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A NEW SHOCK MODEL WITH A CHANGE IN SHOCK SIZE DISTRIBUTION

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For a system that is subject to shocks, it is assumed that the distribution of the magnitudes of shocks changes after the first shock of size at least d_1, and the system fails upon the occurrence of the first shock above a critical level d_2 ($> d_1$). In this paper, the distribution of the lifetime of such a system is studied when the times between successive shocks follow matrix-exponential distribution. In particular, it is shown that the system's lifetime has matrix-exponential distribution when the intershock times follow Erlang distribution. The model is extended to the case when the system fails upon the occurrence of l consecutive critical shocks.

Keywords: matrix-exponential distribution, reliability, shock model

1. INTRODUCTION

One of the main assumptions in studying shock models is that the shock sizes are homogeneous over time in terms of their distribution. Various shock models have been studied in the literature under this most common assumption. The oldest and best-known model is the extreme shock model. Under this model, the system breaks down when there is a single large shock that is above a critical level (see, e.g., [9]). Mallor and Omey [12] introduced a model that can be used to study systems that fail upon the occurrence of a specified number of consecutive shocks. Gut [10] introduced and studied a mixed shock model which appears as a combination of extreme and cumulative shock models. See, e.g., [13] and [16] for other types of mixed shock models. Gong et al. [8] studied a run-based shock model with two critical levels c_1 and c_2. According to the model, the system fails if at least k_1 consecutive shocks with magnitude above c_1 or k_2 consecutive shocks with magnitude above c_2 occur. Eryilmaz and Tekin [6] studied a mixed shock model that combines extreme and run shock models when the interarrival time between the $(i-1)$th and ith shock, and the magnitude of the ith shock are dependent for all i. Zhao et al. [18] proposed a two-stage shock model with a self-healing mechanism as an extension of cumulative shock and delta-shock models. Wang et al. [17] proposed a compound negative binomial distribution with mutative termination conditions and applied this new distribution to study shock models. Gao et al. [7] developed reliability models for

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systems or devices subject to dependent competing soft and hard failure processes with degradation-shock dependences. Qiu and Cui [14] developed a novel mission abort policy for systems experiencing both internal failure and external shocks. Zhao et al. [19] studied multi-state balanced systems in a shock environment.

In most real-life situations, after a shock that is above a certain level, the characteristics, e.g., distribution of the shocks may change. For example, if the shock is a kind of environmental effect, e.g., pressure, temperature, and wind, then a harsher environment might be available after the shock that is significantly different from the previous shocks. Eryilmaz and Kan [5] studied reliability and optimal replacement policy for an extreme shock model when the distribution of the shock size changes after a random number of shocks following a particular distribution. In particular, for a sequence of shock sizes denoted by $Y_1, Y_2, \ldots$, they considered the case when $Y_1, \ldots, Y_M$ are independent and identically distributed (iid) with common distribution G_1 and $Y_{M+1}, Y_{M+2}, \ldots$ are iid with common distribution G_2, where M is a random variable independent of Y_is and follows a particular distribution.

In the present paper, it is assumed that the distribution of the magnitudes of shocks changes after the first shock of size at least d_1, a given level. That is, if N_{d_1} denotes the number of shocks until the first shock of size at least d_1, then it is assumed that $Y_1, \ldots Y_{N_{d_1}}$ have common distribution G_1, and $Y_{N_{d_1}+1}, Y_{N_{d_1}+2}, \ldots$ have another common distribution G_2. In this case, the random variable N_{d_1} denotes the number of shocks until the change point depends on $Y_1, Y_2, \ldots$. Such a model is useful for modeling various real-life systems or processes. In statistical process control, control charts are effectively used to monitor a production process. A control chart with random interarrival times between successive samplings can be modeled by the present shock model. In this case, the random variables $Y_1, Y_2, \ldots$ denote the successive plotting statistics. If a plotting statistic is below the limit d_1, then the process produces an item at a desired quality level. However, if the plotting statistic is above d_1 but below the control limit d_2, then an item with a low quality may be produced by the process. After the first plotting statistic that is above d_1, the process is not stopped but the distribution of the plotting statistics change. Assume that the process is assigned to be out-of-control when there is point above d_2 ($d_2 > d_1$). Thus, the run length N_{d_2} of the control chart is in fact the number of plotting statistics until the first plotting statistic that is above d_2. If we denote by X_i, $i = 1, 2, \ldots$ the times between successive plotted points, then $\sum_{i=1}^{N_{d_2}} X_i$ expresses the time to signal of a monitoring scheme.

In the context of the present shock model, the shock $Y_{N_{d_1}+1}$ actually triggers the change in the shock size distribution. Such a shock can be called as the trigger shock. The concept of trigger shock which changes the degradation rate of the system has been considered by Rafiee et al. [15] in reliability modeling for dependent competing failure processes. Change points not only appear in reliability and quality control but also appear in a variety of scientific fields including economics, medicine, and environment. Depending on the field of study, change points are also known as breakpoints, structural breaks, and regime switching. The random variables $S_1 = \sum_{i=1}^{N_{d_1}} X_i$ and $S_2 = \sum_{i=1}^{N_{d_2}} X_i$ which are the main interests of this paper respectively denote the time until the change point and the time until the failure of the system. The study of distributional properties of the random variables S_1 and S_2 is an important problem to make optimal decisions about the system, process, or structure which appears in the abovementioned fields.

The aim of the paper is to examine distributional properties of random sums $S_1 = \sum_{i=1}^{N_{d_1}} X_i$ and $S_2 = \sum_{i=1}^{N_{d_2}} X_i$ when the random variables $X_1, X_2, \ldots$ have matrix-exponential distribution. The matrix-exponential distributions, as a generalization of the phase-type distributions, are useful and suitable for modeling random quantities involved in stochastic models concerned with finance, insurance, and reliability engineering. There are various

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advantages of using matrix-exponential distributions. They are mathematically tractable and allow us to represent distributions and moments in a matrix form which can be easily evaluated. The class of matrix-exponential distributions also includes well-known distributions such as exponential, Erlang, and hyper-exponential which are suitable for modeling interarrival times between successive shocks. Under some particular assumptions, we show that (S_1, S_2) has a bivariate matrix-exponential distribution.

The paper is organized as follows. In Section 2, we define the model and present preliminary results. Section 3 involves results on distributions of the random variables S_1 and S_2. In Section 4, we present a more general shock model by incorporating the concept of runs. Finally, in Section 5, we present numerical examples.

2. THE MODEL AND PRELIMINARIES

The system is subject to shocks which have random magnitudes denoted by $Y_1, Y_2, \ldots$. The distribution of the magnitudes of shocks changes after the first shock of size at least d_1. That is, $Y_1, \ldots, Y_{N_{d_1}}$ have common distribution G_1, and the shock sizes $Y_{N_{d_1}+1}, Y_{N_{d_1}+2}, \ldots$ have another common distribution G_2, where N_{d_1} denotes the number of shocks until the first shock of size at least d_1. The system fails upon the occurrence of the first shock of size at least d_2 $(d_2 > d_1)$.

Throughout the paper, the following notation will be used.

X_i: The time between the $(i-1)$th and ith shocks, $i = 1, 2, \ldots$

Y_i: The magnitude of the ith shock, $i = 1, 2, \ldots$

d_1: The warning threshold for the system

d_2: The failure threshold for the system

N_{d_1}: The random variable that denotes the number of shocks until the first shock of size at least d_1

N_{d_2}: The random variable that denotes the number of shocks until the first shock of size at least d_2

$p_1 = G_1(d_1)$: The probability that a shock before the change is below d_1

$p_2 = G_1(d_2)$: The probability that a shock before the change is below d_2

$p_3 = G_2(d_2)$: The probability that a shock after the change is below d_2

In the following Lemma, we derive the joint distribution of the random variables N_{d_1} and N_{d_2}.

LEMMA 1: *The joint distribution of the random variables N_{d_1} and N_{d_2} is*

$$P\{N_{d_1} = m, N_{d_2} = n\} = \begin{cases} p_1^{m-1}(p_2 - p_1)p_3^{n-m-1}(1-p_3), & if\ n > m \\ p_1^{n-1}(1-p_2), & if\ n = m. \end{cases} \quad (1)$$

PROOF: For $n > m$, the distribution of the magnitudes of shocks change after the mth shock and

$$\begin{aligned} P\{N_{d_1} = m, N_{d_2} = n\} &= P\{Y_1 < d_1, \ldots, Y_{m-1} < d_1, d_1 \le Y_m < d_2, \\ &\quad Y_{m+1} < d_2, \ldots, Y_{n-1} < d_2, Y_n \ge d_2\} \\ &= G_1^{m-1}(d_1)(G_1(d_2) - G_1(d_1))G_2^{n-m-1}(d_2)(1 - G_2(d_2)) \\ &= p_1^{m-1}(p_2 - p_1)p_3^{n-m-1}(1-p_3). \end{aligned}$$

If $n = m$, then $P\{N_{d_1} = n, N_{d_2} = n\} = G_1^{n-1}(d_1)(1 - G_1(d_2)) = p_1^{n-1}(1-p_2)$. ■

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As an immediate consequence of Lemma 1, we obtain the marginal distribution of N_{d_2}:

COROLLARY 1: *Using* (1)*, the marginal distribution of the number of shocks until the failure of the system is obtained as*

$$P\{N_{d_2}=n\}=\left(\frac{p_3-p_2}{p_3-p_1}\right)p_1^{n-1}(1-p_1)+\left(\frac{p_2-p_1}{p_3-p_1}\right)p_3^{n-1}(1-p_3), \tag{2}$$

for $p_1\neq p_3$*, and*

$$P\{N_{d_2}=n\}=\left(\frac{p_2-p_3}{1-p_3}\right)(n-1)p_3^{n-2}(1-p_3)^2+\left(\frac{1-p_2}{1-p_3}\right)p_3^{n-1}(1-p_3), \tag{3}$$

for $p_1=p_3$.

PROOF: Clearly,

$$\begin{aligned}P\{N_{d_2}=n\}&=\sum_{m=1}^{n-1}P\{N_{d_1}=m,N_{d_2}=n\}+P\{N_{d_1}=n,N_{d_2}=n\}\\&=\sum_{m=1}^{n-1}p_1^{m-1}(p_2-p_1)p_3^{n-m-1}(1-p_3)+p_1^{n-1}(1-p_2)\\&=\frac{(p_2-p_1)(1-p_3)p_3^{n-1}}{p_1}\sum_{m=1}^{n-1}\left(\frac{p_1}{p_3}\right)^m+p_1^{n-1}(1-p_2.)\end{aligned}$$

The result follows because $\sum_{m=1}^{n-1}(p_1/p_3)^m=p_1/(p_3-p_1)[1-(p_1/p_3)^{n-1}]$ if $p_1\neq p_3$ and $\sum_{m=1}^{n-1}(p_1/p_3)^m=n-1$ if $p_1=p_3$. ■

The distributions given by (2) and (3) can be represented as mixtures with two components. A detailed explanation about the corresponding mixture representations is presented in the following remark.

Remark 1: Eq. (2) is a mixture of two geometric distributions with means $1/(1-p_1)$ and $1/(1-p_3)$, and weights $\omega=(p_3-p_2)/(p_3-p_1)$ and $1-\omega=(p_2-p_1)/(p_3-p_1)$, respectively. It should be noted that the weights ω and $1-\omega$ may take negative values. Therefore, Eq. (2) is a generalized mixture of two geometric distributions. Similarly, Eq. (3) is a generalized mixture of negative binomial and geometric distributions, and it can be written as

$$P\{N_{d_2}=n\}=\omega P\{W_1=n\}+(1-\omega)P\{W_2=n\},$$

where $\omega=(p_2-p_3)/(1-p_3)$, $P\{W_1=n\}=(n-1)p_3^{n-2}(1-p_3)^2$, $n=2,3,\ldots$ and $P\{W_2=n\}=p_3^{n-1}(1-p_3), n=1,2,\ldots$

The necessary conditions for making Eqs. (2) and (3) positive mixture are given in the following.

PROPOSITION 1:

(a) *If* $G_1\geq_{st}G_2$*, then* $p_2\leq p_3$ *and hence* (2) *is a positive mixture.*

(b) *If* $G_2\geq_{st}G_1$*, then* $p_3\leq p_2$ *and hence* (3) *is a positive mixture.*

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PROOF: If $G_1 \geq_{st} G_2$, then $p_2 \leq p_3$. On the other hand, we have $p_1 = G_1(d_1) < G_1(d_2) = p_2$ since $d_1 < d_2$. Thus $0 < (p_3 - p_2)/(p_3 - p_1) < 1$ and $0 < (p_2 - p_1)/(p_3 - p_1) < 1$ with $(p_3 - p_2)/(p_3 - p_1) = 1 - (p_2 - p_1)/(p_3 - p_1)$. This implies that (2) is a positive mixture when $p_2 \leq p_3$. The proof of part (b) is similar, and hence omitted. ■

The two dependent lifetime random variables that can be of interest are

$$S_1 = \sum_{i=1}^{N_{d_1}} X_i, \quad S_2 = \sum_{i=1}^{N_{d_2}} X_i.$$

The random variable S_1 denotes the time until the first shock of size at least d_1, i.e., it is the time until the change point. The lifetime of the system is defined by the random variable S_2.

Throughout the paper, the times $X_1, X_2, \ldots$ between successive shocks are assumed to follow matrix-exponential distribution with common cumulative distribution function

$$F(x) = P\{X \leq x\} = 1 + \boldsymbol{\alpha} \exp(\mathbf{S}x)\mathbf{S}^{-1}\mathbf{s}, \tag{4}$$

where $\boldsymbol{\alpha}$ is a p-dimensional row vector, $\mathbf{S}$ is a square matrix of dimension p, and $\mathbf{s}$ is a p-dimensional column vector. We write $X \sim \mathrm{ME}_p(\boldsymbol{\alpha}, \mathbf{S}, \mathbf{s})$ to represent that the random variable X has matrix-exponential distribution. The density corresponding to (4) is $f(x) = \boldsymbol{\alpha} \exp(\mathbf{S}x)\mathbf{s}$.

The Laplace–Stieltjes transform (LST) of X can be computed from

$$\mathcal{L}_X(t) = \boldsymbol{\alpha} (t\mathbf{I} - \mathbf{S})^{-1} \mathbf{s}. \tag{5}$$

For a comprehensive review of matrix-exponential distributions and their sub-class of phase-type distributions, we refer to Bladt and Nielsen [4]. Asmussen and Bladt [1] showed that the LST given by (5) is rational, and it can be written as

$$\mathcal{L}_X(t) = \frac{b_p t^{p-1} + \cdots + b_2 t + b_1}{t^p + a_p t^{p-1} + \cdots + a_2 t + a_1}, \tag{6}$$

where $a_1, \ldots, a_p, b_1, \ldots, b_p$ are real numbers. The representation $(\boldsymbol{\alpha}, \mathbf{S}, \mathbf{s})$ is then given by $\boldsymbol{\alpha} = (b_1, b_2, \ldots, b_p)$,

$$\mathbf{S} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & -a_4 & \cdots & -a_p \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{7}$$

The matrix $\mathbf{S}$ is called the companion matrix.

3. LIFE BEHAVIOR OF THE SYSTEM

We first obtain the joint LST $\mathcal{L}(t_1, t_2) = E(e^{-t_1 S_1 - t_2 S_2})$ for an arbitrary distribution F.

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THEOREM 1: *The joint LST corresponding to* (S_1, S_2) *is*

$$\mathcal{L}(t_1,t_2)=\frac{(1-p_3)(p_2-p_1)\mathcal{L}_X(t_1)(\mathcal{L}_X(t_2))^2+(1-p_2)\mathcal{L}_X(t_1)\mathcal{L}_X(t_2)(1-p_3\mathcal{L}_X(t_2))}{(1-p_1\mathcal{L}_X(t_1)\mathcal{L}_X(t_2))(1-p_3\mathcal{L}_X(t_2))},$$

where $\mathcal{L}_X(t)=E(e^{-tX})$.

PROOF: By conditioning on (N_{d_1}, N_{d_2}),

$$\begin{aligned}\mathcal{L}(t_1,t_2)&=E(e^{-t_1S_1-t_2S_2})=\sum_{m<n}\sum\left[E(e^{-t_1X})\right]^m\left[E(e^{-t_2X})\right]^n P\{N_{d_1}=m,N_{d_2}=n\}\\&\quad+\sum_n\left[E(e^{-t_1X})\right]^n\left[E(e^{-t_2X})\right]^n P\{N_{d_1}=n,N_{d_2}=n\}\\&=\phi(E(e^{-t_1X}),E(e^{-t_2X})),\end{aligned}$$

where $\phi(z_1,z_2)$ is the probability generating function of (N_{d_1}, N_{d_2}) and it can be computed from

$$\begin{aligned}\phi(z_1,z_2)&=\sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty}z_1^mz_2^np_1^{m-1}(p_2-p_1)p_3^{n-m-1}(1-p_3)\\&\quad+\sum_{n=1}^{\infty}(z_1z_2)^np_1^{n-1}(1-p_2)\\&=\frac{(1-p_3)(p_2-p_1)z_1z_2^2+(1-p_2)z_1z_2(1-p_3z_2)}{(1-p_1z_1z_2)(1-p_3z_2)}.\end{aligned}$$

Thus the proof is complete. ■

Using the LST given in Theorem 1, we can obtain the marginal LSTs of the random variables S_1 and S_2.

COROLLARY 2: *The marginal LSTs corresponding to* S_1 *and* S_2 *are given respectively by*

$$\mathcal{L}_{S_1}(t)=\phi(\mathcal{L}_X(t),1)=\frac{(1-p_1)\mathcal{L}_X(t)}{1-p_1\mathcal{L}_X(t)},$$

$$\mathcal{L}_{S_2}(t)=\phi(1,\mathcal{L}_X(t))=\frac{(1-p_2)\mathcal{L}_X(t)+[p_2-p_1-p_3(1-p_1)]\,\mathcal{L}_X^2(t)}{1-(p_1+p_3)\mathcal{L}_X(t)+p_1p_3\mathcal{L}_X^2(t)}.$$

Using the relation $E(S_i)=-\mathcal{L}'_{S_i}(0)$ and Corollary 2, we can obtain the means of the random variables S_1 and S_2. The expressions are presented in the following corollary.

COROLLARY 3: *The means of the random variables* S_1 *and* S_2 *are*

$$E(S_1)=\frac{1}{1-p_1}E(X),\quad E(S_2)=\frac{1-p_1+p_2-p_3}{(1-p_1)(1-p_3)}E(X).$$

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3.1. Erlang Distributed Interarrival Times

In the following, we consider the particular case when the interarrival times between successive shocks follow Erlang distribution with probability density function

$$f(x) = \frac{\lambda^k}{(k-1)!} x^{k-1} e^{-\lambda x}, \quad x \geq 0. \tag{8}$$

DEFINITION 1 [3]: *A non-negative random vector (S_1, S_2) is said to have a bivariate matrix-exponential distribution if the joint LST $E(e^{-t_1 S_1 - t_2 S_2})$ is a rational function in (t_1, t_2), i.e., it can be written as $G(t_1, t_2)/D(t_1, t_2)$, where $G(t_1, t_2)$ and $D(t_1, t_2)$ are polynomial functions in t_1 and t_2.*

In the following proposition, we prove that (S_1, S_2) has bivariate matrix-exponential distribution.

PROPOSITION 2: *If the interarrival times between successive shocks follow Erlang with pdf (8), then (S_1, S_2) has bivariate matrix-exponential distribution with*

$$\mathcal{L}(t_1, t_2) = \frac{G(t_1, t_2)}{D(t_1, t_2)},$$

where the polynomial functions G and D are given by

$$G(t_1, t_2) = [p_2 - p_1 - p_3(1-p_1)]\, \lambda^{3k} + (1-p_2)\lambda^{2k} \sum_{i=0}^{k} \binom{k}{i} \lambda^i t_2^{k-i},$$

and

$$D(t_1, t_2) = p_1 p_3 \lambda^{3k} + \sum_{i=0}^{k} \sum_{j=0}^{2k} \binom{k}{i} \binom{2k}{j} \lambda^{i+j} t_1^{k-i} t_2^{2k-j} - p_3 \lambda^k \sum_{i=0}^{k} \sum_{j=0}^{k} \binom{k}{i} \binom{k}{j} \lambda^{i+j} t_1^{k-i} t_2^{k-j} - p_1 \lambda^{2k} \sum_{i=0}^{k} \binom{k}{i} \lambda^i t_2^{k-i}.$$

PROOF: The proof is a direct application of Theorem 1 when

$$\mathcal{L}_X(t) = E(e^{-tX}) = \left(\frac{\lambda}{\lambda + t}\right)^k. \qquad \blacksquare$$

As a special case of the result in Proposition 2, in the following corollary we obtain the joint LST of S_1 and S_2 when the interarrival times follow exponential distribution.

COROLLARY 4: *If $f(x) = \lambda e^{-\lambda x}, x \geq 0$, then (S_1, S_2) has bivariate matrix-exponential distribution with*

$$\mathcal{L}(t_1, t_2) = \frac{G(t_1, t_2)}{D(t_1, t_2)},$$

where $G(t_1, t_2) = \lambda^2(1-p_2)t_2 + \lambda^3(1-p_1-p_3+p_1p_3)$, and $D(t_1, t_2) = \lambda^2(1-p_3)t_1 + \lambda^2(2-p_1-p_3)t_2 + \lambda(2-p_3)t_1t_2 + \lambda t_2^2 + t_1t_2^2 + \lambda^3(1-p_1-p_3+p_1p_3)$.

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In the following theorem, we prove that the system's lifetime has matrix-exponential distribution for the case when the interarrival times between shocks follow Erlang distribution.

THEOREM 2: *Let the interarrival times between successive shocks follow Erlang distribution with pdf (8). Then* $S_2 \sim \mathrm{ME}_{2k}(\boldsymbol{\gamma}, \mathbf{A}, \mathbf{a})$ *with* $\boldsymbol{\gamma} = (b_1, \ldots, b_{k+1}, 0, \ldots 0)$, *and*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & -a_4 & \cdots & -a_{2k} \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

where

$$b_1 = (1-p_1)(1-p_3)\lambda^{2k},$$
$$b_i = (1-p_2)\binom{k}{k-i+1}\lambda^{2k-i+1}, \quad i = 2, 3, \ldots, k+1,$$

and

$$a_1 = (1-p_1)(1-p_3)\lambda^{2k},$$
$$a_i = \begin{cases} \left[\binom{2k}{2k-i+1} - (p_1+p_3)\binom{k}{k-i+1}\right]\lambda^{2k-i+1}, & i = 2, \ldots, k+1 \\ \binom{2k}{2k-i+1}\lambda^{2k-i+1}, & i = k+2, \ldots, 2k. \end{cases}$$

PROOF: By using Corollary 2, for $\mathcal{L}_X(t) = (\lambda/(\lambda+t))^k$, the LST of S_2 is obtained as

$$\mathcal{L}_{S_2}(t) = \frac{(1-p_2)\lambda^k(\lambda+t)^k + [p_2 - p_1 - p_3(1-p_1)]\,\lambda^{2k}}{(\lambda+t)^{2k} - (p_1+p_3)\lambda^k(\lambda+t)^k + p_1p_3\lambda^{2k}}.$$

Rewriting the terms $(\lambda+t)^k$ and $(\lambda+t)^{2k}$ using binomial expansions, the numerator and denominator of $\mathcal{L}_{S_2}(t)$ can be reexpressed as

$$\begin{aligned} &(1-p_2)\binom{k}{0}\lambda^k t^k + (1-p_2)\binom{k}{1}\lambda^{k+1}t^{k-1} + \cdots \\ &\quad + (1-p_2)\binom{k}{k-1}\lambda^{2k-1}t + (1-p_1)(1-p_3)\lambda^{2k}, \end{aligned} \tag{9}$$

and

$$\begin{aligned} &t^{2k} + \binom{2k}{1}\lambda t^{2k-1} + \binom{2k}{2}\lambda^2 t^{2k-1} + \cdots + \binom{2k}{k-1}\lambda^{k-1}t^{k+1} \\ &\quad + \left[\binom{2k}{k} - (p_1+p_3)\binom{k}{0}\right]\lambda^k t^k + \left[\binom{2k}{k+1} - (p_1+p_3)\binom{k}{1}\right]\lambda^{k+1}t^{k-1} \\ &\quad + \cdots + \left[\binom{2k}{2k-1} - (p_1+p_3)\binom{k}{k-1}\right]\lambda^{2k-1}t + (1-p_1)(1-p_3)\lambda^{2k}. \end{aligned} \tag{10}$$

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Thus $\mathcal{L}_{S_2}(t)$ can be written as

$$\mathcal{L}_{S_2}(t) = \frac{b_{k+1}t^k + \cdots + b_2 t + b_1}{t^{2k} + a_{2k}t^{2k-1} + \cdots + a_2 t + a_1},$$

where $b_1, \ldots, b_{k+1}$ and $a_1, \ldots, a_{2k}$ are respectively the coefficients of the polynomials (9) and (10). Thus the proof is completed. ■

4. A MORE GENERAL MODEL

In this section, we extend the model by utilizing the concept of runs. According to the run shock model, the system fails upon the occurrence of a specified number of consecutive critical shocks (see, e.g., [12]). We present a more general model by modifying only the system's failure criteria. As in the previous model, the distribution of the magnitudes of shocks changes after the first shock of size at least d_1, but now the system fails upon the occurrence of l consecutive shocks of size at least d_2. Clearly, the case l=1 corresponds to the extreme shock model which has been studied in the previous sections.

Before proceeding with the necessary probabilistic derivations, let us define a random variable T_l which is closely related to our derivations. For a sequence binary trials each resulting in either a success "1" or a failure "0," the number of trials until the first l consecutive successes which is denoted by T_l has geometric distribution of order l, and its pmf is given by

$$P\{T_l = x\} = (1, 0, \ldots, 0) \begin{bmatrix} 1-p & p & 0 & \cdots & 0 \\ 1-p & 0 & p & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-p & 0 & 0 & \cdots & 0 \end{bmatrix}^{x-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ p \end{bmatrix},$$

for $x = l, l+1, \ldots$, where the parameter p represents the probability of success at each individual trial. The geometric distribution of order l with parameter p will be denoted by $Geo(l, p)$. The pgf of T_l is given by

$$\psi_l(z) = \frac{(pz)^l(1-pz)}{1 - z + (1-p)p^l z^{l+1}} \tag{11}$$

(see, e.g., Balakrishnan and Koutras [2]).

Let $N_{d_2}(l)$ denote the number of shocks until the first l consecutive shocks of size at least d_2 when the distribution of the shock sizes change after the first shock that is above d_1. In the following Theorem, we obtain the joint distribution of the random variables N_{d_1} and $N_{d_2}(l)$.

THEOREM 3: *The joint distribution of the random variables* N_{d_1} *and* $N_{d_2}(l)$ *is*

(a) $P\{N_{d_1} = m, N_{d_2}(l) = n\} = p_1^{m-1}(1-p_2)(1-p_3)^{l-1}$ *if* $n = m + l - 1$

(b) *For* $n > m + l - 1$,

$$P\{N_{d_1} = m, N_{d_2}(l) = n\} = p_1^{m-1}(p_2 - p_1)P\{T_l = n - m\} + \sum_{i=0}^{\min(l-2, n-m-l-1)} p_1^{m-1}(1-p_2)(1-p_3)^i p_3 P\{T_l = n - m - i - 1\}, \tag{12}$$

where T_l *is a random variable having* $Geo(l, 1-p_3)$ *distribution.*

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PROOF: If $n{=}m{+}l{-}1$, then there is only one case: the first $m{-}1$ shocks are below d_1, and the mth shock and the following $l{-}1$ shocks are all above the level d_2. Thus part (a) is proved. For $n{>}\,m{+}l{-}1$, the mth shock can be either between d_1 and d_2 or above the level d_2. For the first case when the mth shock is between d_1 and d_2, a typical sequence of shocks satisfying the joint event $\{N_{d_1} = m, N_{d_2}(l) = n\}$ is

$$\{Y_1 \le d_1, \ldots Y_{m-1} \le d_1, d_1 < Y_m \le d_2, T_l = n - m\}, \tag{13}$$

where T_l denotes the number of shocks until the first l consecutive shocks above d_2, i.e., $T_l = \min\{n : Y_{n-l+1} > d_2, \ldots, Y_n > d_2\}$. Manifestly, the random variable has $Geo(l, 1-p_3)$ distribution with $1 - p_3 = P\{Y_i > d_2\} = 1 - G_2(d_2)$. If the mth shock is above d_2, then a typical sequence of shocks for the occurrence of $\{N_{d_1} = m, N_{d_2}(l) = n\}$ is

$$\begin{aligned}&\{Y_1 \le d_1, \ldots Y_{m-1} \le d_1, Y_m > d_2, Y_{m+1} > d_2, \ldots, Y_{m+i} > d_2,\\ &\quad d_1 < Y_{m+i+1} \le d_2, T_l = n - m - i - 1\},\end{aligned} \tag{14}$$

for $i = 0, 1, \ldots, \min(l-2, n-m-l-1)$. The proof of part (b) follows by combining the probabilities of the sequences (13) and (14). ■

For the special case when $l = 2$, in the following we obtain the probability generating function of $(N_{d_1}, N_{d_2}(l))$.

PROPOSITION 3: *For $l = 2$, the probability generating function of $(N_{d_1}, N_{d_2}(l))$ is*

$$\begin{aligned}\phi_2(z_1, z_2) = \frac{1}{1 - z_1 z_2 p_1}\Bigg[&(1-p_2)(1-p_3) z_1 z_2^2\\ &+ \frac{(1-p_3)^2 z_2^2 (1 - (1-p_3) z_2)}{1 - z_2 + p_3 (1-p_3)^2 z_2^3}\left\{(p_2 - p_1) z_1 z_2 + (1-p_2) p_3 z_1 z_2^2\right\}\Bigg]\end{aligned} \tag{15}$$

PROOF: For $l = 2$, the joint pmf of N_{d_1} and $N_{d_2}(2)$ becomes $P\{N_{d_1} = m, N_{d_2}(2) = n\} = p_1^{m-1}(1-p_2)(1-p_3)$ if $n{=}m{+}1$, and

$$\begin{aligned}&P\{N_{d_1} = m, N_{d_2}(2) = n\}\\ &\quad = p_1^{m-1}(p_2 - p_1) P\{T_2 = n - m\} + p_1^{m-1}(1-p_2) p_3 P\{T_2 = n - m - 1\},\end{aligned}$$

for $n{>}\,m{+}1$. Thus the pgf of $(N_{d_1}, N_{d_2}(2))$ is

$$\begin{aligned}\phi_2(z_1, z_2) = E(z_1^{N_{d_1}} z_2^{N_{d_2}(2)}) &= \sum_{m=1}^{\infty} z_1^m z_2^{m+1} p_1^{m-1}(1-p_2)(1-p_3)\\ &+ \sum_{m=1}^{\infty}\sum_{n=m+2}^{\infty}\big[z_1^m z_2^n p_1^{m-1}(p_2 - p_1) P\{T_2 = n - m\}\\ &\quad z_1^m z_2^n p_1^{m-1}(1-p_2) p_3 P\{T_2 = n - m - 1\}\big].\end{aligned} \tag{16}$$

Note that

$$\sum_{n=m+2}^{\infty} z_2^n P\{T_2 = n - m\} = z_2^m \psi_2(z_2), \tag{17}$$

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and

$$\sum_{n=m+2}^{\infty} z_2^n P\{T_2 = n-m-1\} = z_2^{m+1}\psi_2(z_2). \tag{18}$$

By using (17) and (18) in (16), we obtain

$$\phi_2(z_1,z_2) = \frac{1}{1-z_1z_2p_1}\left[(1-p_2)(1-p_3)z_1z_2^2 + \psi_2(z_2)\left\{(p_2-p_1)z_1z_2 + (1-p_2)p_3z_1z_2^2\right\}\right].$$

Thus, (15) follows using (11). ■

COROLLARY 5: *For $l=2$, the LST corresponding to the system's lifetime $S_2(l) = \sum_{i=1}^{N_{d_2}(l)} X_i$ is given by*

$$\begin{aligned}\mathcal{L}_{S_{2(2)}}(t) &= \phi_2(1,\mathcal{L}_X(t)) \\ &= \frac{1}{1-p_1\mathcal{L}_X(t)}\left[(1-p_2)(1-p_3)\mathcal{L}_X^2(t)\right. \\ &\quad \left. + \frac{(1-p_3)^2\mathcal{L}_X^2(t)(1-(1-p_3)\mathcal{L}_X(t))}{1-\mathcal{L}_X(t)+p_3(1-p_3)^2\mathcal{L}_X^3(t)}\left\{(p_2-p_1)\mathcal{L}_X(t)+(1-p_2)p_3\mathcal{L}_X^2(t)\right\}\right].\end{aligned}$$

5. NUMERICAL EXAMPLES

Wind turbine is a system that converts the kinetic energy in the wind into electrical energy. The power generated by the wind turbine is dependent on wind speed which is a random variable. The well-known relationship between the wind speed and the power generated by a wind turbine is given in Figure 1 [11]. Figure 1 can be described as follows: if the wind speed in an area is below the cut-in wind speed (v_{ci}), then no electrical power is produced since in this case the power in the wind is not enough. When the wind speed is between the cut-in wind speed and rated wind speed (v_{r}), the wind turbine generates power and in this case there is a cubic relationship between the wind speed and the power. If the wind speed is between the rated wind speed and the cut-out wind speed (v_{co}), then the wind turbine generates power at a constant rate P_{r} which denotes the rated power of the turbine. If the wind speed exceeds the cut-out wind speed, then the turbine is stopped since it is in a danger of mechanical failure and hence no electrical power is produced.

Assume that a wind speed which is above v_{ci} is considered as a shock and after the first wind speed value that exceeds v_{r}, the distribution of the wind speed changes from the distribution G_1 to the distribution G_2. The turbine generates power until the wind speed exceeds the cut-out wind speed v_{co}. If the thresholds d_1 and d_2 are chosen to be $d_1 = v_{\text{r}}$ and $d_2 = v_{\text{co}}$, then the random variable S_2 represents the time that the turbine actively produces energy. Although our generic model is based on the failure of the system, here in this setup we are interested in the time that the turbine actively works without any break due to the wind speed. That is, if the wind speed is above $d_2 = v_{\text{co}}$, then the turbine is stopped to prevent it from failure.

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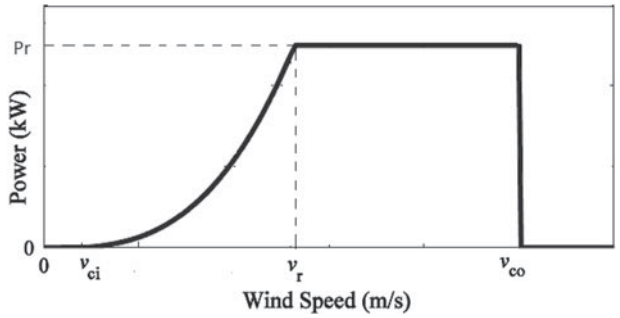


FIGURE **1.** Wind power curve for a wind turbine.

Under this setup, if V denotes the wind speed random variable, then

$$p_1 = P\{V < v_{\rm r} \mid V \geq v_{\rm ci}\} = \frac{G_1(v_{\rm r}) - G_1(v_{\rm ci})}{1 - G_1(v_{\rm ci})}$$

$$p_2 = P\{V < v_{\rm co} \mid V \geq v_{\rm ci}\} = \frac{G_1(v_{\rm co}) - G_1(v_{\rm ci})}{1 - G_1(v_{\rm ci})}$$

$$p_3 = P\{V < v_{\rm co} \mid V \geq v_{\rm ci}\} = \frac{G_2(v_{\rm co}) - G_2(v_{\rm ci})}{1 - G_2(v_{\rm ci})}.$$

Let X_is have Erlang distribution with common pdf $f(x) = \lambda^2 x e^{-\lambda x}, x \geq 0$. Then $X_i \sim \text{ME}_2(\boldsymbol{\alpha}, \mathbf{S}, \mathbf{s})$ with $\boldsymbol{\alpha} = (\lambda^2, 0)$,

$$\mathbf{S} = \begin{bmatrix} 0 & 1 \\ -\lambda^2 & -2\lambda \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Clearly,

$$\mathcal{L}_X(t) = E(e^{-tX}) = \frac{\lambda^2}{t^2 + 2\lambda t + \lambda^2}.$$

Assume first that the system fails upon the occurrence of the first shock that is above d_2. Then, using Theorem 2, the LST corresponding to the system's lifetime is calculated as

$$\mathcal{L}_{S_2}(t) = \frac{b_3 t^2 + b_2 t + b_1}{t^4 + a_4 t^3 + a_3 t^2 + a_2 t + a_1},$$

where

$$b_1 = \lambda^4(1-p_1)(1-p_3), \quad b_2 = 2\lambda^3(1-p_2), \quad b_3 = \lambda^2(1-p_2)$$

$$a_1 = \lambda^4(1-p_1)(1-p_3), \quad a_2 = \lambda^3(4-2(p_1+p_3)), \quad a_3 = \lambda^2(6-(p_1+p_3)), \quad a_4 = 4\lambda.$$

Then $S_2 \sim \text{ME}_4(\boldsymbol{\gamma}, \mathbf{A}, \mathbf{a})$ with $\boldsymbol{\gamma} = (b_1, b_2, b_3, 0)$,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 & -a_4 \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

In Figure 2, we plot the reliability function $P\{S_2 > t\}$ for different choices of the parameters p_1, p_2 and p_3 when $\lambda = 1$. Case 1, Case 2, and Case 3 correspond respectively by $p_1 = 0.5, p_2 = 0.8, p_3 = 0.7; p_1 = 0.5, p_2 = 0.8, p_3 = 0.6$, and $p_1 = 0.5, p_2 = 0.9, p_3 = 0.6$.

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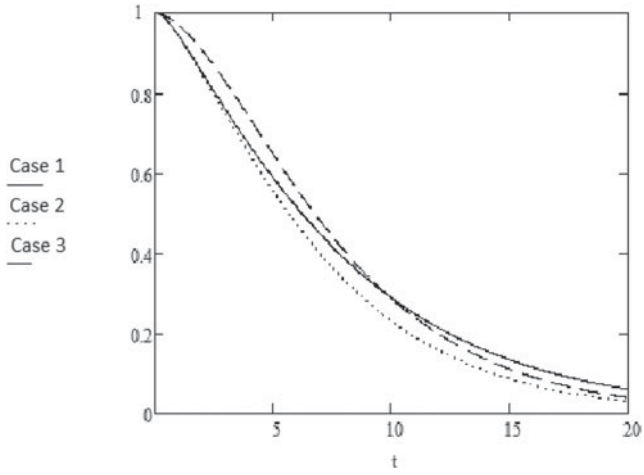


FIGURE **2.** Reliability function of the system.

Next, consider the case the system fails whenever there are two consecutive shocks above d_2. Then, from Corollary 4, the LST takes the following form:

$$\mathcal{L}_{S_2(2)}(t) = \frac{d_5t^4 + d_4t^3 + d_3t^2 + d_2t + d_1}{t^{10} + c_{10}t^9 + c_9t^8 + c_8t^7 + c_7t^6 + c_6t^5 + c_5t^4 + c_4t^3 + c_3t^2 + c_2t + c_1},$$

where

$$\begin{aligned}
d_1 &= \lambda^{10}(1-p_1)(1-p_3)p_3\\
d_2 &= 2\lambda^9(1-p_3)^2\left[1-p_1+(p_2-p_1)(1-2(1-p_3))\right]\\
d_3 &= \lambda^8(1-p_3)^2\left[1-p_1+(p_2-p_1)(4+p_3)\right]\\
d_4 &= 4\lambda^7(1-p_3)^2(p_2-p_1)\\
d_5 &= \lambda^6(1-p_3)^2(p_2-p_1),
\end{aligned}$$

and

$$\begin{aligned}
c_1 &= \lambda^{10}(1-p_1)(1-p_3)^2p_3\\
c_i &= \lambda^{11-i}\left[\binom{10}{11-i} - \binom{8}{9-i}(1+p_1) + \binom{6}{7-i}p_1\right.\\
&\quad \left. + \binom{4}{5-i}p_3(1-p_3)^2 - \binom{2}{3-i}p_1p_3(1-p_3)^2\right], \quad i=2,3\\
c_i &= \lambda^{11-i}\left[\binom{10}{11-i} - \binom{8}{9-i}(1+p_1) + \binom{6}{7-i}p_1 + \binom{4}{5-i}p_3(1-p_3)^2\right], \quad i=4,5\\
c_i &= \lambda^{11-i}\left[\binom{10}{11-i} - \binom{8}{9-i}(1+p_1) + \binom{6}{7-i}p_1\right], \quad i=6,7\\
c_i &= \lambda^{11-i}\left[\binom{10}{11-i} - \binom{8}{9-i}(1+p_1)\right], \quad i=8,9\\
c_{10} &= 10\lambda.
\end{aligned}$$

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TABLE 1. Mean lifetime of the system.

l	p_1	p_2	p_3	$E(S_2)$
1	0	0.8	0.8	10
	0	0.9	0.9	20
	0.5	0.8	0.6	7
	0.5	0.9	0.6	8
2	0	0.8	0.8	60
	0	0.9	0.9	220
	0.5	0.8	0.6	19.5
	0.5	0.9	0.6	20.5

Thus, $S_2(2) \sim \mathrm{ME}_{10}(\boldsymbol{\beta}, \mathbf{B}, \mathbf{b})$ with $\boldsymbol{\beta} = (d_1, d_2, d_3, d_4, d_5, 0, 0, 0, 0, 0)$,

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -c_1 & -c_2 & -c_3 & -c_4 & \cdots & -c_{10} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

In Table 1, we compute the mean lifetime of the system for selected values of the parameters p_1, p_2, p_3, and l when $\lambda = 1$. It should be noted that the case $p_1 = 0$ and $p_2 = p_3$ corresponds to the case when there is no change in shock size distribution. As expected, an increase in l leads to an increase in the mean lifetime.

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