x \in X$, where δ_y is the Dirac measure at $y \in X$. The points x such that $\mathcal{E}_n(x)$ converges in weak* topology are called *quasi-regular points* (cf. [**15**] or [**42**]) and the set of all quasi-regular points is denoted by $Q(f)$.

For $x \in X$, let $p\omega(x)$ denote the set of all limit points of the sequence $\mathcal{E}_n(x)$ in $\mathcal{M}(X)$. A probability measure $\mu \in \mathcal{M}_f(M)$ is *Sinai–Ruelle–Bowen* (*SRB*) (or *physical*) if the set

$$B(\mu) = \{x \in M : p\omega(x) = \{\mu\}\}$$

has positive Lebesgue measure and is *SRB-like* (alternatively, *observable* or *pseudo-physical*) if, for any $\varepsilon > 0$, the set

$$A_\varepsilon(\mu) = \{x \in M : \rho(p\omega(x), \mu) < \varepsilon\}$$

has positive Lebesgue measure.

Abdenur and Andersson showed in [**1**] that on manifolds with $\dim M \geq 2$, for C^0 generic maps $f : M \to M$ (the same for homeomorphisms), Lebesgue almost every $x \in M$ is quasi-regular, but f admits no SRB measure.

However, for any topological dynamical system $f : M \to M$, the SRB-like measure always exists [**12**]. Moreover, let $\mathcal{O}_f$ denote the set of all SRB-like measures for f. Define the basin Δ of attraction of $\mathcal{O}_f$ by putting

$$\Delta = \{x : p\omega(x) \subseteq \mathcal{O}_f\}. \tag{1.1}$$

It was proved in [**12**, Theorem 1.5] that $\mathcal{O}_f$ is the smallest weak* compact subset of $\mathcal{M}(M)$ whose basin of attraction Δ satisfies $\mathscr{L}(\Delta) = 1$, where $\mathscr{L}$ denotes the Lebesgue measure on M. Motivated by [**12**] and with Theorem 1.5 to hand we prove the following theorem.

THEOREM 1.7. *For generic $f \in C(M)$ (the same for generic $f \in H(M)$) with* $\dim M \geq 2$, *the following conditions hold.*

(1) *There is a set $\Lambda \subset M$ with $\mathscr{L}(\Lambda) = 1$ such that:*
   (i) $\Lambda \subset Q(f) \cap \Delta$;
   (ii) $h_{\text{top}}(f, \Lambda) = 0$; *and*

(2) $h_{\text{top}}(f, I(f)) = \infty$.

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2. *Preliminaries*

2.1. *Notions and notations.* Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{N}_0$ the set of natural numbers, integers, and non-negative integers, respectively. A *dynamical system* is a pair (X, f) consisting of a compact metric space (X, d) and a continuous map $f : X \to X$. If, for every pair of non-empty open sets U, V, there is an integer n such that $f^n(U) \cap V \neq \emptyset$, then (X, f) is *transitive*. When, for every pair of non-empty open sets U, V, there exists an integer N such that $f^n(U) \cap V \neq \emptyset$ for every $n > N$, then (X, f) is *topologically mixing*.

We denote the diameter of any subset $A \subset X$ by

$$\operatorname{diam}(A) = \sup_{x,y\in A} \{d(x, y)\}.$$

An open ball, centered at $x \in X$ and with radius $r > 0$, is denoted by $B(x, r)$. For any $n \in \mathbb{N}$, the d_n-distance between $x, y \in X$ is defined as

$$d_n(x, y) := \max_{0\le i\le n-1} \{d(f^i(x), f^i(y))\}.$$

The (n, ε)-Bowen ball centered at x is

$$B_n(x, \varepsilon) := \{y \in X : d_n(x, y) < \varepsilon\}.$$

2.2. *Invariant measures.* We denote by $\mathcal{M}(X)$ the set of Borel probability measures on X. Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be a dense subset of $C(X, \mathbb{R})$. Then

$$\rho(\xi, \tau) = \sum_{j=1}^{\infty} \frac{|\int \varphi_j \, d\xi - \int \varphi_j \, d\tau|}{2^j \|\varphi_j\|}$$

defines a metric on $\mathcal{M}(X)$ for the $weak^*$ topology [**42**], where

$$\|\varphi_i\| = \max\{|\varphi_i(x)| : x \in X\}.$$

Note that

$$\rho(\xi, \tau) \le 2 \quad \text{for any } \xi, \tau \in \mathcal{M}(X). \tag{2.2}$$

Note also that the natural embedding $j : x \mapsto \delta_x$ is continuous and injective. Moreover, (X, f) is compact and $\mathcal{M}(X)$ is Hausdorff, so X is homeomorphic to its image $j(X)$. Thus, without loss of generality, we can choose the following equivalent metric d on X by putting $d(x, y) = \rho(\delta_x, \delta_y)$. This, combined with (2.2), gives the following lemma.

LEMMA 2.1. *Let (X, f) be a dynamical system and let $x \in X$.*

(1) *Let $0 \le k < m < n$ and $x \in X$. Then*

$$\rho(\mathcal{E}_m(x), \mathcal{E}_n(f^k(x))) \le \frac{2}{n}(n - m + k).$$

(2) *Given $\varepsilon > 0$ and $p \in \mathbb{N}$, for every $y \in B_p(x, \varepsilon)$, $\rho(\mathcal{E}_p(y), \mathcal{E}_p(x)) < \varepsilon$.*

(3) *Given $\varepsilon > 0$ and $p, q \in \mathbb{N}$ satisfying $p \le q \le (1 + \varepsilon/2)p$, for every $y \in B_p(x, \varepsilon)$, $\rho(\mathcal{E}_q(y), \mathcal{E}_p(x)) < 2\varepsilon$.*

We say that $\mu \in \mathcal{M}(X)$ is f-invariant if, for any measurable set $E \subset X$,

$$\mu(E) = \mu(f^{-1}(E)),$$

that is, μ is a fixed point of T_f, where $T_f : \mathcal{M}(X) \to \mathcal{M}(X)$ is defined as $T_f(\mu) = \mu \circ f^{-1}$. The set of f-invariant measures in $\mathcal{M}(X)$ is denoted by $\mathcal{M}_f(X)$. A dynamical system (X, f) is *uniquely ergodic* when $\mathcal{M}_f(X)$ is a singleton.

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2.3. *Shadowing property.* A sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called a *δ-pseudo-orbit* of f if

$$d(f(x_n), x_{n+1}) < \delta \quad \text{for any } n \in \mathbb{N}.$$

A δ-pseudo-orbit $\{x_n\}_{n\in\mathbb{N}}$ is *ε-shadowed* by an orbit of $y \in X$ if

$$d(f^n(y), x_n) < \varepsilon \quad \text{for any } n \in \mathbb{N}.$$

We say that $f : X \to X$ has the *shadowing property* if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that any δ-pseudo-orbit is ε-shadowed by some orbit.

2.4. *Topological entropy.* The topological entropy of dynamical systems (X, f) was first defined by Adler *et al* in [**2**] using open covers. Later on, Bowen introduced separated and spanning sets to reformulate this notion [**7**].

Let $Z \subset M$. A set S is (n, ε)-separated for Z if $S \subset Z$ and $d_n(x, y) > \varepsilon$ for any distinct points $x, y \in S$. A set $S \subset Z$ is (n, ε)-spanning for Z if, for any $x \in Z$, there exists $y \in S$ such that $d_n(x, y) \le \varepsilon$.

Define

$$s_n(Z, \varepsilon) = \sup\{|S| : S \text{ is } (n, \varepsilon)\text{-separated for } Z\},$$
$$r_n(Z, \varepsilon) = \inf\{|S| : S \text{ is } (n, \varepsilon)\text{-spanning for } Z\},$$

where $|S|$ denotes the cardinality of S. It is not hard to check that (e.g., see [**42**]):

$$r_n(Z, \varepsilon) \le s_n(Z, \varepsilon) \le r_n(Z, \varepsilon/2). \tag{2.3}$$

Following Bowen, we define *topological entropy* for a compact set $K \subset X$ by

$$h_d(f, K) = \lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log s_n(K, \varepsilon) = \lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log r_n(K, \varepsilon).$$

Bowen also introduced topological entropy for non-compact sets using notation similar to the Hausdorff dimension [**8**]. Let $E \subset X$ and let $\mathcal{G}_n(E, \varepsilon)$ be the collection of all finite or countable covers of E by sets of the form $B_u(x, \varepsilon)$ with $u \ge n$. Put

$$C(E; t, n, \varepsilon, f) := \inf_{\mathcal{C}\in\mathcal{G}_n(E,\varepsilon)} \sum_{B_u(x,\varepsilon)\in\mathcal{C}} e^{-tu}$$

and

$$C(E; t, \varepsilon, f) := \lim_{n\to\infty} C(E; t, n, \varepsilon, f).$$

Then we define

$$h_{\text{top}}(E; \varepsilon, f) := \inf\{t : C(E; t, \varepsilon, f) = 0\} = \sup\{t : C(E; t, \varepsilon, f) = \infty\}.$$

The *Bowen topological entropy* of E is

$$h_{\text{top}}(f, E) := \lim_{\varepsilon\to 0^+} h_{\text{top}}(E; \varepsilon, f). \tag{2.4}$$

It is known that the value of h_d is independent of the choice of metric d (it depends only on the topology on X) and, for every compact f-invariant subset $E \subset X$, $h_{\text{top}}(f, E) = h_d(f, E)$ (see [**35**]).

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2.5. *Symbolic dynamics.* Fix any positive integer $k \geq 2$ and consider the following set $\Sigma_k^+ = \{0, 1, \ldots, k-1\}^{\mathbb{N}_0}$ with the product topology induced by the discrete topology on $\{0, 1, \ldots, k-1\}$. The space Σ_k^+ is always endowed with the *shift map* σ defined by $\sigma(x)_i = x_{i+1}$ for every integer $i \geq 0$. It is not hard to verify that σ is continuous and that Σ_k^+ is a compact metrizable space. We endow it with the (compatible) metric defined by $d(x, y) = 2^{-k}$ when $x \neq y$ and $k = \min\{i : x_i \neq y_i\}$. The dynamical system (Σ_k^+, σ), or simply Σ_k^+ for short, is called a *full shift*. By *subshift* or *shift* we mean any compact and σ-invariant subset of Σ_k^+.

For $i \leq j$ and any $x \in \Sigma_k^+$ we write $x_{[i,j]} = x_i \cdots x_j$ and, similarly, $x_{[i,j)} = x_{[i,j-1]}$. For any $n > 0$ and any subshift X, we denote

$$B_n(X) = \{x_{[i,i+n)} : x \in X, i \geq 0\}.$$

It is well known that, in the case of subshifts, the formula for topological entropy reduces to

$$h_{\text{top}}(\sigma, X) = \lim_{n\to\infty} \frac{1}{n} \log |B_n|.$$

The reader not familiar with symbolic dynamics is referred to [**29**] for more details.

3. *Some auxiliary lemmas*

We start this section with the following simple observation, relating quasi-regular points to the irregular set.

LEMMA 3.1. *Let (X, f) be a dynamical system. Then $Q(f) = X \setminus I(f)$.*

Proof. Let $x \in Q(f)$ be a quasi-regular point for a measure $\mu \in \mathcal{M}(X)$. Observe that $\lim_{n\to\infty} \mathcal{E}_n(x) = \mu$ in a weak* topology if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \lim_{n\to\infty} \int \varphi \, d\mathcal{E}_n(x) = \int \varphi \, d\mu$$

for every $\varphi \in C(X, \mathbb{R})$, which means that $x \notin I(f)$. On the other hand, if $x \notin I(f)$, then, for every $\varphi \in C(X, \mathbb{R})$, the number $L_f(\varphi) = \lim_{n\to\infty} (1/n) \sum_{i=0}^{n-1} \varphi(f^i(x))$ is well defined. Clearly, $L : C(X, \mathbb{R}) \to \mathbb{R}$ is a positive operator, $L(1) = 1$ and $L(\varphi - \psi) \leq \sup_{x\in X} |\varphi(x) - \psi(x)|$, which shows that L is continuous. Then, by the Riesz representation theorem, there is $\mu \in M(X)$ such that $L(\varphi) = \int \varphi \, d\mu$ and so $x \in Q(f)$. □

In our constructions, we will need the following combinatorial lemma which allows us to construct large separated sets.

LEMMA 3.2. *Let (X, d) be a compact metric space and let measure $\mu \in \mathcal{M}_f(X)$ be ergodic with $h_\mu(f) > 0$. Then, for any $\eta > 0$, there exists $\varepsilon > 0$ such that, for each neighborhood F of μ in $\mathcal{M}(M)$, there exists $n_F \in \mathbb{N}$ such that, for any $n \geq n_F$, there exists $\Gamma_n \subset X_{n,F} \cap \text{supp}(\mu)$ that is (n, ε)-separated and satisfies $|\Gamma_n| \geq e^{n(h_\mu(f)-\eta)}$, where $X_{n,F} := \{x \in X : \mathcal{E}_n(x) \in F\}$.*

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Proof. Fix any ergodic measure $\mu \in \mathcal{M}_f(X)$. For any choice of $n \geq 0$ and $\varepsilon, \delta > 0$, let $N_f(n, \varepsilon, \delta)$ denote the minimal number of (n, ε)-Bowen balls which cover the set of μ-measure at least $1 - \delta$. Using the above notation, Katok proved, in [**19**, Theorem 1.1], the following generalization of Bowen's formula for topological entropy: that is,

$$h_\mu(f) = \lim_{\varepsilon \to 0^+} \liminf_{n \to \infty} \frac{\ln N_f(n, \varepsilon, \delta)}{n}. \tag{3.5}$$

Fix any $\eta > 0$ and let $\varepsilon > 0$ be such that

$$h_\mu(f) - \eta < \liminf_{n \to \infty} \frac{\ln N_f(n, \varepsilon, \delta)}{n}. \tag{3.6}$$

Fix any open neighborhood F of μ in $\mathcal{M}(X)$. Since μ is ergodic, the set G_μ of points generic for μ has full μ-measure and, by definition, for every generic point x, $\lim_{n\to\infty} \mathcal{E}_n(x) = \mu$. Note that $G_\mu \subset \bigcup_n D_n$, where

$$D_n = \{x \in G_\mu : \mathcal{E}_k(x) \in F \text{ for all } k \geq n\}.$$

Note that $D_n \subset D_{n+1}$ and $D_n \subset X_{n,F}$, and hence there exists $N_1 \in \mathbb{N}$ such that $\mu(X_{n,F} \cap \operatorname{supp}(\mu)) \geq 1 - \delta$ for every $n > N_1$.

By (3.6), there exists $n_F \geq N_1$ such that, for any $n \geq n_F$,

$$N_f(n, \varepsilon, \delta) \geq e^{n(h_\mu(f)-\eta)}. \tag{3.7}$$

Denote $Z = X_{n,F} \cap \operatorname{supp}(\mu)$ and observe that (3.7) implies that

$$r_n(Z, \varepsilon) \geq e^{n(h_\mu(f)-\eta)}.$$

By inequality (2.3), there exists an (n, ε)-separated set $\Gamma_n \subset Z$ that satisfies

$$|\Gamma_n| \geq e^{n(h_\mu(f)-\eta)}.$$

The proof is complete. □

LEMMA 3.3. *Let (X, f) be a dynamical system. Fix any $\alpha > 0$ and assume that there are $m, k \in \mathbb{N}$ with $\log(m)/k > \alpha$ and a closed set $\Lambda \subset X$, invariant under f^k, such that there is a factor map $\pi : (\Lambda, f^k) \to (\Sigma^+_{m+1}, \sigma)$. Then there exists an ergodic measure μ such that*

$$h_\mu(f) > \alpha, \tag{3.8}$$

and there is an ergodic measure ν such that $\mu \neq \nu$ and $\operatorname{supp}(\nu) \subset \operatorname{supp}(\mu)$.

Proof. Recall that a dynamical system (X, f) is proximal if

$$\liminf_{n \to \infty} d(f^n(x), f^n(y)) = 0$$

for every $x, y \in X$. Equivalently, it means that the only minimal subset of X is the unique fixed point in X (e.g., see [**3**]).

Take any $\gamma > 0$ such that $(1 - \gamma)\log(m)/k > \alpha$. Fix any increasing sequences of positive integers k_n, s_n such that s_n divides s_{n+1}, $k_{n+1} < s_n$ and

$$\sum_{i=1}^{\infty} \frac{k_i}{s_i} < \gamma.$$

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Now, let A be a set consisting of points $x \in \Sigma_{m+1}^+$ such that $x_i = 0$ when $i \pmod{s_n} \geq (s_n - k_n)$ and $x_i \in \{1, \ldots, m\}$ otherwise. Denote $X = \overline{\bigcup_{i=0}^{\infty} \sigma^i(A)}$. Since every element of A has syndetic occurrence of block 0^{k_n} (the next such block occurs after at most $2s_n$ positions), also every element of X has this property. It immediately implies that $0^\infty \in \omega(x)$ for every $x \in X$, and hence $\{0^\infty\}$ is the unique minimal subset of X. Therefore the dynamical system (X, σ) is proximal. Let us estimate its entropy.

$$\begin{aligned}\frac{1}{s_n} \log |B_{s_n}(X)| &\geq \frac{1}{s_n} \log |\{w : w = x_{[0,s_n)]}, x \in A\}| \\ &\geq \frac{1}{s_n} \log m^{(s_n - \sum_{i=1}^{n} (k_i s_n / s_i))} \\ &\geq \left(1 - \sum_{i=1}^{\infty} \frac{k_i}{s_i}\right) \log(m) \geq (1 - \gamma) \log(m).\end{aligned}$$

This shows that $h_{\text{top}}(X) \geq (1 - \gamma) \log(m)$.

Denote $\Gamma = \pi^{-1}(X)$ and let $Z = \bigcup_{i=0}^{\infty} f^i(\Gamma) = \bigcup_{i=0}^{k-1} f^i(\Gamma)$. Then

$$h_{\text{top}}(f|_Z) \geq \frac{1}{k} h_{\text{top}}(f^k|_\Gamma) \geq (1 - \gamma) \log(m)/k > \alpha.$$

Denote by D the smallest invariant subset for f containing the set $\pi^{-1}(0^\infty)$. Then, clearly, $D \subset Z$ and every minimal system E in Z must be a subsystem of D. Put simply, $E \cap Z \neq \emptyset$ and $E \cap Z$ is a union of minimal sets for f^k. But then the only possibility is that $\pi(E \cap Z) = 0^\infty$, which shows that $E \cap Z \subset D$ and, since D is invariant, $E \subset D$.

Let (Y, g) be a system obtained by collapsing D to a single point, where g is an appropriate quotient map obtained from f. Then we have a natural factor map $\eta : (Z, f) \to (Y, g)$. Let $\tilde{\Gamma}$ be obtained from Γ by collapsing $D \cap \Gamma$. Then $\tilde{\Gamma} \subset Y$ and we also obtain an induced map $\tilde{\eta} : (Y, g) \to (X, \sigma)$ such that $\pi = \tilde{\eta} \circ \eta$. Among other things, it implies that $h_{\text{top}}(g) > \alpha$. Let μ be an ergodic measure with $h_\mu(g) > \alpha$ obtained by the variational principle. Note that (Y, g) is proximal, since there are no minimal subsets in it other that the fixed point p obtained by the collapse of D in Z. But then, by the ergodic theorem, $\mu(\{p\}) = 0$. Observe that η is one-to-one everywhere outside D and hence we may view μ as a measure on Z with $\mu(D) = 0$. Clearly, this backward projection by η does not change the entropy of μ. To complete the proof, it is enough to take as ν any ergodic measure supported on some minimal set contained in the invariant set $D \cap \text{supp}(\mu) \neq \emptyset$. □

LEMMA 3.4. *Suppose that a dynamical system (X, f) acting on a compact metric space X has the shadowing property and suppose that there exists $\alpha > 0$ and an ergodic measure $\mu \in \mathcal{M}_f(X)$ with $h_\mu(f) > \alpha$. Then there are $m, k \in \mathbb{N}$, $\log(m)/k > \alpha$ and a closed set $\Lambda \subset X$ invariant under f^k such that there is a factor map $\pi : (\Lambda, f^k) \to (\Sigma_{m+1}^+, \sigma)$.*

Proof. Fix any $\eta > 0$ such that $h_\mu(f) - 4\eta > \alpha$. By Lemma 3.2, there exists $\varepsilon > 0$ and $N \in \mathbb{N}$ such that, for any $n \geq N$, there exists $\Gamma_n \subset \text{supp}(\mu)$ which is (n, ε)-separated and satisfies $|\Gamma_n| \geq e^{n(h_\mu(f) - \eta)}$.

Let $\delta > 0$ be such that every δ-pseudo-orbit is $\varepsilon/3$ traced. Cover $\text{supp}(\mu)$ by open sets $U_1, \ldots, U_s$ such that $\text{diam}(U_i) < \delta$ for every i. Since μ is ergodic, $f|_{\text{supp}(\mu)}$ is transitive

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and thus, for every $1 \le i, j \le s$, there exists a point $z_{ij} \in U_i$ such that $f^{k_{ij}}(z_i) \in U_j$ for some $k_{ij} \in \mathbb{N}$. Denote

$$K = \max_{1 \le i,j \le s} \{k_{ij}\}.$$

Now we choose $n \in \mathbb{N}$ large enough such that

$$2\log(s)/n < \eta, \tag{3.9}$$

$$\frac{n}{n+K} h_\mu(f) < h_\mu(f) - \eta. \tag{3.10}$$

For every $1 \le i, j \le s$, put $\Gamma_n^{ij} = \{z \in \Gamma_n : z \in U_i,\ f^n(z) \in U_j\}$. Fix some $1 \le i, j \le s$ such that $|\Gamma_n^{ij}| \ge |\Gamma_n|/s^2$ and observe that, by (3.9),

$$|\Gamma_n^{ij}| \ge e^{n(h_\mu(f)-2\eta)}.$$

Enumerate elements of Γ_n^{ij}, say, $\Gamma_n^{ij} = \{p_0, \dots, p_{r-1}\}$, where $r = |\Gamma_n^{ij}|$. Using these points, define r finite sequences, putting $l = 0, 1, \dots, r-1$: that is,

$$\eta(l) = (p_l, f(p_l), \dots, f^{n-1}(p_l)), z_{ji}, f(z_{ji}), \dots, f^{k_{ji}-1}(z_{ji}).$$

If we define length $k = |\eta(l)| = n + k_{ij} \le n + K$, then all sequences $\eta(l)$ are δ-pseudo orbits and, since $f^{k_{ji}}(z_{ji}) \in U_i \supset \Gamma_n^{ij}$, we can freely concatenate sequences $\eta(l)$ obtaining other, longer δ-pseudo orbits.

Let Σ_r^+ be the set of element $a = (a_0 a_1 \cdots a_n \cdots)$ such that $a_i \in \{p_0, \dots, p_{r-1}\}$, $i \in \mathbb{Z}^+$. For every $\xi \in \Sigma_r^+$, denote

$$Y_\xi = \{z : d(f^{ik}(z), \xi_i) \le \varepsilon/3 \text{ for } i \in \mathbb{Z}^+\}.$$

By the choice of δ and the shadowing property of (X, f), we see that each set Y_ξ is nonempty. It is also clear that each such set is a closed subset of X. Note that if $\xi \ne \psi$, then there is t such that $\xi_t \ne \psi_t$. But then there is $r < n$ such that $d(f^r(p_{\xi_t}), f^r(p_{\psi_t})) \ge \varepsilon$. This immediately implies that $Y_\xi \cap Y_\psi = \emptyset$. Denote

$$\Lambda = \bigcup_{\xi \in \Sigma_r^+} Y_\xi.$$

Note that $f^k(Y_\xi) \subset Y_{\sigma(\xi)}$. Then Λ is an invariant set for f^k. It is also not hard to see that if $y \in Y_\xi$, $z \in Y_\psi$ and $d(f^l(x), f^l(y)) < \varepsilon/3$ for $l = 0, \dots, ks-1$, then $\xi_i = \psi_i$ for $i = 0, \dots, s-1$. So we can write Λ as the disjoint union of Y_ξ

$$\Lambda = \bigsqcup_{\xi \in \Sigma_r^+} Y_\xi.$$

Therefore, if we define $\pi : \Lambda \to \Sigma_r^+$ as

$$\pi(x) := \xi \quad \text{for all } x \in Y_\xi,$$

then π is a continuous surjection. This shows that Λ is closed and, clearly, also that $\sigma \circ \pi = \pi \circ f^k$. Finally, put $m = r - 1$ and observe that

$$\frac{\log(m)}{k} = \frac{\log(r-1)}{k} \ge \frac{\log(r/2)}{k} = \frac{1}{k} n(h_\mu(f) - 3\eta) \ge h_\mu(f) - 4\eta > \alpha.$$

The proof is complete. □

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Then it is enough to combine the variational principle with Lemmas 3.4 and 3.3 to get the following corollary.

COROLLARY 3.5. *If (X, f) satisfies the shadowing property and $h_{\mathrm{top}}(f) > 0$, then there exists a sequence of ergodic measures $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{M}_f(X)$ such that $h_{\mathrm{top}}(f) = \lim_{n\to\infty} h_{\mu_n}(f)$ and, for each n, there is an ergodic measure ν_n such that $\mu_n \neq \nu_n$ and $\mathrm{supp}(\nu_n) \subset \mathrm{supp}(\mu_n)$, that is, f restricted to $\mathrm{supp}(\mu_n)$ is not uniquely ergodic.*

4. *Proof of Theorem 1.5*

First we prove the following theorem which covers a part of Theorem 1.5.

THEOREM 4.1. *If f satisfies the shadowing property and $h_{\mathrm{top}}(f) > 0$, then*

$$\begin{aligned} h_{\mathrm{top}}(f) &= h_{\mathrm{top}}(f, I(f)) \\ &= \sup\{h_{\mathrm{top}}(f|_\Lambda) : \Lambda \subset X \text{ is a closed invariant set with} \\ &\qquad f|_\Lambda \text{ transitive and not uniquely ergodic}\}. \end{aligned}$$

Proof. First we consider the case of $h_{\mathrm{top}}(f) < \infty$ and divide our proof into the following four steps.

(1) For any $\eta > 0$, find Λ with $f|_\Lambda$ transitive, non-uniquely ergodic and $h_{\mathrm{top}}(f) - \eta < h_{\mathrm{top}}(f|_\Lambda)$ and pick a sufficient number of pseudo-orbits.
(2) Construct an uncountable compact set G.
(3) Prove that $G \subset I(f)$.
(4) Prove that $h_{\mathrm{top}}(f, G) > h_{\mathrm{top}}(f|_\Lambda) - 6\eta$.

Step 1. For any $\eta > 0$, by Corollary 3.5, we select an ergodic measure $\mu \in \mathcal{M}_f(\Lambda)$ such that

$$h_\mu(f) > h_{\mathrm{top}}(f) - \eta$$

and another ergodic measure ν with $\mathrm{supp}(\nu) \subset \mathrm{supp}(\mu)$. Clearly, f restricted to invariant set $\Lambda = \mathrm{supp}\,\mu$ is transitive (e.g., see [**42**]). Denote $d(\mu, \nu) = \alpha$ and let $t \in \mathbb{N}$ be such that

$$t(h_\mu(f) - 3\eta) \geq (t+1)(h_\mu(f) - 4\eta). \tag{4.11}$$

By Lemma 3.2, for μ, there exist $\varepsilon > 0$ such that if we denote $F_1 = \mathcal{B}(\mu, \alpha/(6(t+1))) \subset \mathcal{M}(X)$, then there exists $m_{F_1} \in \mathbb{N}$ such that, for any $n \geq m_{F_1}$, there exists $\Gamma_n^\mu \subset X_{n,F_1} \cap \mathrm{supp}(\mu)$ which is (n, ε)-separated and satisfies $|\Gamma_n^\mu| \geq e^{n(h_\mu(f)-\eta)}$.

Denote $F_2 = \mathcal{B}(\nu, \alpha/(6(t+1)))$ and let $\Gamma_n^\nu = \{z\}$ for some generic point z for ν. Then there exists $m_{F_2} \in \mathbb{N}$ such that $\Gamma_n^\nu \subset X_{n,F_2} \cap \mathrm{supp}(\nu)$ for any $n \geq m_{F_2}$.

By the shadowing property, for

$$\zeta = \min\left\{\frac{\alpha}{12(t+1)}, \frac{\varepsilon}{3}\right\}, \tag{4.12}$$

there exists $\gamma > 0$ such that any γ-pseudo-orbit can be ζ-shadowed by some point.

Since Λ is closed and thus compact, there exists a finite open cover $\{U_i\}_{i=1}^s$ of Λ with $\mathrm{diam}(U_i) < \gamma$ for $i = 1, \ldots, s$. Since $f|_\Lambda$ is transitive, for any $1 \leq i, j \leq s$, there exist

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$n_{ij} \in \mathbb{N}$ and $y_{ij} \in U_i \cap \Lambda$ such that $f^{n_{ij}}(y_{ij}) \in U_j$. Denote

$$T := \max_{1 \le i,j \le s} \{n_{ij}\},$$

$$L := \max\left\{m_{F_1}, m_{F_2}, \frac{48T(t+1)}{\alpha}, \frac{T(h_\mu(f)-3\eta)}{\eta}, \frac{2\ln(s)}{\eta}, 5T\right\}.$$

By the definition, there exist (L, ε)-separated sets $\Gamma_L^\mu \subset X_{L,F_1} \cap \mathrm{supp}(\mu)$ with

$$|\Gamma_L^\mu| \ge e^{L(h_\mu(f)-\eta)}. \tag{4.13}$$

Then, by the pigeonhole principle, we obtain $1 \le i_1, j_1 \le s$ and $\widetilde{\Gamma}_L^\mu \subset \Gamma_L^\mu$ such that $\widetilde{\Gamma}_L^\mu \subset U_{i_1}$, $f^L(\widetilde{\Gamma}_L^\mu) \subset U_{j_1}$ and

$$|\widetilde{\Gamma}_L^\mu| \ge |\Gamma_L^\mu|/s^2 \ge \frac{e^{L(h_\mu(f)-\eta)}}{s^2} \ge e^{L(h_\mu(f)-2\eta)}. \tag{4.14}$$

Take (the unique) $z \in \Gamma_L^\nu$ and let $1 \le i_2, j_2 \le s$ denote indices such that $z \in U_{i_2}$ and $f^L(z) \in U_{j_2}$.

Step 2. Now, for $n \in \mathbb{N}$, let us inductively define

$$S_0 = 0, \quad N_1 = 1, \quad S_n = \sum_{i=1}^{n} N_i \quad \text{and} \quad N_n = \lambda S_{n-1} \quad \text{for } n \ge 2,$$

where $\lambda > ((48(t+1))/\alpha)$ is a fixed integer. Observe that we have an explicit formula $N_n = (\lambda+1)^{n-1}$. For each $m \in \mathbb{N}$ define measure ω_m and integer l_m by the following procedure. If $m \in [S_{2k}(t+1)+1, S_{2k+1}(t+1)]$ for some $k \ge 0$, then we put

$$\omega_m = \mu \quad \text{and} \quad l_m = n_{j_1 i_1}.$$

Otherwise, $m \in [S_{2k+1}(t+1)+1, S_{2k+2}(t+1)]$ for some $k \ge 0$ and then there are uniquely determined integers $r \in [0, N_{2k+1})$ and $\tilde{r} \in [1, t+1]$ such that $m = (S_{2k+1}+r)(t+1)+\tilde{r}$. In this case, we put

$$\omega_m = \begin{cases} \mu & \text{if } \tilde{r} \le t, \\ \nu & \text{if } \tilde{r} = t+1, \end{cases} \qquad l_m = \begin{cases} n_{j_1 i_1} & \text{if } \tilde{r} < t, \\ n_{j_1 i_2} & \text{if } \tilde{r} = t, \\ n_{j_2 i_1}, & \text{if } \tilde{r} = t+1. \end{cases}$$

For each $m \ge 1$, put $\widetilde{\Gamma}_m = \widetilde{\Gamma}_L^{\omega_m}$ and select an arbitrary point $x_m \in \widetilde{\Gamma}_m$. Clearly, each $\widetilde{\Gamma}_m$ is a finite set and, when $\omega_m = \mu$, the choice of x_m is not unique. In what follows, by performing the above procedure, we construct a family of sequences rather than one concrete sequence. Set

$$M_m = mL + \sum_{i=1}^{m} l_i \tag{4.15}$$

and define a sequence $\{w_u\}_{u=0}^\infty$ as a concatenation of the following blocks.

First, put $a = M_{(S_{2k}+r)(t+1)}$ and $b = M_{(S_{2k}+r+1)(t+1)} - 1$ for some $k \ge 0$ and $0 \le r < N_{2k+1}$. Then the segment $\{w_u\}_{u=a}^b$ is defined by

$$\begin{aligned} &x_{(S_{2k}+r)(t+1)+1}, \ldots, f^{L-1}(x_{(S_{2k}+r)(t+1)+1}), y_{j_1 i_1}, \ldots, f^{n_{j_1 i_1}-1}(y_{j_1 i_1}), \\ &x_{(S_{2k}+r)(t+1)+2}, \ldots, f^{L-1}(x_{(S_{2k}+r)(t+1)+2}), y_{j_1 i_1}, \ldots, f^{n_{j_1 i_1}-1}(y_{j_1 i_1}), \\ &\qquad\qquad \vdots \\ &x_{(S_{2k}+r+1)(t+1)}, \ldots, f^{L-1}(x_{(S_{2k}+r+1)(t+1)}), y_{j_1 i_1}, \ldots, f^{n_{j_1 i_1}-1}(y_{j_1 i_1}). \end{aligned} \tag{4.16}$$

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Complementary segments of the sequence, for indices between $a = M_{(S_{2k+1}+r)(t+1)}$ and $b = M_{(S_{2k+1}+r+1)(t+1)} + 1$ for some $k \geq 0$ and $0 \leq r < N_{2k+1}$ have a more complicated structure. We define them by

$$\begin{gathered} x_{(S_{2k+1}+r)(t+1)+1}, \ldots, f^{L-1}(x_{(S_{2k+1}+r)(t+1)+1}), y_{j_1 i_1}, \ldots, f^{n_{j_1 i_1}-1}(y_{j_1 i_1}), \\ \vdots \\ x_{(S_{2k+1}+r)(t+1)+t-1}, \ldots, f^{L-1}(x_{(S_{2k+1}+r)(t+1)+t-1}), y_{j_1 i_1}, \ldots, f^{n_{j_1 i_1}-1}(y_{j_1 i_1}), \\ x_{(S_{2k+1}+r)(t+1)+t}, \ldots, f^{L-1}(x_{(S_{2k+1}+r)(t+1)+t}), y_{j_1 i_2}, \ldots, f^{n_{j_1 i_2}-1}(y_{j_1 i_2}), \\ x_{(S_{2k+1}+r+1)(t+1)}, \ldots, f^{L-1}(x_{(S_{2k+1}+r+1)(t+1)}), y_{j_2 i_1}, \ldots, f^{n_{j_2 i_1}-1}(y_{j_2 i_1}). \end{gathered} \tag{4.17}$$

By the selection of points $\{x_m\}_{m\geq 1}$ and the definition of $y_{i_1 i_1}$, $y_{j_1 i_2}$, $y_{j_2 i_1}$, it is easy to see that $\{w_u\}_{u=0}^{\infty}$ constitutes a γ-pseudo-orbit. Thus, by the shadowing property, $\{w_u\}_{u=0}^{\infty}$ is ζ-shadowed by some point in X.

For $d \geq 1$ and any choice of $z_1 \in \widetilde{\Gamma}_1, \ldots, z_d \in \widetilde{\Gamma}_d$, denote

$$G_d(z_1, \ldots, z_d) := \bigcap_{m=1}^{d} f^{-M_{m-1}} \overline{B_L(z_m, \zeta)},$$

$$G_d := \bigcap_{m=1}^{d} \bigcup_{x_m \in \widetilde{\Gamma}_m} f^{-M_{m-1}} \overline{B_L(x_m, \zeta)}.$$

Observe that both $G_d(z_1, \ldots, z_d)$ and G_d are non-empty closed sets. Moreover, for $d \geq 1$, we have inclusions

$$G_d(z_1, \ldots, z_d) \supset G_{d+1}(z_1, \ldots, z_d, z_{d+1}),$$
$$G_d \supset G_{d+1},$$

which implies that the following sets are closed and non-empty.

$$G(z_1, z_2, \ldots) := \bigcap_{d\geq 1} G_d(z_1, \ldots, z_d),$$
$$G := \bigcap_{d\geq 1} G_d.$$

Let $x_d, y_d \in \widetilde{\Gamma}_d$ with $x_d \neq y_d$. Since x_d and y_d are (L, ε)-separated, there exists $0 \leq l < L$ such that

$$d(f^l(x_d), f^l(y_d)) > \varepsilon,$$

and, therefore, if $x \in B_L(x_d, \zeta)$ and $y \in B_L(y_d, \zeta)$, then

$$\begin{aligned} d(f^l(x), f^l(y)) &\geq d(f^l(x_d), f^l(y_d)) - d(f^l(x_d), f^l(x)) - d(f^l(y_d), f^l(y)) \\ &> \varepsilon - 2\zeta > \frac{\varepsilon}{3} \end{aligned}$$

by the selection of ζ in (4.12). This implies that

$$G_d(z_1, \ldots, z_{d-1}, x_d) \cap G_d(z_1, \ldots, z_{d-1}, y_d) = \emptyset,$$

which shows that G is a closed set that is a disjoint union of closed sets $G(x_1, x_2, \ldots)$ labeled by $(x_1, x_2, \ldots)$ with $x_m \in \widetilde{\Gamma}_m$ for each m. In other words, two different sequences label two different sets.

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Step 3. Next we are going to prove that

$$G \subset I(f). \tag{4.18}$$

Let z be a point corresponding to some closed set $z \in G(x_1, x_2, \ldots)$ with $x_m \in \widetilde{\Gamma}_m$. Then z ζ-shadows some γ-pseudo-orbit $\{w_u\}_{u=0}^{\infty}$ constructed above in Step 2. Since, for every $x, y \in X$, $\rho(\delta_x, \delta_y) = d(x, y)$, it is easily seen that

$$\rho\left(\mathcal{E}_n(z), \frac{\sum_{u=0}^{n-1} \delta_{z_u}}{n}\right) \le \zeta \quad \text{for all } n \in \mathbb{N}. \tag{4.19}$$

Since $x_m \in \widetilde{\Gamma}_m$,

$$\rho(\mathcal{E}_L(x_m), \omega_m) \le \frac{\alpha}{6(t+1)} \tag{4.20}$$

by the selection of open sets F_1 and F_2. Therefore, Lemma 2.1(3) gives

$$\begin{aligned}\rho(\mathcal{E}_{L+l_m}(x_m), \omega_m) &\le \rho(\mathcal{E}_L(x_m), \omega_m) + \rho(\mathcal{E}_L(x_m), \mathcal{E}_{L+l_m}(x_m)) \\ &\le \frac{\alpha}{6(t+1)} + \frac{2T}{L}.\end{aligned}$$

Thus, by the selection of L in (4.13) and Lemma 2.1,

$$\begin{aligned}&\rho(\mathcal{E}_{S_{2k-1}(t+1)L+\sum_{m=1}^{S_{2k-1}(t+1)} l_m}(z), \mu) \\ &\quad\le \frac{N_{2k-1}}{S_{2k-1}}\left(\frac{\alpha}{6(t+1)} + \frac{2T}{L}\right) + 2\frac{S_{2k-2}}{S_{2k-1}} + \zeta \\ &\quad\le \frac{\alpha}{6(t+1)} + \frac{2T}{L} + \frac{2}{\lambda} + \frac{\alpha}{12(t+1)} \\ &\quad\le \frac{\alpha}{6(t+1)} + \frac{\alpha}{24(t+1)} + \frac{\alpha}{24(t+1)} + \frac{\alpha}{12(t+1)} \\ &\quad\le \frac{\alpha}{3(t+1)}.\end{aligned}$$

Next, if we put

$$\beta = \frac{tL + (t-1)n_{j_1 i_1} + n_{j_1 i_2}}{(t+1)L + (t-1)n_{j_1 i_1} + n_{j_1 i_2} + n_{i_2 j_1}},$$

then, by Lemma 2.1, similarly to before, we obtain that

$$\begin{aligned}&\rho(\mathcal{E}_{S_{2k}(t+1)L+\sum_{m=1}^{S_{2k}(t+1)} l_m}(z), \beta\mu + (1-\beta)\nu) \\ &\quad\le \frac{N_{2k}}{S_{2k}}\left[\beta\left(\frac{\alpha}{6(t+1)} + \frac{2T}{L}\right) + (1-\beta)\left(\frac{\alpha}{6(t+1)} + \frac{2T}{L}\right)\right] + \frac{2S_{2k-1}}{S_{2k}} + \zeta \\ &\quad\le \frac{\alpha}{6(t+1)} + \frac{2T}{L} + \frac{2}{\lambda} + \zeta \\ &\quad\le \frac{\alpha}{6(t+1)} + \frac{\alpha}{24(t+1)} + \frac{\alpha}{24(t+1)} + \frac{\alpha}{12(t+1)} \\ &\quad\le \frac{\alpha}{3(t+1)}.\end{aligned}$$

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Now it is enough to observe that

$$d(\mu, \beta\mu + (1-\beta)\nu) = \alpha(1-\beta) \geq \frac{L}{(L+T)(t+1)}\alpha \geq \frac{5\alpha}{6(t+1)} > \frac{\alpha}{3(t+1)} + \frac{\alpha}{3(t+1)},$$

which shows that $\{\mathcal{E}_n(z)\}_{n\geq 1}$ diverges, and so $z \in I(f)$ by Lemma 3.1.

Step 4. Let $h = h_\mu(f) - 5\eta > h_{\text{top}}(f) - 6\eta$. We are going to show that

$$h_{\text{top}}(f, G) \geq h. \tag{4.21}$$

Recall that $h_{\text{top}}(f, G)$ is defined in (2.4) and we will follow the notation introduced there. Since $C(G; s, \sigma, f)$ is a non-increasing function of σ, it is enough to prove that there exists $\widetilde{\sigma} > 0$ such that

$$C(G; h, \widetilde{\sigma}, f) \geq 1. \tag{4.22}$$

We will prove that (4.22) holds for $\widetilde{\sigma} = \varepsilon/6$. By definition,

$$C\left(G; h, \frac{\varepsilon}{6}, f\right) = \lim_{n\to\infty} C\left(G; h, n, \frac{\varepsilon}{6}, f\right)$$

and

$$C\left(G; h, n, \frac{\varepsilon}{6}, f\right) = \inf_{\mathcal{C}\in\mathcal{G}_n(G,\varepsilon/6)} \sum_{B_u(z,\varepsilon/6)\in\mathcal{C}} e^{-hu}.$$

Since G is compact, we can find at least one finite cover $\mathcal{C} \in \mathcal{G}_n(G, \varepsilon/6)$ of G. We may also assume that if $B_u(z, \varepsilon/6) \in \mathcal{C}$, then $B_u(z, \varepsilon/6) \cap G \neq \emptyset$ since, otherwise, we may remove this set from $\mathcal{C}$ and it will still remain a cover of G. It is enough to show that for sufficiently large n (which affects the choice of $\mathcal{C}$),

$$\sum_{B_u(z,\varepsilon/6)\in\mathcal{C}} e^{-hu} \geq 1. \tag{4.23}$$

Let $q > 0$ be an integer such that

$$\frac{q+1}{q} \leq \frac{h_\mu(f) - 3\eta}{h_\mu(f) - 2\eta} \cdot \frac{h_\mu(f) - 4\eta}{h_\mu(f) - 5\eta}. \tag{4.24}$$

We claim that condition (4.23) holds for any $n \geq M_q$.

Let us fix any $n \geq M_q$ and any $\mathcal{C} \in \mathcal{G}_n(G, \varepsilon/6)$. Define the cover $\mathcal{C}'$ in which each ball $B_u(z, \varepsilon/6) \in \mathcal{C}$ is replaced by $B_{M_m}(z, \varepsilon/6)$, where $M_m \leq u < M_{m+1}$, $m \geq q$ and M_m are defined in (4.15). Then, clearly,

$$\sum_{B_u(x,\varepsilon/6)\in\mathcal{C}} e^{-hu} \geq \sum_{B_{M_m}(x,\varepsilon/6)\in\mathcal{C}'} e^{-hM_{m+1}}. \tag{4.25}$$

Let c be the largest value of m for which there exists $B_{M_m}(z, \varepsilon/6) \in \mathcal{C}'$. Define

$$\mathcal{V}_c := \bigcup_{l=1}^{c} \mathcal{W}_l \quad \text{where } \mathcal{W}_l := \prod_{i=1}^{l} \widetilde{\Gamma}_i$$

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and sets $\widetilde{\Gamma}_i$ are as defined in Step 2 just before (4.15). For every $1 \le j \le k$ and $v \in \mathcal{W}_j$, we say that v is a *prefix* of $w \in \mathcal{W}_k$ if the first j coordinates of w coincide with v. Note that each $p \in \mathcal{W}_m$ is a prefix of exactly $|\mathcal{W}_c|/|\mathcal{W}_m|$ words of $\mathcal{W}_c$. If $\mathcal{W} \subset \mathcal{V}_c$ contains a prefix of each word of $\mathcal{W}_c$, then

$$\sum_{m=1}^{c} |\mathcal{W} \cap \mathcal{W}_m| \frac{|\mathcal{W}_c|}{|\mathcal{W}_m|} \ge |\mathcal{W}_c|.$$

To see this, it is enough to note that, for each word of $\mathcal{W}_c$, one of its prefixes must be contained in $\mathcal{W} \cap \mathcal{W}_m$ for some $1 \le m \le c$. Moreover, the number of words in $\mathcal{W}_c$ with this prefix does not exceed $|\mathcal{W}_c|/|\mathcal{W}_m|$. Thus, if $\mathcal{W}$ contains a prefix of each word of $\mathcal{W}_c$, then

$$\sum_{m=1}^{c} |\mathcal{W} \cap \mathcal{W}_m|/|\mathcal{W}_m| \ge 1. \tag{4.26}$$

Note that, since $\mathcal{C}'$ is a cover of G, $\mathcal{C}'$ corresponds to a subset $\mathcal{W} \subset \mathcal{V}_c$ such that $\mathcal{W}$ contains a prefix of each word of $\mathcal{W}_c$. More precisely, as discussed at the end of Step 2, each $x \in B_{M_m}(z, \varepsilon/6) \cap G$ corresponds to a closed set $G(x_1, x_2, \ldots, x_m) \supset G$ and thus corresponds to a point in $\mathcal{W}_m$. Moreover, sequence $x_1, \ldots, x_m$ is uniquely defined since, if the converse were true, it is not hard to see that there would exist $1 \le i \le m$ and $x_i, y_i \in \widetilde{\Gamma}_L^{\mu}$ with $x_i \ne y_i$ such that

$$f^{M_{i-1}}(z) \in B_L\left(x_i, \frac{\varepsilon}{6} + \frac{\varepsilon}{3}\right) \cap B_L\left(y_i, \frac{\varepsilon}{6} + \frac{\varepsilon}{3}\right)$$

and this would lead to

$$d_L(x_i, y_i) \le d_L(x_i, f^{M_{i-1}}(z)) + d_L(y_i, f^{M_{i-1}}(z)) < \varepsilon,$$

which contradicts the fact that x_i and y_i are (L, ε)-separated.

Thus (4.26) gives

$$\sum_{B_{M_m}(x,\varepsilon) \in \mathcal{C}'} \frac{1}{|\mathcal{W}_m|} \ge 1. \tag{4.27}$$

By the selection of the sequence $\widetilde{\Gamma}_i$, we see that the average number of occurrences of $\widetilde{\Gamma}_L^{\nu}$ does not exceed $1/(t+1)$ and, therefore,

$$|\mathcal{W}_m| \ge (e^{L(h_\mu(f)-2\eta)})^{mt/(t+1)}. \tag{4.28}$$

Recall that, by the selection of L,

$$L(h_\mu(f) - 2\eta) \ge (L+T)(h_\mu(f) - 3\eta), \tag{4.29}$$

$$m(L+T) \ge M_m. \tag{4.30}$$

Combining (4.28)–(4.30) and (4.11),

$$|\mathcal{W}_m| \ge e^{m(L+T)(h_\mu(f)-4\eta)} \ge e^{M_m(h_\mu(f)-4\eta)},$$

which, by (4.27), gives

$$\sum_{B_{M_m}(x,\varepsilon) \in \mathcal{C}'} e^{-M_m(h_\mu(f)-4\eta)} \ge 1. \tag{4.31}$$

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On the other hand, by the selection of q in (4.24), definition (4.15) and the inequality (4.29), we can easily check that, for $m \geq q$,

$$\frac{M_{m+1}}{M_m} \leq \frac{(m+1)(L+T)}{mL} \leq \frac{q+1}{q} \cdot \frac{h_\mu(f)-2\eta}{h_\mu(f)-3\eta} \leq \frac{h_\mu(f)-4\eta}{h_\mu(f)-5\eta},$$

which implies that

$$hM_{m+1} = M_{m+1}(h_\mu(f)-5\eta) \leq M_m(h_\mu(f)-4\eta). \tag{4.32}$$

Combining (4.25), (4.31) and (4.32),

$$\sum_{B_u(x,\varepsilon/6)\in\mathcal{C}} e^{-hu} \geq 1,$$

and so the claim holds.

This implies that, for any $n \geq M_q$,

$$C\left(G; h, n, \frac{\varepsilon}{6}, T\right) \geq 1.$$

Therefore (4.22) holds and thus (4.21) is proved.

We have just completed the proof of the case $h_{\text{top}}(f) < \infty$. For the case $h_{\text{top}}(f) = \infty$, it is enough to replace condition $h_{\text{top}}(f) - \eta < h_{\text{top}}(f|_\Lambda)$ in Step 1 by $h_{\text{top}}(f|_\Lambda) > \eta$. All the other arguments in the proof remain unchanged. We leave the details to the reader. □

Now we are ready to prove our first main result. It was proved in [**31**] that if a dynamical system with the shadowing property has a recurrent not minimal point, or a minimal sensitive point, then the entropy must be positive. A similar analysis will be performed here.

Proof of Theorem 1.5. The case of $h_{\text{top}}(f) > 0$ is covered by Theorem 4.1. For the remaining case, assume, on the contrary, that $h_{\text{top}}(f) = 0$ and $I(f) \neq \emptyset$, which by Lemma 3.1 equivalently means that $Q(f) \neq X$. By Lemma 3.1, there is $x \in X$ be such that $\mathcal{E}_n(x)$ diverges. Take two distinct invariant measures μ, ν that are accumulation points of the sequence $\mathcal{E}_n(x)$. First, assume that $\text{supp}(\mu) \neq \text{supp}(\nu)$. Without loss of generality we may assume that $\text{supp}(\mu) \setminus \text{supp}(\nu) \neq \emptyset$. Fix any recurrent point $z \in \text{supp}(\nu)$. We claim that z is a sensitive point. To see that, fix any $y \in \text{supp}(\mu)$ and $\varepsilon > 0$ such that $B_{3\varepsilon}(y) \cap \text{supp}(\nu) = \emptyset$. Fix any $\eta \in (0, \varepsilon)$, put $U = B_\eta(y)$, $V = B_\eta(z)$ and observe that $\mu(U) > 0$ and $\nu(V) > 0$. Thus, there is $\delta > 0$ and increasing sequences n_i, m_i such that $\mathcal{E}_{n_i}(x)(U) > \delta$ and $\mathcal{E}_{m_i}(x)(V) > \delta$. In particular, there is a point $q \in X$ and $0 < n < m$ such that

$$d(q, z) < \eta, \quad d(f^m(q), z) < \eta, \quad d(f^n(q), f^n(z)) \geq \text{dist}(f^n(q), \text{supp}(\nu)) > \varepsilon.$$

Since z is a recurrent point, we can repeat arguments similar to those in Lemma 3.4 and show that $h_{\text{top}}(f) > 0$ (see also [**27**]). This is a contradiction, and hence the only remaining possibility is that $\text{supp}(\mu) = \text{supp}(\nu)$. If $\text{supp}(\mu)$ is not a minimal set, then we can replace ν by a measure supported on a minimal set $Y \subsetneq \text{supp}(\mu)$ and repeat the previous argument to show that $h_{\text{top}}(f) > 0$. Therefore the only possibility is that $\text{supp}(\mu)$ is a minimal set. By [**27**], assumption $h_{\text{top}}(f) = 0$ implies that there is no sensitive point in $\text{supp}(\mu)$ and hence $(\text{supp}(\mu), f|_{\text{supp}(\mu)})$ is an equicontinuous minimal system. But all such systems are uniquely ergodic (e.g., see [**9**]) which gives $\mu = \nu$. This is again a contradiction, which completes the proof. □

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5. *Proof for Theorem 1.7*

We need the following lemma which is a simplified version of the shredding lemma in [**1**]. For the convenience of the reader, we include a proof here. This is also helpful in our subsequent proof of Theorem 1.7, which needs some details for its construction.

LEMMA 5.1. *For every $\varepsilon > 0$, there exists a dense set $\mathcal{C}^\varepsilon \subset H(M)$ (respectively, $\mathcal{C}^\varepsilon \subset C(M)$) such that, for very $f \in \mathcal{C}^\varepsilon$, there are pairwise disjoint open sets $U_1, \ldots, U_N$ such that:*

(i) *each U_j is a trapping region: $f(\overline{U_j}) \subset U_j$, $j = 1, \ldots, N$;*

(ii) *the union of sets U_j occupies, Lebesgue-wise, most of M*

$$\mathscr{L}\left(\bigcup_{j=1}^{N} U_j\right) > 1 - \varepsilon;$$

(iii) *each U_j is contained in a basin of attraction of a periodic cycle of sets, i.e., there are open sets $W_j^1, \ldots, W_j^{k_j}$ such that:*

(a) $\operatorname{diam}(W_j^i) < \varepsilon$ *for* $i = 1, \ldots, k_j$;

(b) $f(\overline{W_j^i}) \subset W_j^{i+1}$ *for* $i = 1, \ldots, k_j - 1$ *and* $f(\overline{W_j^{k_j}}) \subset W_j^1$; *and*

(c) $\overline{U_j} \subset \bigcup_{n\geq 0} f^{-n}(\bigcup_{i=1}^{k_j}(W_j^i))$.

Proof. The case of $H(M)$. We are going to prove that, for any $f \in H(M)$ and $\zeta > 0$, there exists a map $g \in \mathcal{C}^\varepsilon$ such that $d_H(f, g) < 2\zeta$.

Fix any $f \in H(M)$ and any $\zeta, \varepsilon > 0$. Let $0 < \sigma < \min\{\zeta, \varepsilon\}$ be such that $d(x, y) < \sigma$ implies that $d(f(x), f(y)) < \zeta$.

We begin with a decomposition of M (see Definition 1.6), that is, we decompose M into finitely many small pieces $\{R_i\}_{i\in I}$, say, $\operatorname{diam} R_i < \sigma$, each homeomorphic to the simplex

$$\Delta_k = \left\{(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : \sum_{i=1}^{k} x_i \leq 1 \text{ and } x_i \geq 0 \text{ for all } i\right\}.$$

Note that $\operatorname{Int}(f(R_i)) \neq \emptyset$ for every i. We claim that, for each i, there is j such that $\operatorname{Int}(f(R_i)) \cap \operatorname{Int}(R_j) \neq \emptyset$. Assume, on the contrary, that it is not the case: that is, that $\operatorname{Int}(f(R_i)) \subset \bigcup_{k\in I} \partial R_k$. Denote $U_1 = \operatorname{Int} f(R_i)$ and observe that either $U_1 \subset \partial R_1$ or there is $\alpha > 0$ such that $U_2 = U_1 \cap \{x : d(x, \partial R_1) > \alpha\} \neq \emptyset$ and, clearly, also $U_2 \subset \bigcup_{k\geq 2} \partial R_k$. As before, either $U_2 \subset \partial R_2$ or there is a non-empty open (in M) set $U_3 \subset \bigcup_{k\geq 3} \partial R_k$. Since I is a finite set, there is $k \in I$ and a non-empty open set $U_k \subset M$ such that $U_k \subset \partial R_k$, which is a contradiction. Indeed the claim holds, and so we can define a function $\tau : I \to I$ (not necessarily surjective) such that $\operatorname{Int}(f(R_i)) \cap \operatorname{Int}(R_{\tau(i)}) \neq \emptyset$ for every $i \in I$.

Once τ is chosen, we select for each $i \in I$ a point $p_i \in \operatorname{Int}(R_i)$ such that $f(p_i) \in \operatorname{Int}(R_{\tau(i)})$. For $\delta > 0$, we write R_i^δ for the set $\{x \in R_i : d(x, \partial R_i) > \delta\}$. Since f is a homeomorphism, we can choose $\delta > 0$ small enough so that each $p_i \in R_i^\delta$, $f(p_i) \in R_{\tau(i)}^\delta$ and, moreover, $\mathscr{L}(\bigcup_{i\in I} R_i^\delta) > 1 - \varepsilon$.

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Take $\delta' > 0$ small enough to satisfy $\overline{f(B_{\delta'}(p_i))} \subset R^{\delta}_{\tau(i)}$ for every $i \in I$. We are going to construct a homeomorphism $h : M \to M$ that sends R^{δ}_i into $B_{\delta'}(p_i)$. This can easily done by using the linear structure in each R_i induced by the charts $\eta_i : R_i \to \Delta_k$.

To see this, fix any $x \in R_i$ and denote $\hat{x} = \eta_i(x)$. For any $x \neq p_i$, let $\hat{q}^x_i$ denote the unique point in $\partial\Delta_k$ on the ray starting at $\hat{p}_i$ and passing through $\hat{x}$. Clearly, the following map $\alpha : R_i \to \mathbb{R}$ is continuous.

$$\alpha(x) = \begin{cases} 0, & x = p_i, \\ \dfrac{d(\hat{x}, \hat{p}_i)}{d(\hat{p}_i, \hat{q}^x_i)}, & x \neq p_i. \end{cases}$$

Now fix any (sufficiently large) $\beta \geq 1$ and define a homeomorphism $h_i : M \to M$ by

$$h_i(x) = \begin{cases} x & \text{if } x \notin R_i \cup \{p_i\}, \\ p_i, & x = p_i, \\ \eta_i^{-1}(\hat{p}_i + \alpha(x)^{\beta}(\hat{q}^x_i - \hat{p}_i)) & \text{otherwise.} \end{cases}$$

Note that $\alpha|_{\partial R_i} = 1$ and do $h_i|_{\partial R_i} = id$. Denote by h the composition of all the maps h_i in any order. Since

$$h_i(R_i) = h_i^{-1}(R_i) = R_i, \tag{5.33}$$

we easily see that all h_i commute, which means that the order of composition is irrelevant. It is also not hard to see that h is a homeomorphism which is close to identity: that is,

$$d_H(h, \mathrm{id}) \leq \max_{i \in I} \mathrm{diam}(R_i) < \sigma.$$

Denote $g = f \circ h$ and observe that $g \in H(M)$ since $h \in H(M)$. It directly follows from the definition of h_i that, for any $\delta' > 0$, taking $\beta > 1$ sufficiently large, we can ensure inclusion

$$\overline{h(R^{\delta}_i)} \subset B_{\delta'}(p_i)$$

and, consequently,

$$\overline{g(R^{\delta}_i)} \subset R^{\delta}_{\tau(i)}. \tag{5.34}$$

Note that since I is a finite set, there are a finite number of periodic orbits $O_1, \ldots, O_N$ under the action of τ. Moreover, the trajectory of every $l \in I$ ends up in one of the orbits O_i after at most $|I|$ iterations of τ. Thus, if we denote $\widehat{O}_i = \bigcup_{k=1}^{|I|} \tau^{-k}(O_i)$ for $i = 1, \ldots, N$, then $I = \bigsqcup_{i=1}^{N} \hat{O}_i$, where $\bigsqcup$ denotes the disjoint union. For each $i = 1, \ldots, N$, fix a point $z_i \in O_i$ and define

$$U_j = \bigcup_{i \in \hat{O}_j} R^{\delta}_i \quad \text{and} \quad W^s_i = R^{\delta}_{\tau^s(z_i)} \tag{5.35}$$

for $s = 1, 2, \ldots, k_i$, where $k_i = |O_i|$.

Observe that

$$g(\overline{U_j}) = g\left(\overline{\bigcup_{i \in \hat{O}_j} R^{\delta}_i}\right) = \bigcup_{i \in \hat{O}_j} g(\overline{R^{\delta}_i}) = \bigcup_{i \in \hat{O}_j} \overline{g(R^{\delta}_i)} \subset \bigcup_{i \in \hat{O}_j} R^{\delta}_{\tau(i)} \subset \bigcup_{i \in \hat{O}_j} R^{\delta}_i = U_j,$$

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which is (i). It is also clear that

$$\mathscr{L}\Big(\bigcup_{j=1}^{N} U_j\Big) = \mathscr{L}\Big(\bigcup_{j=1}^{N}\bigcup_{i\in\hat{O}_j} R_i^{\delta}\Big) = \mathscr{L}\Big(\bigcup_{i\in\bigcup_{j=1}^{N}\hat{O}_j} R_i^{\delta}\Big) = \mathscr{L}\Big(\bigcup_{i\in I} R_i^{\delta}\Big) > 1-\varepsilon,$$

which is exactly (ii). Note that $\operatorname{diam} W_i^s \le \max_{i\in I}\operatorname{diam}(R_i) < \sigma < \varepsilon$, which gives (iii)(a). By (5.34),

$$g(\overline{W_i^s}) = g(\overline{R^{\delta}_{\tau^s(z_i)}}) = \overline{g(R^{\delta}_{\tau^s(z_i)})} \subset R^{\delta}_{\tau^{s+1}(z_i)}.$$

But if $s < k_i$, then $R^{\delta}_{\tau^{s+1}(i)} = W_i^{s+1}$ and, when $s = k_i$, then $\tau^{s+1}(z_i) = \tau(z_i)$ and so, in this case, $R^{\delta}_{\tau^{s+1}(z_i)} = W_i^1$. This proves (iii)(b). Finally, for every $i \in \hat{O}_j$, there is an $1 \le n \le |I|$ such that $\tau^n(i) \in O_j$. Then

$$g^n(\overline{R_i^{\delta}}) \subset R^{\delta}_{\tau^n(i)} \subset \bigcup_{i=1}^{k_j} W_j^i,$$

which implies that $\overline{R_i^{\delta}} \subset g^{-n}(\bigcup_{i=1}^{k_j} W_j^i)$. But then $\overline{U_j} \subset \bigcup_{n\ge 0} g^{-n}(\bigcup_{i=1}^{k_j} W_j^i)$, which is (iii)(c).

Finally, observe that by (5.33), for every $x \in M$, $d(x, h(x)) < \sigma$ and $d(x, h^{-1}(x)) < \sigma$, which implies, by definition of σ, that, for every $x \in M$,

$$d(f^{-1}(x), h^{-1}f^{-1}(x)) < \sigma < \zeta \quad \text{and} \quad d(f(x),\, f(h(x))) < \zeta.$$

It means that $d_H(f, g) < 2\zeta$, which completes the proof in the case of $H(M)$.

The case of $C(M)$. Observe that, in the construction for $H(M)$ we, in fact, only require that if $f(R_i) \cap R_j \ne \emptyset$, then there exists a point $p_i \in \operatorname{Int} R_i$ such that $f(p_i) \in \operatorname{Int} R_j$. Assume for a moment that this condition holds. This allows us to define τ and the same proof can be repeated. For any such $f \in C(M)$, we define $h \in H(M)$ as above and put $g = f \circ h$ obtaining all desired sets U_i, W_i^j. It is also clear that $g \in C(M)$ and $d_C(f, g) < \zeta$ since metric d_C is less restrictive than d_H.

Observe that, by [**23**, Lemma 3.11], for any $\zeta > 0$, we can find a map $\hat{f}$ such that if

$$\hat{f}(R_i) \cap R_j \ne \emptyset \quad \Longrightarrow \quad \hat{f}(\operatorname{Int} R_i) \cap \operatorname{Int} R_j \ne \emptyset$$

and $d_C(f, \hat{f}) < \zeta$. Then we can find g for $\hat{f}$, as sketched above, and $d_C(f, g) < 2\zeta$. The proof is complete. □

Proof of Theorem 1.7. We divide the proof into four intermediate steps, to make it more readable.

Step 1. Construction of $\Lambda \subset M$ with full $\mathscr{L}$-measure.

Observe that each $\mathcal{C}^{\varepsilon}$ provided by Lemma 5.1 is open and dense,and hence the following set is residual.

$$\mathcal{R} = \bigcap_{n\in\mathbb{N}} \mathcal{C}^{1/n}.$$

Fix $f \in \mathcal{R}$ and, for each $n \in \mathbb{N}$, let V_n be the union of the open set U_i corresponding to $C^{1/n}$. Denote

$$\Lambda = \bigcap_{n\ge 1}\bigcup_{m\ge n} V_m \tag{5.36}$$

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and observe that, since $\mathscr{L}(V_n) > 1 - (1/n)$ for each V_n, $\mathscr{L}(\Lambda) = 1$.

Step 2. $\Lambda \subset Q(f) \cap \Delta$. The proof that $\Lambda \subset Q(f)$ is exactly the same as in [**1**, Theorem 3.6] and, therefore, we leave the details to the reader. For the rest of the proof, fix any $x \in \Lambda$ and take $\mu \in \mathcal{M}_f(X)$ such that $\lim_{n\to\infty} \mathcal{E}_n(x) = \mu$. We claim that μ is a SRB-like measure.

By definition, it is enough to show that, for every $\epsilon > 0$, the set

$$A_\varepsilon(\mu) = \{y \in X : d(p\omega(y), \mu) < \varepsilon\}$$

has positive Lebesgue measure. Take an integer $n > 0$ such that $1/n < \epsilon$. By definition of Λ, there exists an $m \geq n$ such that $x \in V_m$.

By the definition of V_m, which was obtained in Lemma 5.1 by (5.35), there exist $|I(m)|$ open sets $R^\delta_{m,i}$ with $\operatorname{diam}(R^\delta_{m,i}) < 1/m < \epsilon$ and a function $\tau : I(m) \to I(m)$ such that

$$V_m = \bigsqcup_{1\leq i\leq |I(m)|} R^\delta_{m,i} \quad \text{and} \quad f(\overline{R^\delta_{m,i}}) \subset R^\delta_{m,\tau(i)} \quad \text{for } 1 \leq i \leq |I(m)|.$$

Suppose that $x \in R^\delta_{m,j}$ and let $d_m = \max_{1\leq i\leq |I(m)|} \operatorname{diam}(R^\delta_{m,i})$.

Observe that if we fix any $z, w \in R^\delta_{m,j}$ and any $l \geq 1$, then

$$f^l(z) \in f^l(R^\delta_{m,j}), \quad f^l(w) \in f^l(R^\delta_{m,j}) \quad \text{and} \quad f^l(R^\delta_{m,j}) \subset R^\delta_{m,\tau^l(j)}.$$

Therefore, for any $z, w \in R^\delta_{m,j}$ and any $l \geq 1$,

$$d(f^l(z), f^l(w)) \leq d_m < \frac{1}{m} < \epsilon. \tag{5.37}$$

This, by Lemma 2.1, implies that, for any $y \in R^\delta_{m,j}$ and any $n \geq 1$ (recall that $x \in R^\delta_{m,j}$),

$$\rho(\mathcal{E}_n(x), \mathcal{E}_n(y)) \leq d_m < \epsilon.$$

By the definition of μ and $p\omega(y)$, there exists an integer $k > 0$ such that

$$d(\mu, \mathcal{E}_k(x)) < \tfrac{1}{2}(\varepsilon - d_m) \quad \text{and} \quad d(p\omega(y), \mathcal{E}_k(y)) < \tfrac{1}{2}(\varepsilon - d_m),$$

which yields that

$$\begin{aligned} d(p\omega(y), \mu) &\leq d(p\omega(y), \mathcal{E}_k(y)) + d(\mathcal{E}_k(y), \mathcal{E}_k(x)) + d(\mathcal{E}_k(x), \mu) \\ &< \tfrac{1}{2}(\varepsilon - d_m) + d_m + \tfrac{1}{2}(\varepsilon - d_m) = \varepsilon. \end{aligned}$$

This implies that $y \in A_\varepsilon(\mu)$, and hence, since $y \in R^\delta_{m,j}$ was arbitrary, $R^\delta_{m,j} \subset A_\varepsilon(\mu)$. Recall that $R^\delta_{m,j}$ is open and, therefore,

$$\mathscr{L}(A_\varepsilon(\mu)) \geq \mathscr{L}(R^\delta_{m,j}) > 0,$$

which completes our proof.

Step 3. $h_{\text{top}}(f, \Lambda) = 0$.

Recall the process of construction from Lemma 5.1. For any $n \in \mathbb{N}$, there are $|I(n)|$ small open sets $R^\delta_{n,i}$ with $\operatorname{diam}(R^\delta_{n,i}) < 1/n$ such that (5.34) holds. So if, for each i, we fix a point $x_{n,i} \in R^\delta_{n,i}$, then, by (5.37) for any $l \geq 1$, the set $\{x_{n,i}\}_{i=1}^{I(n)}$ constitutes an $(l, 1/n)$-spanning set of V_n.

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On the other hand, we can replace $I(n)$ by its subsequence and, since diameters of R_i^δ can be arbitrarily small, without loss of generality, we may assume that $|I(n)| < |I(n+1)|$ for each $n \geq 1$ and, therefore,

$$|I(n)| \geq n \quad \text{for } n \geq 1. \tag{5.38}$$

Fix any $\sigma > 0$ and select an integer $k \in \mathbb{N}$ such that $1/k < \sigma$. Since $\Lambda = \bigcap_{n\geq 1} \bigcup_{m\geq n} V_m$, in particular, $\Lambda \subset \bigcup_{m\geq k} V_m$. For any $m, n \in \mathbb{N}$, V_m can be covered by $|I(m)|$ Bowen balls $\{B_{|I(n+m)|}(x_{m,i}, 1/m)\}_{i=1}^{|I(m)|}$, and so

$$\begin{aligned} C(\Lambda; t, n, \sigma, f) &= \inf_{\mathcal{C}\in\mathcal{G}_n(\Lambda,\sigma)} \sum_{B_u(x,\sigma)\in\mathcal{C}} e^{-tu} \\ &\leq \sum_{m\geq k} |I(m)| e^{-t|I(n+m)|} \\ &\leq \sum_{m\geq k} |I(n+m)| e^{-t|I(n+m)|} \\ &\leq \sum_{s\geq |I(n+k)|} s e^{-ts} \\ &= \frac{|I(n+k)| e^{-t|I(n+k)|}}{1-e^{-t}} + \frac{e^{-t(|I(n+k)|+1)}}{(1-e^{-t})^2}. \end{aligned}$$

By our assumptions, when $n \to \infty$, then also $|I(n)| \to \infty$. Thus we see that, for any $t > 0$,

$$C(\Lambda; t, \sigma, f) := \lim_{n\to\infty} C(\Lambda; t, n, \sigma, f) = 0,$$

which implies that

$$h_{\text{top}}(\Lambda; \sigma, f) := \inf\{t : C(\Lambda; t, \sigma, f) = 0\} = 0.$$

This proves that

$$h_{\text{top}}(f, \Lambda) = \lim_{\sigma\to 0} h_{\text{top}}(\Lambda; \sigma, f) = 0.$$

Step 4. On one hand it is known (see [**23**] and [**21**, **22**], respectively) that shadowing is a generic property in $C(M)$ and in $H(M)$. On the other hand, [**22**, **45**] show that, for generic maps in $C(M)$ (respectively, $H(M)$), $h_{\text{top}}(f) = \infty$. Thus, by Theorem 1.5,

$$h_{\text{top}}(f, I(f)) = \infty,$$

which completes the proof. □

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