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Thicknesses of knots

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Abstract

In this paper we define a set of radii called thickness for simple closed curves denoted by K, which are assumed to be differentiable. These radii capture a balanced view between the geometric and the topological properties of these curves. One can think of these radii as representing the thickness of a rope in space and of K as the core of the rope. Great care is taken to define our radii in order to gain freedom from small pieces with large curvature in the curve. Intuitively, this means that we tend to allow the surface of the ropes that represent the knots to deform into a non smooth surface. But as long as the radius of the rope is less than the thickness so defined, the surface of the rope will remain a two manifold and the rope (as a solid torus) can be deformed onto K via strong deformation retract. In this paper we explore basic properties of these thicknesses and discuss the relationship amongst them.

1. *Introduction*

In this paper we examine the concept of thickness of a C^1 simple closed curve. The concept of 'thickness' for a curve is a much discussed topic, but little has appeared in the literature. Moffatt [**1**] studied a thickness of knots derived from a magnetic flux energy in the physics literature. This work indicated that the ratio of the thickness of a simple closed curve to its length is a dimensionless quantity which can be maximized over the conformations of the curve. This maximum value is called the 'thickness' of a knot(-type) and is a knot invariant. These new concepts are indeed very interesting and it seems necessary that a sound mathematical theory should be developed for the thickness of a curve.

A thickness for C^2 curves appeared in reference [**2**]. A key result in this work relates this thickness for C^2 curves to the maximum value of the curvature and the minimum of the distance between double critical points in the curve. (The points x and y in a

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C^1 curve α are double critical points if the cord joining them is perpendicular to the tangent vectors of α at x and y.) Thus, this thickness is understood in the sense that one can compute it by considering the curvature of the curve and the set of double critical points. Given a realization of a certain knot as a C^2 curve, one can indeed compute its thickness (in principle) and then attempt to take the supremum of the thickness over all the realizations of the knot as a C^2 curve to obtain a maximum thickness in the sense suggested by Moffatt. It is not clear whether one can realize a knot with this maximum thickness as a C^2 curve and this makes it necessary to generalize the concepts in [**2**] to a wider class of curves. In this paper we study possible generalizations to C^1-curves.

In Section 2 we will introduce two ϵ-parameter families of thicknesses for a C^1 curve K, denoted $T_\epsilon(K)$ and $t_\epsilon(K)$ respectively. We show that $t_0(K) = T_0(K)$ are only defined for C^2 curves, and are equal to the C^2-thickness defined in [**2**] (and this thickness is thus part of the families). If $0 < \epsilon < 2\pi/3$, then both $T_\epsilon(K)$ and $t_\epsilon(K)$ are defined for C^1 curves and we can therefore define thickness without reference to curvature. We prove in Section 4 that $t_0(K)$ and $T_0(K)$ are continuous functions in the space of C^2 knots of unit length, but neither $t_\epsilon(K)$ nor $T_\epsilon(K)$ are continuous in the space of C^1 knots. This seems to be the price for generalizing the definition of thickness to C^1 knots, and it is an outstanding question whether one can define a thickness for C^1 curves which is continuous in the space of unit length C^1 curves. Different thicknesses defined in [**4**] for curves which are not C^2 have been shown to be only lower semi-continuous (and not continuous).

It is desirable that the thickness of a curve reflects the topological properties of the curve (viewed as a knot). To this end, we require the following. For any number $0 < c < t_\epsilon(K)$ (or $T_\epsilon(K)$), the c-neighbourhood of K should be a solid torus which can be deformed to K via a strong deformation retract. Thus, the boundary surface of this c-neighbourhood is a surface. This property is true for the thickness defined in [**2**] and we will require our generalizations to C^1 curves to behave in the same way. In this sense, any thickness will capture the topology of the embedded curve; the complement of the embedded curve K will be isotopic to the complement of the c-neighbourhood of K.

Once the notion of the thickness of a knotted space curve is defined, one then defines the thickness of a knot type by taking the supremum of the thickness over all embeddings of the same knot type, where all embedded curves have unit length. An attempt to measure the thickness of knot types experimentally with ropes was made in reference [**3**]. In these experiments one thinks of the thickness as a constant, namely the radius of the rope, while the length of the rope is shortened as much as possible (keeping the same knot type). If one measures the length of the rope used to tie the knot, then the ratio of the radius of the rope over the total length of the rope used will be dimensionless and will (in the ideal situation) be independent of the particular thickness of the rope. This ratio is the experimentally measured thickness of a knot. Intuitively, it seems that the maximum of this ratio will be realized when the rope is bent tightly (depending on the quality of the rope there may be a fold in its surface), or when it is constrained by a 'contact' on the chord between double critical points. Both the $t_\epsilon(K)$ and $T_\epsilon(K)$ thicknesses allow for folds in the boundary of the $t_\epsilon(K)$-neighbourhood (or $T_\epsilon(K)$-neighbourhood). Lastly, we compute approximations to the maximum thickness of polygonal knots by simulated

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annealing. In particular, we consider a discrete analogue of the $t_\epsilon(K)$ thickness for polygonal knots and estimates of the maximal thickness for a polygonal knot (which is a trefoil) are computed in Section 5.

2. *The definitions of thickness*

In this section we consider some definitions of thickness and discuss their relative merits. Let K be a C^1 knot of length L parametrized by arc length and with parametric equation $\alpha(t)$. The c-neighbourhood of K is defined as $K(c) = \{x \in \mathbf{R}^3 : d(x, K) \leqslant c\}$. If $K(c)$ is a solid torus which can be deformed onto K via strong deformation retract, then $K(c)$ may be pictured as a closed rope with K as its core. The supremum over all such cs defines a thickness for the curve. The definition above is sound, but it has a serious drawback. One can find examples of curves K such that $K(c)$ is not homotopic to K for some values of c smaller than the above supremum. This property is at odds with our intuitive perception that $K(c)$ should be a model of a rope with core K. Any thinner rope within $K(c)$ and core K should also be of the same knot type K. This results in the following definition of a maximum thickness.

Definition of maximum thickness. Let K be a C^1 or piecewise linear simple closed curve. The maximum thickness of K, denoted by $t_m(K)$, is defined as the supremum over all cs such that for any $0 < c' < c$ the c'-neighbourhood of K is homotopic to K via strong deformation retract.

As we will see, this is indeed the maximum possible thickness that one can define (see Theorem 4·1). Calculation of $t_m(K)$ also seems difficult, since it depends not only on the geometry but also on the topology of K. The above definition requires us to find values of c_0 such that $K(c)$ can be strongly retracted onto K for any $c < c_0$. It is thus natural to consider the question: under which conditions does K have a natural structure which ensures that $K(c)$ can be retracted to K by strong deformation retract? If K is a C^2-curve, then one can find such a structure and the following definition is taken from reference [**2**].

Definition of disk thickness. Let K be a C^2 curve. A number $r > 0$ is said to be 'nice' if for any x, y on K, we have $D_x(r) \bigcap D_y(r) = \varnothing$, where $D_x(r)$ and $D_y(r)$ are disks of radius r centred at x and y on K and which are normal to K at x and y respectively. The disk thickness of K is the supremum over all the nice rs and will be denoted by $t_D(K)$.

For any $0 < r < t_D(K)$, $K(r)$ has a natural circle bundle structure $D^2 \times S^1$. Since each $D_x(r)$ is disjoint from $D_y(r)$ if $x \neq y$, $K(r)$ can be retracted onto K by retracting each $D_x(r)$ to x. It is possible to explicitly relate $t_D(K)$ to properties of K and this results in a formula for it [**2**]. The minimum over the distances between all doubly critical points on K is denoted by $D(K)$ for short.

THEOREM 2·1 ([**2**]). $t_D(K) = \min\{\frac{1}{2}D(K), \frac{1}{\kappa}\}$ *where* κ *is the maximum curvature of* K.

Since $t_D(K)$ depends on the curvature of K, it becomes undefined if K is not C^2. Moreover, it is dependent on the maximum curvature in the knot, so it has a tendency to lose the big picture of the knot. The requirement that the normal disks

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are disjoint in the definition of $t_D(r)$ is important; this ensures that there is a strong deformation retract of $K(r)$ onto K. But this is not a necessary condition. Given a C^1 knot K, one can show that for small enough c, $K(c)$ is homotopic to K by strong deformation retract (see Lemma 3·2). In fact, if normal disks $D_x(r)$ and $D_y(r)$ are disjoint for some $r > 0$ whenever the minimum arc length separation between x and y exceeds some number $\epsilon > 0$, then a strong deformation retract of $K(r)$ onto K is guaranteed (see Theorem 3·1 and Lemma 3·2). This observation leads to the following generalization of the disk thickness (where we define $s(x, y)$ to be the length of the shorter arc on a curve K between the points x and y on K).

Definition of T_ϵ-thickness. Let K be a C^1 curve. Fix an $\epsilon \in [0, 2\pi/3)$. Then c is an ϵ-nice number if $D_x(c) \bigcap D_y(c) = \varnothing$ for all $x, y \in K$ such that $s(x, y) \geqslant \epsilon c$. The ϵ-thickness of K is defined as

$$T_\epsilon(K) = \sup\{t\colon\ t \text{ is } \epsilon\text{-nice}\}. \tag{1}$$

Note that $T_0(K) = t_D(K)$.

It does not seem possible to prove a formula for the T_ϵ-thickness in terms of curvature or other properties of the curve (such as doubly critical points). This makes the explicit calculation of the T_ϵ-thickness of a given C^1-curve difficult, if not intractable. It seems desirable to define the thickness of a curve in terms of an explicit formula. We can do this as follows. Let $\alpha(t)$ be an arc length parameterized equation of K and let $x = \alpha(t_0)$, $y = \alpha(t_0 + h)$ be two points on K separated by an arc of length h along K. Let $d(x, y)$ be the distance between x and y and $\theta(x, y)$ be the smaller angle between $\alpha'(t_0)$ (the tangent vector of K at x) and $y - x = \alpha(t_0 + h) - \alpha(t_0)$. If K is a C^2 curve and $|\alpha''(t_0)| \neq 0$, then we can approximate the centre of curvature O of K at x by the point O' defined as follows. Let L be the intersecting line of the normal plane of K at x and the plane normal to $y - x$ at $(x + y)/2$. Notice that if h is small enough, then the existence of L is guaranteed by the condition $|\alpha''(t_0)| \neq 0$. Now define O' to be the point on L that is closest to x, then O' is uniquely determined. This is illustrated in Fig. 1, where the page is the plane determined by $y - x$ and $\alpha'(t_0)$ so that L is perpendicular to the page. It is obvious that O' must be on the page since it is closest to x. Note that K is not necessarily planar, it only cuts through the page at x and y. Solving for R in Fig. 1 gives $R = d(x, y)/2 \sin \theta(x, y)$. As we shall see in Remark 3 and Theorem 4·1, $1/R \to |\alpha''(t_0)|$ as $h \to 0$. Also, as $h \to 0$, then R approaches the curvature of K at x, and the vector $O' - x$ becomes parallel to $\alpha''(t_0)$. To see this, note that the plane determined by $\alpha'(t_0)$ and $y - x = \alpha(t_0 + h) - \alpha(t_0)$ is the same plane as the plane determined by $\alpha'(t_0)$ and $(y - x)/h^2 - \alpha'(t_0)/h = ((\alpha(t_0 + h) - \alpha(t_0))/h - \alpha'(t_0))/h$. Note that $((\alpha(t_0 + h) - \alpha(t_0))/h - \alpha'(t_0))/h$ and $O' - x$ point to the same side of $\alpha'(t_0)$. Now let $h \to 0$, then this becomes $\alpha''(t_0)$. With the comments in Remark 3 and Theorem 4·1, this shows that $O' \to O$ as $y \to x$. The computation of R can also be carried out if K is only C^1 since $\alpha(t)$ exists.

The t_0-thickness of K is the infimum of R for all points x and y in K. Similar to the previous definition we can increase the thickness by taking the infimum only over pairs of points separated at least distance ϵR along the curve K, i.e. the t_ϵ-thickness of K is the infimum of R over all pairs of points (x, y) such that $s(x, y) \geqslant \epsilon R$.

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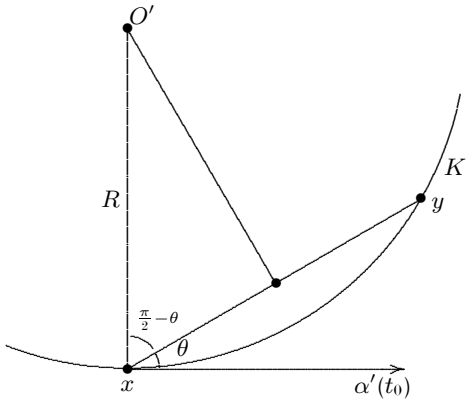


Fig. 1. O', x, y and $\alpha'(t_0)$ are on the page. If $\theta = \theta(x,y) \neq 0$, then $R = d(x,y)/2\sin\theta(x,y)$.

Definition of t_ϵ-thickness. Let k be a C^1 curve. Fix an $\epsilon \in [0, 2\pi/3)$, then the ϵ-thickness of K is defined as

$$t_\epsilon(K) = \inf_{x,y\in K}\left\{\frac{d(x,y)}{2\sin\theta(x,y)}\,\middle|\,\frac{2\sin\theta(x,y).s(x,y)}{d(x,y)} \geqslant \epsilon\right\}. \tag{2}$$

The condition $2\sin\theta(x,y).s(x,y)/d(x,y) \geqslant \epsilon$ is called the controlling condition and the function $2\sin\theta(x,y).s(x,y)/d(x,y)$ is called the controlling function. The set of all points $(x,y) \in K \times K$ defined by the controlling condition is denoted by M_ϵ.

The following are some important observations about the ϵ-thickness.

Remark 1. If $\epsilon > 0$, then $t_\epsilon(K)$ is defined for K and $t_\epsilon(K) > 0$. To see this, we observe that the controlling function $2\sin\theta(x,y).s(x,y)/d(x,y)$ is a continuous function over the compact set $K \times K$. It equals 0 if $x = y$. Since M_ϵ is a closed subset of $K \times K$, it is also compact. M_ϵ is not empty as long as $\epsilon \leqslant 2\pi/3$ (see Remark 4 below and the note after the proof of Lemma 3·5). In addition, $\sin\theta(x,y)$ is bounded away from 0 on M_ϵ. Hence, $d(x,y)/2\sin\theta(x,y)$ is a continuous function on M_ϵ and there exists $(x_0,y_0) \in M_\epsilon$ such that

$$t_\epsilon(K) = \inf_{(x,y)\in M_\epsilon}\left\{\frac{d(x,y)}{2\sin\theta(x,y)}\right\} = \frac{d(x_0,y_0)}{2\sin\theta(x_0,y_0)}. \tag{3}$$

Now $x_0 \neq y_0$, thus $d(x_0,y_0) > 0$ (K is a simple closed curve), and so $t_\epsilon(K) > 0$.

Remark 2. $t_\epsilon(K)$ is a non-decreasing function of ϵ for each fixed K. This is because $M_{\epsilon_1} \subset M_{\epsilon_2}$ whenever $\epsilon_1 > \epsilon_2$.

Remark 3. If $\epsilon = 0$ and K is a C^2 curve, then $M_0 = K \times K$ and $\sin\theta(x,y)$ is not bounded away from 0. One can show that

$$\lim_{y\to x}\frac{2\sin\theta(x,y)}{d(x,y)} = \kappa(x) \tag{4}$$

where $\kappa(x)$ is the curvature of K at x. Furthermore, this limit converges uniformly on K. This will be proved in Theorem 4·1 when we show the continuity of $t_0(K)$. Thus, $2\sin\theta(x,y)/d(x,y)$ can be extended to a continuous function on $K \times K$. It then becomes clear that $T = \max\{2\sin\theta(x,y)/d(x,y)\} > 0$ exists and so $t_0(K) = 1/T > 0$ is also defined.

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Remark 4. The condition $\epsilon < 2\pi/3$ is essential in the definition of the t_ϵ- and T_ϵ-thicknesses. This can be seen directly for the T_ϵ-thickness. Consider the unit circle with three points on it separated by arc-length $2\pi r/3$. Now push the three points slightly inward so that the resulting curve remains smooth and the normal planes at the three points all contain the y-axis. For most values of ϵ the thickness of the new curve is defined by the minimal normal disks around these three points. By making the deformation of the circle very small, we obtain a curve whose thickness is as close to 1 as we choose. But for some $\epsilon_0 > 2\pi/3$ these three points will not be used in the calculation of the thickness. As a result, we will get a thickness of one. Thus we will be able to find a $c > 0$ that is close enough to 1 so that the c-neighbourhood of the curve is no longer a solid torus. The details of the above are tedious and long and it will not serve any purpose here, instead, we leave it as an exercise for the interested reader.

3. *The main theorems*

In this section we examine the definitions of $T_\epsilon(K)$ and $t_\epsilon(K)$. In particular, we show that for any value of c, $0 < c < T_\epsilon(K)$ (or $0 < c < t_\epsilon(K)$), K is a strong deformation retract of its neighbourhood $K(c)$. As in the previous section, we take K to be a C^1 embedding of length L of the circle in $\mathscr{R}^3$.

Theorem 3·1. *$T_\epsilon(K)$ defines a thickness of K, i.e. $K(c)$ is a solid torus which is homotopic to K via a strong deformation retract for every $c < T_\epsilon(K)$.*

We prove Theorem 3·1 in the four lemmas (Lemmas 3·2 to 3·5). In Lemma 3·2 we prove that there exists a c_0 such that $K(c)$ is a solid torus which is homotopic to K via strong deformation retract for all values of c in $(0, c_0]$. Lemma 3·2 seems to be a known result [**5**], but we could not find a published proof. We therefore present a proof of this result here. We then make a connection between this result and $T_\epsilon(K)$ by showing that for every $\epsilon > 0$ there exists an interval $(0, c_0)$ of ϵ-nice numbers. This implies that $T_\epsilon(K) > 0$ if $\epsilon > 0$ (and if $\epsilon < 2\pi/3$). Lastly, we show that the boundary of $K(c)$ is a surface and that $K(c)$ is a solid torus which is homotopic to K via strong deformation retract.

Lemma 3·2. *Let K be a C^1 curve. There exists $c_0 > 0$ such that $K(c)$ is a solid torus which is homotopic to K via strong deformation retract if $0 < c \leqslant c_0$.*

Proof. The goal of the proof is to define a product structure $K \times D(x)$ on $K(c)$, where $D(x)$ is a planar disk containing x in its interior such that $D(x) \bigcap K = x$. This product structure can then be used to define a strong deformation retract from $K(c)$ to K. Let $\alpha(t)$ be an arc-length parametrization of K. Define

$$||s - t|| = \min_{z \in \mathscr{Z}} \{|s - t + zL|\}. \tag{5}$$

Since $\alpha'(t)$ is continuous on a compact set S^1, it is also uniformly continuous. For $\epsilon > 0$ there exists a $\delta_0 > 0$ such that $|\alpha'(t) - \alpha'(s)| < \epsilon/3$ whenever $||s - t|| < \delta_0$. Since K is not singular, there exists a positive number

$$\delta_1 = \inf_{||s-t|| \geqslant \delta_0} |\alpha(s) - \alpha(t)|. \tag{6}$$

Choose m such that $L/m < \delta_0$. Partition K into m arcs of equal length $\ell_1, \ell_2, \ldots, \ell_m$

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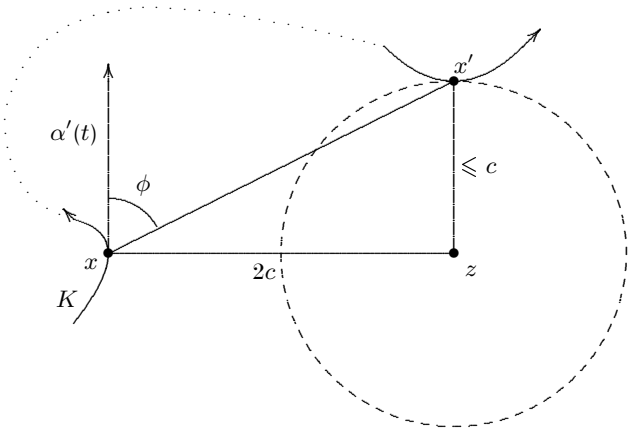


Fig. 2. z is in $E_x(c)$ and a distance $2c$ from x. Since there exists a point x' in K such that $d(z, x') \leqslant c$, the distance between x and x' is at most $3c$ and ϕ is at least 1 radian.

by selecting points $x_0, x_1, \ldots, x_m = x_0$ along K, such that the endpoints of ℓ_i is x_{i-1} and x_i. Define $d(\ell_i, \ell_j)$ to be minimum distance between any point in ℓ_i and any point in ℓ_j. Let $s(i, j) = \min\{|i - j|, m - |i - j|\}$. Define

$$h = \min_{i,j} \{d(\ell_i, \ell_j) | s(i, j) > 1\}.$$

Then h is the minimum separation between points on non-adjacent segments ℓ_i and ℓ_j and $h \leqslant \min\{L/4, \delta_1\}$. Notice that $|x_i - x_j| \geqslant h$ if $i \neq j$. Define Σ_x to be the normal plane to K at x. Let $c \leqslant c_0 < h/4$. For any $x \in K$, let $E_x(c)$ be the path component of $\Sigma_x \bigcap K(c)$ which contains x. We now prove the following.

Claim 1. $E_x(c)$ is star-convex with respect to x, i.e. each ray in Σ_x starting from x intersects $\partial E_x(c)$ only once.

Claim 2. $E_x(c) \bigcap K = \{x\}$.

Claim 3. $E_{x_i} \bigcap E_{x_j} = \varnothing$ if $i \neq j$.

We first show that if $y \in E_x(c)$, then $|x - y| < 2c$. Assume that there is a $z_1 \in E_x(c)$ and $|x - z_1| \geqslant 2c$. Since $E_x(c)$ is path-connected, there is a path P from z_1 to x in $E_x(c)$ and a point $z \in P$ such that $|x - z| = 2c$. Suppose that $x = \alpha(t)$ and let $x' = \alpha(s)$ such that $|z - x'| = d(z, K) \leqslant c$. Then we note that $|x - x'| \leqslant |x - z| + |z - x'| \leqslant 3c < \delta_1$; thus $||t - s|| < \delta_0$, by (6). Without loss of generality, let $s < t$; then $0 < t - s < \delta_0$. The situation is illustrated in Fig. 2. (This drawing is really a caricature of the true situation: if we assume that $z - x$ and $\alpha'(t)$ define the plane of the page in Fig. 2, then x' may not lie in the same plane at all.) Define the angle ϕ as $\theta(x, x')$, i.e. the angle between $\alpha'(t)$ and the line segment from x to x'. By projecting the angle ϕ to the same plane as the triangle xzx', we note that $\phi \geqslant 1$ radian, as a natural consequence of the constraints on the lengths of the sides of the triangle xzx'.

Now $\alpha(t) - \alpha(s) = \int_s^t \alpha'(u)du$ and, by taking the inner product of this with $\alpha'(t)$, we get

$$\alpha'(t) . (\alpha(t) - \alpha(s)) = \int_s^t \alpha'(t) . \alpha'(u)du = \int_s^t \cos\theta(u)du, \tag{7}$$

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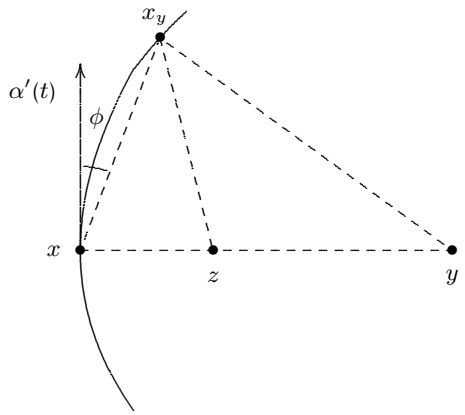


Fig. 3. The distance between x and x_y is less than $3c$; we can bound ϕ as in the text. Notice that, as in Figs 1 and 2, it may be that x_y is not in the same plane as $\alpha'(t)$ and $y - x$.

where $\theta(u)$ is the angle between $\alpha'(t)$ and $\alpha'(u)$. Since $||t - u|| < \delta_0$, we have

$$\sin(\theta(u)/2) = |\alpha'(t) - \alpha'(u)|/2 < \epsilon/6, \tag{8}$$

hence

$$\int_s^t \cos\theta(u)du = \int_s^t (1 - 2\sin^2(\theta(u)/2))du > (t - s)(1 - \epsilon^2/18). \tag{9}$$

It follows that $|\alpha(t) - \alpha(s)| \cos\phi > (1 - \epsilon^2/18)(t - s)$ and $\cos\phi > 1 - \epsilon^2/18 \approx 1$ if we choose ϵ to be a number small enough to begin with. On the other hand we note from Fig. 2 that $1 \leqslant \phi \leqslant \pi/2$. This means that $\cos\phi < 0{\cdot}55$ and we have a contradiction.

We can now prove the first claim: let y be a point in $E_x(c)$. We need to show that the line segment between x and y is in $E_x(c)$. If $|x - y| \leqslant c$, then there is nothing to prove. So suppose that $c < |x - y| < 2c$. Let $x_y \in K$ such that $|y - x_y| \leqslant c$ (x_y exists, since $y \in K(c)$). Let ϕ be the angle between $x_y - x$ and $\alpha'(t)$ (as above, we take $x = \alpha(t)$). This situation is illustrated in Fig. 3. Now note that $|x - x_y| \leqslant |x - y| + |y - x_y| < 3c < \delta_1$. Thus, we can repeat the argument above for x and x_y to find that $\cos\phi > 1 - \epsilon^2/18$, hence ϕ is close to 0 if ϵ is small. Since $\angle x x_y y \geqslant \angle y x x_y$ and $\angle y x x_y$ is close to $\pi/2$, it follows that $\angle x y x_y$ must be small. Thus for any z on the line segment xy ($z \neq y$), we have $|z - x_y| \leqslant |y - x_y| < c$, i.e. z is an interior point of $E_x(c)$. This proves Claim 1.

As a by-product we also get Claim 2: if $y \in K$ and $y \in E_x(c)$, then we can take $x_y = y$ in Fig. 3. This collapses the triangle to a line segment, and then $\phi = \pi/2$, which contradicts the fact that $\phi \leqslant \sqrt{\epsilon}$. Lastly, refer to the points x_i and suppose that $E_{x_i}(c) \bigcap E_{x_j}(c) \neq \varnothing$. Then $|x_i - x_j| < 4c < h$, which is a contradiction, since we chose the x_i such that they are at least a distance h apart. This gives Claim 3.

We can now complete the proof of Lemma 3·2: by the three claims above, we conclude that each $E_x(c)$ is a topological disk which is planar and star convex with respect to x. Let Σ'_x be a plane through x, such that the angle between Σ'_x and Σ_x is less than $\epsilon/2$, i.e. Σ'_x is almost perpendicular to K at x. The above statements and their proofs remain true if $E_x(c)$ is replaced by the path component $P_x(c)$ of $\Sigma'_x \bigcap K(c)$ containing x.

We will now endow $K(c)$ with a product structure $K \times P_x(c)$, where the $P_x(c)$ are

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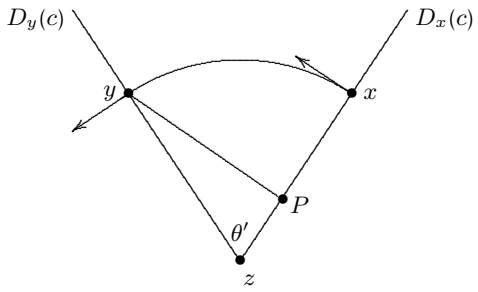


Fig. 4. z is an intersection point of $D_x(c)$ and $D_y(c)$. x, y and z are on the plane of the page but not necessarily T_x and T_y. The angle θ between T_x and T_y is greater than or equal to the angle θ' shown here. $y - P$ is perpendicular to $x - z$ and we have shown that $d(y, P) > \frac{3}{4}c\epsilon$.

disjoint topological planar disks whose angle with the tangent T_x changes continuously as x moves along K. Consider the planes Σ_{x_i} and $\Sigma_{x_{i+1}}$. If these planes are parallel, and $x \in \ell_{i+1}$, then let Σ'_x be the plane parallel to Σ_{x_i} containing x. If Σ_{x_i} and $\Sigma_{x_{i+1}}$ are not parallel, then define Σ'_x to be the plane determined by $x \in \ell_{i+1}$ and the intersection line $\Sigma_{x_i} \bigcap \Sigma_{x_{i+1}}$. (Notice that the minimum distance between x_i (or x_{i+1}) and this intersection line is greater than c.) If $x = x_i$, then we pick $\Sigma'_{x_i} = \Sigma_{x_i}$. Now let $P_x(c)$ be $K(c) \bigcap \Sigma'_x$. The disjoint union of the geometric disks $P_x(c)$ in the planes Σ'_x is the solid torus K_c with the necessary product structure.

Lemma 3·3. *For each $0 < \epsilon < 2\pi/3$ there exists an interval $(0, c_0)$ of ϵ-nice numbers, so $T_\epsilon(K) > 0$.*

Proof. Let δ_0 and δ_1 be defined as in the proof of Lemma 3·2. Let $c_0 < \min\{\delta_0, \delta_1/2\}$. We show that for any $c \in (0, c_0)$, for some value of c_0, the normal disks $D_x(c)$ and $D_y(c)$ are disjoint if $s(x, y) \geqslant \epsilon c$. The proof is by contradiction. Suppose there exist $x = \alpha(s)$ and $y = \alpha(t)$ and a value of $c \in (0, c_0)$ such that $s(x, y) \geqslant \epsilon c$ and $D_x(c) \bigcap D_y(c) \neq \varnothing$. Then $|x - y| \leqslant 2c < \delta_1$ and so $s(x, y) < \delta_0$ by definition of δ_1. Thus $\epsilon c \leqslant s(x, y) < \delta_0$. Next we like to bound the $d(y, \Sigma_x)$ from below. We have $d(y, \Sigma(x)) = (\alpha(t) - \alpha(s)) . \alpha'(s) = \int_s^t \alpha'(s)\alpha'(u)du = \int_s^t \cos\theta(u)$ where $\theta(u)$ again is the angle between $\alpha'(s)$ and $\alpha'(u)$. By (9), we obtain $d(y, \Sigma(x)) \geqslant (1 - \epsilon^2/18)(t - s) \geqslant \frac{3}{4}c\epsilon$ since $\epsilon < 2\pi/3$ and $t - s \geqslant c\epsilon$. Let $P \in z - x$ such that $y - P \perp z - x$, then $d(y, P) \geqslant d(y, \Sigma(x)) \geqslant \frac{3}{4}c\epsilon$. Let θ be the angle between $\alpha'(t)$ and $\alpha'(s)$ and let θ' be the angle as shown in Fig. 4, then $\sin\theta \geqslant \sin\theta' = d(y, P)/d(y, z) \geqslant 3\epsilon/4$ (remember that $d(y, z) \leqslant c$). But on the other hand, by (8), $\sin\theta \leqslant 2\sin(\theta/2) < \epsilon/3$, a contradiction. □

For any point $p \in \partial K(c)$, let $N(p)$ be the set of points on K that have distances exactly c from p, in other words, $N(p) = B(p, c) \bigcap K$ where $B(p, c)$ is the geometric 3-ball that has radius c and which is centred at p. Let $\mathscr{H}(N(p))$ be the convex hull of $N(p)$. If $p \in \mathscr{H}(N(p))$ then p is called a critical point. The corresponding c value will then be called a critical value. By theorem 3·1 in [**6**], $\partial K(c)$ is a two manifold as long as it does not contain critical points.

Lemma 3·4. *$\partial K(c)$ contains no critical points for any $c < T_\epsilon(K)$ if $\epsilon < 2\pi/3$.*

Proof by contradiction. Assume this is not true, i.e. there exists a $c < T_\epsilon(K)$ such that $\partial K(c)$ contains at least one critical point. Let p be such a critical point and

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$N(p) = B(p,c) \bigcap K$ as defined above. Since $p \in \mathscr{H}(N(p))$, $N(p)$ contains at least two points. There must be two points x and y in $N(p)$ such that $s(x,y) \geqslant 2\pi c/3$, otherwise $N(p)$ will be contained entirely in a hemisphere and p (which is the centre of the sphere) will not be in $\mathscr{H}(N(p))$. But then $p \in \text{int}\,(D_x(c+\delta)) \bigcap \text{int}\,(D_y(c+\delta))$ for any $\delta > 0$. If δ is small enough, we will have and $s(x,y) \geqslant 2\pi c/3 > \epsilon(c+\delta)$, since $\epsilon < 2\pi/3$. Thus c is not ϵ-nice, this contradicts the definition of $T_\epsilon(K)$. □

We are now ready to tie the above lemmas into a proof for Theorem 3·1.

Lemma 3·5. *For any $c < T_\epsilon(K)$, $K(c)$ is a solid torus which is homotopic to K via strong deformation retract.*

Proof. Let c_0 be the smaller of the two c_0s in Lemmas 3·2 and 3·3. Then $K(c)$ is a solid torus which is homotopic to K via strong deformation retract for all $c \leqslant c_0$. If $T_\epsilon(K) > c > c_0$, then, by Lemma 3·4, $[c_0, c]$ contains no critical values. By theorem 4·1 of [**6**] $K(c) \backslash K^\circ(c_0)$ is homeomorphic to $[0,1] \times \partial K(c_0)$. So $K(c)$ can be deformed onto $\partial K(c_0)$ via strong deformation retract and then $K(c_0)$ can be deformed onto K. This completes the proof of the lemma and of Theorem 3·1.

Note. Since $K(c)$ is a 3-ball if c is large enough, there must exist critical values of c. At any of the given critical points, one of them will have two points x and y on K such that $s(x,y) \geqslant 2\pi c/3$. Note that then $d(x,y) = 2c \sin\theta(x,y)$ and $2\sin\theta(x,y)s(x,y)/d(x,y) \geqslant 2c\pi/3 \times 2\sin\theta(x,y)/d(x,y) = 2\pi/3 \geqslant \epsilon$. Hence $M_\epsilon \neq \varnothing$ for any $\epsilon \in (0, 2\pi/3]$.

Theorem 3·6. *$t_\epsilon(K)$ also defines a thickness of K, i.e. $K(c)$ is a solid torus which is homotopic to K via a strong deformation retract for every $c < t_\epsilon(K)$.*

Proof. In light of Theorem 3·1, it suffices to show that $t_\epsilon(K) \leqslant T_\epsilon(K)$. We will show this for each $\epsilon > 0$. By letting $\epsilon \searrow 0$, we also obtain $t_0(K) \leqslant T_0(K)$. (In Theorem 4·2 we show that $t_0(K) = T_0(K)$.)

Let x and y be two points on K and suppose that $c < t_\epsilon(K)$. We must show that if $s(x,y) \geqslant c\epsilon$, then $D_x(c) \bigcap D_y(c) = \varnothing$. Let Σ be the plane bisecting the line-segment $y - x$. If the intersection between $D_x(c)$ and $D_y(c)$ is not empty, then one, say $D_x(c)$, intersects Σ (see Fig. 1). Thus, there is a point p in $\Sigma \bigcap D_x(c)$ such that $d(p,x) \leqslant c$. But $d(p,x) \geqslant d(O',x) = d(x,y)/2\sin\theta(x,y)$ by the definition of O' (see Fig. 1). Thus $c \geqslant d(x,y)/2\sin\theta(x,y) \geqslant t_\epsilon(K)$, by the definition of $t_\epsilon(K)$. This is a contradiction.

4. *Properties of thicknesses*

In this section we prove several results involving $t_\epsilon(K)$ and $T_\epsilon(K)$. In particular, we first discuss the question of continuity. In general, $t_\epsilon(K)$ and $T_\epsilon(K)$ are not continuous functions on the space of closed C^1-curves. However Theorem 4·1 shows that $t_0(K)$ is continuous in C^2. An example of a curve which illustrates the discontinuity of $T_\epsilon(K)$ is given in Fig. 5.

Let C^k be the space of knots of unit lengths with continuous kth derivatives. The distance between two elements K_1 and K_2 with arc-length parametric equations $\alpha(t)$

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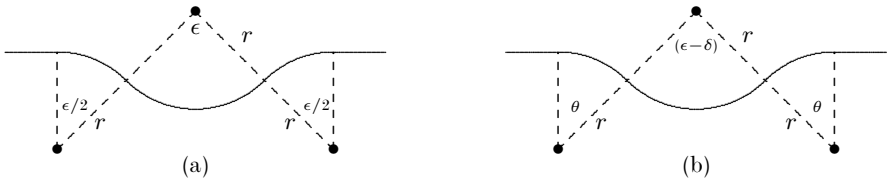


Fig. 5. A knotted arc can be constructed which contains the curve in (a) such that its T_ϵ-thickness is exactly r. By slightly perturbing the curve (let $\epsilon \to (\epsilon - \delta)$, where δ is a very small angle) we obtain (b), and in this case the T_ϵ-thickness is not determined by any pair of points on this piece of the curve. The angle θ is equal to $(\epsilon - \delta)/2$. (This is not a trivial exercise, but we nevertheless leave it to the reader to fill in the details).

and $\beta(t)$ in C^k is defined to be

$$d(K_1, K_2) = \inf \sup_{0 \leqslant t \leqslant 1} \sum_{j=0}^{k} |\alpha^{(j)}(t) - \beta^{(j)}(t)|, \tag{10}$$

where the infimum is taken over all possible parametrizations of both curves. It can be shown that (C^k, d) is a metric space. Note that if $\epsilon > 0$ then $t_\epsilon(K)$ and $T_\epsilon(K)$ are functions from C^1 to $\mathscr{R}$ and $t_0(K) = T_0(K)$ is a function from C^2 to $\mathscr{R}$.

THEOREM 4·1. *$t_0(K)$ is continuous in C^2.*

Proof. Let $K \in C^2$ and $\alpha(t)$ be an arc-length parametrized equation of K. As claimed in Remark 3 in Section 2, we first show that

$$\lim_{h \to 0} \frac{2 \sin \theta(\alpha(t), \alpha(t+h))}{d(\alpha(t), \alpha(t+h))} = |\alpha''(t)| \tag{11}$$

and the convergence to the limit is uniform. Let $\alpha(t) = (x_1(t), x_2(t), x_3(t))$, then by the mean value theorem,

$$\alpha(t+h) - \alpha(t) = \alpha'(t)h + z\tfrac{1}{2}\alpha''(t)h^2 + \tfrac{1}{2}(\alpha - \alpha''(t))h^2, \tag{12}$$

where $\alpha = (x_1''(\eta_1), x_2''(\eta_2), x_3''(\eta_3))$ and $|t - \eta_i| \leqslant h$. Since $\alpha''(t)$ is continuous on a compact set, it is also uniformly continuous. So $|\alpha - \alpha''(t)| \to 0$ uniformly as $h \to 0$. Thus, computing $d^2(\alpha(t), \alpha(t+h))$ from the above, we get that

$$\frac{h^2 - d^2(\alpha(t), \alpha(t+h))}{h^3} \longrightarrow 0 \tag{13}$$

uniformly as $h \to 0$. Define $d = d(\alpha(t), \alpha(t+h))$ and $\theta = \theta(\alpha(t), \alpha(t+h))$. Then

$$\begin{aligned} \frac{\sin(\theta/2)}{d} &= \frac{1}{2d}\left|\frac{\alpha(t+h) - \alpha(t)}{d} - \alpha'(t)\right| \\ &= \frac{1}{2}\left|\frac{1}{d}\left(\frac{h}{d} - 1\right)\alpha'(t) + \frac{h^2}{2d^2}\alpha''(t) + \frac{h^2}{2d^2}(\alpha - \alpha''(t))\right|. \end{aligned} \tag{14}$$

Using (13), we see that as $h \to 0$, $1/d((h/d) - 1) \to 0$ uniformly, hence $(\sin(\theta/2))/d \to |\alpha''(t)|/4$ uniformly. Notice that from this we also have $\sin \theta/2 \to 0$ uniformly as $h \to 0$. This proves (11) since

$$\lim_{\theta \to 0} \frac{\sin \theta}{2 \sin(\theta/2)} = 1. \tag{15}$$

Now assume that K has unit length and let K' be a second closed C^2 curve of unit

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length. For any given $\epsilon > 0$, we need to show that there exists a $\delta_1 > 0$ such that $|t_0(K) - t_0(K')| < \epsilon$ if $d(K, K') < \delta_1$. Let K_n be any closed C^2 curve of unit length such that $d(K, K_n) < 1/n$. It suffices to show that

$$\frac{2\sin\theta(\beta_n(t), \beta_n(t+h))}{d(\beta_n(t), \beta_n(t+h))} \longrightarrow \frac{2\sin\theta(\alpha(t), \alpha(t+h))}{d(\alpha(t), \alpha(t+h))} \tag{16}$$

uniformly as $n \to \infty$, where $\beta_n(t)$ is an arc-length parametrized equation of K_n such that

$$\sup_{0\leqslant t\leqslant 1} \sum_{j=0}^{2} |\alpha^{(j)}(t) - \beta_n^{(j)}(t)| < \frac{1}{n}. \tag{17}$$

The following uniformly convergent limits can be verified:

$$\lim_{n\to\infty} \sin\theta(\beta_n(t), \beta_n(t+h)) = \sin\theta(\alpha(t), \alpha(t+h)) \tag{18}$$

and

$$\lim_{n\to\infty} d(\beta_n(t), \beta_n(t+h)) = d(\alpha(t), \alpha(t+h)). \tag{19}$$

Let $\epsilon > 0$ be any given number, then there exists $\delta > 0$ and $N_0 > 0$ such that if $0 \leqslant h \leqslant \delta$ and $n > N_0$, then for any $0 \leqslant t \leqslant 1$ we have

$$\left| \frac{2\sin\theta(\alpha(t), \alpha(t+h))}{d(\alpha(t), \alpha(t+h))} - |\alpha''(t)| \right| < \frac{\epsilon}{3}, \tag{20}$$

$$\left| \frac{2\sin\theta(\beta_n(t), \beta_n(t+h))}{d(\beta_n(t), \beta_n(t+h))} - |\beta_n''(t)| \right| < \frac{\epsilon}{3} \tag{21}$$

and

$$|\alpha''(t) - \beta_n''(t)| < \frac{\epsilon}{3}, \tag{22}$$

where (21) can be proved similar to the proof of (11) with the help of (17) for uniform bounds. Thus,

$$\begin{aligned} &\left| \frac{2\sin\theta(\beta_n(t), \beta_n(t+h))}{d(\beta_n(t), \beta_n(t+h))} - \frac{2\sin\theta(\alpha(t), \alpha(t+h))}{d(\alpha(t), \alpha(t+h))} \right| \\ &\quad \leqslant \left| \frac{2\sin\theta(\beta_n(t), \beta_n(t+h))}{d(\beta_n(t), \beta_n(t+h))} - |\beta_n''(t)| \right| \\ &\qquad + |\beta_n''(t) - \alpha''(t)| \\ &\qquad + \left| |\alpha''(t)| - \frac{2\sin\theta(\alpha(t), \alpha(t+h))}{d(\alpha(t), \alpha(t+h))} \right| \leqslant \epsilon. \end{aligned} \tag{23}$$

On the other hand, if $\frac{1}{2} \geqslant h \geqslant \delta$, then there exists $N_1 > 0$ such that if $n > N_1$, then

$$\left| \frac{2\sin\theta(\beta_n(t), \beta_n(t+h))}{d(\beta_n(t), \beta_n(t+h))} - \frac{2\sin\theta(\alpha(t), \alpha(t+h))}{d(\alpha(t), \alpha(t+h))} \right| < \epsilon, \tag{24}$$

by (18) and (19). Let $N = \max\{N_0, N_1\}$, then

$$\left| \frac{2\sin\theta(\beta_n(t), \beta_n(t+h))}{d(\beta_n(t), \beta_n(t+h))} - \frac{2\sin\theta(\alpha(t), \alpha(t+h))}{d(\alpha(t), \alpha(t+h))} \right| < \epsilon \tag{25}$$

for all t and h if $n > N$. □

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There is a relation between the thicknesses. In particular, we note that by definition, $t_m(K)$ is the largest and that $t_\epsilon(K)$ and $T_\epsilon(K)$ are non-decreasing functions of ϵ. In fact, $t_\epsilon(K) \leqslant T_\epsilon(K)$.

THEOREM 4·2. $t_D(K) = t_0(K) = T_0(K) \leqslant t_\epsilon(K) \leqslant T_\epsilon(K) \leqslant t_m(K)$. *Furthermore,* $t_\epsilon(K)$ *and* $T_\epsilon(K)$ *are nondecreasing functions of* ϵ *for each fixed* K.

Proof. The only part that we must still show is $t_D(K) = t_0(K)$. Note that $t_0(K) \leqslant t_D(K)$ by Theorem 2·1 since $d(x,y)/2\sin\theta(x,y) \to 1/\kappa(x)$ when $y \to x$ and $d(x,y)/2\sin\theta(x,y) = d(x,y)/2$ when x and y are doubly critical points. So we need to show that $t_0(K) \geqslant t_D(K)$. Let $x_0, y_0 \in K$ such that $t_0(K) = d(x_0,y_0)/2\sin\theta(x_0,y_0)$. x_0 and y_0 exist since $2\sin\theta(x,y)/d(x,y)$ is a continuous function on the compact set $K \times K$. If $x_0 = y_0$, then we have $t_0(K) = 1/\kappa(x_0)$, hence $t_0(K) \geqslant \min_{x\in K}(1/\kappa(x)) \geqslant t_D(K)$ by Theorem 2·1 and we are done.

Now we need to consider the case that $t_0(K)$ cannot be obtained at a point (x,y) with $x = y$. So let (x_0,y_0) where $x_0 \neq y_0$ be a realizing point of $t_0(K)$. We need to prove that $\theta(x_0,y_0) = \theta(y_0,x_0) = \pi/2$. Suppose that $\theta_0 = \theta(x_0,y_0) \geqslant \theta(y_0,x_0)$; we first show that $\theta_0 = \pi/2$. Assume not, so that $\theta_0 < \pi/2$ (notice that we must have $\theta_0 > 0$) hence $\cos\theta_0 > 0$. We need also to observe that $2\sin\theta_0/d(x_0,y_0) > \kappa(x_0)$ since $2\sin\theta_0/d(x_0,y_0)$ is greater than the maximum curvature of K.

Let $\alpha(t_0) = x_0$ and assume that $\alpha'(t_0).(y_0 - \alpha(t_0)) = d(x_0,y_0)\cos\theta_0$ (the other case is $\alpha'(t_0).(\alpha(t_0) - y_0) = d(x_0,y_0)\cos\theta_0$). Also let $r(t) = |y_0 - \alpha(t)|$ and let $\theta(t)$ be the angle between $\alpha'(t)$ and $y_0 - \alpha(t)$ so we have $\alpha'(t).(y_0 - \alpha(t)) = r(t)\cos\theta(t)$. Since $f(t) = r(t)/\sin\theta(t)$ obtains its minimal value at $t = t_0$,

$$f'(t) = \frac{-\cos\theta(t)\sin\theta(t) - r(t)\cos\theta(t)\theta'(t)}{\sin^2\theta(t)} \tag{26}$$

(notice that $r'(t) = -\cos\theta(t)$) must be 0 when $t = t_0$. And it follows that

$$\theta'(t_0) = -\frac{\sin\theta_0}{r(t_0)}, \tag{27}$$

since $\cos\theta_0 > 0$.

On the other hand, from $\cos\theta(t) = \alpha'(t).(y_0 - \alpha(t))/r(t)$ we have, after taking the derivative,

$$\begin{aligned} -\sin\theta(t)\theta'(t) &= \frac{1}{r^2(t)}((\alpha''(t).(y_0 - \alpha(t)) - 1)r(t) - r'(t)\alpha'(t).(y_0 - \alpha(t)), \\ &= \frac{1}{r^2(t)}((\alpha''(t).(y_0 - \alpha(t)) - 1)r(t) + r(t)\cos^2\theta(t)) \\ &= \frac{1}{r(t)}(\alpha''(t).(y_0 - \alpha(t)) - \sin^2\theta(t)). \end{aligned} \tag{28}$$

where we have used (27). So

$$\theta'(t_0) = \frac{\sin\theta_0}{r(t_0)} - \frac{1}{\sin\theta_0 r(t_0)}\alpha''(t).(y_0 - \alpha(t)). \tag{29}$$

Combining the two equations (27) and (29) yields

$$2\sin^2\theta_0 = \alpha''(t_0).(y_0 - \alpha(t_0)). \tag{30}$$

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Let γ be the angle between $\alpha''(t_0)$ and $y_0 - \alpha(t_0)$, then $2\sin^2\theta_0 = \alpha''(t_0).(y_0 - \alpha(t_0)) = |\alpha''(t_0)|r(t_0)\cos\gamma$, hence $\pi/2 > \gamma$. But $\alpha''(t_0)$ and $\alpha'(t_0)$ are perpendicular to each other hence $\gamma \geqslant \pi/2 - \theta_0$ so $\cos\gamma \leqslant \cos(\pi/2 - \theta_0) = \sin\theta_0$ and finally we have

$$\kappa(x_0) = \frac{2\sin^2\theta_0}{r(t_0)\cos(\gamma)} = |\alpha''(t_0)| \geqslant \frac{2\sin\theta_0}{r(t_0)}, \tag{31}$$

which is the contradiction we want.

We also need to show that $\theta(y_0, x_0) = \pi/2$. Let $y_0 = \alpha(t_1)$, $\theta(t) = \theta(x_0, \alpha(t))$, $\theta_1(t) = \theta(\alpha(t), x_0)$ and $r_1(t) = |x_0 - \alpha(t)|$. Thus the function

$$f(t) = \frac{r_1(t)}{2\sin\theta(t)} \tag{32}$$

is minimized at $t = t_1$ hence we have $f'(t_1) = 0$. But $r_1'(t) = -\cos\theta_1(t)$ hence we have

$$f'(t) = -\frac{\cos\theta_1(t)\sin\theta(t) + r_1(t)\cos\theta(t)\theta'(t)}{2\sin^2\theta(t)}. \tag{33}$$

Thus (notice that we have shown $\theta(t_1) = \pi/2$):

$$f'(t_1) = -\cos\theta_1(t_1)/2 = 0. \tag{34}$$

So $\theta_1(t_1) = \theta(y_0, x_0) = \pi/2$. □

Examples of the thicknesses of some curves are given below.

Example 1. Let K be a circle of radius r then, as expected for any $2\pi/3 \geqslant \epsilon \geqslant 0$, $t_\epsilon(K) = T_\epsilon(K) = t_m(k) = t_D(K) = r$.

Example 2. Let K be the ellipse $x^2/16 + y^2/9 = 1$, then $t_0(K) = t_D(K) = 9/4 = 2{\cdot}25$, which is given by the reciprocal of the maximal curvature. Moreover $t_{1{\cdot}34}(K) \approx 2{\cdot}40$, $T_{1{\cdot}34}(K) \approx 2{\cdot}50$, $t_{1{\cdot}80}(K) \approx 2{\cdot}53$, $T_{1{\cdot}80}(K) \approx 2{\cdot}75$ and $t_{2{\cdot}09}(K) \approx 2{\cdot}63$, $T_{2{\cdot}09}(K) \approx 2{\cdot}90$. Also notice that in this case $t_m(K) = 3$. This shows that in general $t_\epsilon(K) \neq T_\epsilon(K) \neq t_m(K)$.

When K is C^1 but not necessarily twice differentiable, $t_0(K)$ may not be defined hence Theorem 2·1 no longer makes sense. However, $D(K)$ is still defined and it is interesting to see how it is related to the other thicknesses.

Theorem 4·3. *If* $\epsilon \leqslant 2$, *then* $D(K)/2 \geqslant T_\epsilon(K) \geqslant t_\epsilon(K)$.

Proof. Assume that $t_\epsilon(K) > D(K)/2$. Let δ be any positive number less than $(1/\epsilon - 1/2)D(K)$. Then for $c = D(K)/2 + \delta$, we have $\epsilon c = \epsilon(D(K)/2 + \delta) < D(K) \leqslant s(x, y)$, where $x, y \in K$ are two points such that $d(x, y) = D(K)$. This contradicts the assumption that $t_\epsilon(K) > D(K)/2$. □

On the other hand, as the following example shows, Theorem 4·3 is not true in general if $\epsilon > 2$.

Example 3. K contains a piece as shown in Fig. 6, which is a line segment of length 2 with two quarter circles of radius r attached at each end, which are joined to two other quarter circles as shown. We leave it to the reader to verify that $T_\epsilon(K)$ is not determined by any two points on this piece if $\epsilon > 2$ and a small enough r is chosen. One can now complete K with a suitable curve so that $T_\epsilon(K)$ is much larger than 1. But $D(K) \leqslant 2(1 + r)$.

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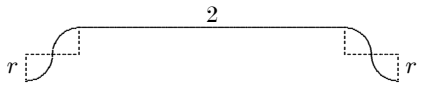


Fig. 6. If K contains a line segment of length 2 with two quarter circles attached at each end as shown, then, if $\epsilon > 2$ and r is small enough, $T_\epsilon(K)$ is not determined by any two points on this segment.

Example 4. K contains U, the upper half of the ellipse $x^2/4+y^2/0{\cdot}01 = 1$ so the two vertices of the ellipse on the x-axis are a pair of doubly critical points. But a (computer aided) calculation reveals that the maximal value of $2\sin\theta(x,y)s(x,y)/d(x,y)$ is about 2·003. So $t_\epsilon(K)$ will not be obtained at any two points on U if $\epsilon > 2{\cdot}003$. One can thus construct a K from U such that $D(K) = 2$ but $t_\epsilon(K) > 2$.

Example 5. A circle of length L has thickness $L/2\pi$. We show now that this is a maximum on the thickness of curves of length L and that the circle uniquely maximizes the thickness.

THEOREM 4·4. *$T_\epsilon(K) \leqslant L/2\pi$ for any $0 \leqslant \epsilon < 2\pi/3$ and the equality holds only when K is a circle. The same holds for $t_\epsilon(K)$ and $t_m(K)$.*

Note. The following proof is for $T_\epsilon(K)$ and $t_\epsilon(K)$. More care is needed for $t_m(K)$. We leave it to the reader by providing a hint that the proof below can be modified using a result from [**6**].

Proof. Let L be the length of K. If K is a circle of radius $L/2\pi$, then the thickness of K is $L/2\pi$ for any $0 < \epsilon < 2\pi/3$. If K has thickness $c \geqslant L/2\pi$, then $K(c-\delta)$ is a solid torus homotopic to K via strong deformation retract. Thus K cannot be deformed to a single point within $K(c-\delta)$. Fix a point X on K and let Y be any point on K. Let l_Y be the line segment joining X and Y. As Y moves along K, the union of these l_Y defines a cone shape path connected set U. Now l_Y has to intersect $\partial K(c-\delta)$ at a point Z_δ for some Y_δ; otherwise U is in $K(c-\delta)$ and K can be deformed to X within U and hence within $K(c-\delta)$. For $\delta = 1/n$, we will then have a sequence of points $(Y_{1/n}, Z_{1/n})$ on $S^1 \times S^1$. This sequence will have a convergent subsequence since $S^1 \times S^1$ is compact. Let (Y_0, Z_0) be the limit point and let $B(Z_0)$ be the ball of radius c centred at Z_0. Clearly, $B(Z_0)$ does not intersect K at an interior point and the line segment XY_0 contains a diameter of $B(Z_0)$ hence the length of K is greater than or equal to $2\pi c \geqslant L$. The equality is true only when X and Y_0 are both on $\partial B(Z_0)$ and K is the great circle on $B(Z_0)$. Since K is of length L, $c = L/2\pi$ and K is a circle.

Example 6. The thickness of a curve can be used to define a knot invariant, if we normalize it with the length of the curve. This normalization makes it scale invariant and we can define the thickness of a knot type $\mathscr{K}$ as $\sup t_\epsilon(K)/L(K)$ (or $\sup T_\epsilon(K)/L(K)$), where the supremum is taking over all curves K of knot type $\mathscr{K}$ and length $L(K)$. In [**7**], a knot energy called projection energy is defined and discussed. A relationship between $t_D(K) = t_0(K)$ and the crossover number of K is established using this energy. It is not our intention to repeat that work here, but we do want to point out the following relationship between the projection energy and

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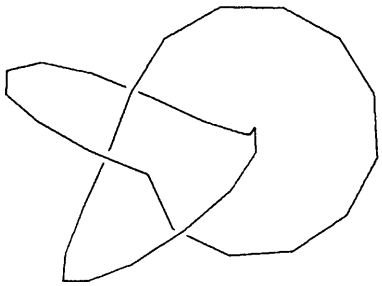


Fig. 7. A polygonal trefoil with 32 edges and t_ϵ-thickness 0·0316, with $\epsilon = 1{\cdot}0$. This polygon was produced by simulated annealing to maximize the thickness.

$t_0(K)$:

$$E_{\mathrm{Proj}}(K) = \int_{K\times K} \frac{\sin^2\theta(x,y)}{d^2(x,y)}\,dxdy, \tag{35}$$

where θ and d are as defined before. Let $c(K)$ be the crossing number of K. It is known that $E_{\mathrm{Proj}}(K) \geqslant 4\pi c(K)$. This gives the following immediate result (a better estimate is given in [**7**]):

Theorem 4·5.

$$\frac{t_0(K)}{L(K)} \leqslant \frac{1}{\sqrt{4\pi c(K)}}.$$

For instance, if K is a trefoil and is of unit length, then its thickness is at most $1/\sqrt{12\pi} \approx 0{\cdot}163$.

5. *The thickness of polygonal knots*

In this section, we numerically approximate the t_ϵ-thickness for the trefoil by approximating it on polygonal knots. We introduce a discrete analogue to the t_ϵ-thickness defined in Section 2 by taking the infimum only over the vertices of the knot. In particular, let $V = \{v_0, v_1, \ldots, v_{n-1}\}$ be a set of vertices such that $|v_i - v_{i+1}| = 1$ for each i (where we define $v_n = v_0$). We obtain an equilateral polygonal knot P by joining consecutive vertices with unit length line segments. We define the t_ϵ-thickness of P as in (2):

$$t_\epsilon(P) = \min_{x,y\in V}\left\{\frac{d(x,y)}{2\sin\theta(x,y)}\,\middle|\,\frac{2\sin\theta(x,y)\,.\,s(x,y)}{d(x,y)} \geqslant \epsilon\right\}. \tag{36}$$

The angle $\theta(x,y)$ needs to be redefined here to make this definition appropriate. We define the tangent to P at vertex v_i to be the vector through v_i and parallel to $v_{i+1} - v_{i-1}$. The angle θ is then defined as in Fig. 1.

The purpose of the simulations below is to examine the behaviour of the t_ϵ-thickness with increasing ϵ. We observe that the thickness of a knotted C^1 curve remains analytically uncomputable, and we settle for a polygonal approximation as the best that we can do at this stage. The approximation defined above is not a thickness of polygons, for example, a square of edge length one has $t_0(\square) = \sqrt{2}/2$, but any polygonal thickness of the square should not exceed $\frac{1}{2}$. Sufficient criteria for a polygonal knot energy to approximate the energy of a smooth curve approximated by the polygon is given in [**8**].

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Table 1. *t_ϵ-Thickness of the trefoil*

N	$\epsilon = 0{\cdot}5$	$\epsilon = 1{\cdot}0$	$\epsilon = 1{\cdot}5$	$\epsilon = 2{\cdot}0$	$\epsilon = 2{\cdot}09$
32	0·0316	0·0316	0·0315	0·0316	0·0316
64	0·0305	0·0305	0·0303	0·0305	0·0306
128	0·305	0·0302	0·0302	0·0302	0·0302

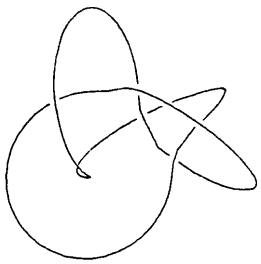


Fig. 8. A polygonal figure-eight knot with 128 edges and t_ϵ-thickness approximately $0{\cdot}0148$ if $\epsilon \leqslant 0{\cdot}52$ and t_ϵ-thickness approximately $0{\cdot}0234$ if $\epsilon \geqslant 0{\cdot}53$.

A Metropolis Monte Carlo algorithm which samples along a Markov chain in the state space of equilateral polygons of fixed knot type was adapted with simulated annealing to estimate the maximum values of the t_ϵ-thicknesses. We computed thicknesses for equilateral polygons of unit total length (each edge has length $1/N$) with $N = 32$, $N = 64$ and $N = 128$ for $\epsilon = 0{\cdot}5, 1{\cdot}0, 1{\cdot}5, 2{\cdot}0, 2{\cdot}09$. The results are in Table 1.

We note in Table 1 that $t_\epsilon(K)$ seems to be independent of ϵ. This fact is plausible for the following reason. The purpose of $t_\epsilon(K)$ (as well as $T_\epsilon(K)$) is to define a thickness of K that is not affected by short arcs on K with high curvature (which does not affect the knot's appearance at large). However, when K approaches its maximal thickness conformation, such short arcs of high curvature have been 'smoothed out' and it is reasonable to conjecture that all the thicknesses would agree on this maximal state. Of course this seems to be a conjecture too hard to prove at this point. The data in Table 1 suggests that the maximal thickness of the trefoil knot is approximately $0{\cdot}030$.

On the other hand there are knot conformations (which are not close to the maximum thickness) which have a marked dependence on the value of ϵ. In Figs 8 and 9 we illustrate knots (the Fig. 8 knot) which are not near the maximal thickness conformation for a Fig. 8 knot. The knot in Fig. 8 appears quite smooth, while the knot in Fig. 9 is not very smooth. In the knot in Fig. 8 the t_ϵ-thickness shows a dependency on ϵ as follows. For $\epsilon \leqslant 0{\cdot}52$ its t_ϵ-thickness is approximately $0{\cdot}0148$ and for $\epsilon \geqslant 0.53$ its t_ϵ-thickness is approximately $0{\cdot}0248$. For the knot in Fig. 9 the dependence on ϵ is more pronounced. If $\epsilon = 0{\cdot}5$ then $t_\epsilon(K) = 0{\cdot}0103$, if $\epsilon = 1{\cdot}0$ then $t_\epsilon(K) = 0{\cdot}0128$, if $\epsilon = 1{\cdot}5$ then $t_\epsilon(K) = 0{\cdot}0149$ and if $\epsilon \geqslant 2$ then $t_\epsilon(K) = 0{\cdot}0192$. Thus, the t_ϵ-thickness increases with ϵ, as small features in the knot are ignored.

6. *Conclusion*

In this paper we have considered the notion of a thickness of simple, closed C^1 curves. We showed that 'thickness' can be defined from several points of view, each giving rise to a different definition of (or to a family of) thickness. We studied in

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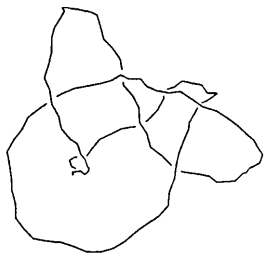


Fig. 9. A polygonal figure eight knot with 128 edges. The t_ϵ-thickness of this knot increases from 0·0103 if $\epsilon = 0{\cdot}5$ to a maximum of 0·0192 for $\epsilon \geqslant 2$.

particular two different families of thicknesses; the t_ϵ-thickness and the T_ϵ-thickness. We proved that the T_ϵ-thickness of a knot K defines a solid torus of radius $c < T_\epsilon(K)$ which is homotopic to K via strong deformation retract, as long as $\epsilon < 2\pi/3$. This result also proves the same about the t_ϵ-thickness: it defines a solid torus of radius $c < t_\epsilon(K)$ which is homotopic to K via strong deformation retract. We also showed that the circle is the unique curve which maximizes the thicknesses T_ϵ, t_ϵ and t_m.

In addition, we noted that $t_0(K)$ is the thickness of C^2-curves which was considered in [**2**]. In particular, we extend the results in that paper by proving that this thickness is continuous in the space of C^2-curves.

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