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# Interpolation by analytic functions on $c_0$

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*Abstract*

Let $B$ be the open unit ball of $c_0$. We give a geometric characterization of the sequences $\{x_n\} \subset bB$ with the property that, given any bounded sequence $\{\alpha_n\} \subset \mathbb{C}$, there is a continuous function $f: \bar{B} \to \mathbb{C}$, analytic in $B$, such that $f(x_n) = \alpha_n$ for all $n$ and such that $\sup_{x \in B}|f(x)| = \sup_{n \in \mathbb{N}}|\alpha_n|$.

## 1. *Introduction and main results*

In the present paper we study interpolation for polydisc algebras in infinite dimensions. Denote by $A_B$ the algebra of all bounded continuous functions on the closed unit ball $\bar{B}$ of $c_0$, analytic in the open ball $B$, with sup norm, and by $A_U$ the subalgebra of $A_B$ of those functions which are uniformly continuous on $B$ (see [2]). It is known that there are sequences $\{x_n\} \subset bB$ such that, given any sequence $\{\alpha_n\} \subset \mathbb{C}$ with $\sup|\alpha_n| = 1$, there is an $f \in A_B$ such that $f(x_n) = \alpha_n$ for all $n \in \mathbb{N}$ and $\|f\| = 1$ (see [1]). We call such sequences *norm-preserving interpolation sequences* (for $A_B$).

It is not difficult to show that there are no norm-preserving interpolation sequences for $A_U$, since this space is separable. To see this, we must appeal to the following classical result on polynomials on $c_0$ (see e.g. [3]). Let $P$ be an arbitrary $n$-homogeneous polynomial on $c_0$, and let $\epsilon > 0$. Then there is a polynomial $Q: c_0 \to \mathbb{C}$ of the form $Q(x) = \sum_{j=1}^{k} a_j \phi_j^n(x)$, where $a_j \in \mathbb{C}$ and $\phi_j \in l_1$ for each $j$, such that $\sup\{|P(x) - Q(x)|: x \in B\} < \epsilon$. Now let $f \in A_U$ and $\epsilon > 0$ be arbitrary. The uniform continuity of $f$ implies that for some $R \in (0, 1)$, $\sup\{|f(x) - f(Rx)|: x \in B\} < \epsilon$. Let $\sum_{n=0}^{\infty} P_n$ be the Taylor series of $f$ at 0, where each $P_n: c_0 \to \mathbb{C}$ is an $n$-homogeneous continuous polynomial. By Cauchy's inequality, the series $f = \sum_{n=0}^{\infty} P_n$ converges uniformly on the set $\{x \in c_0: \|x\| \leqslant R\}$. Thus we can find an integer $M$ such that $\sup\{|f(x) - \sum_{n=0}^{M} P_n(x)|: \|x\| \leqslant R\} < \epsilon$, and so there are polynomials $Q_0, \ldots, Q_M$ of the type described above such that $\sup\{|f(x) - \sum_{n=0}^{M} Q_n(Rx)|: x \in B\} < \epsilon$. Clearly each such polynomial $Q_n$ can be approximated by polynomials from a countable set (using the separability of $l_1$ and taking $a_j = b_j + ic_j$ with $b_j$ and $c_j$ rational). Therefore the separability of $A_U$ is proved.

In this paper we give a geometric characterization of norm-preserving interpolation sequences for $A_B$.

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Denote by $\{e_n\}$ the canonical basis in $c_0$. Let $x \in bB$. Write

$$x = \sum_{i=1}^{\infty} x_i e_i = \sum_{i=1}^{\infty} \langle x | e_i \rangle e_i.$$

We call the set $S(x) = \{n \in \mathbb{N} : x_n \neq 0\}$ the *support* of $x$ and the set $S_T(x) = \{n \in \mathbb{N} : |x_n| = 1\}$ the *toroidal support* of $x$. We call $x_T = \Sigma_{i \in S_T(x)} x_i e_i$ the *toroidal part* of $x$. The set

$$F(x) = \{x_T + y : \|y\| \leqslant 1, S(x_T) \cap S(y) = \varnothing\}$$

is called the *closed complex face* associated with $x$, or simply the *face* of $x$. The face of $x$ is a closed convex subset of $bB$ which contains $x$. It is the set of all $y$ obtained from $x$ by keeping fixed those coordinates of absolute value 1 and varying other coordinates.

The notion of face is central in the study of norm-preserving interpolation because of the following consequence of the maximum principle. Let $f \in A_B$. If $|f(x)| = \|f\|$ for some $x \in bB$ then $f$ is constant on $F(x)$ (see [1]). This implies that if $\{x_n\} \subset bB$ is a norm-preserving interpolation sequence then $F(x_i) \cap F(x_j) = \varnothing$ whenever $i \neq j$. As we will see below this condition is not sufficient for a sequence to be a norm-preserving interpolation sequence.

*Definition.* A sequence $F_n$ of faces is *strongly separated* if

(A) $F_i \cap F_j = \varnothing$ whenever $i \neq j$, and also

(B) if $\{y_n\}$ is a sequence such that $y_n \in F_n$ for each $n$ then $\{y_n\}$ has no cluster point.

It is easy to see that (B) is equivalent to

(B′) There is a sequence $\{r_n\}$ of positive numbers such that if $U_n = (F_n + r_n B) \cap \bar{B}$, then for every $x \in \bar{B}$ there is a neighbourhood of $x$ which intersects at most finitely many sets $U_i$.

Our main result is the following

THEOREM. *The following are equivalent*:

(*a*) $\{x_n\}$ *is a norm-preserving interpolation sequence*;

(*b*) *the sequence* $\{F(x_n)\}$ *of associated faces is strongly separated.*

*Let either condition be satisfied. Then*

(*c*) *given a complex Banach space* $Y$ *and a sequence* $\{y_n\} \subset Y$ *with* $\sup \|y_n\| = 1$, *there is a continuous map* $f : \bar{B} \to Y$, *analytic in* $B$, *such that*

(i) $f | F(x_n) = y_n \quad (n \in \mathbb{N})$

(ii) $\|f(x)\| < 1 \quad (x \in B \setminus \bigcup_{n=1}^{\infty} F(x_n))$.

Clearly (*c*) implies (*a*). It is easy to see that (*a*) implies (*b*): we already know that (*a*) implies (A). To see that (*a*) implies (B), assume that (*a*) holds and (B) does not. That is, assume that there is a strictly increasing sequence $\{n_k\} \subset \mathbb{N}$ and a sequence $\{z_k\} \in F_{n_k}$ $(k \in \mathbb{N})$ which converges to $x$, say. Let $\alpha_n = 0$ $(n \in \mathbb{N} \setminus \{n_k\})$, $\alpha_{n_k} = 1$ $(k \in \mathbb{N}$, $k$ odd), $\alpha_{n_k} = -1$ $(k \in \mathbb{N}$, $k$ even). By (*a*), there is a function $f \in A_B$ with $\|f\| = 1$ such that $f(x_n) = \alpha_n$ $(n \in \mathbb{N})$. In particular, $f | F_{n_k} = \alpha_{n_k}$ $(k \in \mathbb{N})$ which implies that $f(z_k) = \alpha_{n_k}$ $(k \in \mathbb{N})$. By the continuity of $f$ we get

$$1 = \lim \{f(z_k) : k \text{ odd}\} = f(x) = \lim \{f(z_k) : k \text{ even}\} = -1,$$

a contradiction. This proves that (*a*) implies (*b*).

So the non-trivial part is to prove that (*b*) implies (*c*).

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## 2. *Functions that peak on faces*

We say that a function $f \in A_B$ *peaks* on a set $F \subset bB$ if $f|F = 1$ and $|f(x)| < 1$ $(x \in bB \backslash F)$. We say that $f$ is a *strong peaking function* for $F$ if $f|F = 1$ and if given $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x)| < 1-\delta$ whenever $x \in \bar{B}$ and $\operatorname{dist}(x, F) > \epsilon$.

LEMMA 1. *Let $0 < \theta < \pi/6$ and let $P = \{\zeta \in \mathbb{C} : |\zeta| \leqslant 1, |\arg \zeta| \leqslant \theta\}$. Assume that $F_1$, $F_2$ are two faces with $F_1 \cap F_2 = \varnothing$. There is an $f \in A_U$ such that*

(i) *$f$ is a strong peaking function for $F_1$;*
(ii) *$f|F_2 = 0$;*
(iii) *$f(\bar{B}) \subset P$.*

Denote by $\Delta$ the open unit disc in $\mathbb{C}$. Let $S \subset \mathbb{N}$ be a finite set. We define the projection $P_S$ by $P_S(\Sigma_{i=1}^{\infty} x_i e_i) = \Sigma_{i \in S} x_i e_i$. Let $S = \{i_1, i_2, \ldots, i_n\}$ where $i_1 < i_2 < \ldots < i_n$. Denote by $A_n$ the polydisc algebra, that is the algebra of all continuous functions on $\bar{\Delta}^n$ which are analytic on $\Delta^n$. We denote by $A_S$ the closed subalgebra of $A_U$ of all functions $f$ of the form $f(x) = \Phi(x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ where $\Phi \in A_n$.

*Proof of Lemma* 1. Choose $x_1, x_2$ such that $F(x_1) = F_1$ and $F(x_2) = F_2$. Let $S_1 = S_T(x_1)$, $S_2 = S_T(x_2)$ and $S = S_1 \cap S_2$. Since $F_1 \cap F_2 = \varnothing$, it follows that $S \neq \varnothing$ and that $P_S(x_1) \neq P_S(x_2)$. Since the set consisting of two points on the distinguished boundary of the polydisc is a peak interpolation set for the polydisc algebra (see [**4**]), it follows that there is $\Phi_1 \in A_S$ such that $\Phi_1(x_1) = 1$, $\Phi_1(x_2) = 0$, and $|\Phi_1(x) - \frac{1}{2}| < \frac{1}{2}$ if $P_S(x) \notin \{P_S(x_1), P_S(x_2)\}$. Suppose that $S = S_1$. Put $f = \Phi_1^{2\theta/\pi}$. Clearly $f$ satisfies (iii). Since $F(P_S x_1) = F_1$ and since $F_2 = F(x_2) \subset F(P_S x_2)$ it follows that $f$ satisfies (i) and (ii). Assume now that $S \neq S_1$. Choose $\Phi_2 \in A_{S_1 \backslash S}$ such that $\Phi_2(P_{S_1 \backslash S} x_1) = 1$ and $|\Phi_2(x) - \frac{1}{2}| < \frac{1}{2}$ if $P_{S_1 \backslash S} x \neq P_{S_1 \backslash S} x_1$. Putting $f = \Phi_1^{\theta/\pi} . \Phi_2^{\theta/\pi}$ it is easy to check that $f$ satisfies (i)–(iii). This completes the proof.

Write $D = \{\zeta \in \mathbb{C} : |\zeta - \frac{1}{2}| \leqslant \frac{1}{2}\}$. It is easy to see that for any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if $\zeta \in D$ and $|\zeta| > 1 - \delta(\epsilon)$ then $|1 - \zeta| < \epsilon$, and such that if $\zeta \in D$ and $|1 - \zeta| > 1 - \delta(\epsilon)$ then $|\zeta| < \epsilon$.

LEMMA 2. *Assume that $F_1, F_2$ are two faces with $F_1 \cap F_2 = \varnothing$. There is an $f \in A_U$ such that*

(i) *$f(\bar{B}) \subset D$;*
(ii) *$f|F_1 = 1, f|F_2 = 0$;*
(iii) *given $\epsilon > 0$, there is a $\delta > 0$ such that*

$$|f(x)| < 1 - \delta \quad (x \in \bar{B}, \operatorname{dist}(x, F_1) > \epsilon),$$

*and*

$$|f(x)| > \delta \quad (x \in \bar{B}, \operatorname{dist}(x, F_2) > \epsilon).$$

*Proof.* Let $0 < \theta < \pi/6$ and let $P = \{\zeta \in \mathbb{C} : |\arg \zeta| \leqslant \theta\}$. Observe that

(A) $P + (-P) \subset \bar{\Delta}$;
(B) if $0 < \delta < 1$ and $z, v \in P$, where $|z| < 1 - \delta$, then $|(v - z) - (-1)| > \delta$.

By Lemma 1, there are $h_1, h_2 \in A_U$ such that

(*a*) $h_i$ is a strong peaking function for $F_i$, $i = 1, 2$;
(*b*) $h_1|F_2 = 0, \quad h_2|F_1 = 0$;
(*c*) $h_i(\bar{B}) \subset P$, $i = 1, 2$.

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Define $h = h_1 - h_2$. By (A), $\|h\| \leqslant 1$. Further, by $(a)$ and $(b)$, we have $h(F_1) = 1$, $h(F_2) = -1$. By $(a)$ and (B) it follows that given $\epsilon > 0$ there is some $\delta > 0$ such that

$$|h(x) - 1| > \delta \ (x \in \bar{B}, \operatorname{dist}(x, F_1) > \epsilon),$$

and

$$|h(x) - (-1)| > \delta \ (x \in \bar{B}, \operatorname{dist}(x, F_2) > \epsilon).$$

Put $f = (1+h)/2$. It is easy to see that $f$ has all the required properties. This completes the proof.

LEMMA 3. *Suppose that $\{F_j\}$ is a strongly separated sequence of faces. For each $j \in \mathbb{N}$, there is a $\Phi_j \in A_B$ such that*

(i) $\Phi_j(\bar{B}) \subset D$;

(ii) $\Phi_j | F_i = 0 \ (i \in \mathbb{N}, i \neq j)$;

(iii) *$\Phi_j$ is a strong peaking function for $F_j$.*

*Proof.* It clearly suffices to prove this for $j = 1$. By Lemma 2, for each $j \geqslant 2$ there is an $f_j \in A_U$ such that $(a)$ $f_j(\bar{B}) \subset D$, $(b)$ $f_j | F_1 = 1, f_j | F_j = 0$, and $(c)$ given $\epsilon_j > 0$ there is some $\delta_j > 0$ such that $|f_j(x)| > \delta_j$ for all $x \in \bar{B}$ with $\operatorname{dist}(x, F_j) > \epsilon_j$. Since the sequence $\{F_j\}$ is strongly separated, each $F_j$ has a neighbourhood $U_j = (F_j + \epsilon_j B) \cap \bar{B}$ with the following property: every $x \in \bar{B}$ has a neighbourhood intersecting at most a finite number of the sets $U_j$. Let $\delta_j$ be associated with $\epsilon_j$ as in $(c)$ above, so that $x \in \bar{B} \setminus U_j$ implies that $|f_j(x)| > \delta_j$ for all $j \geqslant 2$. This, together with $(a)$, implies that there is a sequence $\{n_j\}$ of positive integers such that

$$|f_j(x)^{1/n_j} - 1| < 1/2^j \quad (x \in \bar{B} \setminus U_j, j \geqslant 2) \tag{1}$$

and such that

$$\sum_{j=1}^{\infty} \frac{1}{n_j} < \tfrac{1}{2}. \tag{2}$$

Define $\Psi(x) = \prod_{j=2}^{\infty} f_j(x)^{1/n_j}$ for $x \in \bar{B}$. Since each $x \in \bar{B}$ has a neighbourhood intersecting at most a finite number of sets $U_j$, it follows by (1) that the product converges locally uniformly on $\bar{B}$. Since each $f_j$ is bounded by 1 on $\bar{B}$ the same holds for $\Psi$. Since $f_j \in A_U$ for all $j \in \mathbb{N}$ it follows that $\Psi \in A_B$. By (2) and by $(a)$ we have $|\arg \Psi(x)| < \pi/4$ for $x \in \bar{B}$. Let $g$ be a strong peaking function for $F_1$. Then $h = (\frac{1}{2}(1+g))^{\frac{1}{2}}$ is also a strong peaking function for $F_1$ which satisfies $|\arg h(x)| < \pi/4$ for $x \in \bar{B}$. Since $\Psi | F_1 = 1$ and $\|\Psi\| = 1$ it follows that $h\Psi$ is a strong peaking function for $F_1$. Clearly $\operatorname{Re}(h\Psi)(x) \geqslant 0$ for $x \in \bar{B}$. Let $\phi$ map $\{\zeta \in \bar{\Delta} : \operatorname{Re} \zeta > 0\}$ conformally onto $D$ so that $\phi(0) = 0, \phi(1) = 1$. Put $\Phi_1 = \phi \circ (h\Psi)$. Then $\Phi_1 \in A_B$, and it is easy to see that $\Phi_1$ has all the required properties. In particular, $\Phi_1 | F_j = 0$ for $j \geqslant 2$ since by construction $\Psi | F_j = 0$ for $j \geqslant 2$. This completes the proof.

*Remark.* Note that by (i)–(iii) we have $\|\Phi_j\| = \|1 - \Phi_j\| = 1$ for $j \in \mathbb{N}$.

Note also that by the properties of $D$, given $\epsilon > 0$ there is $\delta(\epsilon) > 0$ such that for each $j \in \mathbb{N}$

$$\left.\begin{aligned} &|1 - \Phi_j(z)| < \epsilon \quad \text{whenever } z \in \bar{B} \text{ and } |\Phi_j(z)| > 1 - \delta(\epsilon), \\ &|\Phi_j(z)| < \epsilon \quad \text{whenever } z \in \bar{B} \text{ and } |1 - \Phi_j(z)| > 1 - \delta(\epsilon). \end{aligned}\right\} \tag{3}$$

## 3. *Proof of the theorem*

In this section we complete the proof of the theorem by proving that $(b)$ implies $(c)$. Let $Y$ be a complex Banach space. We denote by $A_B(Y)$ the Banach space of all

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bounded continuous maps from $\bar{B}$ to $Y$, analytic in $B$, with sup norm. Assume that $\{x_n\} \subset bB$ and that the sequence $F_n = F(x_n)$ is strongly separated. Let $\{y_n] \subset Y$ be a sequence satisfying $\sup \|y_n\| = 1$.

*Part* 1. Assume that there are open sets $U_{ij} \subset \bar{B}$ for all $i,j \in \mathbb{N}$ and a sequence $\{f_n\} \in A_B(Y)$ such that

$$\left.\begin{aligned}
&\text{(i) } U_{1m} \supset U_{2m} \supset \ldots \supset F_m, \quad \textstyle\bigcap_{j=1}^{\infty} U_{jm} = F_m \quad (m \in \mathbb{N}),\\
&\text{(ii) } U_{1i} \cap U_{1j} = \varnothing \quad (i,j \in \mathbb{N}, i \neq j),\\
&\text{(iii) } f_n \mid F_m = (1/2^n) y_m \quad (m,n \in \mathbb{N}),\\
&\text{(iv) } \|f_n\| \leqslant 1/2^n + 1/2^{n+1} \quad (n \in \mathbb{N}),\\
&\text{(v) if } x \in U_{nm} \ (n,m \in \mathbb{N}), \text{ then}\\
&\qquad \|f_i(x)\| \leqslant 1/2^i + (1/n)(1/2^{n+2}) \ (1 \leqslant i \leqslant n-1),\\
&\text{(vi) } \|f_n(x)\| < 1/2^{n+2} \quad (x \in \bar{B} \setminus \textstyle\bigcup_{i=1}^{\infty} U_{ni}).
\end{aligned}\right\} \tag{H}$$

Define $$f = \sum_{i=1}^{\infty} f_i.$$

Since $f_i \in A_B(Y)$ for $i \in \mathbb{N}$, it follows by (iv) that $f \in A_B(Y)$. By (iii) we have $f \mid F_m = y_m$ for each $m$.

Let $x \in \bar{B} \setminus \bigcup_{i=1}^{\infty} F_i$. If $x \in \bar{B} \setminus \bigcup_{i=1}^{\infty} U_{1i}$ then $x \notin \bigcup_{i=1}^{\infty} U_{ni}$ for each $n \in \mathbb{N}$, and by (vi) we have

$$\|f(x)\| \leqslant \sum_{n=1}^{\infty} \|f_n(x)\| < \sum_{n=1}^{\infty} 1/2^{n+2} < 1.$$

Let $x \in \bigcup_{i=1}^{\infty} U_{1i}$. Then there are $m,n \in \mathbb{N}$ such that $x \in U_{nm} \setminus U_{n+1,m}$. So $x \notin \bigcup_{i=1}^{\infty} U_{n+1,i}$, and consequently $x \notin \bigcup_{i=1}^{\infty} U_{ji}$ for every $j \geqslant n+1$. By (vi) it follows that

$$\|f_i(x)\| \leqslant 1/2^{i+2} \quad (i \geqslant n+1). \tag{4}$$

Further, since $x \in U_{nm}$, (v) implies that

$$\|f_i(x)\| \leqslant 1/2^i + (1/n)(1/2^{n+2}) \quad (1 \leqslant i \leqslant n-1),$$

and consequently

$$\sum_{i=1}^{n-1} \|f_i(x)\| \leqslant \sum_{i=1}^{n-1} 1/2^i + 1/2^{n+2}. \tag{5}$$

By (iv) we have

$$\|f_n(x)\| \leqslant 1/2^n + 1/2^{n+1}. \tag{6}$$

By (4), (5) and (6) it follows that

$$\|f(x)\| \leqslant \sum_{i=1}^{n-1} \|f_i(x)\| + \|f_n(x)\| + \sum_{i=n+1}^{\infty} \|f_i(x)\| < \left(\sum_{i=1}^{n-1} 1/2^i + 1/2^{n+2}\right) + (1/2^n + 1/2^{n+1}) + \sum_{i=n+1}^{\infty} 1/2^{i+2} = \sum_{i=1}^{\infty} 1/2^i = 1.$$

Thus we have proved that $\|f(x)\| < 1$ whenever $x \in \bar{B} \setminus \bigcup_{n=1}^{\infty} F_n$.

It remains therefore to prove (H).

*Part* 2. Let $\Phi_j \in A_B$ be a sequence satisfying (i)–(iii) in Lemma 3. Define $G_1 = \Phi_1$ and $G_j = (1-\Phi_1)\ldots(1-\Phi_{j-1}).\Phi_j$ for $j \geqslant 2$.

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Since the sequence $\{F_j\}$ is strongly separated, each $F_j$ has a neighbourhood $U_j = (F_j + r_j B) \cap \bar{B}$ with the property that every $x \in \bar{B}$ has a neighbourhood intersecting at most a finite number of the sets $U_j$. Since $\Phi_j$ is a strong peaking function for $F_j$ for each $j$, there is a sequence $\{\eta_j\}$ of positive numbers such that $|\Phi_j(x)| < 1-\eta_j$ $(x \notin U_j, j \in \mathbb{N})$. Let $\{n_k\}$ be any sequence of positive integers such that

$$\sum_{j=1}^{\infty} (1-\eta_j)^{n_j} < \infty. \tag{7}$$

Then the function

$$g = \sum_{j=1}^{\infty} G_j^{n_j} y_j \tag{8}$$

is continuous on $\bar{B}$ and analytic in $B$. To see this, observe that $|G_j(x)| < 1-\eta_j$ $(x \notin U_j$, $j \in \mathbb{N})$. Since each $x \in \bar{B}$ has a neighbourhood that intersects at most finitely many sets $U_j$ and since $\sup \|y_n\| = 1$ it follows by (7) that the series (8) converges locally uniformly on $\bar{B}$. Since $G_i \in A_B$ for each $i$, it follows that $g$ is continuous on $\bar{B}$ and analytic in $B$.

Let $\{\epsilon_p\}$ be a decreasing sequence of positive numbers with $\epsilon_p < \frac{1}{2}$ for each $p$, converging to zero and satisfying

$$\frac{\epsilon_p}{1-\epsilon_p} < \frac{1}{p}\frac{1}{2^{p+2}} \quad (p \in \mathbb{N}). \tag{9}$$

Further, let $\{\delta_p\}$ be a decreasing sequence of positive numbers with $\delta_p < \frac{1}{2}$ for each $p$, converging to zero, such that for each $p$

$$\left.\begin{aligned} |1-\zeta| &< \epsilon_p \quad (\zeta \in D, |\zeta| > 1-\delta_p), \\ |\zeta| &< \epsilon_p \quad (\zeta \in D, |1-\zeta| > 1-\delta_p). \end{aligned}\right\} \tag{10}$$

For each $p \in \mathbb{N}$ choose a strictly increasing sequence $\{m_{pi}\}$ of positive integers such that

$$\sum_{i=1}^{\infty} (1-\eta_i)^{m_{pi}} < \infty \tag{11}$$

and

$$\sum_{i=1}^{\infty} (1-\delta_p)^{m_{pi}} < \tfrac{1}{4}. \tag{12}$$

For each $p \in \mathbb{N}$ define

$$f_p = \frac{1}{2^p} \sum_{i=1}^{\infty} G_i^{m_{pi}} y_i$$

and

$$U_{pi} = \{x \in \bar{B} : |G_i(x)| > 1-\delta_p\} \quad (i \in \mathbb{N}).$$

By the preceding discussion, (11) implies that each $f_p$ is continuous on $\bar{B}$ and analytic in $B$. We will show that each $f_p$ is bounded, which will imply that $f_p \in A_B(Y)$ for each $p$.

By the properties of $\Phi_j$ we have

$$U_{1m} \supset U_{2m} \supset \ldots \supset F_m, \quad \bigcap_{j=1}^{\infty} U_{jm} = F_m \quad (m \in \mathbb{N}).$$

We show that $U_{1i} \cap U_{1j} = \varnothing$ if $i \neq j$. Assume that $i < j$. Let $x \in U_{1i}$. This means that $|G_i(x)| > 1-\delta_i$, that is,

$$|(1-\Phi_1(x)) \ldots (1-\Phi_{i-1}(x))\, \Phi_i(x)| > 1-\delta_1.$$

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Since $\|1-\Phi_j\| = 1$ for $j \leqslant i-1$ it follows that $|\Phi_i(x)| > 1-\delta_1$. Thus by (10) we have that $|1-\Phi_i(x)| < \epsilon_1$. Since $i < j$ this implies

$$|G_j(x)| = |(1-\Phi_1(x)) \dots (1-\Phi_{j-1}(x))\, \Phi_j(x)| < \epsilon_1$$

so that $|G_j(x)| < \frac{1}{2}$. On the other hand, if $x \in U_{1j}$ then $|G_j(x)| > 1-\delta_1 > \frac{1}{2}$. This proves that $U_{1i} \cap U_{1j} = \varnothing$.

*Part* 3. Let $p \in \mathbb{N}$ and let $x \in \bar{B} \setminus \bigcup_{i=1}^{\infty} U_{pi}$. By definition of $U_{pi}$ we have $|G_i(x)| < 1-\delta_p$ for each $i$ and so by (12)

$$|f_p(x)| = \left|\frac{1}{2^p} \sum_{i=1}^{\infty} G_i^{m_{pi}}(x)\, y_i\right| \leqslant \frac{1}{2^p} \sum_{i=1}^{\infty} (1-\delta_p)^{m_{pi}} < \frac{1}{2^p}\frac{1}{4} = \frac{1}{2^{p+2}}$$

which proves (vi) in (H).

By the properties of $\Phi_i$, (iii) in (H) is satisfied.

Further, let $p, k \in \mathbb{N}$, and let $x \in U_{pk}$. This means that $|G_k(x)| > 1-\delta_p$, that is,

$$|(1-\Phi_1(x)) \dots (1-\Phi_{k-1}(x))\, \Phi_k(x)| > 1-\delta_p.$$

It follows that each factor has absolute value greater than $1-\delta_p$. So if $1 \leqslant j \leqslant k-1$ we have $|1-\Phi_j(x)| > 1-\delta_p$. This implies that $|\Phi_j(x)| < \epsilon_p$, which further implies that $|G_j(x)| < \epsilon_p$. Moreover $|\Phi_k(x)| > 1-\delta_p$ implies that $|1-\Phi_k(x)| < \epsilon_p$, which further implies that $|G_j(x)| < \epsilon_p$ for $j \geqslant k+1$. Thus we have proved that

$$|G_j(x)| < \epsilon_p \quad (j \in \mathbb{N}, j \neq k).$$

By (9) it follows that, for $l \leqslant p-1$,

$$\sum_{i=1, i\neq k}^{\infty} |G_i(x)|^{m_{li}} < \sum_{i=1, i\neq k}^{\infty} (\epsilon_p)^{m_{li}} < \sum_{i=1}^{\infty} (\epsilon_p)^i = \frac{\epsilon_p}{1-\epsilon_p} < \frac{1}{p}\frac{1}{2^{p+2}}$$

which, by the definition of $f_l$, implies that

$$\|f_l(x)\| < \left(\frac{1}{2}\right)^l + \frac{1}{2^{p+2}} \quad (l \leqslant p-1).$$

This shows that (v) in (H) is satisfied.

Fix $p \in \mathbb{N}$. If $x \notin \bigcup_{i=1}^{\infty} U_{pi}$ then, by (vi), $\|f_p(x)\| < 1/2^{p+2}$. Let $x \in \bigcup_{i=1}^{\infty} U_{pi}$. Since the $U_{pi}$ with $i \in \mathbb{N}$ are pairwise disjoint there is a $j \in \mathbb{N}$ such that $x \in U_{pj}$ and $x \notin U_{pk}$ for $k \neq j$. In particular, $|G_k(x)| \leqslant 1-\delta_p$ for $k \neq j$. By (12) it follows that

$$\|f_p(x)\| = \frac{1}{2^p}\left\|\sum_{i=1}^{\infty} G_i(x)^{m_{pi}} y_i\right\| < \frac{1}{2^p}\left[\sum_{i=1}^{j-1} (1-\delta_p)^{m_{pi}} + 1 + \sum_{i=j+1}^{\infty} (1-\delta_p)^{m_{pi}}\right]$$
$$< \frac{1}{2^p} + \frac{1}{2^p} \sum_{i=1}^{\infty} (1-\delta_p)^{m_{pi}} < \frac{1}{2^p} + \frac{1}{2^{p+2}}.$$

This proves (iv) in (H) and completes the proof of the theorem.

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