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Sharp bounds for higher Steklov–Dirichlet eigenvalues on domains with spherical holes

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Abstract. We consider a mixed Steklov–Dirichlet eigenvalue problem on a smooth bounded domain having a spherical hole. In this article, we take Dirichlet condition on the boundary of the spherical hole and Steklov condition on the other boundary component/s. Under certain symmetry assumptions on multiconnected domains in $\mathbb{R}^n$ having a spherical hole, we obtain isoperimetric inequalities for the k-th Steklov–Dirichlet eigenvalues for each $k \in \{2, 3, \ldots, n+1\}$. We provide examples to emphasise the fact that the symmetry assumptions, on the family of domains considered, are crucial. We also extend Theorem 3.1 of "Gavitone et al. (2023), An isoperimetric inequality for the first Steklov–Dirichlet Laplacian eigenvalue of convex sets with a spherical hole, *Pacific Journal of Mathematics*, 320(2): 241–259" not only from Euclidean domains to domains in space forms but also from convex domains to star-shaped domains. In particular, we obtain sharp lower and upper bounds for the first Steklov–Dirichlet eigenvalue on the family of all bounded star-shaped domains on the hemisphere as well as on the hyperbolic space.

1 Introduction

Let (M, g_M) be a Riemannian manifold and let D denote the Levi-Civita connection of (M, g_M). For a smooth vector field X on M, the divergence of X denoted as $\mathrm{div}(X)$ is defined as $trace(DX)$. For a smooth function $f : M \to \mathbb{R}$, the gradient of f at a point $p \in M$ denoted as $(\nabla f)(p)$ is defined by $\langle \nabla f(p), v \rangle_{g_M} = df(p)(v)$ for all $v \in T_pM$. Here, T_pM is the tangent space to M at the point p. The Laplace–Beltrami operator Δ is defined by $\Delta f = \mathrm{div}(\nabla f)$. The Riemannian manifolds, we consider in this article, are going to be space forms, that is, the complete simply connected Riemannian manifolds with constant sectional curvature. It is well known that they are one of the following spaces: (i) the Euclidean space $\mathbb{E}^n$, (ii) the unit sphere $S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$ with induced Riemannian metric $\langle \cdot, \cdot \rangle$ from the Euclidean space $\mathbb{R}^{n+1}$, and (iii) the hyperbolic

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space $\mathbb{H}^n := \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n} x_i^2 - x_{n+1}^2 = -1 \text{ and } x_{n+1} > 0\}$ with the Riemannian metric induced from the bilinear form $(x, y) := \sum_{i=1}^{n} x_i\, y_i - x_{n+1}\, y_{n+1}$. Here, $x = (x_1, x_2, \ldots, x_{n+1}), y = (y_1, y_2, \ldots, y_{n+1}) \in \mathbb{R}^{n+1}$. For details about the standard geometric terminologies used in this article, the reader may refer to [13].

Finding sharp bounds for Steklov–Dirichlet eigenvalues among a constrained family of doubly connected domains is an interesting problem which has caught attention of many authors [21, 25] owing to its physical significance. In this article, we consider a mixed Steklov–Dirichlet eigenvalue problem on domains with a spherical hole. Let Ω_{out} denote a bounded domain in the Riemannian manifold M, with smooth boundary denoted by $\partial\Omega_{\text{out}}$. Let $B_r \subset M$ be a ball of radius r such that $\overline{B_r} \subset \Omega_{\text{out}}$. Consider the following Steklov–Dirichlet eigenvalue problem on $\Omega := \Omega_{\text{out}} \backslash \overline{B_r}$:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega = \Omega_{\text{out}} \backslash \overline{B_r}, \\ u = 0 & \text{on } \partial B_r, \\ \frac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\Omega_{\text{out}}. \end{cases} \tag{1.1}$$

It is known (cf. [1]) that the boundary value problem (1.1) has a discrete spectrum $0 < \sigma_1 \le \sigma_2 \le \sigma_3 \le \cdots \nearrow \infty$, counted with multiplicity. For each $1 \le k < \infty$, the k-th eigenvalue of (1.1), i.e., σ_k, admits the following variational characterization:

$$\sigma_k(\Omega) = \min_{E \in \mathcal{H}_{k,0}(\Omega)} \max_{u(\neq 0) \in E} \frac{\int_\Omega \|\nabla u\|^2\, dv}{\int_{\partial\Omega_{\text{out}}} u^2\, ds}, \tag{1.2}$$

where $\mathcal{H}_{k,0}(\Omega)$ is the collection of all the k-dimensional subspaces of the Sobolev space $\tilde{H}_0^1(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial B_r\}$. The expression $R(u) := \dfrac{\int_\Omega \|\nabla u\|^2\, dv}{\int_{\partial\Omega_{\text{out}}} u^2\, ds}$ is called a Rayleigh quotient. If v is an eigenfunction corresponding to a Steklov Dirichlet eigenvalue σ, then $\sigma = R(v) = \dfrac{\int_\Omega \|\nabla v\|^2\, dv}{\int_{\partial\Omega_{\text{out}}} v^2\, ds}$.

Various bounds for the first Steklov–Dirichlet eigenvalue relating the geometry of the underlying domains have been obtained for bounded domains in $\mathbb{R}^2$. Using the theory of conformal mapping, Hersch and Payne [19] obtained sharp upper bounds for the first Steklov–Dirichlet eigenvalue on planar doubly connected bounded domains. In [12], using circular symmetrization and a distortion theorem for conformal mappings of an annulus, lower bounds for the first Steklov–Dirichlet eigenvalue were derived for doubly connected planar domains. For bounded domains in $\mathbb{R}^n$, Banuelos et al. [6] obtained some classical inequalities comparing eigenvalues of the mixed Steklov–Dirichlet problem and mixed Steklov–Neumann problem.

Many isoperimetric bounds and monotonicity results for the first Steklov–Dirichlet eigenvalue have been derived for various domains in $\mathbb{R}^n$ [16, 20]. For instance, in [30], it was proved that, if $n \ge 3$ and Ω_{out} is a ball of fixed radius, the first Steklov–Dirichlet eigenvalue $\sigma_1(\Omega)$ is maximum when Ω_{out} and B_r are concentric. This result was later proved for planar annular domains and for domains contained in two-point homogeneous spaces [14, 27]. Generalizing this, Paoli et al. [26] proved that concentric annular domain locally minimizes first Steklov–Dirichlet eigenvalue when Ω_{out} varies over nearly spherical sets of fixed volume. Further, if Ω_{out} is an

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open, bounded, and convex set contained in a suitable ball, Gavitone et al. [15] proved that, under a volume constraint, $\sigma_1(\Omega)$ is maximum for concentric annular domain. In particular, they proved the following result.

Theorem 1.1 [15, Theorem 1.1] *Fix $n \geq 2$. Let Ω_{out} be an open bounded convex set in $\mathbb{R}^n$. Let B_r be a ball in $\mathbb{R}^n$ of radius r such that $\overline{B_r} \subset \Omega_{\text{out}} \subset B_{\bar{r}}$, where $B_{\bar{r}}$ is the ball in $\mathbb{R}^n$, centered at the origin, with radius $\bar{r}$ defined in terms of r and n, as*

$$\bar{r} = \begin{cases} re^{\sqrt{2}} & \text{if } n = 2, \\ r\left[\dfrac{(n-1)+(n-2)\sqrt{2(n-1)}}{n-1}\right]^{\frac{1}{n-2}} & \text{if } n \geq 3. \end{cases}$$

Then, $\sigma_1(\Omega_{\text{out}} \backslash \overline{B_r}) \leq \sigma_1(B_{R,r})$, where $B_{R,r}$ is the concentric annular domain with inner radius r and outer radius R such that $\text{Vol}\,(\Omega_{\text{out}} \backslash \overline{B_r}) = \text{Vol}\,(B_{R,r})$.

In [15], the authors have also obtained sharp lower and upper bounds for $\sigma_1(\Omega)$ in terms of the minimal and maximal distances between the center of Ω_{out} and its boundary $\partial\Omega_{\text{out}}$. Similar results have been proved for Steklov eigenvalues on star-shaped domains contained in unit hemisphere [29].

In this article, we generalize the above-mentioned results in three directions:

(1) First eigenvalue on star-shaped domains in non-Euclidean space forms: The aim is to find sharp lower and upper bounds for the first Steklov–Dirichlet eigenvalue of doubly connected star-shaped domains in non-Euclidean space forms (Theorem 2.1). These bounds are proved using a similar approach to that used in [15], with appropriate changes due to the difference in the geometries of the non-Euclidean space forms.

(2) Higher eigenvalues on symmetric Euclidean domains (Isoperimetric Inequality): The aim is to find optimal shapes for the higher Steklov–Dirichlet eigenvalues (Theorem 5.1) under a volume constraint and with certain symmetry assumptions on Euclidean domains. To prove this result, we plug suitable test functions into the variational characterization of the eigenvalue and in order to use this characterization, we develop a substantial list of auxiliary results concerning the behavior of a few relevant functions appearing therein. In Remark 5.2, we justify by giving suitable examples that the symmetry assumption in the statement of Theorem 5.1 is crucial.

(3) First eigenvalue on star-shaped Euclidean domains (Isoperimetric Inequality): We further remark that Theorem 1.1 of [15] can easily be extended from convex Euclidean domains to star-shaped Euclidean domains (Theorem 2.2).

We bring to the readers' notice that the eigenvalue optimization problems considered in [2, 7–11, 17, 18, 22] are for the first Dirichlet eigenvalue of the Laplacian, the p-Laplacian, ($1 < p < \infty$), and are of certain general linear elliptic operators. In [23], the authors deal with the second Dirichlet eigenvalue of the Laplacian. In these problems, the authors considered the Dirichlet eigenvalue problem on doubly connected domains of the form $B \backslash P$ under a volume constraint, where B is a disk, while P is either a smaller disk or a domain having a dihedral symmetry with some monotonicity assumption on its boundary. In each of these articles, it was proved that

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the concentric configuration, among the admissible family of domains, maximizes the first Dirichlet eigenvalue of the operator under consideration.

We further mention here similar problems considered for the Neumann Eigenvalues (cf. [3–5, 30]).

This article is organized as follows: In Section 2, we discuss the geometry of star-shaped domains in non-Euclidean space forms. We then derive sharp bounds for the first Steklov–Dirichlet eigenvalue on doubly connected star-shaped domains with a spherical hole (Theorem 2.1). We also state a generalization of Theorem 1.1 from convex domains to star-shaped domains (Theorem 2.2). This theorem can be proved following the ideas in [15]. In Section 3, we study the behavior of Steklov–Dirichlet eigenvalues and their corresponding eigenfunctions in annular domains. Certain integral inequalities on domains with given symmetry assumptions are proved in Section 4. In Section 5, we obtain isoperimetric bounds for higher eigenvalues (Theorem 5.1). The "order 4 symmetry assumption" on the domains in the statement of Theorem 5.1 is crucial. We illustrate this fact by providing example of a domain without any symmetry for which the theorem does not hold true. We also provide examples of symmetric domains having other symmetries but not having the symmetry of order 4 for which the theorem fails. These examples are explained in Remark 5.2.

We will be using the following notations in the rest of the article. For $s \in M$ and $r > 0$, $B_r(s)$ would denote the open geodesic ball in the Riemannian manifold (M, g) with center s and radius r. When there is no confusion about the center s, we can denote $B_r(s)$ by B_r. For $M = S^n$, we take $r \in (0, \frac{\pi}{2}]$. By $B_{R,r}$ we would mean the concentric annular domain in M with inner radius r and outer radius R. By $\tilde{\Omega}_{out}$, we mean a star-shaped domain in a space form, centered at a point p in its interior. For us, $\tilde{\Omega} := \tilde{\Omega}_{\text{out}} \backslash \overline{B_{r_1}(p)}$ when $\overline{B_{r_1}(p)} \subset \tilde{\Omega}_{\text{out}}$ for $r_1 > 0$. For $M = \mathbb{E}^n$, without loss of generality, B_r would mean $B_r(\underline{0})$, the Euclidean ball with radius r centered at the origin. Let $0 < R_1 < R_2$. Consider $B_{R_2} \backslash \overline{B_{R_1}}$, the concentric annular domain in $\mathbb{R}^n$ centered at the origin with inner radius R_1 and outer radius R_2. We set $\Omega_0 := B_{R_2} \backslash \overline{B_{R_1}}$. For us, Ω_{out} would mean a bounded domain in $\mathbb{R}^n$ having a symmetry of order 4 with respect to the origin. For us, $\Omega := \Omega_{\text{out}} \backslash \overline{B_{R_1}}$ when $\overline{B_{R_1}} \subset \Omega_{\text{out}}$.

2 Sharp bounds for the first Steklov–Dirichlet eigenvalue on star-shaped domains

In this section, we consider the Steklov–Dirichlet problem on star-shaped domains in space forms. Let $M = \mathbb{R}^n, S^n$, or $\mathbb{H}^n$ with Riemannian metric $g_M = dr^2 + \sin_M^2 r \, dg_{S^{n-1}}$, where $dg_{S^{n-1}}$ is the standard metric on the $(n-1)$-dimensional unit sphere S^{n-1}, and

$$\sin_M r := \begin{cases} \sin r & \text{for } r \in [0, \frac{\pi}{2}] \quad \text{when } M = S^n, \\ r & \text{for } r \in [0, \infty) \quad \text{when } M = \mathbb{R}^n, \\ \sinh r & \text{for } r \in [0, \infty) \quad \text{when } M = \mathbb{H}^n. \end{cases}$$

We wish to mention here that $\sin_M r$ is an increasing function of r.

Let $\tilde{\Omega}_{\text{out}} \subset M$ be a bounded domain having smooth boundary $\partial\tilde{\Omega}_{\text{out}}$. We further assume that $\tilde{\Omega}_{\text{out}}$ is a star-shaped domain with respect to a point p in its interior.

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Then, for every unit vector $u \in T_pM$, there exists a unique point $q \in \partial\tilde{\Omega}_{\text{out}}$ such that $q = \exp_p(r_u u)$ for some real number $r_u > 0$. Here, T_pM is the tangent space of the manifold M at the point $p \in M$; and $\exp_p$ is the exponential map defined from T_pM to M. Let $\|u\| := \sqrt{\langle u, u\rangle_{g_M}}$, and let

$$U_pM := \{u \in T_pM \,|\, \|u\| = 1\}.$$

Then, $\tilde{\Omega}_{\text{out}}$ and $\partial\tilde{\Omega}_{\text{out}}$ can be represented as

$$\tilde{\Omega}_{\text{out}} = \{\exp_p(tu) \,|\, u \in U_pM, 0 \le t < r_u\}, \quad \partial\tilde{\Omega}_{\text{out}} = \{\exp_p(r_u u) \,|\, u \in U_pM\}.$$

Note that r_u is the distance from p to $\partial\tilde{\Omega}_{\text{out}}$ in the direction $u \in U_pM$. Define

$$R_{\min} := \min_{u \in U_pM} r_u \quad \text{and} \quad R_{\max} := \max_{u \in U_pM} r_u.$$

Let ∂_r denote the radial vector field emanating from p. For $q \in \partial\tilde{\Omega}_{\text{out}}$, let $\nu(q)$ denote the unit outward normal to $\partial\tilde{\Omega}_{\text{out}}$ at q; and $\theta(q)$ be the angle between $\partial_r(q)$ and $\nu(q)$. Since $\partial\tilde{\Omega}_{\text{out}}$ is a level set of the function $\xi(r, u) = r - r_u$, we get $\nu(q) = \frac{\nabla\xi}{\|\nabla\xi\|}$. Let $\overline{\nabla} r_u$ represent the component of ∇r_u tangential to S^{n-1}. Then, in normal coordinates, the radial component of gradient of ξ satisfies

$$\langle \nabla\xi(q), \partial_r(q)\rangle_{g_M} = 1. \text{ And, } \|\nabla\xi\| = \sqrt{1 + \frac{1}{\sin^2_M(r_u)}\|\overline{\nabla} r_u\|^2}.$$

For any point $q \in \partial\tilde{\Omega}_{\text{out}}$, define

$$\cos\theta_u := \cos(\theta(q)) = \langle \nu(q), \partial_r(q)\rangle_{g_M} = \frac{1}{\|\nabla\xi\|}\langle \nabla\xi(q), \partial_r(q)\rangle_{g_M} = \frac{1}{\sqrt{1 + \frac{\|\overline{\nabla} r_u\|^2}{\sin^2_M(r_u)}}}.$$

Since $\tilde{\Omega}_{\text{out}}$ is a star-shaped bounded domain, $\cos(\theta(q)) > 0$ and therefore $\theta(q) < \frac{\pi}{2}$ for all $q \in \partial\tilde{\Omega}_{\text{out}}$. By the compactness assumption of $\partial\tilde{\Omega}_{\text{out}}$, there exists α such that $0 \le \theta_u = \theta(q) \le \alpha < \frac{\pi}{2}$ for all $q \in \partial\tilde{\Omega}_{\text{out}}$. Let $a = \tan^2\alpha$. Thus, for all $q \in \partial\tilde{\Omega}_{\text{out}}$,

$$\frac{\|\overline{\nabla} r_u\|^2}{\sin^2_M(r_u)} = \tan^2(\theta(q)) \le \tan^2(\alpha) = a.$$

For $s \in M$ and $r > 0$, let $B_r(s)$ denote the open geodesic ball in M with center s and radius r. When there is no confusion about the center s, we can denote $B_r(s)$ by B_r. For $M = S^n$, we take $r \in (0, \frac{\pi}{2}]$. Clearly, for any $s \in S^n$, $B_{\frac{\pi}{2}}(s)$ is nothing but the open hemisphere of S^n centered at s. Consider the following problem on $\tilde{\Omega}_{\text{out}}\backslash\overline{B_{r_1}}$:

$$\begin{cases} \Delta u = 0 & \text{in } \tilde{\Omega}_{\text{out}}\backslash\overline{B_{r_1}}, \\ u = 0 & \text{on } \partial B_{r_1}, \\ \frac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\tilde{\Omega}_{out}. \end{cases} \tag{2.1}$$

Theorem 2.1 *Let $M = \mathbb{R}^n, S^n$, or $\mathbb{H}^n$. Let $p \in M$, $r_1 > 0$, and $\tilde{\Omega}_{\text{out}} \subset M$ be a star-shaped domain with respect to p such that $\overline{B_{r_1}(p)} \subset \tilde{\Omega}_{\text{out}}$. If $M = S^n$, we further assume*

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that $\tilde{\Omega}_{\text{out}} \subset B_{\frac{\pi}{2}}(p)$. *Then, we have the following bounds for* $\sigma_1(\tilde{\Omega}_{\text{out}} \backslash \overline{B_{r_1}})$ *in terms of* $a, R_{\max}, R_{\min}$, $\sigma_1(B_{R_{\min}} \backslash \overline{B_{r_1}})$, *and* $\sigma_1(B_{R_{\max}} \backslash \overline{B_{r_1}})$:

$$\frac{1}{\sqrt{1+a}} \left(\frac{\sin_M^{n-1}(R_{\min})}{\sin_M^{n-1}(R_{\max})} \right) \sigma_1(B_{R_{\min}} \backslash \overline{B_{r_1}}) \leq \sigma_1(\tilde{\Omega}_{\text{out}} \backslash \overline{B_{r_1}}) \leq \left(\frac{\sin_M^{n-1}(R_{\max})}{\sin_M^{n-1}(R_{\min})} \right) \sigma_1(B_{R_{\max}} \backslash \overline{B_{r_1}}),$$

where $B_{R_{\max}}$ *and* $B_{R_{\min}}$ *are balls centered at* p *with radii* $R_{\max}$ *and* $R_{\min}$, *respectively.*

Proof Note that for any $s \in M$, $\sin_M^{n-1}(r)$ is the volume density of the geodesic sphere $S(s,r)$ in M, and du is the spherical volume density on the unit sphere $U_s M$. It is well known (see [24, p. 385]) that the surface measure dS and volume measure dV on M are given by

$$dS = \sec(\theta_u) \sin_M^{n-1}(r)\, du, \quad dV = \sin_M^{n-1}(r)\, dr\, du.$$

Let f be a positive eigenfunction of (2.1) corresponding to $\sigma_1(\tilde{\Omega}_{\text{out}} \backslash \overline{B_{r_1}})$. Then, substituting $\|\nabla f\|^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{\sin_M^2 r} \|\overline{\nabla} f\|^2$, $dV = \sin_M^{n-1}(r)\, dr\, du$, and using $r_u \geq R_{\min}$, we obtain

$$\begin{aligned}
\int_{(\tilde{\Omega}_{\text{out}} \backslash \overline{B_{r_1}})} \|\nabla f\|^2\, dV &= \int_{U_p M} \int_{r_1}^{r_u} \left[\left(\frac{\partial f}{\partial r} \right)^2 + \frac{1}{\sin_M^2 r} \|\overline{\nabla} f\|^2 \right] \sin_M^{n-1} r\, dr\, du \\
&\geq \int_{U_p M} \int_{r_1}^{R_{\min}} \left[\left(\frac{\partial f}{\partial r} \right)^2 + \frac{1}{\sin_M^2 r} \|\overline{\nabla} f\|^2 \right] \sin_M^{n-1} r\, dr\, du \\
&= \int_{(B_{R_{\min}} \backslash \overline{B_{r_1}})} \|\nabla f\|^2\, dV.
\end{aligned}$$

Now, since $dS = \sec(\theta_u)\, \sin_M^{n-1}(r)\, du$, $\sec(\theta_u) \leq \sec(\alpha)$, and $r_u \leq R_{\max}$, we have

$$\begin{aligned}
\int_{\partial \tilde{\Omega}_{\text{out}}} f^2\, dS &= \int_{U_p M} f^2 \sec(\theta_u)\, \sin_M^{n-1}(r_u)\, du \leq \sec(\alpha)\, \sin_M^{n-1}(R_{\max}) \int_{U_p M} f^2\, du \\
&= \left(\sqrt{1 + \tan^2(\alpha)} \right) \frac{\sin_M^{n-1}(R_{\max})}{\sin_M^{n-1}(R_{\min})} \int_{U_p M} f^2 \sin_M^{n-1}(R_{\min})\, du \\
&= \left(\sqrt{1+a} \right) \frac{\sin_M^{n-1}(R_{\max})}{\sin_M^{n-1}(R_{\min})} \int_{\partial B_{R_{\min}}} f^2\, dS.
\end{aligned}$$

Thus, using the restriction of f on the set $B_{R_{\min}} \backslash \overline{B_{r_1}}$ as a test function in the variational characterization (1.2) of $\sigma_1\left(B_{R_{\min}} \backslash \overline{B_{r_1}}\right)$, we can write

$$\begin{aligned}
\frac{\sin_M^{n-1}(R_{\min})}{\left(\sqrt{1+a}\right) \sin_M^{n-1}(R_{\max})} \sigma_1\left(B_{R_{\min}} \backslash \overline{B_{r_1}}\right) &\leq \frac{\sin_M^{n-1}(R_{\min})}{\left(\sqrt{1+a}\right) \sin_M^{n-1}(R_{\max})} \frac{\int_{\left(B_{R_{\min}} \backslash \overline{B_{r_1}}\right)} \|\nabla f\|^2\, dV}{\int_{\partial B_{R_{\min}}} f^2\, dS} \\
&\leq \frac{\int_{\left(\tilde{\Omega}_{\text{out}} \backslash \overline{B_{r_1}}\right)} \|\nabla f\|^2\, dV}{\int_{\partial \tilde{\Omega}_{\text{out}}} f^2\, dS}.
\end{aligned}$$

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Therefore,

$$\frac{\sin_M^{n-1}(R_{\min})}{\left(\sqrt{1+a}\right)\,\sin_M^{n-1}(R_{\max})}\sigma_1(B_{R_{\min}}\backslash\overline{B_{r_1}}) \le \frac{\displaystyle\int_{(\tilde{\Omega}_{\text{out}}\backslash\overline{B_{r_1}})}\|\nabla f\|^2\,dV}{\displaystyle\int_{\partial\tilde{\Omega}_{\text{out}}} f^2\,dS} = \sigma_1(\tilde{\Omega}_{\text{out}}\backslash\overline{B_{r_1}}).$$

Similar to the previous calculation, we now take h to be a positive eigenfunction corresponding to $\sigma_1(B_{R_{\max}}\backslash\overline{B_{r_1}})$. Then, using $dV = \sin_M^{n-1}(r)\,dr\,du$ and $r_u \le R_{\max}$, we get

$$\begin{aligned}\int_{(\tilde{\Omega}_{\text{out}}\backslash\overline{B_{r_1}})}\|\nabla h\|^2\,dV &= \int_{U_pM}\int_{r_1}^{r_u}\left[\left(\frac{\partial h}{\partial r}\right)^2+\frac{1}{\sin_M^2(r)}\|\overline{\nabla} h\|^2\right]\sin_M^{n-1}(r)\,dr\,du\\ &\le \int_{U_pM}\int_{r_1}^{R_{\max}}\left[\left(\frac{\partial h}{\partial r}\right)^2+\frac{1}{\sin_M^2(r)}\|\overline{\nabla} h\|^2\right]\sin_M^{n-1}(r)\,dr\,du\\ &= \int_{(B_{R_{\max}}\backslash\overline{B_{r_1}})}\|\nabla h\|^2\,dV.\end{aligned}$$

Now, using $dS = \sec(\theta_u)\,\sin_M^{n-1}(r_u)\,du$, $\sec(\theta_u)\ge 1$, and $R_{\min}\le r_u$, we get

$$\begin{aligned}\int_{\partial\tilde{\Omega}_{\text{out}}} h^2\,dS &= \int_{U_pM} h^2\,\sec(\theta_u)\,\sin_M^{n-1}(r_u)\,du \ge \int_{U_pM} h^2\,\sin_M^{n-1}(r_u)\,du\\ &\ge \sin_M^{n-1}(R_{\min})\int_{U_pM} h^2\,du\\ &= \frac{\sin_M^{n-1}(R_{\min})}{\sin_M^{n-1}(R_{\max})}\int_{U_pM} h^2\,\sin_M^{n-1}(R_{\max})\,du\\ &= \frac{\sin_M^{n-1}(R_{\min})}{\sin_M^{n-1}(R_{\max})}\int_{\partial B_{R_{\max}}} h^2\,dS.\end{aligned}$$

Using the variational characterization (1.2) of $\sigma_1(\tilde{\Omega}_{out}\backslash\overline{B_{r_1}})$ once again, we get

$$\sigma_1(\tilde{\Omega}_{\text{out}}\backslash\overline{B_{r_1}}) \le \frac{\displaystyle\int_{(\tilde{\Omega}_{\text{out}}\backslash\overline{B_{r_1}})}\|\nabla h\|^2\,dV}{\displaystyle\int_{\partial\tilde{\Omega}_{\text{out}}} h^2\,dS} \le \frac{\sin_M^{n-1}(R_{\max})}{\sin_M^{n-1}(R_{\min})}\frac{\displaystyle\int_{(B_{R_{\max}}\backslash\overline{B_{r_1}})}\|\nabla h\|^2\,dV}{\displaystyle\int_{\partial B_{R_{\max}}} h^2\,dS}.$$

Since h is an eigenfunction corresponding to $\sigma_1(B_{R_{\max}}\backslash\overline{B_{r_1}})$, we conclude that

$$\sigma_1(\tilde{\Omega}_{\text{out}}\backslash\overline{B_{r_1}}) \le \frac{\sin_M^{n-1}(R_{\max})}{\sin_M^{n-1}(R_{\min})}\sigma_1(B_{R_{\max}}\backslash\overline{B_{r_1}}).$$ ∎

The following theorem extends Theorem 1.1 of [15] from bounded convex domains to bounded star-shaped domains of $\mathbb{R}^n$, $n\ge 2$. The proof follows using exactly the same arguments as in [15].

For $n\ge 2$ and $R_1>0$, define $\overline{R_1}$ in terms of R_1 and n, as

$$\overline{R_1} := \begin{cases} R_1 e^{\sqrt{2}} & \text{if } n=2,\\ R_1\left[\dfrac{(n-1)+(n-2)\sqrt{2(n-1)}}{n-1}\right]^{\frac{1}{n-2}} & \text{if } n\ge 3.\end{cases}$$

Let B_R be a ball in $\mathbb{R}^n$ of radius $R>0$ centered at the origin.

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Theorem 2.2 *Fix $n \geq 2$. Let $\tilde{\Omega}_{out}$ be a bounded domain in $\mathbb{R}^n$ which is star shaped with respect to the origin. For any $R_1 > 0$ for which $\overline{B_{R_1}} \subset \tilde{\Omega}_{\text{out}} \subset B_{\overline{R_1}}$, we get*

$$\sigma_1(\tilde{\Omega}_{\text{out}} \backslash \overline{B_{R_1}}) \leq \sigma_1(B_{R_2} \backslash \overline{B_{R_1}}),$$

where $R_2 > 0$ is such that $\overline{B_{R_1}} \subset B_{R_2}$ and $\text{Vol}\,(\tilde{\Omega}_{\text{out}}) = \text{Vol}\,(B_{R_2})$.

3 The Steklov–Dirichlet problem on concentric annular domains

In this section, we study some properties of the mixed Steklov–Dirichlet eigenvalues and their corresponding eigenfunctions on concentric annular domains in $\mathbb{R}^n$. We first use separation of variables method in order to compute the eigenfunctions, where we need the following proposition which describes the eigenvalues and the eigenfunctions of $\Delta_{S^{n-1}}$, Laplace operator on $n-1$-dimensional unit sphere [28, Sections 22.3 and 22.4].

Proposition 3.1 *The set of all eigenvalues of $\Delta_{S^{n-1}}$ is $\{l\,(l+n-2) : l \in \mathbb{N} \cup \{0\}\}$. The eigenfunctions corresponding to each eigenvalue $l(l+n-2)$ are the spherical harmonics of degree l and thus, the multiplicity of the eigenvalue $l(l+n-2)$ equals the dimension of $\mathcal{H}_l$, the space of harmonic homogeneous polynomials of degree l on $\mathbb{R}^n$.*

Let $\Omega_0 = B_{R_2} \backslash \overline{B_{R_1}}$, where B_{R_1} and B_{R_2} are concentric balls in $\mathbb{R}^n$ of radius R_1 and R_2, respectively, $0 < R_1 < R_2$. Without loss of generality, we assume that both balls B_{R_1} and B_{R_2} are centered at the origin. Now, we find eigenvalues and corresponding eigenfunctions of the following Steklov–Dirichlet eigenvalue problem on domain Ω_0 and study some of their properties:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_0 = B_{R_2} \backslash \overline{B_{R_1}}, \\ u = 0 & \text{on } \partial B_{R_1}, \\ \frac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial B_{R_2}. \end{cases} \tag{3.1}$$

Let $u(r, \omega) = f(r)g(\omega)$ be a smooth function, where f is a radial function defined on $[R_1, R_2]$ and g is an eigenfunction of $\Delta_{S^{n-1}}$ corresponding to the eigenvalue $l(l+n-2)$. Now,

$$\begin{aligned} \Delta u(r, \omega) &= g(\omega)\left(-f''(r) - \frac{n-1}{r} f'(r)\right) + \frac{f(r)}{r^2} \Delta_{S^{n-1}} g(\omega) \\ &= g(\omega)\left(-f''(r) - \frac{n-1}{r} f'(r) + \frac{f(r)}{r^2} l(l+n-2)\right). \end{aligned}$$

If u is a solution of (3.1), then f satisfies the following ordinary differential equation and boundary constraints:

$$\begin{aligned} &-f''(r) - \tfrac{n-1}{r} f'(r) + \tfrac{l(l+n-2)}{r^2} f(r) = 0 \ \text{ for } \ r \in (R_1, R_2), \\ &f(R_1) = 0, \ f'(R_2) = \sigma f(R_2). \end{aligned} \tag{3.2}$$

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The eigenfunctions of (3.2) are given below:

$$f_0(r) = \begin{cases} \frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} & \text{if } n > 2 \\ \ln(r) - \ln(R_1) & \text{if } n = 2, \end{cases} \quad \text{and,} \quad f_l(r) = r^l - \frac{R_1^{n+2l-2}}{r^{n+l-2}}, \text{ for } l \geq 1. \tag{3.3}$$

These eigenfunctions correspond to the following eigenvalues:

$$\sigma_{(0)}(\Omega_0) = \begin{cases} \frac{(n-2)R_1^{n-2}}{R_2^{n-1} - R_2 R_1^{n-2}} & \text{for } n > 2 \\ \frac{1}{R_2(ln(R_2) - ln(R_1))} & \text{for } n = 2 \end{cases}$$

and

$$\sigma_{(l)}(\Omega_0) = \frac{l\, R_2^{n+2l-2} + (n+l-2)R_1^{n+2l-2}}{R_2^{n+2l-1} - R_2 R_1^{n+2l-2}}, \text{ for } l \geq 1, \text{ respectively.}$$

Here, $\sigma_{(i)}(\Omega_0)$, $i \in \mathbb{N} \cup \{0\}$ denote the $(i+1)$-th Steklov–Dirichlet eigenvalue of Ω_0 counted without multiplicity.

Remark 3.2 For $2 \leq i \leq n+1$, $\sigma_i(\Omega_0) = \sigma_2(\Omega_0) = \sigma_{(1)}(\Omega_0)$. That is, $\sigma_{(1)}(\Omega_0)$ has multiplicity n and the corresponding eigenfunctions are $f_1(r)\,\frac{x_i}{r}$, $i = 1, 2, \ldots n$. Further, by substituting the value of f_1 from (3.3) in the variational characterization of $\sigma_2(\Omega_0)$, we get

$$\sigma_2(\Omega_0) = \sigma_{(1)}(\Omega_0) = R\left(\sum_{n=1}^{n}\left(\frac{f_1(r)x_i}{r}\right)\right) = \frac{\sum_{i=1}^{n}\int_{\Omega_0}\left\|\nabla\left(\frac{f_1(r)x_i}{r}\right)\right\|^2 dV}{\sum_{i=1}^{n}\int_{\partial B_{R_2}}\left(\frac{f_1(r)x_i}{r}\right)^2 dS}$$
$$= \frac{\int_{\Omega_0}\left((f_1'(r))^2 + \frac{(n-1)}{r^2}f_1^2(r)\right) dV}{\int_{\partial B_{R_2}} f_1^2(r)\, dS}.$$

Using the definition of $f_1(r)$, we get

$$(f_1'(r))^2 + \frac{(n-1)}{r^2}f_1^2(r) = n + n\,(n-1)\frac{R_1^{2n}}{r^{2n}}.$$

Then,

$$\int_{\Omega_0}\left((f_1'(r))^2 + \frac{(n-1)}{r^2}f_1^2(r)\right) dV = \int_{R_1}^{R_2}\int_{S^{n-1}}\left((f_1'(r))^2 + \frac{(n-1)}{r^2}f_1^2(r)\right) r^{n-1}\, dr\, du$$
$$= \int_{R_1}^{R_2}\int_{S^{n-1}}\left(nr^{n-1} + n\,(n-1)\frac{R_1^{2n}}{r^{n+1}}\right) dr\, du$$
$$= \frac{(R_2^n - R_1^n)(R_2^n + (n-1)R_1^n)}{R_2^n}\,\mathrm{Vol}(S^{n-1}).$$

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And

$$\int_{\partial B_{R_2}} f_1^2(r)\, dS = R_2^{n-1} f_1^2(R_2)\,\mathrm{Vol}(S^{n-1}) = \frac{(R_2^n - R_1^n)^2}{R_2^{n-1}}\,\mathrm{Vol}(S^{n-1}).$$

Hence,

$$\sigma_2(\Omega_0) = \frac{\int_{\Omega_0}\left((f_1'(r))^2 + \frac{(n-1)}{r^2} f_1^2(r)\right) dV}{\int_{\partial B_{R_2}} f_1^2(r)\, dS} = \frac{R_2^n + (n-1)R_1^n}{R_2^{n+1} - R_2 R_1^n}. \tag{3.4}$$

Lemma 3.3 *For $f_1 : [R_1, \infty) \to \mathbb{R}$, the eigenfunction given in (3.3), define two real-valued functions F and G on $[R_1, \infty)$ as follows:*

$$F(r) := (f_1'(r))^2 + \frac{(n-1)}{r^2} f_1^2(r), \quad \text{and} \quad G(r) := 2 f_1(r) f_1'(r) + \frac{n-1}{r} f_1^2(r).$$

Then, both F and G are nonnegative functions. The function F is a decreasing function of r while G is an increasing function of r.

Proof We have $f_1(r) = r - \frac{R_1^n}{r^{n-1}}$. Therefore, $f_1'(r) = 1 + (n-1)\left(\frac{R_1}{r}\right)^n$. Substituting these in the definition of F and G, we get

$$F(r) = n\left(1 + (n-1)\left(\frac{R_1}{r}\right)^{2n}\right), \quad G(r) = (n+1)\, r - 2\,\frac{R_1^n}{r^{n-1}} - (n-1)\,\frac{R_1^{2n}}{r^{2n-1}}.$$

Consequently, for all $r \in (R_1, \infty)$, $F'(r) = -2\,n\,(n-1)\,\frac{R_1^{2n}}{r^{2n+1}} < 0$, while $G'(r) = (n+1) + 2\,(n-1)\,\frac{R_1^n}{r^n} + (2n-1)\,(n-1)\,\frac{R_1^{2n}}{r^{2n}} > 0$. Hence, proved. ∎

4 Integral inequalities related to the Steklov–Dirichlet eigenfunctions on annular domains

Let $E_1, E_2, \ldots, E_n$ be the standard orthonormal basis of $\mathbb{R}^n$ and $(x_1, x_2, \ldots, x_n)$ be the standard normal coordinate system on $\mathbb{R}^n$. For $s \in \mathbb{N}$ and $i, j \in \{1, 2, \ldots, n\}$, let $R_{i,j}^{\frac{2\pi}{s}}$ denote the rotation, in the anti-clockwise direction, around the origin by an angle $\frac{2\pi}{s}$ in the coordinate plane (x_i, x_j).

Definition 4.1 A domain $W \subset \mathbb{R}^n$ is said to have a *symmetry of order s* with respect to the origin if $R_{i,j}^{\frac{2\pi}{s}}(W) = W$ for all $i, j \in \{1, 2, \ldots, n\}$.

Definition 4.2 A domain $W \subset \mathbb{R}^n$ is said to be *centrally symmetric* with respect to the origin if $-x \in W$ whenever $x \in W$.

The following proposition is useful in choosing the test functions to be substituted in the variational characterization of the Steklov–Dirichlet eigenvalues on a domain with a symmetry of order 4.

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Proposition 4.1 *Let $g : (0, \infty) \to \mathbb{R}$ be a smooth function, and $W \subset \mathbb{R}^n$ be a bounded domain with smooth boundary.*

(1) *If W has a symmetry of order 2, then for each $i = 1, 2, \ldots n$, we have*

$$(4.1) \qquad (a) \int_{x\in W} g(\|x\|)\, x_i \, dV = 0, \quad (b) \int_{x\in \partial W} g(\|x\|)\, x_i \, dS = 0.$$

(2) *If W has symmetry of order 4, then for each $i, j = 1, 2, \ldots n$, $i \neq j$, we have*

$$(4.2) \qquad (a) \int_{x\in W} g(\|x\|)\, x_i\, x_j \, dV = 0, \quad (b) \int_{x\in \partial W} g(\|x\|)\, x_i\, x_j \, dS = 0.$$

Proof

(1) (a) The domain W has a symmetry of order 2. Therefore, the transformation $x = R_{i,j}^{\frac{2\pi}{2}}(y) = -y$ for $j \neq i$ would give us

$$\int_{x\in W} g(\|x\|)\, x_i \, dV = -\int_{y\in W} g(\|y\|)\, y_i \, dV.$$

Thus, $\int_{x\in W} g(\|x\|)\, x_i \, dV = 0$, for $i = 1, 2, \ldots n$.

(b) Now, since ∂W is also symmetric of order 2, the proof of part (b) follows by the same arguments used in part (a).

(2) (a) If the domain W has symmetry of order 4 then the transformation $x = R_{i,j}^{\frac{2\pi}{4}}(y)$ would yield us

$$\int_{x\in W} g(\|x\|)\, x_i\, x_j \, dV = -\int_{y\in W} g(\|y\|)\, y_i\, y_j \, dV.$$

Thus, $\int_W g(\|x\|)\, x_i\, x_j \, dV = 0$.

(b) The proof of this part is similar to that of part (a). ∎

Now, for the eigenfunction f_1 given in (3.3), we get

$$\left\langle \nabla\left(\frac{f_1(\|x\|)}{\|x\|}x_i\right), E_j\right\rangle = \frac{\partial}{\partial x_j}\left(\frac{f_1(\|x\|)}{\|x\|}x_i\right) = \begin{cases} \frac{f_1'(\|x\|)}{\|x\|^2}x_j\, x_i - \frac{f_1(\|x\|)}{\|x\|^3}x_j\, x_i, & \text{for } j \neq i, \\ \frac{f_1'(\|x\|)}{\|x\|^2}x_i^2 + \frac{f_1(\|x\|)}{\|x\|^3}(\|x\|^2 - x_i^2), & \text{for } j = i. \end{cases}$$

Consequently, we get

$$\left\langle \nabla\left(\frac{f_1(\|x\|)}{\|x\|}x_i\right), \nabla\left(\frac{f_1(\|x\|)}{\|x\|}x_j\right)\right\rangle = \begin{cases} \left(\frac{(f_1'(\|x\|))^2}{\|x\|^2} - \frac{(f_1(\|x\|))^2}{\|x\|^4}\right)x_i x_j, & \text{for } j \neq i, \\ \left(\frac{(f_1'(\|x\|))^2}{\|x\|^2}x_i^2 - \frac{f_1^2(\|x\|)}{\|x\|^4}x_i^2 + \frac{f_1^2(\|x\|)}{\|x\|^2}\right), & \text{for } j = i. \end{cases}$$

(4.3)

And $$\left\langle \nabla\left(f_1(\|x\|)\right), \nabla\left(\frac{f_1(\|x\|)}{\|x\|}x_j\right)\right\rangle = \frac{(f_1'(\|x\|))^2}{\|x\|}x_j.$$

(4.4)

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We recall that $\Omega_0 = B_{R_2}\backslash\overline{B_{R_1}}$ is the concentric annular domain, and that f_1 is the radial eigenfunction corresponding to the Steklov–Dirichlet eigenvalue $\sigma_1(\Omega_0)$ given in (3.3). Now, from equalities (4.3) and (4.4) and Proposition 4.1, we get the following result.

Corollary 4.2 *For the domain $\Omega := \Omega_{\text{out}}\backslash\overline{B_{R_1}}$, where Ω_{out} is a smooth domain which has a symmetry of order 4 with respect to the origin and is such that* $\text{Vol}\,(\Omega_{\text{out}}) = \text{Vol}\,(B_{R_2})$*, we get the following integral equalities for each $i, j = 1, 2, \ldots n$, $i \neq j$,*

(1) $$\int_{x\in\partial\Omega} f_1(\|x\|)\frac{f_1(\|x\|)}{\|x\|}x_i\,dS = 0,$$

(2) $$\int_{x\in\partial\Omega} \frac{f_1(\|x\|)}{\|x\|}x_i\frac{f_1(\|x\|)}{\|x\|}x_j\,dS = 0,$$

(3) $$\int_{x\in\Omega}\left\langle\nabla f_1(\|x\|), \nabla\left(\frac{f_1(\|x\|)}{\|x\|}x_i\right)\right\rangle dV = 0,$$

(4) $$\int_{x\in\Omega}\left\langle\nabla\left(\frac{f_1(\|x\|)}{\|x\|}x_i\right), \nabla\left(\frac{f_1(\|x\|)}{\|x\|}x_j\right)\right\rangle dV = 0.$$

Proof Please note that any domain having a symmetry of order 4 also has a symmetry of order 2. Equalities (4.1) and (4.4) imply (1) and (3). Similarly, using equalities (4.2) and (4.3), we get the integral equalities (2) and (4). ∎

Remark 4.3 It is easy to see that $\text{Vol}\,(\Omega) = \text{Vol}\,(\Omega_0)$.

Lemma 4.4 *Let W be a bounded smooth domain in $\mathbb{R}^n$ having a symmetry of order 4. Let $g : (0, \infty) \to \mathbb{R}$ be a positive continuous radial function. Then, there exists a constant $A > 0$, independent of i, such that*

$$\int_{x\in W} g(\|x\|)\,x_i^2\,dV = A \text{ for each } i = 1, 2\ldots n.$$

Proof Clearly, $\int_{x\in W} g(\|x\|)\,x_i^2\,dV > 0$ for each $i = 1, 2\ldots n$. Now, for a fixed $i \neq 1$, if we take the transformation $x = R_{1,i}^{\frac{2\pi}{4}}(y)$, we obtain $\int_{x\in W} g(\|x\|)\,x_i^2\,dV = \int_{y\in W} g(\|y\|)\,y_1^2\,dV$. That is, the value of the integral is independent of the choice of i. This value is the constant $A > 0$ as desired. Hence, proved. ∎

Lemma 4.5 *The following inequality holds for $F : [R_1, \infty) \to \mathbb{R}$ defined in Lemma 3.3:*

$$\int_{\Omega=(\Omega_{\text{out}}\backslash\overline{B_{R_1}})} F(\|x\|)\,dV \leq \int_{\Omega_0=(B_{R_2}\backslash\overline{B_{R_1}})} F(\|x\|)\,dV. \tag{4.5}$$

Proof The domain $\Omega = \Omega_{\text{out}}\backslash\overline{B_{R_1}}$ is a disjoint union of the domains $(\Omega \cap \Omega_0)$ and $\Omega\backslash(\Omega \cap \Omega_0)$. Therefore,

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$$\int_{\Omega} F(\|x\|)\,dV = \int_{\Omega\cap\Omega_0} F(\|x\|)\,dV + \int_{\Omega\setminus(\Omega\cap\Omega_0)} F(\|x\|)\,dV$$
$$= \int_{\Omega_0} F(\|x\|)\,dV - \int_{\Omega_0\setminus(\Omega\cap\Omega_0)} F(\|x\|)\,dV + \int_{\Omega\setminus(\Omega\cap\Omega_0)} F(\|x\|)\,dV.$$

Since F is a decreasing function of r (cf. Lemma 3.3), we have

$$F(\|x\|) \geq F(R_2) \quad \text{for } x \in \Omega_0\setminus(\Omega\cap\Omega_0), \tag{4.6}$$

$$\text{and } F(\|x\|) \leq F(R_2) \quad \text{for } x \in \Omega\setminus(\Omega\cap\Omega_0). \tag{4.7}$$

Therefore,

$$\int_{\Omega} F(\|x\|)\,dV \leq \int_{\Omega_0} F(\|x\|)\,dV - \int_{\Omega_0\setminus(\Omega\cap\Omega_0)} F(R_2)\,dV + \int_{\Omega\setminus(\Omega\cap\Omega_0)} F(R_2)\,dV.$$

Since Vol (Ω) = Vol (Ω_0) (cf. Remark 4.3), we have Vol $(\Omega_0\setminus(\Omega\cap\Omega_0))$ = Vol $(\Omega\setminus(\Omega\cap\Omega_0))$. As a consequence, we get the desired inequality. ∎

Let $\partial\Omega_{\text{out}}$ and ∂B_{R_2} denote the boundaries of Ω_{out} and B_{R_2}, respectively. For each $u \in S^{n-1}$, define $R_u = \sup\{r \,|\, r\,u \in \partial\Omega_{\text{out}}\}$. Since $\overline{B_{R_1}} \subset \Omega_{\text{out}}$ and Ω_{out} is a smooth bounded domain in $\mathbb{R}^n$ containing $\overline{B_{R_1}}$, we can say that, for each $u \in S^{n-1}$, there exists at least one real number $r > R_1$ such that $r\,u \in \partial\Omega_{\text{out}}$. Thus, $R_1 < R_u < \infty$ and $R_u u \in \partial\Omega_{\text{out}}$ for each $u \in S^{n-1}$. Therefore, the set $C = \{R_u\,u \,|\, u \in S^{n-1}\}$ is a subset of $\partial\Omega_{\text{out}}$. Let $C_{in} := \{r\,u \,|\, 0 \leq r < R_u,\ u \in S^{n-1}\}$. Then, C_{in} is a domain in $\mathbb{R}^n$ such that $\Omega_{\text{out}} \subset C_{\text{in}}$. These sets will be used in the proof of the following lemma.

Lemma 4.6 *The following inequality holds for the radial eigenfunction f_1 (given in (3.3)) corresponding to the Steklov–Dirichlet eigenvalue $\sigma_1(\Omega_0)$:*

$$\int_{x\in\partial\Omega_{\text{out}}} f_1^2(\|x\|)\,dS \geq \int_{x\in\partial B_{R_2}} f_1^2(\|x\|)\,dS. \tag{4.8}$$

Proof Since $C \subset \partial\Omega_{\text{out}}$, we get $\displaystyle\int_{x\in\partial\Omega_{\text{out}}} f_1^2(\|x\|)\,dS \geq \int_{x\in C} f_1^2(\|x\|)\,dS$.

Note that if x is an element of set $C = \{R_u u \,|\, u \in S^{n-1}\}$, then $\|x\| = R_u$ and surface measure on C is $dS = \sec(\theta_u)\,{R_u}^{n-1}\,du \geq {R_u}^{n-1}\,du$. Thus, we get the following inequality:

$$\int_{x\in C} f_1^2(\|x\|)\,dS = \int_{u\in S^{n-1}} f_1^2(R_u)\,\sec(\theta_u)\,R_u^{n-1}\,du \geq \int_{u\in S^{n-1}} f_1^2(R_u)\,R_u^{n-1}\,du.$$

Recall that, for the Steklov–Dirichlet eigenfunction f_1 on Ω_0, $f_1(R_1) = 0$. Therefore, we can write

$$\int_{u\in S^{n-1}} f_1^2(R_u)\,R_u^{n-1}\,du = \int_{u\in S^{n-1}} \left(f_1^2(R_u)\,R_u^{n-1} - f_1^2(R_1)\,R_1^{n-1}\right)\,du$$
$$= \int_{u\in S^{n-1}} \left(\int_{R_1}^{R_u} \frac{d}{dr}\left(f_1^2(r)\,r^{n-1}\right)\,dr\right)\,du$$

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$$= \int_{u\in S^{n-1}} \int_{R_1}^{R_u} \left(2\, r^{n-1} f_1(r) f_1'(r) + (n-1)\, r^{n-2} f_1^2(r)\right) dr\, du$$
$$= \int_{u\in S^{n-1}} \int_{R_1}^{R_u} \left(2 f_1(r) f_1'(r) + \frac{(n-1)}{r} f_1^2(r)\right) r^{n-1}\, dr\, du.$$

We recall here that $C_{in} = \{r\,u \mid 0 \le r < R_u,\ u \in S^{n-1}\}$ is a domain in $\mathbb{R}^n$ containing Ω_{out}. Therefore, in the spherical coordinates $C_{in}\backslash\overline{B_{R_1}} = \{r\,u \mid R_1 < r < R_u,\, u \in S^{n-1}\}$. And hence, we can rewrite the above integral as follows:

$$\int_{u\in S^{n-1}} \int_{R_1}^{R_u} \left(2 f_1(r) f_1'(r) + \frac{(n-1)}{r} f_1^2(r)\right) r^{n-1}\, dr\, du$$
$$= \int_{x\in\left(C_{\text{in}}\backslash\overline{B_{R_1}}\right)} \left(2 f_1(\|x\|) f_1'(\|x\|) + \frac{(n-1)}{\|x\|} f_1^2(\|x\|)\right) dV$$
$$\ge \int_{x\in\left(\Omega_{\text{out}}\backslash\overline{B_{R_1}}\right)} \left(2 f_1(\|x\|) f_1'(\|x\|) + \frac{(n-1)}{\|x\|} f_1^2(\|x\|)\right) dV$$
$$= \int_{\Omega=\left(\Omega_{\text{out}}\backslash\overline{B_{R_1}}\right)} G(\|x\|)\, dV,$$

where $G(r) = 2f_1(r) f_1'(r) + \frac{(n-1)}{r} f_1^2(r)$ is as defined in Lemma 3.3. From all of the above integral equalities and inequalities, we get

$$\int_{x\in\partial\Omega_{\text{out}}} f_1^2(\|x\|)\, dS \ge \int_{x\in\Omega} G(\|x\|)\, dV. \tag{4.9}$$

Clearly,

$$\int_{x\in\Omega} G(\|x\|)\, dV = \int_{x\in\Omega\cap\Omega_0} G(\|x\|)\, dV + \int_{x\in\Omega\backslash(\Omega\cap\Omega_0)} G(\|x\|)\, dV$$
$$= \int_{x\in\Omega_0} G(\|x\|)\, dV - \int_{x\in\Omega_0\backslash(\Omega\cap\Omega_0)} G(\|x\|)\, dV$$
$$+ \int_{x\in\Omega\backslash(\Omega\cap\Omega_0)} G(\|x\|)\, dV.$$

Now, since G is an increasing function of r (cf. Lemma 3.3), we get

$$G(\|x\|) \le G(R_2)\ \forall\ x \in \Omega_0\backslash(\Omega\cap\Omega_0) \ \text{ and } \ G(\|x\|) \ge G(R_2)\ \forall\ x \in \Omega\backslash(\Omega\cap\Omega_0). \tag{4.10}$$

Therefore, we get

$$\int_{x\in\Omega} G(\|x\|)\, dV \ge \int_{x\in\Omega_0} G(\|x\|)\, dV - \int_{x\in\Omega_0\backslash(\Omega\cap\Omega_0)} G(R_2)\, dV + \int_{x\in\Omega\backslash(\Omega\cap\Omega_0)} G(R_2)\, dV.$$

Since Vol $(\Omega_0\backslash(\Omega\cap\Omega_0))$ = Vol $(\Omega\backslash(\Omega\cap\Omega_0))$, the last two terms of the previous inequality add up to zero. As a result, we have

$$\int_{x\in\Omega} G(\|x\|)\, dV \ge \int_{x\in\Omega_0} G(\|x\|)\, dV$$
$$= \int_{x\in\Omega_0} \left(2 f_1(\|x\|) f_1'(\|x\|) + \frac{(n-1)}{\|x\|} f_1^2(\|x\|)\right) dV$$

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$$= \int_{u\in S^{n-1}} \int_{R_1}^{R_2} \left(2 f_1(r) f_1'(r) + \frac{(n-1)}{r} f_1^2(r) \right) r^{n-1}\, dr\, du$$
$$= \int_{u\in S^{n-1}} f_1^2(R_2)\, R_2^{n-1}\, du = \int_{x\in\partial B_{R_2}} f_1^2(\|x\|)\, dS.$$

Here, we have once again used the fact that $f_1(R_1) = 0$. We have thus proved that $\int_{x\in\partial\Omega_{out}} f_1^2(\|x\|)\, dS \geq \int_{x\in\partial B_{R_2}} f_1^2(\|x\|)\, dS$. ∎

5 Bounds for higher eigenvalues

Let W_1 and W_2 be smooth bounded domains in $\mathbb{R}^n$ such that $\overline{W_2} \subset W_1$. Let Γ_1, Γ_2 denote the boundaries of W_1, W_2 respectively. Let $W := W_1 \backslash \overline{W_2}$. We consider the following Steklov–Dirichlet eigenvalue problem on W:

$$\begin{cases} \Delta u = 0 & \text{in } W := W_1 \backslash W_2 \\ u = 0 & \text{on } \Gamma_2, \\ \frac{\partial u}{\partial \nu} = \sigma u & \text{on } \Gamma_1. \end{cases} \tag{5.1}$$

Let $\sigma_k(W)$ denote the k-th Steklov–Dirichlet eigenvalue of (5.1) on W. We recall here that B_R denotes a ball of radius R centered at the origin. We now state our main theorem which provides the isoperimetric bounds for higher Steklov Dirichlet eigenvalues.

Theorem 5.1 *Fix $0 < R_1 < R_2$. Let Ω_{out} be a bounded smooth domain in $\mathbb{R}^n$ having a symmetry of order 4 with respect to the origin such that $\overline{B_{R_1}} \subset \Omega_{\text{out}}$ and $\text{Vol}\,(\Omega_{\text{out}}) = \text{Vol}\,(B_{R_2})$. Then, for $2 \leq k \leq n+1$,*

$$\sigma_k(\Omega_{\text{out}} \backslash \overline{B_{R_1}}) \leq \sigma_k(B_{R_2} \backslash \overline{B_{R_1}}) = \sigma_2(B_{R_2} \backslash \overline{B_{R_1}}). \tag{5.2}$$

Proof We recall here the notations used in the previous section, viz., $\Omega = \Omega_{\text{out}} \backslash \overline{B_{R_1}}$, $\Omega_0 = B_{R_2} \backslash \overline{B_{R_1}}$. From the hypothesis of the theorem, it follows that $\text{Vol}\,(\Omega) = \text{Vol}\,(\Omega_0)$.

Since $\sigma_i(\Omega_0) = \sigma_2(\Omega_0)\ \forall\ i \in \{2, 3, \ldots, n+1\}$, it is enough to prove that $\sigma_{n+1}(\Omega) \leq \sigma_{n+1}(\Omega_0) = \sigma_2(\Omega_0)$. In order to prove this inequality, we will find a suitable test function to be plugged in the variational characterization (1.2) of $\sigma_{n+1}(\Omega)$.

Consider the following $(n+1)$-dimensional subspace of $\check{H}_0^1(\Omega)$:

$$E = \text{span}\left\{ f_1, \frac{x_1}{\|x\|} f_1 \ldots \frac{x_n}{\|x\|} f_1 \right\},$$

where f_1 is the Steklov–Dirichlet eigenfunction corresponding to $\sigma_1(\Omega_0)$. That is, f_1 is a solution of (3.2) and its expression is given in (3.3) for $l = 1$. Now, for any $u \in E \backslash \{0\}$, there exist $c_0, c_1, \ldots c_n \in \mathbb{R}$, not simultaneously equal to zero, such that

$$u = c_0 f_1 + c_1 \frac{x_1}{\|x\|} f_1 + \cdots + c_n \frac{x_n}{\|x\|} f_1.$$

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Then, using Corollary 4.2, we get

$$\frac{\int_\Omega \|\nabla u\|^2\, dV}{\int_{\partial\Omega_{\text{out}}} u^2 dS} = \frac{c_0^2 \int_\Omega \|\nabla f_1(\|x\|)\|^2\, dV + \sum_{i=1}^n c_i^2 \int_\Omega \left\|\nabla\left(\frac{f_1(\|x\|)}{\|x\|} x_i\right)\right\|^2 dV}{c_0^2 \int_{\partial\Omega_{\text{out}}} f_1^2(\|x\|)\, dS + \sum_{i=1}^n c_i^2 \int_{\partial\Omega_{\text{out}}} \frac{f_1^2(\|x\|)}{\|x\|^2} x_i^2\, dS}. \tag{5.3}$$

Now, by using Lemma 4.4 and equation (4.3), it follows that there exist constants $A_1, A_2 > 0$ such that for all $i = 1, 2, \ldots, n$,

$$\int_{\partial\Omega_{\text{out}}} \left(\frac{f_1(\|x\|)}{\|x\|} x_i\right)^2 dS = \int_{\partial\Omega_{\text{out}}} \frac{f_1^2(\|x\|)}{\|x\|^2} x_i^2\, dS = A_1,$$

$$\int_\Omega \left\|\nabla\left(\frac{f_1(\|x\|)}{\|x\|} x_i\right)\right\|^2 dV = \int_\Omega \left(\frac{(f_1'(\|x\|))^2}{\|x\|^2} x_i^2 - \frac{f_1^2(\|x\|)}{\|x\|^4} x_i^2 + \frac{f_1^2(\|x\|)}{\|x\|^2}\right) dV = A_2.$$

Therefore,

$$n A_1 = \sum_{i=1}^n \int_{\partial\Omega_{\text{out}}} \left(\frac{f_1(\|x\|)}{\|x\|} x_i\right)^2 dS = \int_{\partial\Omega_{\text{out}}} f_1^2(\|x\|)\, dS$$

and

$$\begin{aligned} n A_2 &= \sum_{i=1}^n \int_\Omega \left(\frac{(f_1'(\|x\|))^2 x_i^2}{\|x\|^2} - \frac{f_1^2(\|x\|) x_i^2}{\|x\|^4} + \frac{f_1^2(\|x\|)}{\|x\|^2}\right) dV \\ &= \int_\Omega \left((f_1'(\|x\|))^2 + \frac{(n-1) f_1^2(\|x\|)}{\|x\|^2}\right) dV. \end{aligned}$$

As a consequence, for all $i \in \{1, 2, \ldots, n\}$, we have

$$\int_{\partial\Omega_{\text{out}}} \left(\frac{f_1(\|x\|)}{\|x\|} x_i\right)^2 dS = A_1 = \frac{1}{n} \int_{\partial\Omega_{\text{out}}} f_1^2(\|x\|)\, dS \tag{5.4}$$

$$\int_\Omega \left\|\nabla\left(\frac{f_1(\|x\|)\, x_i}{\|x\|}\right)\right\|^2 dV = A_2 = \frac{1}{n} \int_\Omega \left((f_1'(\|x\|))^2 + \frac{(n-1)}{\|x\|^2} f_1^2(\|x\|)\right) dV. \tag{5.5}$$

Now, from (5.3)–(5.5), we get

$$\frac{\int_\Omega \|\nabla u\|^2\, dV}{\int_{\partial\Omega_{\text{out}}} u^2\, dS} = \frac{c_0^2 \int_\Omega \|\nabla f_1(\|x\|)\|^2\, dV + A_2 \sum_{i=1}^n c_i^2}{c_0^2 \int_{\partial\Omega_{\text{out}}} f_1^2(\|x\|)\, dS + A_1 \sum_{i=1}^n c_i^2} \le \max\left\{\frac{\int_\Omega \|\nabla f_1(\|x\|)\|^2\, dV}{\int_{\partial\Omega_{\text{out}}} f_1^2(\|x\|)\, dS}, \frac{A_2}{A_1}\right\}. \tag{5.6}$$

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Now,

$$\frac{A_2}{A_1} = \frac{\displaystyle\int_\Omega \left((f_1'(\|x\|))^2 + \frac{(n-1)}{\|x\|^2} f_1^2(\|x\|) \right) dV}{\displaystyle\int_{\partial\Omega_{\text{out}}} f_1^2(\|x\|)\, dS}$$
$$\geq \frac{\displaystyle\int_\Omega (f_1'(\|x\|))^2\, dV}{\displaystyle\int_{\partial\Omega_{\text{out}}} f_1^2(\|x\|)\, dS} = \frac{\displaystyle\int_\Omega \|\nabla f_1(\|x\|)\|^2\, dV}{\displaystyle\int_{\partial\Omega_{\text{out}}} f_1^2(\|x\|)\, dS}.$$

Then, from the inequality (5.6), we get

$$\frac{\displaystyle\int_\Omega \|\nabla u\|^2\, dV}{\displaystyle\int_{\partial\Omega_{\text{out}}} u^2\, dS} \leq \frac{A_2}{A_1} = \frac{\displaystyle\int_\Omega \left((f_1'(\|x\|))^2 + \frac{(n-1)}{\|x\|^2} f_1^2(\|x\|) \right) dV}{\displaystyle\int_{\partial\Omega_{\text{out}}} f_1^2(\|x\|)\, dS}. \tag{5.7}$$

Next, using the expression for $\sigma_2(\Omega_0)$ from (3.4), and using Lemmas 4.5 and 4.6, we get

$$\frac{A_2}{A_1} = \frac{\displaystyle\int_\Omega \left((f_1'(\|x\|))^2 + \frac{(n-1)}{\|x\|^2} f_1^2(\|x\|) \right) dV}{\displaystyle\int_{\partial\Omega_{\text{out}}} f_1^2(\|x\|)\, dS}$$
$$\leq \frac{\displaystyle\int_{\Omega_0} \left((f_1'(\|x\|))^2 + \frac{(n-1)}{\|x\|^2} f_1^2(\|x\|) \right) dV}{\displaystyle\int_{\partial B_{R_2}} f_1^2(\|x\|)\, dS} = \sigma_2(\Omega_0). \tag{5.8}$$

Therefore, from the variational characterization (1.2) and inequalities (5.7) and (5.8), we conclude that

$$\sigma_{(n+1)}(\Omega) \leq \max_{u(\neq 0)\in E} \frac{\displaystyle\int_\Omega \|\nabla u\|^2\, dV}{\displaystyle\int_{\partial\Omega_{\text{out}}} u^2\, dS} \leq \sigma_2(\Omega_0). \tag{5.9}$$

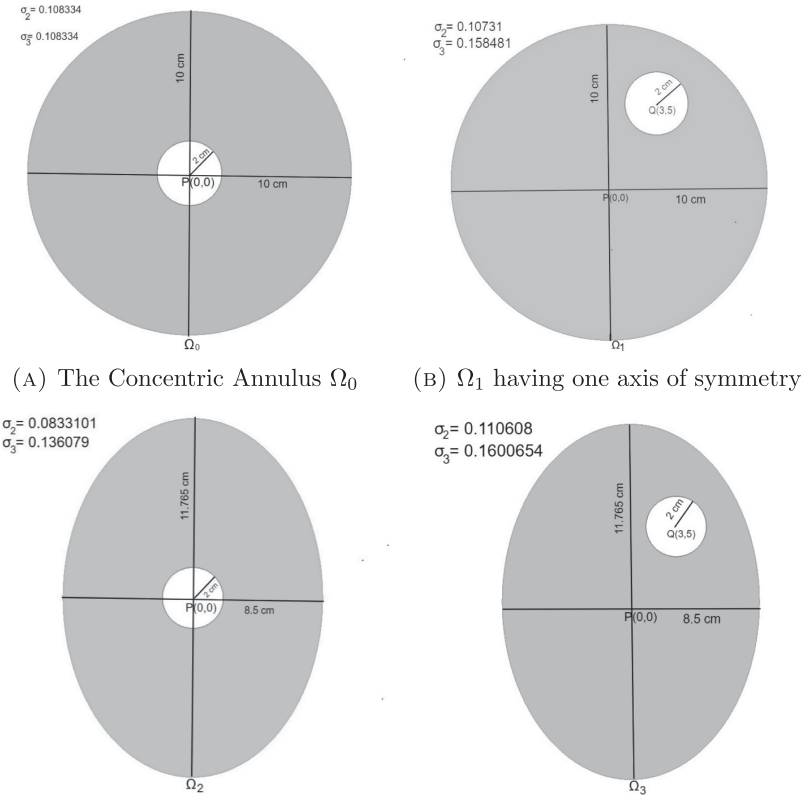
This completes the proof of Theorem 5.1. ∎

Remark 5.2 We now illustrate that the "symmetry of order 4" assumption in the statement of Theorem 5.1 cannot be dropped. In Figure 1, we give examples of three doubly connected planar domains, Ω_1, Ω_2, and Ω_3 (say), having the same area and such that they do not enjoy a symmetry of order 4. We will show that Theorem 5.1 does not hold true for these domains. These three domains are of the form $\Omega_i = \Omega_{\text{out}}^i \backslash \overline{B_0}$, where B_0 is the common inner circular hole for each Ω_i ($1 \leq i \leq 3$).

The domain Ω_1 has only one axis of symmetry while the domain Ω_2 has a symmetry of order 2. The domain Ω_3 has no symmetry at all.

Using FreeFem++, we found that $\sigma_3(\Omega_1)$ and $\sigma_3(\Omega_2)$ are higher than $\sigma_2(\Omega_0)$, where Ω_0 is the concentric annulus having the same area as that of Ω_i ($1 \leq i \leq 2$),

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(A) The Concentric Annulus Ω_0 (B) Ω_1 having one axis of symmetry

(C) Ω_2 having symmetry of order 2 (D) Ω_3 without any symmetry

Figure 1: Comparison of $\sigma_2(\Omega_i), \sigma_3(\Omega_i)(1 \le i \le 3)$ with $\sigma_2(\Omega_0) = \sigma_3(\Omega_0)$.

Table 1: Comparison of $\sigma_2(\Omega_0) = \sigma_3(\Omega_0)$ with $\sigma_2(\Omega_i), \sigma_3(\Omega_i),\ i = 1, 2, 3.$

The Steklov–Dirichlet Eigenvalues	Ω_0 Ball \Ball, concentric	Ω_1 Ball \Ball, non-concentric	Ω_2 Ellipse \Ball, concentric	Ω_3 Ellipse \Ball, non-concentric
σ_2	0.108334	0.10731	0.0833101	0.110608
σ_3	0.108334	0.158481	0.136079	0.1600654

and having the same circular hole B_0. This shows that Theorem 5.1 may not hold for σ_{n+1} even for domains with some symmetry if they do not have a symmetry of order 4.

The domain Ω_3 is an example of a domain which has no symmetry at all, and for which both $\sigma_2(\Omega_3)$ and $\sigma_3(\Omega_3)$ are higher than $\sigma_2(\Omega_0)$. This example shows that Theorem 5.1 does not hold for σ_2 as well (in addition to σ_3) if we drop the order 4 symmetry assumption.

The values of these Steklov–Dirichlet eigenvalues for Ω_i, $0 \le i \le 3$, are also listed in Table 1.

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