μf (mode) $\geqslant \frac{1}{2}$. Note also that the theorem holds for $A = +\infty$ and the proof remains the same. This means that we can apply our result to probability distributions with probability density functions in $[0, +\infty)$. For example, let us apply the theorem to the chi-squared distribution. The corresponding probability density function with k degrees of freedom for $x \geqslant 0$ is

$$f(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}.$$

The mean μ is equal to k and the mode is equal to $k - 2$ for $k \geqslant 3$. Thus from the theorem we have $\mu = k \geqslant \frac{1}{2f(k-2)}$ which gives (for $k \geqslant 3$)

$$2k(k-2)^{\frac{k}{2}-1} e^{-\frac{k-2}{2}} \geqslant 2^{k/2}\Gamma\left(\tfrac{k}{2}\right).$$

If we let $k = 2n$, then for $n \geqslant 2$ we have

$$\Gamma(n) = (n-1)! \leqslant 2n(n-1)^{n-1} e^{-(n-1)}.$$

This is an upper bound for the factorial function which remind us of Stirling's bounds (although Stirling's upper bound for $n!$ is better).

A final remark, which the referee kindly pointed out to us, is that the same method, but with Mx replaced by $M^n x^n$ in the definition of $I(f)$, can be used to prove that

$$\int_0^A x^n f(x)\,dx \geqslant \frac{1}{(n+1)M^n} \text{ for all } n \geqslant 1.$$

The above inequality gives lower bounds for all the moments of f. The proof is left as an exercise to the reader.

Acknowledgment: I would like to thank the anonymous referee for helpful suggestions which lead to an improvement of the article.

Reference

1. Douglas Quadling, Some thoughts on survival, *Math. Gaz.*, **93** (July 2009) pp. 323-327.

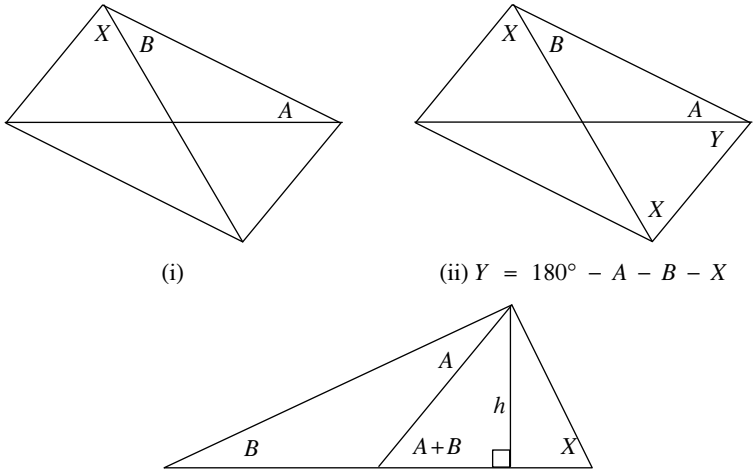
doi:10.1017/mag.2015.95

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99.34 A special parallelogram

The diagonals of a parallelogram subdivide it into two pairs of congruent triangles. In this note we investigate whether these triangles can all have 'nice' angles, where by 'nice' we mean an integer number of degrees. Since, in Figure 1(i), one of the triangles has to have a pair of non-obtuse angles A, B, this is equivalent to seeking integer-degree solutions for A, B, X with $0 < A, B \leqslant 90°$.

https://doi.org/10.1017/mag.2015.96 Published online by Cambridge University Press



(i) (ii) $Y = 180° - A - B - X$

(iii)

FIGURE 1

Two families of solutions are immediate:

- $(A, B, X) = (A,\ 90° - A,\ 90° - A)$ which gives a rhombus;
- $(A, B, X) = (A,\ A,\ 90° - A)$ which gives a rectangle.

We also note, from Figure 1(ii), that if (A, B, X) is a solution then so too is $(B, A,\ 180° - A - B - X)$, arising from the other diagonal of the same parallelogram.

The basis for our search is the cotangent identity

$$\cot X = \cot B - 2\cot(A + B). \qquad (*)$$

[This may be proved either through a double application of the sine rule to the two triangles in Figure 1(i) or, more quickly, by dropping the perpendicular in Figure 1(iii) to show that

$$h\cot B - h\cot(A + B) = h\cot(A + B) + h\cot X.]$$

We used (*) to conduct a spreadsheet search for 'nice' solutions for X with $1° \leqslant A, B \leqslant 89°$. We did a separate search with $B = 90°$, when (*) reduces to $\cot X = 2\tan A$ and with $X = 90°$ when (*) becomes

$$\cot A = \cot B + 2\tan B.$$

And the result of the search?

Apart from the families of rhombuses and rectangles noted above and a 'near miss' $(A, B, X) = (26°,\ 6°,\ 9.00008°)$, there is only one exact solution $(A, B, X) = (30°,\ 15°,\ 30°)$. These angles make this special parallelogram easy to construct using ruler and compasses: a particularly attractive way is to use two equilateral triangles inside a square, as shown in Figure 2. The internal angles of the parallelogram are 45° and 135° and the ratio of its side-lengths is the striking '1-2-3' combination $1 : 2\cos 15° = 1 : \sqrt{2 + \sqrt{3}}$.

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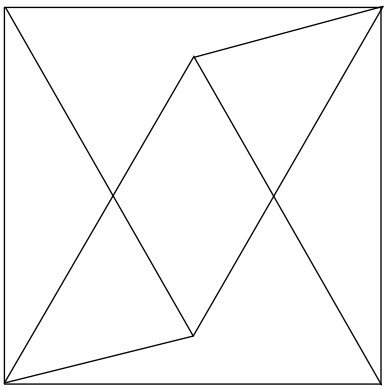


FIGURE 2

Acknowledgement

I am deeply grateful to Dr. M. N. Deshpande for kindly sending me a copy of his delightful recent problem book, [1]. With the notation above, Problem No. 4.8 asks for the value of X when $A = 30°$ and $A + B = 45°$; the answer $(X = 30°)$ set me wondering whether there were any other nice solutions.

Reference

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doi:10.1017/mag.2015.96

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99.35 On the asymptotes of the Kiepert hyperbola

The asymptotes of the Kiepert rectangular hyperbola may be employed as Cartesian axes for the general scalene triangle ABC (see [1]).

The origin of coordinates is the Kiepert centre $K(0,\ 0)$ and the vertices $A\left(a, \frac{1}{a}\right), B\left(b, \frac{1}{b}\right), C\left(c, \frac{1}{c}\right)$, centroid $G\left(g, \frac{1}{g}\right)$ and orthocentre $H\left(h, \frac{1}{h}\right)$ lie on the curve $xy = 1$. Thus $3g = a + b + c$, $\frac{3}{g} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ with $abch = -1$ for rectangularity. Note also that g, h and two of the parameters a, b, c may be taken as positive.

We first see that the midpoint of any side is also the midpoint of the pair of asymptote meets with this side (produced).

The meets

The equation of the side BC is $bcy + x = b + c$ and the respective asymptote meets (M, N say) are clearly $M(b + c,\ 0), N\left(0, \frac{1}{b} + \frac{1}{c}\right)$, whence A' is the midpoint of both BC and MN.

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