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Complexity and varieties for infinitely generated modules

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1. *Introduction*

In the past fifteen years the theory of complexity and varieties of modules has become a standard tool in the modular representation theory of finite groups. Moreover the techniques have been used in the study of integral representations [**8**] and have been extended to the representation theories of objects such as groups of finite virtual cohomological dimension [**1**], infinitesimal subgroups of algebraic groups and restricted Lie algebras [**14**, **16**]. In all cases some sort of finiteness condition on the module category has been required to make the theory work. Usually this comes in the form of stipulating that all modules under consideration be finitely generated. While the restrictions have been efficient for most applications to date, there are very good reasons for wanting to develop a theory that will accommodate infinitely generated modules. One reason might be the possibility of extending the techniques of representations to other classes of infinite groups. Another reason is that some recent work has revealed a few of the defects of the finiteness requirement. One such problem can be summarized as follows.

Let G be a finite group and k a field of characteristic p dividing $|G|$. In the paper of Carlson, Donovan and Wheeler [**10**], they consider the quotient category of finitely generated kG-modules by the subcategory of modules of less than maximal complexity. In this quotient the endomorphism ring of the trivial module is a direct product of local rings corresponding to the conjugacy classes of elementary abelian subgroups of maximal rank (i.e. the components of the maximal ideal spectrum of $H^*(G, k)$ of maximal dimension). There is no decomposition of the trivial module corresponding to the idempotents in this endomorphism ring, because idempotents fail to split in the quotient category. On the other hand, a direct sum of a suitable number of copies of the trivial module does have such a decomposition (see also [**9**]). Hence the quotient category suffers from a failure of the Krull–Schmidt property, and with the failure, the module theory becomes hard work. The problem here really is with the finite generation requirement. For if the category had infinite direct sums then the idempotents in the endomorphism ring could be made to split, and the Krull–Schmidt theorem would be recovered. We therefore reach the somewhat startling conclusion that in the larger quotient category, in which we allow infinitely generated modules, the trivial module decomposes as a direct sum of infinitely generated modules, corresponding to the conjugacy classes of elementary abelian subgroups of maximal rank.

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Our aim in this paper is to provide definitions of the complexity and the varieties of modules that do not depend on the modules being finitely generated. It is certainly natural to ask that the subcategory of all modules having a fixed complexity c or smaller be closed under direct sums and summands. Its intersection with the subcategory of finitely generated modules should be the finitely generated modules of complexity c or smaller. In addition, the corresponding stable category should be triangulated, meaning that if two terms of a short exact sequence are in the subcategory then so is the third. Finally the functor on the quotient categories, induced by the inclusions of the finite categories into the infinite, should be fully faithful.

We actually propose several definitions for complexity. All but one of them turn out to be the same. But even the odd one, the minimum category, coincides with the others if we restrict ourselves to countably generated modules. The most easily testable of the conditions defining complexity is the condition for membership of the subcategory $\mathscr{E}_c$, which says that, if we allow adding finite dimensional projective summands, any finite subset of a module of complexity c is contained in a finitely generated submodule of complexity at most c. Other definitions are given in terms of a condition which assures that the functors on the quotient categories are fully faithful, filtered colimits of finite dimensional modules and, of course, dimensions of varieties. Each of these seems to provide a unique insight into the nature of the subcategories.

As in the case of finitely generated modules, the varieties give the most information about modules and their complexities. However, there is a big difference in the case of infinitely generated modules. It is that the variety of a module M is a collection $\mathscr{V}_G(M)$ of closed homogeneous subvarieties of $V_G(k)$, rather than a single closed subvariety $V_G(M) \subseteq V_G(k)$. In the finitely generated case, this collection is exactly the collection of closed homogeneous subvarieties of $V_G(M)$, but for an infinitely generated module this collection need not have a unique maximal element. However, $\mathscr{V}_G(M)$ does have a dimension, namely the maximal dimension of an element. This turns out to be equal to the complexity of the module.

The paper is organized roughly as follows. As one motivation for the extension of complexity, we present in Section 3 an account of how to split idempotents when we are allowed infinite direct sums. The two main definitions of complexity and the proof of their equivalence is given next. Section 5 gives the definition in terms of filtered colimits, and shows how this implies the statement about the minimum subcategory. Next it is shown how the splitting of idempotents allows us to recover the Krull–Schmidt theorem in the quotient categories. The final sections contain the definitions of the varieties and the analog for infinitely generated modules of the Alperin–Evens theorem.

In an appendix we have included several examples of what happens in the category of infinitely generated modules. The category has many more pitfalls than we had originally anticipated, and we have tried to illustrate some of the strange behaviour with the examples. In addition, it is actually possible to see how the splitting of the idempotents works in some easy cases using diagrams for modules. As one example we show how, in the complexity quotient category, the trivial module for the dihedral group decomposes as the direct sum of two infinitely generated modules. The interested reader may wish to use the appendix as a companion to the text.

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2. *Definitions and preliminaries*

Most of the concepts used in this paper involve standard constructions of categories, cohomology rings and representations. However, for the benefit of the reader we begin with a brief review of some of the definitions. For general reference, we refer the reader to the books by Benson[**5**] and Evens[**13**] for information on cohomology and representations. For more details on triangulated categories and quotient categories see Happel[**15**].

Throughout the paper we assume that G is a finite group and that k is an algebraically closed field of characteristic $p > 0$. The cohomology ring $H^*(G,k)$ is a finitely generated graded commutative k-algebra, and it has a maximal ideal spectrum $V_G(k)$ which is a homogeneous affine variety. By a theorem of Quillen, $V_G(k)$ has dimension equal to the p-rank of G and the components of $V_G(k)$ correspond to the conjugacy classes of maximal elementary abelian p-subgroups of G. In fact, the minimal prime ideals of $H^*(G,k)$ are precisely the radicals of the kernels of the restriction maps

$$\mathrm{res}_{G,E}: H^*(G,k) \to H^*(E,k)$$

for maximal elementary abelian p-subgroups E of G. These last facts do not play a large role in this paper, but they are crucial to the decompositions of the trivial module in the quotient categories that are a major motivation for our writing.

For a finitely generated kG-module M, let $I_G(M) \subseteq H^*(G,k)$ be the annihilator in $H^*(G,k)$ of the cohomology ring $\mathrm{Ext}^*_{kG}(M,M)$. Notice that $I_G(M)$ also annihilates $\mathrm{Ext}^*_{kG}(M,N)$ for all kG-modules N. This is because $\mathrm{Ext}^*_{kG}(M,N)$ is an $\mathrm{Ext}^*_{kG}(M,M)$-module. The variety of M is the closed subset $V_G(M) \subseteq V_G(k)$ consisting of all maximal ideals (points of $V_G(k)$) which contains $I_G(M)$. So $V_G(M)$ is a closed homogeneous subvariety, and it has dimension at most the p-rank of G.

The complexity of M is, in fact, the dimension of $V_G(M)$. The original definition of Alperin was in terms of the polynomial rate of growth of the cohomology of M, but this is equivalent to our definition. An important fact about varieties which we need is that if

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence, then the variety of any one of the modules, M, M' or M'', is a subset of the union of the varieties of the other two. Hence if two of the modules in the sequence have complexity at most c, then the third also has complexity at most c. It is a consequence of this fact that the stable category of all modules of complexity at most c is triangulated.

We write $\mathbf{mod}\,(kG)$ for the category of all finitely generated kG-modules, and $\mathbf{stmod}\,(kG)$ for the corresponding stable category. That is, $\mathbf{stmod}\,(kG)$ has the same objects as $\mathbf{mod}\,(kG)$, but for M and N in $\mathbf{mod}\,(kG)$

$$\mathrm{Hom}_{\mathbf{stmod}\,(kG)}(M,N) = \mathrm{Hom}_{kG}(M,N)/\mathrm{PHom}_{kG}(M,N),$$

where $\mathrm{PHom}_{kG}(M,N)$ is the set of all homomorphisms from M to N which factor through a projective module. The module category is abelian while the stable category is triangulated. The basic properties of triangulated categories can be summarized as follows.

An additive category $\mathscr{C}$ is triangulated if it has triangles which satisfy certain axioms. First we need a translation functor $T: \mathscr{C} \to \mathscr{C}$, which is an additive self-

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equivalence of the category $\mathscr{C}$. Then a triangle in $\mathscr{C}$ is a sextuple (X, Y, Z, u, v, w) of objects X, Y, Z and morphisms

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX.$$

We should emphasize that not every sextuple is a triangle. The triangles are a distinguished set of sextuples with some special properties. A morphism of triangles is a triple f, g, h such that the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Tf} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX' \end{array}$$

commutes. The axioms for triangles assert that every sextuple isomorphic to a triangle is a triangle. Every morphism $u: X \to Y$ is in a unique triangle, up to isomorphism. If $\mathrm{Id}_X : X \to X$ is the identity then the third object in the triangle of Id_X is the zero object. A sextuple (X, Y, Z, u, v, w) is a triangle if and only if $(Y, Z, TX, v, w, -Tu)$ is also a triangle. Given two triangles (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w'), and maps $f: X \to X'$, $g: Y \to Y'$ satisfying $gu = u'f$, there must exist $h: Z \to Z'$ such that (f, g, h) is a morphism of triangles. Finally we have the 'octahedral axiom', which we do not explicitly state. In the case of the stable category, it is basically the third isomorphism theorem.

It is actually not very difficult to prove that the stable category of kG-modules modulo projectives is triangulated. The key observation is that short exact sequences in the module category correspond to triangles in the stable category. That is, if

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is exact then there is a triangle $(A, B, C, \alpha, \beta, \gamma)$ in the stable category. Indeed if $\alpha: M \to N$ is a morphism in the stable category then we can find projective modules P and Q and representatives $\alpha': M \to N \oplus P$, $\alpha'': M \oplus Q \to P$ such that α' is injective and α'' is surjective. That is, we have exact sequences

$$0 \to M \xrightarrow{\alpha'} N \oplus P \xrightarrow{\beta} V \to 0$$

and

$$0 \to U \xrightarrow{\gamma} M \oplus Q \xrightarrow{\alpha''} N \to 0.$$

Finally, it can be shown that $V \cong \Omega^{-1}(U) \oplus P'$ for some projective module P'. Here $\Omega^{-1}(U)$ is defined to be the cokernel of the map $\theta: U \to Q'$ where Q' is an injective hull for U. Thus Ω^{-1} is the translation functor for the stable category and

$$U \xrightarrow{\gamma} M \xrightarrow{\alpha} N \xrightarrow{\beta} \Omega^{-1}(U)$$

is a triangle.

Finally we should mention the quotient construction. Suppose that $\mathscr{C}$ is a triangulated category and $\mathscr{M} \subseteq \mathscr{C}$ is a triangulated full subcategory. Then we can form the quotient category $\mathscr{C}/\mathscr{M} = \mathscr{Q}$ by a localization process. The objects in $\mathscr{Q}$ are the same as those in $\mathscr{C}$. For morphisms in $\mathscr{Q}$ we invert any morphism in $\mathscr{C}$ if the third object of the triangle of that morphism is in $\mathscr{M}$. So if M and N are objects in $\mathscr{Q}$, then a morphism from M to N is represented by a diagram $M \xleftarrow{s} U \xrightarrow{t} N$ of objects and morphisms such that the third object in the triangle of the morphism $s: U \to M$ is in $\mathscr{M}$. Thus the morphism has the form ts^{-1}. Now if $f: V \to U$ is also invertible then we must allow that

$$ts^{-1} = (tf)(sf)^{-1}.$$

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So a morphism is actually an equivalence class of pairs (t, s). The equivalence relation is generated by $(t, s) \sim (tf, sf)$ as above. The quotient category is universal for functors taking the morphisms s described above to isomorphisms.

In fact, the localization process may be performed using either left or right fractions. In other words, we can represent morphisms in $\mathscr{C}/\mathscr{M}$ by diagrams $M \xrightarrow{t'} U' \xleftarrow{s'} N$ where the third object in the triangle of the morphism $s' : U' \to N$ is in $\mathscr{M}$. To see this, observe that the opposite category of a triangulated category is triangulated, and use the universal property of the quotient.

3. *Splitting of idempotents*

If M is an object in an additive category, and $e \in \mathrm{End}(M)$ is idempotent, we say that e splits if $M \cong M_1 \oplus M_2$ in such a way that

$$e = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{pmatrix}.$$

Here, we are viewing $\mathrm{End}(M)$ as the ring of 2×2 matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

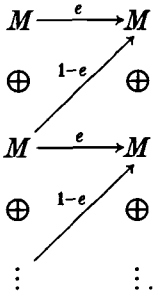
with $\alpha \in \mathrm{End}(M_1)$, $\beta \in \mathrm{Hom}(M_2, M_1)$, $\gamma \in \mathrm{Hom}(M_1, M_2)$ and $\delta \in \mathrm{End}(M_2)$.

LEMMA 3·1. *Let $\mathscr{C}$ be a triangulated category with countable direct sums, and with the property that a countable direct sum of triangles is a triangle. If e is an idempotent in the endomorphism ring of an object M in $\mathscr{C}$, then e splits.*

Proof. This is essentially proposition 3·2 of Bökstedt and Neeman [**6**]. For the convenience of the reader, we recall their proof here. Suppose that $e \in \mathrm{End}(M)$ is idempotent. Let $\hat{M} = M \oplus M \oplus \dots$ be a countable direct sum of copies of M, and consider the endomorphism

$$\phi = \begin{pmatrix} e & 1-e & 0 & \dots \\ 0 & e & 1-e & \\ 0 & 0 & e & \\ \vdots & & & \ddots \end{pmatrix} \in \mathrm{End}(\hat{M}).$$

We may view this endomorphism schematically as



Complete this to a triangle

$$M_1 \to \hat{M} \xrightarrow{\varphi} \hat{M} \to T(M_1)$$

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(where T is the translation functor) to define the object M_1. Similarly, swapping the roles of e and $1-e$, we get an endomorphism $\psi \in \mathrm{End}(\hat{M})$, which we complete to a triangle

$$M_2 \to \hat{M} \xrightarrow{\psi} \hat{M} \to T(M_2)$$

to define the object M_2.

Because

$$\begin{pmatrix} 1-e & e \\ -e & 1-e \end{pmatrix}\begin{pmatrix} 1-e & 0 \\ 0 & e \end{pmatrix}\begin{pmatrix} 1-e & -e \\ e & 1-e \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 1-e & e \\ -e & 1-e \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & 1-e \end{pmatrix}\begin{pmatrix} 1-e & -e \\ e & 1-e \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we may rechoose a decomposition of $\hat{M} \oplus \hat{M}$ as a direct sum of copies of M in such a way that $\phi \oplus \psi$ is the endomorphism

$$\begin{array}{ccc} M \oplus M & \xrightarrow{a} & M \oplus M \\ \oplus & \overset{b}{\nearrow} & \oplus \\ M \oplus M & \xrightarrow{a} & M \oplus M \\ \oplus & \overset{b}{\nearrow} & \oplus \\ \vdots & & \vdots \end{array}$$

where $a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. It follows that there is a triangle

$$M \longrightarrow \hat{M} \oplus \hat{M} \xrightarrow{\phi \oplus \psi} \hat{M} \oplus \hat{M} \longrightarrow T(M)$$

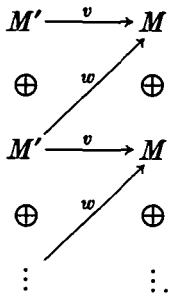
and so $M_1 \oplus M_2 \cong M$. It is clear that, with respect to this direct sum decomposition, e has the desired form. ∎

Let M be an object in a quotient category $\mathscr{C}/\mathscr{C}'$, where $\mathscr{C}'$ is a full triangulated subcategory of a triangulated category $\mathscr{C}$. Suppose that $\mathscr{C}$ and $\mathscr{C}'$ are closed under countable direct sums, and that a countable direct sum of triangles is a triangle. Then an idempotent $e \in \mathrm{End}_{\mathscr{C}/\mathscr{C}'}(M)$ may be represented by maps $M \xleftarrow{u} M' \xrightarrow{v} M$, where u and v are arrows in $\mathscr{C}$, and u has the property that when completed to a triangle, the third vertex lies in $\mathscr{C}'$. We may arrange this in such a way that the same map u is used to represent $1-e$ by maps $M \xleftarrow{u} M' \xrightarrow{w} M$. Then the diagram used to split M in the proof of the above lemma is as follows.

$$\begin{array}{ccccc} M & \xleftarrow{u} & M' & \xrightarrow{v} & M \\ \oplus & & \oplus & \overset{w}{\nearrow} & \oplus \\ M & \xleftarrow{u} & M' & \xrightarrow{v} & M \\ \oplus & & \oplus & \overset{w}{\nearrow} & \oplus \\ \vdots & & \vdots & & \vdots \end{array}$$

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Since the map u is an isomorphism in $\mathscr{C}/\mathscr{C}'$, we may as well leave out these maps and just complete the following diagram to a triangle:



4. *Complexity*

We define $\mathscr{M}_c$ to be the subcategory of **stmod** (kG) consisting of the modules of complexity at most c. This is a triangulated subcategory, and so we may form quotient categories **stmod** $(kG)/\mathscr{M}_c$ and $\mathscr{M}_c/\mathscr{M}_{c'}$ for $c' < c$.

Let **Mod** (kG) be the category of all kG-modules (both finitely and infinitely generated), and **StMod** (kG) be the corresponding stable category. This is triangulated in the same way as **stmod** (kG) (see Section 2), using the following lemma.

LEMMA 4·1. *Let S be a Sylow p-subgroup of G. For a (possibly infinitely generated) kG-module M, the following conditions are equivalent*:

(i) *The module M is projective.*
(ii) *The module M is injective.*
(iii) *The restriction $M\downarrow_S$ is projective.*
(iv) *The restriction $M\downarrow_S$ is injective.*
(v) *Every S-fixed element $m \in M$ can be written as $\sum_{g\in S} g(m')$ for some $m' \in M$.*

Proof. One proves that (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (v) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (ii). ∎

Actually, projective modules are the same as injective modules for an arbitrary finite dimensional self-injective algebra, but the proof is a little more messy.

We define various subcategories of **StMod** (kG) which are possible contenders for the category of modules of complexity at most c, and prove that these subcategories almost coincide.

(i) We define $\underline{\mathrm{Min}}_c$ to be the smallest full triangulated subcategory of the category **StMod** (kG) containing $\mathscr{M}_c$ and closed under taking direct summands and countable direct sums. The intersection of any collection of such subcategories is also one, and so there is a unique smallest one.

(ii) We define $\underline{\mathscr{U}}_c$ to be the full subcategory of **StMod** (kG) consisting of those modules M with the property that any map $\phi: X \to M$ from a finitely generated module X factors through a finitely generated module of complexity at most c. Equivalently, given any such map ϕ, there exists a map $\phi: Y \to X$ in **stmod** (kG) such that $\phi \circ \psi = 0$ in **StMod** (kG) and ψ passes down to an isomorphism in **stmod** $(kG)/\mathscr{M}_c$.

The property defining $\underline{\mathscr{U}}_c$ is exactly the condition which will be needed for the results of Section 6.

(iii) We define $\underline{\mathscr{E}}_c$ to be the full subcategory of **StMod** (kG) consisting of those

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modules M with the property that given any finite subset S of M, there is a finitely generated projective module P and a finitely generated submodule M' of $M \oplus P$ such that M' is in $\mathscr{M}_c$ (i.e. M' has complexity at most c), and M' contains the elements $(s, 0) \in M \oplus P$ with $s \in S$.

THEOREM 4·2. *We have*

$$\underline{\mathrm{Min}}_c \subseteq \underline{\mathscr{U}}_c = \underline{\mathscr{E}}_c.$$

Proof. (i) $\underline{\mathrm{Min}}_c \subseteq \underline{\mathscr{U}}_c$: The subcategory $\underline{\mathscr{U}}_c$ is clearly closed under taking direct summands and countable direct sums. Given a triangle in $\mathbf{StMod}(kG)$ with two corners in $\underline{\mathscr{U}}_c$, represent it in $\mathbf{Mod}(kG)$ by a short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

with M_2 and M_3 in $\mathscr{U}_c$. We claim that M_1 is also in $\mathscr{U}_c$, so that $\underline{\mathscr{U}}_c$ is triangulated and hence contains $\underline{\mathrm{Min}}_c$. Given a map $X \to M_1$ with X finitely generated, we may factor the composite $X \to M_1 \to M_2$ as a composite $X \to Z \to M_2$ with Z in $\mathscr{M}_c$. By adding a suitable projective to Z, we may assume $X \to Z$ is injective. Then there is an induced map $Z/X \to M_3$, which we may factor as a composite $Z/X \to Z' \to M_3$ with Z' in $\mathscr{M}_c$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & Z' & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0. \end{array}$$

Add a suitable projective module P to Z and make $Z \oplus P \to Z'$ surjective with kernel Z'' in $\mathscr{M}_c$. Then $X \to M_1$ factors through Z''.

(ii) $\underline{\mathscr{E}}_c \subseteq \underline{\mathscr{U}}_c$: if M is a kG-module representing an object in $\underline{\mathscr{E}}_c$, then given a map ϕ: $X \to M$ with X finitely generated, say by a set S, then we may find a projective module P and a finitely generated submodule M' of $M \oplus P$ such that M' is in $\mathscr{M}_c$, and the elements of $M \oplus P$ of the form $(\phi(S), 0)$ are in M'. This means that ϕ factors as $X \to M' \hookrightarrow M \oplus P \to M$, and so M lies in $\underline{\mathscr{U}}_c$.

(iii) $\underline{\mathscr{U}}_c \subseteq \underline{\mathscr{E}}_c$: If M is a kG-module representing an object in $\underline{\mathscr{U}}_c$ and S is a finite subset of M, let X be the submodule of M generated by S. Then the inclusion $X \xrightarrow{i} M$ factors as $X \xrightarrow{\alpha} Z \xrightarrow{\beta} M$, where Z is a finitely generated module of complexity at most c. Let $Z \xrightarrow{\sigma} Z'$ be the cokernel of α, and let $Z' \xrightarrow{\gamma} P$ be the injective hull of Z'. This is a projective module since kG is self-injective. The map $(\beta, \gamma\sigma)\colon Z \to M \oplus P$ is injective, and $(\beta, \gamma\sigma) \circ \alpha = (i, 0)$, so the image of Z is a submodule of $M \oplus P$ containing the elements $(s, 0)$ with $s \in S$. ∎

Definition 4·3. We define the complexity of a (possibly infinitely generated) kG-module M to be c if M is in $\underline{\mathscr{U}}_c$ but not in $\underline{\mathscr{U}}_{c-1}$.

5. *Filtered colimits*

In this section, we characterize the modules of complexity at most c as the modules which may be written as filtered colimits of finitely generated modules of complexity at most c. The examples in the Appendix show that we may not assume that the maps in the filtered colimit are inclusions. We begin by recalling some formalities on filtered colimits.

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If I is a small indexing category ('small' means that the class of objects is a set, and 'indexing' has the empty meaning), then a *diagram* of type I in a category $\mathscr{C}$ is a covariant functor $F: I \to \mathscr{C}$. A *colimit* of F is an object $X \in \mathscr{C}$ together with a natural transformation λ_X from F to the constant functor $c_X: I \to \mathscr{C}$ with value X, and universal in the sense that given a natural transformation λ to another constant functor c_Y, there is a unique map $X \to Y$ whose composite with λ_X gives λ. If the colimit of F exists, we write $\varinjlim_I F$ for it. If all such colimits exist, they form a functor $\varinjlim_I : \mathscr{C}^I \longrightarrow \mathscr{C}$, where $\mathscr{C}^I$ denotes the category whose objects are the functors from I to $\mathscr{C}$ and whose arrows are the natural transformations. In this case, we have an adjunction

$$\mathrm{Hom}_{\mathscr{C}}(\varinjlim_I F, Y) \cong \mathrm{Hom}_{\mathscr{C}^I}(F, c_Y).$$

If $\sigma: J \to I$ is a covariant functor between indexing categories, then composition gives a functor $\mathscr{C}^I \to \mathscr{C}^J$, and hence a natural transformation

$$\varinjlim_I (- \circ \sigma) \longrightarrow \varinjlim_I (-).$$

We recall (Artin–Mazur[3], Appendix) that a small category I is *right filtering* if

(*a*) given any pair of objects i, i' in I, there is an object i'' and morphisms $i \to i''$ and $i' \to i''$, and

(*b*) if $i \rightrightarrows i'$ are two morphisms in I then there is a morphism $i' \to i''$ such that the composites $i \to i''$ are equal.

A *filtered colimit* is a colimit of a functor $F: I \to \mathscr{C}$ from a small right filtering category I to a category $\mathscr{C}$.

If $\mathscr{C}$ is a category of sets or modules, then the elements of $\varinjlim_I F$ are equivalence classes of elements in the disjoint union of the $F(i)$, as i runs over the objects in I. The equivalence relation is given as follows. If $\alpha \in F(i)$ and $\alpha' \in F(i')$, then $\alpha \sim \alpha'$ if there are arrows $\rho: i \to i''$, $\rho': i' \to i''$ such that $F(\rho)(\alpha) = F(\rho')(\alpha')$. The module structure is defined similarly: for example, if $\alpha \in F(i)$ and $\alpha' \in F(i')$ then we choose arrows $\rho: i \to i''$ and $\rho': i' \to i''$, and instead of adding α and α', we add $F(\rho)(\alpha)$ and $F(\rho')(\alpha')$ in $F(i'')$.

Filtered colimits of modules possess the following property not shared by all colimits: an element $\alpha \in F(i)$ goes to zero in $\varinjlim_I F$ if and only if there is an arrow $\rho: i \to i'$ in I such that $F(\rho)(\alpha) = 0$ in $F(i')$.

A functor $\sigma: J \to I$ between indexing categories is *right cofinal* if:

(*c*) for every object i in I there is an object j in J and a morphism $i \to \sigma j$ in I, and

(*d*) if i and j are objects in I and J, and $i \rightrightarrows \sigma j$ are morphisms in I, then there is a morphism $j \to j'$ in J such that the composites $i \rightrightarrows \sigma j \to \sigma j'$ are equal.

LEMMA 5·1. *Composing with a right cofinal functor preserves filtered colimits. In other words, if I and J are small right filtering categories, $\sigma: J \to I$ is right cofinal, and $F: I \to \mathscr{C}$ is a diagram of type I in $\mathscr{C}$, then the natural map*

$$\varinjlim_J (F \circ \sigma) \longrightarrow \varinjlim_I F$$

is an isomorphism.

Proof. See proposition 1·8 of the Appendix of Artin–Mazur[3].

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LEMMA 5·2. *Suppose that $\mathscr{C}$ is a triangulated subcategory of* **StMod** (kG) *which is closed under taking countable direct sums. Suppose that we are given a diagram*

$$M_0 \xrightarrow{\rho_0} M_1 \xrightarrow{\rho_1} \dots$$

in **Mod** (kG) *with the property that each M_i, regarded as an object in* **StMod** (kG), *lies in $\mathscr{C}$. Then the colimit of the M_i in* **Mod** (kG), *regarded as an object in* **StMod** (kG), *lies in $\mathscr{C}$.*

Proof. The argument for this is given in §2 of Bökstedt and Neeman [6], and goes as follows. Let $M = \bigoplus_i M_i$, and consider the endomorphism

$$\phi = \begin{pmatrix} 1 & 0 & 0 \dots \\ -\rho_0 & 1 & 0 \\ 0 & -\rho_1 & 1 \\ \vdots & & \ddots \end{pmatrix}$$

of M. Then ϕ is injective, and its cokernel is isomorphic to the required colimit. It follows that if we complete ϕ to a triangle in $\mathscr{C}$

$$M \to M \to M' \to T(M),$$

then M' is isomorphic, in **StMod** (kG), to the required colimit. ∎

We define $\underline{\mathscr{C}_c}$ to be the full subcategory of **StMod** (kG) consisting of filtered colimits of modules in $\mathscr{M}_c$.

THEOREM 5·3. *We have*

$$\underline{\mathrm{Min}_c} \subseteq \underline{\mathscr{U}_c} = \underline{\mathscr{C}_c} = \underline{\mathscr{E}_c}.$$

Furthermore, $\underline{\mathrm{Min}_c}$ consists of the countably generated modules in $\underline{\mathscr{U}_c} = \underline{\mathscr{E}_c}$.

Proof. We have already proved much of this in Theorem 4·2. The statement that $\underline{\mathscr{U}_c} = \underline{\mathscr{C}_c}$ follows from proposition 2·1 of Lenzing [17] with a little work, but for the benefit of the reader we include a self contained proof here. The following statements complete the proof of the theorem:

(i) $\underline{\mathscr{U}_c} \subseteq \underline{\mathscr{C}_c}$. Given an object in $\underline{\mathscr{U}_c}$, represent it by a module M in **Mod** (kG). Let I be the category whose objects are the morphisms $X \to M$ with X finitely generated, and whose morphisms are the commutative triangles. (Actually, to avoid set theoretic problems, namely to ensure that I is small, we should really replace I by a small skeletal subcategory.) Let J be the full subcategory of I whose objects are the morphisms $X \to M$ with X finitely generated of complexity at most c. Then M is the filtered colimit of the functor $I \to$ **Mod** (kG) which sends $X \to M$ to X. The functor $J \to I$ is right cofinal, by definition of $\mathscr{U}_c$, and so by Lemma 5·1, M is also the filtered colimit of the same functor from J to $\overline{\textbf{Mod}}\,(kG)$.

(ii) $\underline{\mathscr{C}_c} \subseteq \underline{\mathscr{E}_c}$. If S is a finite subset of a filtered colimit $M = \varinjlim_I F$ with $F(i)$ in $\mathscr{M}_c$ for all i in I, then S lies in the image of some $F(i) \to M$. The kernel of $F(i) \to M$ is equal to the kernel of $F(\rho)$ for some arrow $\rho: i \to j$ in I. Let K be the kernel of $F(j) \to M$, and $K \to P$ be an injective map with P a projective module. Then $\mathrm{Im}\,(F(\rho)) \cap K = 0$, so $K \to P$ lifts to a map $F(j) \to P$ in such a way that the composite $F(i) \to F(j) \to P$ is zero. It follows that the map $F(j) \to M \oplus P$ is injective, and the elements $(s, 0) \in M \oplus P$ with $s \in S$ lie in the image.

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(iii) Every countably generated module in $\mathscr{C}_c$ lies in $\underline{\mathrm{Min}}_c$. If M is a countably generated module in **Mod** (kG) representing an object in $\underline{\mathscr{C}}_c$, then we may express M as a colimit of a diagram $M_0 \to M_1 \to \ldots$ of modules which, when regarded as objects in **StMod** (kG), lie in $\mathscr{M}_c$. By Lemma 5·2, the colimit M, regarded as an object in **StMod** (kG), lies in $\underline{\mathrm{Min}}_c$. ∎

We end this section with some further properties of filtered colimits. If I is a right filtering category, and $F_1, F_2, F_3 : I \to$ **Mod** (kG) are functors, then a short exact sequence

$$0 \to F_1 \xrightarrow{\alpha} F_2 \xrightarrow{\beta} F_3 \to 0$$

consists of natural transformations $\alpha : F_1 \to F_2$ and $\beta : F_2 \to F_3$ with the property that for all objects i in I,

$$0 \longrightarrow F_1(i) \xrightarrow{\alpha(i)} F_2(i) \xrightarrow{\beta(i)} F_3(i) \longrightarrow 0$$

is a short exact sequence of modules.

PROPOSITION 5·4. *If* $0 \to F_1 \xrightarrow{\alpha} F_2 \xrightarrow{\beta} F_3 \to 0$ *is a short exact sequence of functors* $I \to$ **Mod** (kG) *as above, then*

$$0 \longrightarrow \varinjlim_I F_1 \xrightarrow{\alpha_*} \varinjlim_I F_2 \xrightarrow{\beta_*} \varinjlim_I F_3 \longrightarrow 0$$

is also a short exact sequence.

Proof. This is Grothendieck's axiom AB5, which is well known to be satisfied by module categories.

PROPOSITION 5·5. *A filtered colimit* $M = \varinjlim_I F$ *of projective kG-modules is a projective kG-module.*

Proof. In general, a filtered colimit of flat modules is flat. For a group algebra, or more generally for a perfect ring, flat modules are the same as projective modules (Bass [4]).

COROLLARY 5·6. *A (possibly infinitely generated) kG-module is projective if and only if it has complexity zero.*

6. *The Krull–Schmidt theorem*

In this section, we prove that if $c' < c$, then the functor

$$\mathscr{M}_c / \mathscr{M}_{c'} \to \underline{\mathscr{U}}_c / \underline{\mathscr{U}}_{c'}$$

induced by the inclusion functor $\mathscr{M}_c \to \underline{\mathscr{U}}_c$ is faithful. Our object is to define a subcategory $\mathscr{M}_{c,c'}$ of $\underline{\mathscr{U}}_c / \underline{\mathscr{U}}_{c'}$ by taking direct summands of objects in $\mathscr{M}_c / \mathscr{M}_{c'}$. Then we prove the Krull–Schmidt theorem in $\mathscr{M}_{c,c-1}$, and deduce a cancellation theorem for $\mathscr{M}_c / \mathscr{M}_{c-1}$.

LEMMA 6·1. *Let* $c' < c$. *Suppose that* $U \to M$ *is a morphism in* $\mathscr{U}_c$ *representing an isomorphism in* $\underline{\mathscr{U}}_c / \underline{\mathscr{U}}_{c'}$, *and with* M *in* $\mathscr{M}_c$. *Then there is a morphism* $M' \to U$ *in* $\underline{\mathscr{U}}_c$ *representing an isomorphism in* $\underline{\mathscr{U}}_c / \underline{\mathscr{U}}_{c'}$, *and with* M' *in* $\mathscr{M}_c$.

Proof. Begin by completing $U \to M$ to a triangle

$$U \to M \to W \to \Omega^{-1} U$$

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in $\underline{\mathcal{U}_c}$, so that W is in $\underline{\mathcal{U}_{c'}}$. Then $M \to W$ factors as $M \to Y \to W$ with Y in $\mathcal{M}_{c'}$. Complete $M \to Y$ to a triangle

$$M' \to M \to Y \to \Omega^{-1}M'.$$

Then M' is in $\mathcal{M}_c$ and $M' \to M$ represents an isomorphism in $\mathcal{M}_c/\mathcal{M}_{c'}$. The composite $M' \to M \to W$ is zero, so $M' \to M$ factors as $M' \to U \to M$, and $M' \to U$ represents an isomorphism in $\underline{\mathcal{U}_c}/\underline{\mathcal{U}_{c'}}$.

PROPOSITION 6·2. *If $c' < c$, then the functor*

$$\mathcal{M}_c/\mathcal{M}_{c'} \to \underline{\mathcal{U}_c}/\underline{\mathcal{U}_{c'}}$$

induced by the inclusion functor $\mathcal{M}_c \to \underline{\mathcal{U}_c}$ is fully faithful.

Proof. To say that a functor is fully faithful means that it induces isomorphisms on hom sets.

Suppose that $M \xleftarrow{s} U \xrightarrow{t} N$ is a morphism in $\underline{\mathcal{U}_c}/\underline{\mathcal{U}_{c'}}$, between objects M and N in $\mathcal{M}_c$ (i.e. U is in $\underline{\mathcal{U}_c}$, and the third vertex of the triangle containing $U \to M$ is in $\underline{\mathcal{U}_{c'}}$). By the lemma, there is a homomorphism $M' \xrightarrow{\phi} U$ in $\underline{\mathcal{U}_c}$ which represents an isomorphism in $\underline{\mathcal{U}_c}/\underline{\mathcal{U}_{c'}}$, and with M' in $\mathcal{M}_c$. So $M \xleftarrow{s\phi} M' \xrightarrow{t\phi} N$ represents the same morphism in $\underline{\mathcal{U}_c}/\underline{\mathcal{U}_{c'}}$.

If $M \leftarrow M' \to N$ and $M \leftarrow M'' \to N$ represent the same morphism in $\underline{\mathcal{U}_c}/\underline{\mathcal{U}_{c'}}$, with both M' and M'' in $\mathcal{M}_c$, then another application of the lemma shows that they represent the same morphism in $\mathcal{M}_c/\mathcal{M}_{c'}$. ∎

Definition 6·3. We define $\mathcal{M}_{c,c'}$ to be the smallest subcategory of $\underline{\mathcal{U}_c}/\underline{\mathcal{U}_{c'}}$, which contains $\mathcal{M}_c/\mathcal{M}_{c'}$ and which is closed under taking direct summands.

LEMMA 6·4. *The category $\mathcal{M}_{c,c'}$ is triangulated.*

Proof. Given a morphism in $\mathcal{M}_{c,c'}$, represent it by a morphism $\phi: M \to N$ in $\underline{\mathcal{U}_c}$. Complete it to a triangle

$$L \to M \xrightarrow{\phi} N \to \Omega^{-1}(L)$$

in $\underline{\mathcal{U}_c}$. Choose modules M' and N' in $\underline{\mathcal{U}_c}$ so that $M \oplus M'$ and $N \oplus N'$ are isomorphic in $\underline{\mathcal{U}_c}/\underline{\mathcal{U}_{c'}}$ to objects in $\mathcal{M}_c$. We set

$$\tilde{\phi} = \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}: M \oplus M' \to N \oplus N'.$$

If we complete $\tilde{\phi}$ to a triangle in $\underline{\mathcal{U}_c}$, then the third corner is isomorphic in $\underline{\mathcal{U}_c}$ to $L \oplus M' \oplus \Omega(N')$. Thus L is isomorphic in $\underline{\mathcal{U}_c}/\underline{\mathcal{U}_{c'}}$ to a direct summand of an object in $\mathcal{M}_c$, and so L is in $\mathcal{M}_{c,c'}$.

PROPOSITION 6.5. *The endomorphism ring of a module in $\mathcal{M}_c/\mathcal{M}_{c-1}$ is an Artin algebra.*

Proof. Let M be a finitely generated module of complexity at most c, and let $I_G(M)$ be the annihilator in $H^*(G,k)$ *of* $\mathrm{Ext}^*_{kG}(M,M)$. Then $\mathrm{Ext}^*_{kG}(M,M)$ is a finitely generated module over the image of

$$H^*(G,k) = \mathrm{Ext}^*_{kG}(k,k) \xrightarrow{\otimes M} \mathrm{Ext}^*_{kG}(M,M).$$

This is a graded central subring isomorphic to $H^*(G,k)/I_G(M)$, and has Krull dimension at most c. Let S be the multiplicative subset of $H^*(G,k)/I_G(M)$ consisting

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of the homogeneous elements which do not belong to any prime ideal of dimension c. It is proved in §3 of Carlson, Donovan and Wheeler[**10**] that the endomorphism ring of M in $\mathscr{M}_c/\mathscr{M}_{c-1}$ is equal to the ring of degree zero elements in the homogeneous localization $[\mathrm{Ext}^*_{kG}(M,M)S^{-1}]_0$. This is finitely generated as a module over $[(H^*(G,k)/I_G(M))S^{-1}]_0$. Every prime ideal in the latter ring is maximal, so it is a finite direct sum of Artinian local rings. It follows that $[\mathrm{Ext}^*_{kG}(M,M)S^{-1}]_0$ is an Artin algebra.

COROLLARY 6·6. *The endomorphism ring of an indecomposable module in $\mathscr{M}_{c,c-1}$ is a local Artin algebra.*

Proof. By the above proposition, the endomorphism ring of any module in $\mathscr{M}_{c,c-1}$ is an Artin algebra. It follows from Lemma 3·1 that idempotents in $\mathscr{M}_{c,c-1}$ split, because $\underline{\mathscr{U}_c}/\underline{\mathscr{U}_{c-1}}$ has countable direct sums.

THEOREM 6·7. *The Krull–Schmidt theorem holds in the category $\mathscr{M}_{c,c-1}$.*

Proof. The Krull–Schmidt theorem holds in any additive category with the following properties:

(i) the endomorphism ring of an indecomposable object is a local ring, and

(ii) every object decomposes as a finite direct sum of indecomposables.

COROLLARY 6·8. *If $M_1 \oplus M_2 \cong M_1 \oplus M_3$ in $\mathscr{M}_c/\mathscr{M}_{c-1}$ then $M_2 \cong M_3$.*

Proof. If $M_1 \oplus M_2 \cong M_1 \oplus M_3$ in $\mathscr{M}_c/\mathscr{M}_{c-1}$ then the same holds in $\mathscr{M}_{c,c-1}$, so by the theorem, $M_2 \cong M_3$ in $\mathscr{M}_{c,c-1}$. It follows from Proposition 6·2 that the functor $\mathscr{M}_c/\mathscr{M}_{c-1} \to \mathscr{M}_{c,c-1}$ is fully faithful, and so $M_2 \cong M_3$ in $\mathscr{M}_c/\mathscr{M}_{c-1}$.

7. *Varieties*

Suppose that M is a (possibly infinitely generated) kG-module, and N is a finitely generated kG-module. Then $\mathrm{Ext}^*_{kG}(N,M)$ is an $H^*(G,k)$-module which is finitely generated whenever M is finitely generated as a kG-module.

LEMMA 7·1. *Suppose that $U \neq 0$ is a finitely generated graded $H^*(G,k)$-submodule of $\mathrm{Ext}^*_{kG}(N,M)$. Then there exists a map $\phi: X \to M$ with X a finitely generated kG-module such that*

$$U \subseteq \mathrm{Im}\,(\phi_*: \mathrm{Ext}^*_{kG}(N,X) \to \mathrm{Ext}^*_{kG}(N,M))$$

with equality in all large enough degrees. In particular $\sqrt{\mathrm{Ann}\,(U)} = \sqrt{\mathrm{Ann}\,(\mathrm{Im}\,(\phi_*))}$.

Proof. Choose non-zero homogeneous elements $\zeta_1, \ldots, \zeta_t$ generating U, with $\zeta_i \in \mathrm{Ext}^{n_i}_{kG}(N,M)$. Represent each ζ_i by a cocycle $\hat{\zeta}_i: \Omega^{n_i}(N) \to M$. Then for $j > 0$ the map

$$\widehat{\mathrm{Ext}}^{j-n_i}_{kG}(N,N) \cong \mathrm{Ext}^j_{kG}(N, \Omega^{n_i}(N)) \xrightarrow{(\hat{\zeta}_i)_*} \mathrm{Ext}^j_{kG}(N,M)$$

is a multiplication by ζ_i. So let $X = \bigoplus_{i=1}^t \Omega^{n_i}(N)$ and $\phi = \sum_{i=1}^t \hat{\zeta}_i$. Then the image of ϕ_* contains U, and in degrees which are at least as big as the maximum of the n_i, the image is equal to U. In smaller degrees there can be superfluous elements coming from elements of negative degree in $\mathrm{Ext}^*_{kG}(N,N)$.

The statement about radicals of annihilators is proved as follows. If $0 \neq U \subseteq U'$ is an inclusion of finitely generated graded $H^*(G,k)$-modules with one dimensional quotient, then $\mathrm{Ann}\,(U'/U)$ consists of all elements of positive degrees, and therefore contains $\mathrm{Ann}\,(U)$. Since

$$\mathrm{Ann}\,(U)\,\mathrm{Ann}\,(U'/U) \subseteq \mathrm{Ann}\,(U') \subseteq \mathrm{Ann}\,(U),$$

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this means that $\sqrt{\mathrm{Ann}\,(U)} = \sqrt{\mathrm{Ann}\,(U')}$. Applying this argument inductively, we see that this holds whenever U has finite codimension in U'. ∎

Definition 7·2. We define $\mathscr{V}_G(M)$ to be the set of closed homogeneous subvarieties of $V_G(k)$ which are contained in subvarieties of the form $V_G(\mathrm{Ann}\,(U))$, where U is a finitely generated graded submodule of $\mathrm{Ext}^*_{kG}(N, M)$ for some finitely generated kG-module N.

We remark that if M is itself finitely generated, then $\mathscr{V}_G(M)$ consists of the closed homogeneous subvarieties of $V_G(M)$, and hence has a unique maximal element. In general, $\mathscr{V}_G(M)$ need not have a unique maximal element (for example, take an infinite direct sum of finitely generated modules with different varieties of the same dimension), but it is closed under finite unions:

LEMMA 7·3. *The set $\mathscr{V}_G(M)$ has the following properties*:

(i) *if $V \in \mathscr{V}_G(M)$ and $W \subseteq V$ is closed and homogeneous then $W \in \mathscr{V}_G(M)$.*

(ii) *$V_1, \ldots, V_t \in \mathscr{V}_G(M)$ then $V_1 \cup \ldots \cup V_t \in \mathscr{V}_G(M)$.*

Proof. (i) This is clear from the definition.

(ii) Choose finitely generated kG-modules N_i, and choose finitely generated graded $H^*(G, k)$-submodules $U_i \subseteq \mathrm{Ext}^*_{kG}(N_i, M)$, so that $V_i \subseteq W_i = V_G(\mathrm{Ann}\,(U_i))$. Then $U = U_1 \oplus \ldots \oplus U_t$ is a finitely generated graded $H^*(G, k)$-submodule of $\mathrm{Ext}^*_{kG}(\bigoplus_i N_i, M)$, and $\mathrm{Ann}\,(U) = \mathrm{Ann}\,(U_1) \cap \ldots \cap \mathrm{Ann}\,(U_t)$. Thus $V_1 \cup \ldots \cup V_t \subseteq W_1 \cup \ldots \cup W_t = V_G(\mathrm{Ann}\,(U))$. ∎

If ζ is a non-zero element of $H^n(G, k)$, we represent ζ by a cocycle $\hat{\zeta}\colon \Omega^n(k) \to k$, and write L_ζ for the kernel of $\hat{\zeta}$. Then L_ζ is a finitely generated kG-module with the property that $V_G(L_\zeta)$ is the closed homogeneous hypersurface $V_G\langle\zeta\rangle$ determined by regarding ζ as a homogeneous element of the coordinate ring of $V_G(k)$. There is a triangle

$$L_\zeta \to \Omega^n(k) \xrightarrow{\hat{\zeta}} k \to \Omega^{-1}(L_\zeta)$$

in **stmod** (kG).

LEMMA 7·4. *Suppose that $\theta\colon X \to M$ is a homomorphism of kG-modules, with X finitely generated, and that U is the image of*

$$\theta_*\colon \mathrm{Ext}^*_{kG}(X, X) \to \mathrm{Ext}^*_{kG}(X, M).$$

Let $\zeta \in H^*(G, k)$ be an element which annihilates U. Then θ factors as a composition $X \to X \otimes \Omega^{-1}(L_\zeta) \to M$.

Proof. This follows from the fact that the composite

$$\Omega^n(k) \otimes X \xrightarrow{\hat{\zeta} \otimes 1} X \to M$$

is zero in **StMod** (kG).

THEOREM 7·5. *Let $\mathscr{V}$ be a collection of closed homogeneous subvarieties of $V_G(k)$ satisfying conditions* (i) *and* (ii) *of Lemma* 7·3. *Then for a kG-module M the following conditions are equivalent*:

(i) *$\mathscr{V}_G(M) \subseteq \mathscr{V}$.*

(ii) *Every map $\phi\colon X \to M$ from a finitely generated module X factors through a finitely generated module M' with $V_G(M') \in \mathscr{V}$.*

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(iii) *M is a filtered colimit $\varinjlim_I F$ of finitely generated kG-modules $F(i)$ with $V_G(F(i)) \in \mathscr{V}$.*

(iv) *Given any finite subset S of M, there is a finitely generated projective module P and a finitely generated submodule M' of $M \oplus P$ such that $V_G(M') \in \mathscr{V}$, and M' contains the elements $(s,0) \in M \oplus P$ with $s \in S$.*

Proof. The equivalence of (ii), (iii) and (iv) is exactly the same as the proof of Theorems 4·2 and 5·3. So we shall prove that (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii).

First suppose that $M = \varinjlim_I F$ with $V_G(F(i)) \in \mathscr{V}$ for all i in I. If U is a finitely generated graded $H^*(G,K)$-submodule of $\mathrm{Ext}^*_{kG}(N,M)$ with N finitely generated, then by Lemma 7·1, we may replace U by $\mathrm{Im}(\phi_*)$, where $\phi: X \to M$ with X finitely generated. This does not change $\sqrt{\mathrm{Ann}(U)}$, and hence does not change $V_G(\mathrm{Ann}(U))$. Since U is Noetherian, we may choose a finitely generated submodule M_0 of M, containing $\mathrm{Im}(\phi)$, in such a way that the kernel of $\mathrm{Ext}^*_{kG}(N,X) \to \mathrm{Ext}^*_{kG}(N,M)$ is the same as the kernel of $\mathrm{Ext}^*_{kG}(N,X) \to \mathrm{Ext}^*_{kG}(N,M_0)$. The inclusion $M_0 \hookrightarrow M$ factors as $M_0 \to F(i) \to M$ for some i in I, and so

$$\mathrm{Im}(\phi_*) = \mathrm{Ext}^*_{kG}(N,X) \to \mathrm{Ext}^*_{kG}(N,F(i))).$$

Since $V_G(F(i)) \in \mathscr{V}$, we have $V_G(\mathrm{Ann}(\mathrm{Im}(\phi_*))) \in \mathscr{V}$, and so $V_G(\mathrm{Ann}(U)) \in \mathscr{V}$. Since this is true for all such U, we have $\mathscr{V}_G(M) \subseteq \mathscr{V}$.

Conversely, suppose that $\mathscr{V}_G(M) \subseteq \mathscr{V}$. We wish to show that every map $X \to M$, with X finitely generated, factors through a finitely generated module M' with $V_G(M') \in \mathscr{V}$. Let U be the image of

$$\mathrm{Ext}^*_{kG}(X,X) \to \mathrm{Ext}^*_{kG}(X,M).$$

Then $V_G(\mathrm{Ann}(U)) \in \mathscr{V}_G(M) \subseteq \mathscr{V}$. So we may choose homogenous elements

$$\zeta_1, \ldots, \zeta_t \in \mathrm{Ann}(U) \subseteq H^*(G,k)$$

so that
$$V_G\langle\zeta_1\rangle \cap \ldots \cap V_G\langle\zeta_t\rangle \in \mathscr{V}.$$

For each i, we represent ζ_i by a cocycle $\hat{\zeta}_i: \Omega^{n_i}(k) \to k$. Tensoring the triangle

$$\Omega^{n_t}(k) \xrightarrow{\hat{\zeta}_t} k \to \Omega^{-1}(L_{\zeta_t}) \to \Omega^{n_t-1}(k)$$

with X, we obtain a triangle

$$\Omega^{n_t}(X) \to X \to X \otimes \Omega^{-1}(L_{\zeta_t}) \to \Omega^{n_t-1}(X)$$

in **StMod** (kG). Since $\zeta_t \in \mathrm{Ann}(U)$, we may apply the lemma to see that $X \to M$ factors as $X \to X \otimes \Omega^{-1}(L_{\zeta_t}) \to M$. Each ζ_j still annihilates the image of

$$\mathrm{Ext}^*_{kG}(X \otimes \Omega^{-1}(L_{\zeta_t}), X \otimes \Omega^{-1}(L_{\zeta_t})) \to \mathrm{Ext}^*_{kG}(X \otimes \Omega^{-1}(L_{\zeta_t}), M).$$

So we may use the lemma repeatedly to see that $X \to M$ factors as

$$X \to X \otimes \Omega^{-1}(L_{\zeta_t}) \otimes \ldots \otimes \Omega^{-1}(L_{\zeta_t}) \to M.$$

Furthermore, $V_G(X \otimes \Omega^{-1}(L_{\zeta_1}) \otimes \ldots \otimes \Omega^{-1}(L_{\zeta_t})) \subseteq V_G\langle\zeta_1\rangle \cap \ldots \cap V_G\langle\zeta_t\rangle \in \mathscr{V}$. ∎

Definition 7·6. We define the dimension of $\mathscr{V}_G(M)$ to be the maximal dimension of an element $V \in \mathscr{V}_G(M)$.

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COROLLARY 7·7. *The complexity of a (possibly infinitely generated) kG-module M is equal to the dimension of $\mathscr{V}_G(M)$.*

Proof. This follows from the theorem together with Theorem 5·3.

8. *Further properties of the varieties*

In contrast with the finitely generated case, it is not true in general that if H is a subgroup of G then

$$\mathscr{V}_H(M\downarrow_H) = (\mathrm{res}^*_{G,H})^{-1}\mathscr{V}_G(M).$$

An example of this is given in the appendix. However, one inclusion still holds:

PROPOSITION 8·1. *If H is a subgroup of G then*

$$\mathscr{V}_H(M\downarrow_H) \subseteq (\mathrm{res}^*_{G,H})^{-1}\mathscr{V}_G(M).$$

Proof. If $U \subseteq \mathrm{Ext}^*_{kH}(N, M\downarrow_H) \cong \mathrm{Ext}^*_{kG}(N\uparrow^G, M)$ is finitely generated as an $H^*(H,k)$-module, then it is also finitely generated as an $H^*(G,k)$-module, since $H^*(H,k)$ is finitely generated as a module over the image of the restriction map. We have

$$\mathrm{Ann}_{H^*(G,k)}(U) \subseteq \mathrm{res}^{-1}_{G,H}\,\mathrm{Ann}_{H^*(H,k)}(U),$$

so that
$$V_G(\mathrm{Ann}_{H^*(G,k)}(U)) \supseteq \mathrm{res}^*_{G,H}\,V_H(\mathrm{Ann}_{H^*(H,k)}(U)).$$

This is true for all such U, which proves the proposition. ∎

It is also false in general, in contrast with the finitely generated case, that

$$\mathscr{V}_G(M_1 \otimes M_2) = \mathscr{V}_G(M_1) \cap \mathscr{V}_G(M_2).$$

An example of this is given in the appendix. Again, one inclusion still holds:

PROPOSITION 8·2. *If M_1 and M_2 are kG-modules then*

$$\mathscr{V}_G(M_1 \otimes M_2) \subseteq \mathscr{V}_G(M_1) \cap \mathscr{V}_G(M_2).$$

Proof. For the exterior tensor product we have

$$\mathscr{V}_{G\times G}(M_1 \otimes M_2) = \{V_1 \times V_2 \mid V_1 \in \mathscr{V}_G(M_1),\, V_2 \in \mathscr{V}_G(M_2)\}.$$

Now apply the previous proposition to the diagonal inclusion of G in $G \times G$.

PROPOSITION 8·3. *If $\{M_i, i \in I\}$ is a collection of kG-modules and $M = \bigoplus_{i\in I} M_i$, then $\mathscr{V}_G(M)$ is the closure under finite unions of $\bigcup_{i\in I} \mathscr{V}_G(M_i)$.*

Proof. The main point here is that if $\theta: X \to M$ is a homomorphism with X finitely generated, then $\theta(X) \subseteq \bigoplus_{i\in J} M_i$ for some finite subset $J \subseteq I$. Hence, by Lemma 7·1, any finitely generated submodule of $\mathrm{Ext}^*_{kG}(N,M)$, for N finitely generated, is contained in

$$\mathrm{Ext}^*_{kG}(N, \bigoplus_{i\in J} M_i) = \bigoplus_{i\in J} \mathrm{Ext}^*_{kG}(N, M_i)$$

for J a finite subset of I.

PROPOSITION 8·4. *Suppose that $0 \to M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \to 0$ is a short exact sequence of kG-modules. Then for $\{i,j,k\} = \{1,2,3\}$ we have*

$$\mathscr{V}_G(M_i) \subseteq \{V_j \cup V_k \mid V_j \in \mathscr{V}_G(M_j),\, V_k \in \mathscr{V}_G(M_k)\}.$$

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Proof. The proof is based on the long exact sequence for cohomology. We shall treat the case $i = 2$; the other two cases are similar. Suppose that U_2 is a finitely generated $H^*(G,k)$-submodule of $\mathrm{Ext}^*_{kG}(N,M_2)$, and I_2 is its annihilator in $H^*(G,k)$. We wish to show that $V_G(I_2) \subseteq V_1 \cup V_3$ for some $V_1 \in \mathscr{V}_G(M_1)$ and $V_3 \in \mathscr{V}_G(M_3)$. Let I_3 be the annihilator in $H^*(G,k)$ of the submodule $U_3 = \beta_*(U_2) \subseteq \mathrm{Ext}^*_{kG}(N,M_3)$. Let $U_2' = U_2 \cap \mathrm{Ker}\,(\beta_*)$. Then U_2' is finitely generated, and so $U_2' = \alpha_*(U_1)$ for some finitely generated $H^*(G,k)$-submodule $U_1 \subseteq \mathrm{Ext}^*_{kG}(N,M_1)$. Namely, we can take U_1 to be the $H^*(G,k)$-submodule generated by inverse images under α_* of a set of generators for U_2'. Let I_1 be the annihilator in $H^*(G,k)$ of U_1. Then we have an exact sequence $U_1 \to U_2 \to U_3 \to 0$ of $H^*(G,k)$-modules, and so $I_2 \supseteq I_1.I_3$. So setting $V_1 = V_G(I_1)$ and $V_3 = V_G(I_3)$, we have $V_G(I_2) \subseteq V_1 \cup V_3$ as required. ∎

It follows from Proposition 8·4 that for an object in the quotient category $\underline{\mathscr{U}_c}/\underline{\mathscr{U}_{c-1}}$, the collection of elements of $\mathscr{V}_G(M)$ with the property that all irreducible components have dimension c is well defined. We write $\mathscr{V}_G^{(c)}(M)$ for this collection. For an object in $\mathscr{M}_{c,c-1}$, $\mathscr{V}_G^{(c)}(M)$ is a finite set with a unique maximal element, which we write as $V_G^{(c)}(M)$.

Proposition 8·5. *If M is an object in $\mathscr{M}_{c,c-1}$ and $\mathscr{V}_G^{(c)}(M) = \{V_1, \ldots, V_t\}$ is a finite set, then M is isomorphic to $M_1 \oplus \ldots \oplus M_t$ in $\mathscr{M}_{c,c-1}$ with $\mathscr{V}_G^{(c)}(M_i) = \{V_i\}$ for $1 \leqslant i \leqslant t$.*

Proof. This follows directly from theorem 3·2 of Carlson and Wheeler [**11**] and the Krull–Schmidt theorem 6·7 for the category $\mathscr{M}_{c,c-1}$.

9. *The Alperin–Evens theorem*

In [**2**], Alperin and Evens proved that the complexity of a finitely generated module is equal to the maximal complexity of a restriction to an elementary abelian p-subgroup of G. In this section, we prove this without the finite generation hypothesis.

Let K be an algebraically closed field of infinite transcendence degree over k. It follows from the Noether normalization lemma that we may choose a finite set of elements $\zeta_1, \ldots, \zeta_s$ in the same degree in $H^*(G,k)$, generating a subring over which the whole of $H^*(G,k)$ is finitely generated as a module. Choose independent transcendentals $X_{ij} \in K$, with $1 \leqslant i,j \leqslant s$. Set

$$\xi_i = \sum_{j=1}^{s} X_{ij}\,\zeta_j.$$

Proposition 9·1. *Let M be a (not necessarily finitely generated) kG-module. Then the following are equivalent:*

(i) *M has complexity at most c, and*

(ii) *$L_{\xi_1} \otimes_K \ldots \otimes_K L_{\xi_c} \otimes_K (K \otimes_k M)$ is projective.*

Proof. If V is a closed homogeneous subvariety of $V_G(k)$, denote by KV the subvariety of $V_G(K)$ defined by the same equations. Then for any such $V \subseteq V_G(k)$ of dimension at most c, we have

$$V_G\langle\xi_1\rangle \cap \ldots \cap V_G\langle\xi_c\rangle \cap KV = \{0\}.$$

The proposition therefore follows from Corollary 5·6 and Corollary 7·7.

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THEOREM 9·2. *The complexity of a (not necessarily finitely generated) kG-module M is equal to the maximal complexity of a restriction of M to an elementary abelian p-subgroup E of M.*

Proof. This follows from Proposition 9·1 and Chouinard's theorem [**12**]. Chouinard proves without any finite generation hypothesis, that a kG-module is projective if and only if its restriction to every elementary abelian p-subgroup is projective.

Appendix: Examples

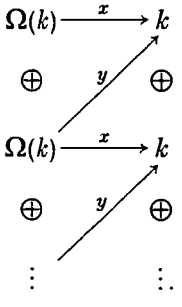
We begin with an example of the construction described in Section 3, to show how idempotents in the endomorphism ring of the trivial module in a quotient category give rise to a direct sum decomposition when we pass to infinitely generated modules.

Let $G = D_8$, the dihedral group of order eight, and k be an algebraically closed field of characteristic two. The finitely generated indecomposable modules were classified by Ringel [**18**], and we shall use his string-type and band-type schemata for describing modules. Thus $\Omega(k)$ is the string-type module $M(a^{-1}b^{-1}a^{-1}bab)$:



Here, we have used single and double lines to distinguish the actions of the two chosen elements of the group algebra squaring to zero. We look at the quotient category $\mathcal{Q}$ of the stable category of kG-modules by the subcategory of modules of complexity one. Note in particular that all band-type modules are in this subcategory.

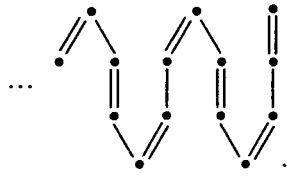
The cohomology ring is $H^*(G,k) = k[x,y,z]/(xy)$, where $\deg(x) = \deg(y) = 1$ and $\deg(z) = 2$. The element x (resp. y) is represented by the cocycle $\hat{x}$ (resp. $\hat{y}$): $\Omega(k) \to k$ sending the left-hand top basis element to 1 (resp. 0) and the right-hand one to 0 (resp. 1). So $\zeta = x+y$ is represented by a cocycle $\hat{\zeta}$: $\Omega(k) \to k$ whose kernel L_ζ is a band-type module, and hence is periodic, and so $\hat{\zeta}$ is an isomorphism in $\mathcal{Q}$. The endomorphism ring of the trivial module in $\mathcal{Q}$ is the ring of degree zero elements in the homogeneous localization of $H^*(G,k)$ given by inverting all elements outside the ideals (x) and (y) (Carlson, Donovan and Wheeler [**10**], theorem 3·5). The idempotents in this ring are $\zeta^{-1}x$ and $\zeta^{-1}y$. So to construct the summand of the trivial module k corresponding to $\zeta^{-1}x$, we form the map



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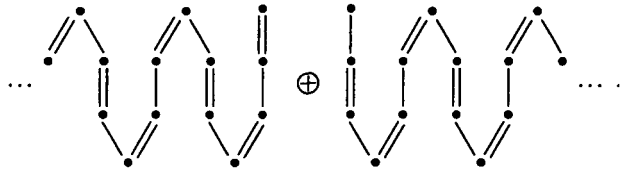
This map is surjective, so when we complete it to a triangle, the third vertex is just the kernel. This is the infinitely generated module

$$M(\dots ba^{-1}b^{-1}a^{-1}baba^{-1}b^{-1}a^{-1}bab)$$



Thus the trivial module decomposes as a direct sum

$$M(\dots ba^{-1}b^{-1}a^{-1}baba^{-1}b^{-1}a^{-1}bab) \oplus M(a^{-1}b^{-1}a^{-1}baba^{-1}b^{-1}a^{-1}baba^{-1}\dots)$$



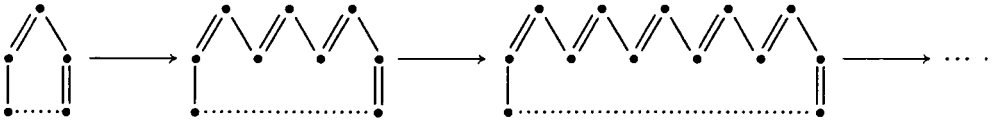
The map from this to the trivial module is the obvious one, which sends the right-hand end vertex of the left summand and the left-hand end vertex of the right summand to 1 and the rest of the basis elements to zero. The kernel is an infinite string module M_0, which has complexity one. In fact, every finitely generated submodule of M_0 is contained in a finitely generated periodic submodule.

The syzygy of the above kernel, $\Omega(M_0)$, is the module $M(\dots ba^{-1}ba^{-1}\dots)$:



This module has no finitely generated periodic submodules, despite the fact that it is the colimit of the following sequence of finitely generated periodic modules:

$$M(aba^{-1}b^{-1}, \mathrm{id}) \to M(a(ba^{-1})^3b^{-1}, \mathrm{id}) \to M(a(ba^{-1})^5b^{-1}, \mathrm{id}) \to \dots$$



The modules here are modules of band type, obtained by identifying the two end vertices joined by a dotted line. Each homomorphism has a one-dimensional kernel spanned by the basis element obtained by identifying these two end vertices. The quotient by this one-dimensional kernel is embedded into the middle of the diagram for the next module in the sequence. Each of the modules in this sequence is periodic, but the colimit has no finitely generated periodic submodules.

The same phenomenon does not appear to happen for the Klein four group in characteristic two, but does happen for all other non-cyclic abelian p-groups in characteristic p. Let

$$G = \langle x, y \,|\, [x, y] = x^{p^\alpha} = y^{p^\beta} = 1 \rangle$$

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with $p^a \geqslant 3$. Consider the directed system of modules

$$M_2 \xrightarrow{\phi_2} M_4 \xrightarrow{\phi_4} M_6 \xrightarrow{\phi_6} \dots,$$

where M_n is generated by $a_{n,1}, \dots, a_{n,n}, b_n$ with relations

$$\begin{aligned}(y-1)\,a_{n,i} &= (x-1)^{p^a-1}\alpha_{n,i+1}, \quad 1 \leqslant i \leqslant n-1,\\ (y-1)\,a_{n,n} &= (x-1)^{p^a-2}b_n,\\ (y-1)\,b_n &= 0.\end{aligned}$$

Then M_n is periodic, because $M_n \downarrow_{\langle x\rangle}$ is free. The map ϕ_n is defined by

$$\begin{aligned}\phi_n(a_{n,i}) &= a_{n+2,i} \quad 1 \leqslant i \leqslant n\\ \phi_n(b_n) &= (x-1)\,a_{n+2,n+1}.\end{aligned}$$

Let M be the direct limit of this system of modules. Then M has no periodic submodules.

Similarly, if $$G = \langle x, y, z\rangle \cong (\mathbb{Z}/2)^3$$

and $p = 2$, we consider the directed system

$$M_2 \xrightarrow{\phi_2} M_4 \xrightarrow{\phi_4} M_6 \xrightarrow{\phi_6} \dots,$$

where M_n is generated by $a_{n,1}, \dots, a_{n,n}$ with relations

$$\begin{aligned}(z+1)\,a_{n,i} &= (x+1)(y+1)\,a_{n,i+1}, \quad 1 \leqslant i \leqslant n-2,\\ (z+1)\,a_{n,n-1} &= (x+1)\,a_{n,n},\\ (z+1)\,a_{n,n} &= 0.\end{aligned}$$

Then M_n is periodic, because $M_n \downarrow_{\langle x,y\rangle}$ is free. The map ϕ_n is defined by

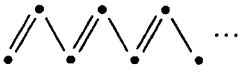
$$\begin{aligned}\phi_n(a_{n,i}) &= a_{n+2,i}, \quad 1 \leqslant i \leqslant n-1,\\ \phi_n(a_{n,n}) &= (y+1)\,a_{n+2,n}.\end{aligned}$$

Let M be the direct limit of this system of modules. Then again M has no periodic submodules.

We now turn to some examples which demonstrate the problems with the restriction and tensor product theorems (see Section 8). Let G be the Klein four group and k be an algebraically closed field of characteristic two. Again we use the notation of Ringel [**18**]. Let M_1 and M_2 be the infinite string modules

$$M_1 = M(ab^{-1}ab^{-1}ab^{-1}\dots) \quad M_2 = M(b^{-1}ab^{-1}ab^{-1}\dots)$$

$M_1 = M(ab^{-1}ab^{-1}ab^{-1}\dots)$ $\qquad$ $M_2 = M(b^{-1}ab^{-1}ab^{-1}\dots)$

The module M_2 has complexity one; indeed it is the union of finitely generated periodic submodules. The module M_1, despite the fact that it is periodic ($M_1 \cong \Omega(M_1)$), has complexity two. The set $\mathscr{V}_G(M_1)$ consists of all subsets of $V_G(k)$. However, there is a subgroup H of order two in G with the property that $M_1 \downarrow_H$ is free. So

$$\mathscr{V}_H(M_1 \downarrow_H) \neq (\mathrm{res}^*_{G,H})^{-1}\mathscr{V}_G(M_1).$$

See Proposition 8·1.

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Let L be the module $M(b)$. Then L is periodic, and is induced from a one-dimensional module for H. Since $M_1 \downarrow_H$ is free, so is $M_1 \otimes L$. So we have

$$\mathscr{V}_G(M_1 \otimes L) \neq \mathscr{V}_G(M_1) \cap \mathscr{V}_G(L).$$

See Proposition 8·2.

In some sense, this example shows that the definition of $\mathscr{V}_G(M)$ is not quite right as it stands. The variety $\mathscr{V}_G(M_1)$ in the above example should really be the collection of homogeneous subvarieties of the open subset of $V_G(k)$ given by removing the line $\operatorname{res}^*_{G,H} V_H(k)$.

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