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On the unique predual problem for Lipschitz spaces

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Abstract

For any metric space X, the predual of $\mathrm{Lip}(X)$ is unique. If X has finite diameter or is complete and convex—in particular, if it is a Banach space—then the predual of $\mathrm{Lip}_0(X)$ is unique.

1. *Lipschitz spaces*

Let (X, ρ) be a metric space. The *Lipschitz number* of a function $f : X \to \mathbb{R}$ is the quantity

$$L(f) = \sup_{p \neq q} \frac{|f(p) - f(q)|}{\rho(p, q)};$$

f is *Lipschitz* if $L(f) < \infty$. Note that the Lipschitz number of any constant function is zero. We define $\mathrm{Lip}(X)$ to be the space of bounded Lipschitz functions from X into $\mathbb{R}$, equipped with the norm $\|f\|_L = \max(L(f), \|f\|_\infty)$.

If X is a pointed metric space, i.e., it is equipped with a distinguished "base point" e, then we define $\mathrm{Lip}_0(X)$ to be the space of all (possibly unbounded) Lipschitz functions from X into $\mathbb{R}$ which vanish at e, with norm $L(\cdot)$. Since the zero function is the only constant function which vanishes at the base point, Lipschitz number is a norm on this space, not merely a seminorm. The Banach space structure of $\mathrm{Lip}_0(X)$ does not depend on the choice of e; if e' is another base point then the map $f \mapsto f - f(e') \cdot 1_X$ is an isometric linear map taking the Lipschitz functions which vanish at e to the Lipschitz functions which vanish at e'.

Lip and Lip_0 spaces are closely related. Write $\cong$ for isometric isomorphism.

PROPOSITION 1·1 (**9**, proposition 1·7·1, theorem 1·7·2, and proposition 1·7·3). *Given any metric space X, let Y be the pointed set $X \cup \{e\}$ equipped with the metric which sets*

$$\rho^Y(p, q) = \min(\rho^X(p, q), 2)$$

for all $p, q \in X$ and $\rho^Y(p, e) = 1$ for all $p \in X$. Then $\mathrm{Lip}(X) \cong \mathrm{Lip}_0(Y)$. Lip *spaces are effectively the special case of* Lip_0 *spaces for which the greatest element of the unit ball is a multiplicative identity.*

The $\mathrm{Lip}(X)$ construction is insensitive to distances greater than 2 because if $\rho(p, q) > 2$ then the slope $|f(p) - f(q)|/\rho(p, q)$ is dominated by $\|f\|_\infty$. Thus the metric can be truncated at 2 without affecting the norm on $\mathrm{Lip}(X)$. The idea of Proposition 1·1 is that

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once this is done, we can attach a base point whose distance to every point of X is 1 and extend any Lipschitz function on X to be zero at the base point. The Lipschitz number of the extended function now includes the slopes $|f(p) - f(e)|/\rho(p, e) = |f(p)|$, whose collective contribution to this Lipschitz number exactly matches the sup norm of the original function. Thus Lip spaces are effectively the special case of Lip_0 spaces which arises when the base point is exactly one unit away from every other point. This condition is characterized by the property that the greatest element of the unit ball is the function which is 0 at the base point and constantly 1 elsewhere, i.e., a multiplicative identity.

Every Lip_0 space is a dual Banach space. This was first shown by Arens and Eells in [**1**], but was later rediscovered in varying degrees of generality by a number of authors. (I know of four essentially different methods of constructing the predual.) In [**9**] I termed the natural preduals "Arens–Eells spaces", but in their celebrated paper [**6**] Godefroy and Kalton renamed them "Lipschitz-free spaces". As a result these spaces now go by two different names in the literature.

The most important feature of the Arens–Eells spaces is a universal property: for any pointed metric space X there is a natural isometric embedding $\iota : X \to \text{Æ}(X)$ of X in its Arens–Eells space, which takes the base point of X to the origin of $\text{Æ}(X)$, and which is universal among all nonexpansive base point preserving maps from X into any Banach space. This fact is due to me and first appeared in [**9**], although it could have been deduced from results in an earlier paper of Kadets [**7**] (which I unfortunately was not aware of). Indeed, one can simply define $\text{Æ}(X)$ to be this universal space, whose existence and uniqueness is routine, and then easily check that its dual space is isometrically isomorphic to $\mathrm{Lip}_0(X)$. This was pointed out in [**4**].

In [**9**] I asked whether the predual of $\mathrm{Lip}_0(X)$ is unique. Since then this problem has been noted by various authors, but the only previous work in this direction I know of is in [**5**], where it is shown that $\mathrm{Lip}_0(X) \cong L^\infty[0, 1]$ when X is a separable metric tree. This implies that the predual of $\mathrm{Lip}_0(Y)$ is unique if Y is isometrically contained in such a space. The aim of this paper is to show that the predual of $\mathrm{Lip}_0(X)$ is unique in a much more extensive class of cases. First, for any X the predual of $\mathrm{Lip}(X)$ is unique. Second, the predual of $\mathrm{Lip}_0(X)$ is unique if the diameter of X is finite. This case is important because it is precisely then that $\mathrm{Lip}_0(X)$ has a natural algebraic structure. (If $\mathrm{diam}(X) = \infty$ then the square of the function $f(p) = \rho(p, e)$ is not Lipschitz.) But $\mathrm{Lip}_0(X)$ is also interesting when $\mathrm{diam}(X) = \infty$, most notably when X is itself a Banach space. In this case as well the predual of $\mathrm{Lip}_0(X)$ turns out to be unique. The general infinite diameter case will not be settled here.

2. *Lip spaces*

The main fact about $\text{Æ}(X)$ we need is the following.

PROPOSITION 2·1 (**9**, theorem 2·2·2). *Let X be a pointed metric space and let (f_λ) be a bounded net in $\mathrm{Lip}_0(X)$. Then $f_\lambda \to f \in \mathrm{Lip}_0(X)$ weak* (relative to the predual $\text{Æ}(X)$) if and only if $f_\lambda \to f$ pointwise.*

Give $\mathrm{Lip}_0(X)$ the pointwise partial order, i.e., $f \leqslant g$ if $f(p) \leqslant g(p)$ for all $p \in X$. Then the pointwise join or meet of any two Lipschitz functions is again Lipschitz; indeed, it is easy to see that the join or meet of any bounded family of Lipschitz functions is again Lipschitz [**9**, proposition 1·5·5]. This lattice structure is related to the weak* topology.

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LEMMA 2·2. *Let X be a pointed metric space, let (f_λ) be a bounded net in* $\mathrm{Lip}_0(X)$*, and let $f \in \mathrm{Lip}_0(X)$. Suppose that $f_\lambda \to f$ weak* relative to the predual* $\text{Æ}(X)$*. Then $f = \bigvee_\lambda \bigwedge_{\kappa \geqslant \lambda} f_\kappa$.*

Proof. Since (f_λ) is bounded, $g = \bigvee_\lambda \bigwedge_{\kappa \geqslant \lambda} f_\kappa$ is a well-defined element of $\mathrm{Lip}_0(X)$. Now given $p \in X$ and $\epsilon > 0$, we can find λ_0 such that $|f_\kappa(p) - f(p)| \leqslant \epsilon$ for all $\kappa \geqslant \lambda_0$. Thus $\bigwedge_{\kappa \geqslant \lambda} f_\kappa \leqslant f(p) + \epsilon$ for any λ, and $\bigwedge_{\kappa \geqslant \lambda_0} f_\kappa \geqslant f(p) - \epsilon$. This shows that $|g(p) - f(p)| \leqslant \epsilon$. Since p and ϵ were arbitrary, we conclude that $g = f$.

The next result on our way to the uniqueness results is also of independent interest. Say that a linear functional $\phi \in \mathrm{Lip}_0(X)^*$ is *positive*, and write $\phi \geqslant 0$, if $f \geqslant 0$ implies $\phi(f) \geqslant 0$. Say that ϕ is *normal* if it satisfies $\phi(f_\lambda) \to \phi(f)$ whenever (f_λ) is a bounded increasing net in $\mathrm{Lip}_0(X)$ and $f = \bigvee f_\lambda$.

THEOREM 2·3. *Let $X \in \mathcal{M}_0$ and suppose $\phi \in \mathrm{Lip}_0(X)^*$ is positive. Then ϕ is weak* continuous relative to the predual* $\text{Æ}(X)$ *if and only if it is normal.*

Proof. The forward direction is easy because if (f_λ) is a bounded increasing net then $f_\lambda \to \bigvee f_\lambda$ boundedly pointwise and hence weak* (Proposition 2·1). For the reverse direction, assume ϕ is normal; since the intervals of the form (a, ∞) and $(-\infty, a)$ generate the topology on $\mathbb{R}$ it will suffice to show that for every $a \in \mathbb{R}$ the sets $\phi^{-1}((-\infty, a])$ and $\phi^{-1}([a, \infty))$ are weak* closed. We just consider the first case, as the second is similar. By the Krein–Smulian theorem, it will suffice to show that $\phi^{-1}((-\infty, a])$ contains the limits of all bounded weak* convergent nets.

Thus let $(f_\lambda) \subseteq \phi^{-1}((-\infty, a])$ be bounded and suppose it converges weak* to $f \in \mathrm{Lip}_0(X)$. By Lemma 2·2 we have $f = \bigvee_\lambda \bigwedge_{\kappa \geqslant \lambda} f_\kappa$. Then $\phi(f_\lambda) \leqslant a$ for all λ, so by positivity we know that $\phi(\bigwedge_{\kappa \geqslant \lambda} f_\kappa) \leqslant a$ for all λ. But by normality

$$\phi\left(\bigwedge_{\kappa \geqslant \lambda} f_\kappa\right) \longrightarrow \phi(f),$$

so that $\phi(f) \leqslant a$ as well. Thus $f \in \phi^{-1}((-\infty, a])$, as desired.

Now we prove the uniqueness result for $\mathrm{Lip}(X)$. The proof follows the strategy used by Sakai to prove uniqueness of von Neumann algebra preduals [**8**]. The key property of $\mathrm{Lip}(X)$ used in the proof, whose analog in other Lip_0 spaces fails, is the fact that if $f \in \mathrm{Lip}(X)$ satisfies $f \geqslant 0$ and $\|f\|_L \leqslant 1$, then $\|f - 1_X\|_L \leqslant 1$ also. Another way to say this is that the unit ball $[\mathrm{Lip}(X)]_1$ and its positive part $[\mathrm{Lip}(X)]_1^+$ are related by

$$[\mathrm{Lip}(X)]_1^+ \subseteq [\mathrm{Lip}(X)]_1 + 1_X.$$

A dual Banach space V^* has a *unique predual* if $V^* \cong W^*$ implies $V \cong W$, for any Banach space W. An a priori stronger condition is that every isometric isomorphism between V^* and another dual space W^* is weak* continuous. (It would then have to be a weak* homeomorphism and its adjoint would take $W \subseteq W^{**}$ isometrically onto $V \subseteq V^{**}$.) In this case V^* is said to have a *strongly unique predual*. It is not known whether there are any spaces whose predual is unique but not strongly unique.

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THEOREM 2·4. *Let X be a metric space and let Y be as in Proposition* 1·1. *Then $\text{Æ}(Y)$ is the strongly unique Banach space predual of* $\text{Lip}(X)$.

Proof. Let V be a Banach space and suppose $\phi : \text{Lip}(X) \to V^*$ is a surjective isometric isomorphism. Write $\langle v, f \rangle$ for the pairing of $v \in V$ with $f \in \text{Lip}(X)$, i.e., $\langle v, f \rangle = \phi(f)(v)$.

This pairing induces an alternate weak* topology on $\text{Lip}(X)$, which within this proof I will refer to as the "a-weak*" topology. The standard weak* topology indueced by $\text{Æ}(Y)$ will be termed the "s-weak*" topology. One thing we know about this a-weak* topology is that $[\text{Lip}(X)]_1$ is compact relative to it. Moreover, the comment made just before the theorem yields

$$[\text{Lip}(X)]_1^+ = ([\text{Lip}(X)]_1 + 1_X) \cap [\text{Lip}(X)]_1,$$

so that the positive part of the unit ball is an intersection of two a-weak* compact sets and is therefore itself a-weak* compact. By the Krein–Smulian theorem, this implies that $\text{Lip}(X)^+$, the positive cone of $\text{Lip}(X)$, is a-weak* closed. It then follows from a standard separation theorem for topological vector spaces that for any $f \in \text{Lip}(X)$, $f \not\geqslant 0$, there exists an a-weak* continuous linear functional which separates f from $\text{Lip}(X)^+$. That is, there exists $v \in V$ such that $\langle v, f \rangle < 0$ but $\langle v, g \rangle \geqslant 0$ for all $g \in \text{Lip}(X)^+$. This means that for any $f \not\geqslant 0$ there is a positive linear functional $v \in V$, i.e., an element $v \in V^+$, such that $\langle v, f \rangle < 0$. (Here the order on V is defined via its pairing with $\text{Lip}(X)$.) Applying this to $-f$, we also see that for any $f \neq 0$ there is a $v \in V^+$ such that $\langle v, f \rangle \neq 0$.

We do not know that V^+ spans V, and indeed this can fail; see Example 2·5 below. However, it follows from the previous paragraph that $\text{span}(V^+)$ is dense in V.

Now suppose (f_λ) is a bounded increasing net in $\text{Lip}(X)$ and let $f = \bigvee f_\lambda$. Fix a subnet (f_{λ_κ}) which converges a-weak*, say to $g \in \text{Lip}(X)$. Then for any $v \in V^+$ the values $\langle v, f_{\lambda_\kappa} \rangle$ increase to $\langle v, g \rangle$. But also, just by positivity, we have $\langle v, f_{\lambda_\kappa} \rangle \leqslant \langle v, f \rangle$ for every κ and every $v \in V^+$. So $\langle v, g \rangle \leqslant \langle v, f \rangle$ for every $v \in V^+$. By what we showed earlier about the abundance of positive elements of V, this implies that $g \leqslant f$; but also, since $\langle v, f_{\lambda_\kappa} \rangle \leqslant \langle v, g \rangle$ for all κ and v we similarly have $f_{\lambda_\kappa} \leqslant g$ for all κ, and therefore $f = \bigvee f_{\lambda_\kappa} \leqslant g$. So $f = g$, and we conclude that every a-weak* convergent subnet of (f_λ) converges a-weak* to f, which implies that f_λ itself converges a-weak* to f.

Thus, the pairing of any element of V^+ with $\text{Lip}(X)$ is normal, and therefore $\phi^* : V^{**} \to \text{Lip}(X)^*$ takes $V^+ \subseteq V^{**}$ into $\text{Æ}(Y) \subseteq \text{Lip}(X)^*$ by Theorem 2·3. Since $\text{span}(V^+)$ is dense in V, it follows that $\phi^*(V) \subseteq \text{Æ}(Y)$. This implies that $\phi^*(V) = \text{Æ}(Y)$ and we conclude that the predual is strongly unique.

In connection with the previous proof, note that Arens-Eells spaces in general are not spanned by their positive elements. Here we use a construction of $\text{Æ}(X)$ as the closed subspace of $\text{Lip}_0(X)^*$ generated by the point evaluations $\delta_p : f \mapsto f(p)$.

Example 2·5. Work on the unit interval $[0, 1]$ augmented by a base point as in Proposition 1·1; denote this space $[0, 1]^e$. Let m be the sum of elementary "molecules" $m = \sum_{n=0}^{\infty} (\delta_{2^{-2n}} - \delta_{2^{-2n-1}})$. Since $\|\delta_{2^{-2n}} - \delta_{2^{-2n-1}}\| = 2^{-2n-1}$ in $\text{Æ}([0, 1]^e)$, this sum is absolutely convergent.

Suppose we could write $m = m^+ - m^-$ with $m^+, m^- \in \text{Æ}([0, 1]^e)$ both positive. For each n find a function $f_n \in \text{Lip}[0, 1]$ which satisfies $0 \leqslant f_n \leqslant 1$ and which takes the value 1 at the points $1, 2^{-2}, \ldots, 2^{-2n}$, the value 0 at the points $2^{-1}, 2^{-3}, \ldots, 2^{-2n-1}$, and is constantly

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0 on $[0, 2^{-2n-1}]$. Then

$$\langle m^+, 1_{[0,1]}\rangle \geqslant \langle m^+, f_n\rangle \geqslant \langle m, f_n\rangle = n+1$$

for any n. Thus $\langle m^+, 1_{[0,1]}\rangle$ cannot be finite, a contradiction.

3. *Lip_0 spaces*

The next lemma was proven by Uri Bader [**2**]. Its proof uses a result of Dixmier [**3**] according to which, for any Banach space V, a closed subspace of V^* is a predual of V in the natural way if and only if the weak topology it induces on V makes the unit ball $[V]_1$ compact Hausdorff. (Thus V has a unique predual if and only if all closed subspaces of V^* with this property are isometrically isomorphic, and it has a strongly unique predual if and only if V^* has exactly one closed subspace with this property.)

LEMMA 3·1. *Let V be a dual Banach space with a strongly unique predual and let V_0 be a weak* closed codimension* 1 *subspace of V. Then V_0 has a strongly unique predual.*

Proof. Let V_* be the predual of V, identified uniquely with a subspace of V^*, and let $L \subseteq V^*$ be the annihilator of V_0. Since V_0 is weak* closed, the 1 dimensional subspace L is contained in V_*. Thus the dual of V_0 can be identified with V^*/L and its canonical predual is V_*/L.

Suppose $W \subseteq V^*/L$ satisfies Dixmier's criterion for V_0; we must show that $W = V_*/L$. Let $\pi : V^* \to V^*/L$ be the quotient map and let $\tilde{W} = \pi^{-1}(W) \subseteq V^*$. Also fix $z_0 \in V \setminus V_0$. Given any two distinct points $x+az_0$ and $y+bz_0$ of $V = V_0+\mathbb{R}\cdot z_0$, if $a \neq b$ then any nonzero element of $L \subseteq \tilde{W}$ separates them, and if $a = b$ then $(x+az_0)-(y+bz_0) = x-y \neq 0$ and again elements of $\tilde{W}$ separate them. So the weak topology on V induced by $\tilde{W}$ is Hausdorff. Given any bounded net $(x_\lambda + a_\lambda z_0)$ in V, we can pass to a subnet such that (a_λ) converges and $(\phi(x_\lambda))$ converges for every $\phi \in W$. It then follows that $(\psi(x_\lambda + a_\lambda z_0)) = (\pi(\psi)(x_\lambda) + a_\lambda \psi(z_0))$ converges for every $\psi \in \tilde{W}$. Thus every net in the unit ball of V has a convergent subnet for the weak topology induced by $\tilde{W}$. We have shown that $\tilde{W}$ satisfies Dixmier's criterion for V, so by strong uniqueness we must have $\tilde{W} = V_*$ and $W = \pi(\tilde{W}) = V_*/L$.

THEOREM 3·2. *Let X be a pointed metric space whose diameter is finite. Then* $\mathrm{Lip}_0(X)$ *has a strongly unique predual.*

Proof. For sufficiently small $r > 0$ the rescaled metric $\rho'(p,q) = r\rho(p,q)$ satisfies $\rho'(p,e) \leqslant 1$ for all $p \in X$. Let X' be the resulting metric space. The identity map ι from $\mathrm{Lip}_0(X)$ to $\mathrm{Lip}_0(X')$ is then a surjective isomorphism satisfying $L(\iota(f)) = \frac{1}{r}L(f)$ for all $f \in \mathrm{Lip}_0(X)$. Thus $\mathrm{Lip}_0(X) \cong \mathrm{Lip}_0(X')$.

It therefore suffices to show that the predual of $\mathrm{Lip}_0(X')$ is strongly unique. But $\mathrm{Lip}_0(X')$ is a codimension 1 subspace of $\mathrm{Lip}(X')$, whose predual is strongly unique by Theorem 2·4; moreover, it is the kernel of the evaluation functional δ_e and hence is weak* closed. The inclusion of $\mathrm{Lip}_0(X')$ into $\mathrm{Lip}(X')$ is isometric because for any $f \in \mathrm{Lip}_0(X')$ and $p \in X'$ we have $|f(p)| \leqslant |f(p)-f(e)|/\rho(p,e)$ (since $f(e) = 0$ and $\rho(p,e) \leqslant 1$), which shows that $\|f\|_\infty \leqslant L(f)$ and hence $\|f\|_L = L(f)$. The desired conclusion now follows from Lemma 3·1.

A metric space X is *convex* if for every distinct $p, q \in X$ there exists a third distinct point r such that $\rho(p,q) = \rho(p,r) + \rho(r,q)$. If X is complete and convex, then for any distinct

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$p, q \in X$ there is an isometric embedding of the interval $[0, a] \subset \mathbb{R}$ into X which takes 0 to p and a to q, where $a = \rho(p, q)$.

Given any pointed metric space X and any closed subset $K \subseteq X$ containing the base point, let $\mathcal{I}(K) = \{f \in \mathrm{Lip}_0(X) : f|_K = 0\}$. This is a weak* closed subspace (relative to the predual $\text{Æ}(X)$) and we have $\mathrm{Lip}_0(X)/\mathcal{I}(K) \cong \mathrm{Lip}_0(K)$ [**9**, corollary 4·2·7].

THEOREM 3·3. *Let X be a complete convex metric space. Then $\mathrm{Lip}_0(X)$ has a strongly unique predual.*

Proof. For each $n \in \mathbb{N}$ let X_n be the closed ball of radius n about the base point in X. It follows from Theorem 3·2 that each $\mathrm{Lip}_0(X_n)$ has a strongly unique predual. Also, we have a sequence of quotient maps $\pi_n : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(X)/\mathcal{I}(X_n) \cong \mathrm{Lip}_0(X_n)$.

Suppose $W \subseteq \mathrm{Lip}_0(X)^*$ satisfies Dixmier's criterion. Then it is a predual of $\mathrm{Lip}_0(X)$ and gives rise to a weak* topology. I claim that the subspaces $\mathcal{I}(X_n)$ are closed in this topology; granting this, it follows that for each n the space $W_n = \{\phi \in W : \phi|_{\mathcal{I}(X_n)} = 0\} \subseteq (\mathrm{Lip}_0(X)/\mathcal{I}(X_n))^* \cong (\mathrm{Lip}_0(X_n))^*$ satisfies Dixmier's criterion for $\mathrm{Lip}_0(X_n)$ and therefore equals $\text{Æ}(X_n) \subseteq \mathrm{Lip}_0(X_n)^*$. As $\bigcup_{n=1}^{\infty} \text{Æ}(X_n)$ is a dense subspace of $\text{Æ}(X)$, this implies that $\text{Æ}(X) \subseteq W$ and hence that the two spaces are equal, establishing strong uniqueness.

To prove the claim, fix n and let $h \in \mathrm{Lip}_0(X)$ be the function $h(p) = \min(\rho(p, e), n)$. We will show that $f \in [\mathrm{Lip}_0(X)]_1$ belongs to $\mathcal{I}(X_n)$ if and only if $L(f \pm h) \leqslant 1$. This implies that

$$[\mathcal{I}(X_n)]_1 = [\mathrm{Lip}_0(X)]_1 \cap ([\mathrm{Lip}_0(X)]_1 + h) \cap ([\mathrm{Lip}_0(X)]_1 - h)$$

and hence that the unit ball of $\mathcal{I}(X_n)$ is weak* closed relative to any predual. By the Krein–Smulian theorem, this is enough.

Suppose $f \notin \mathcal{I}(X_n)$. Then for some $p \in X_n$ either $f(p) > 0$ or $f(p) < 0$. In the former case, $L(f + h) > 1$ because the slope

$$\frac{|(f + h)(p) - (f + h)(e)|}{\rho(p, e)} = \frac{f(p) + \rho(p, e)}{\rho(p, e)}$$

exceeds 1, and in the latter case $L(f - h) > 1$ for a similar reason.

Conversely, suppose $f \in \mathcal{I}(X_n)$; we must show that $L(f \pm h) \leqslant 1$. Fix $p, q \in X$. If $\rho(p, e), \rho(q, e) \geqslant n$ then $h(p) = h(q)$,

$$\frac{|(f \pm h)(p) - (f \pm h)(q)|}{\rho(p, q)} = \frac{|f(p) - f(q)|}{\rho(p, q)} \leqslant 1$$

and, if $\rho(p, e), \rho(q, e) \leqslant n$ then $f(p) = f(q) = 0$ and

$$\frac{|(f \pm h)(p) - (f \pm h)(q)|}{\rho(p, q)} = \frac{|h(p) - h(q)|}{\rho(p, q)} \leqslant 1.$$

In the remaining case, say $\rho(p, e) > n$ and $\rho(q, e) < n$. By completeness and convexity we can find a point r such that $\rho(p, q) = \rho(p, r) + \rho(r, q)$ and $\rho(r, e) = n$, and so

$$\begin{aligned}
&\frac{|(f \pm h)(p) - (f \pm h)(q)|}{\rho(p, q)} \\
&\leqslant \max\left(\frac{|(f \pm h)(p) - (f \pm h)(r)|}{\rho(p, r)}, \frac{|(f \pm h)(r) - (f \pm h)(q)|}{\rho(r, q)}\right) \\
&= \max\left(\frac{|f(p) - f(r)|}{\rho(p, r)}, \frac{|h(r) - h(q)|}{\rho(r, q)}\right) \\
&\leqslant \max(L(f), L(h)) = 1
\end{aligned}$$

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using the elementary inequality $(b+d)/(a+c) \leqslant \max(b/a, d/c)$ for $a, c > 0$ and $b, d \geqslant 0$. We conclude that $L(f \pm h) \leqslant 1$, as desired.

COROLLARY 3·4. *Let X be a Banach space. Then $\mathrm{Lip}_0(X)$ has a strongly unique predual.*

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