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Embedding theorems for groups with an integer-valued length function

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(1) In this paper we are concerned with normalized integer-valued length functions on a group G, that is, mappings $l: G \to \mathbb{Z}$ satisfying three axioms:

(A 1′) $l(1) = 0$

(A 2) $l(x) = l(x^{-1})$ for all $x \in G$

(A 4) $d(x, y) > d(x, z)$ implies that $d(x, z) = d(y, z)$ for all x, y and z in G, where $d(x, y) = \frac{1}{2}(l(x) + l(y) - l(xy^{-1}))$.

These axioms were first considered by Lyndon (6), where their significance is discussed. Lyndon's axiom A 1, which we shall not use, stated that $l(x) = 0$ if and only if $x = 1$. His axiom A 3 was that $d(x, y) \geqslant 0$ for all x and y in G, but it was noted in (2) that this follows from A 1′, A 2 and A 4. In particular, taking $x = y$, we find that $l(x) \geqslant 0$ for all x in G. The other major axioms used in (6) were:

(A 0) $l(x^2) > l(x)$, provided that $x \neq 1$,

and (A 5) $d(x, y) + d(x^{-1}, y^{-1}) > l(x) = l(y)$ implies that $x = y$.

In order to prove certain embedding theorems, which we shall state shortly, Lyndon also introduced three further axioms:

(C 0) $d(x, y)$ is always an integer;

(C 1) if $l(x)$ is even, then $l(x^2) > l(x)$, unless $x = 1$.

(C 2) for no x in G is $l(x^2) = l(x) + 1$.

We shall use another axiom, which first appeared as a hypothesis in (3)

(C 3) if $l(x)$ is odd, then $l(x^2) \geqslant l(x)$.

Finally, we shall need an axiom considered by Hurley (4):

(N 1*) G is generated by $\{x \in G | l(x) \leqslant 1\}$.

When we speak of a length function on a group G, we mean an integer-valued function satisfying A 1′, A 2 and A 4. Any other axioms needed will be imposed as hypotheses. Note that, by A 2, $d(x, y) = d(y, x)$.

Lyndon proposed the idea of a length function in order to carry out the cancellation arguments of Nielsen in an abstract setting. However, in the last section of (6), it is shown that (integer-valued) length functions are not as abstract as they might appear. Specifically, the following results were obtained:

(1) *Let G be a group with length function l satisfying* A 0 *and* C 0. *Then G is a subgroup*

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of a free group F, with basis S, such that l is the restriction to G of the natural length function on F relative to S.

(2) *Let G be a group with length function l satisfying* A 5, C 1 *and* C 2. *Then G is a subgroup of a group F, which is a free product of a certain family of its subgroups, such that l is the restriction to G of the natural length function on F induced by the given free decomposition.*

In this paper we prove the following result:

(3) *Let G be a group with length function l. Then G is a subgroup of a group F which has a length function $\bar{l}$ satisfying* N 1*, *such that l is the restriction of $\bar{l}$ to G.*

Lyndon's first embedding theorem is an immediate consequence of our construction, but the second result cannot be obtained without some extra work. In (2), the author established a connexion between length functions and actions of groups on trees, and our method is to use this connexion, together with the theory of Bass and Serre concerning such actions (8). Indeed, one of the aims of this paper is to spell out in more detail the connexion between length functions and the Bass–Serre theory.

If G is an HNN-group, there is a mapping $l: G \to \mathbb{Z}$, namely $l(g)$ is the number of occurrences of stable letters and their inverses in a Britton-reduced word representing g (see (7), ch. IV, section 2). This mapping is a length function on G, a fact which can be verified directly, and which also follows from results in Hurley (4). We shall prove it later as a consequence of our main development. We refer to l as the natural length on G corresponding to the given decomposition of G as an HNN-group. In the course of proving the embedding result (3), we obtain the following.

(4) *Let G be a group with length function l satisfying* C 0 *and* C 3. *The G is a subgroup of a group F, which has a decomposition as an HNN-group, such that l is the restriction to G of the natural length on F corresponding to this decomposition.*

A summary of the constructions of Bass and Serre was given in (1), and we shall use the notation developed there, with some changes which we now describe. Given a graph of groups $(\mathscr{G}, Y)$ and a maximal tree T of Y, $\pi(\mathscr{G}, Y, T)$ is the group with presentation

$$\langle q_e, G_y | \text{rel}\, G_y,\ q_e a^e q_{\bar{e}}^{-1} = a^{\bar{e}},\ q_e q_{\bar{e}} = 1,\ q_f = 1 \rangle,$$

where e runs through $E(Y)$, y runs through $V(Y)$, f runs through $E(T)$, and for each $e \in E(Y)$, a runs through G_e. We recall that, for each $e \in E(Y)$, $\bar{e}$ is the opposite edge and $a \mapsto a^e$ is the map $G_e \to G_{t(e)}$ in the graph of groups $(\mathscr{G}, Y)$. Note that we are using the letter e to denote an edge, and y to denote a vertex. In (1), y was used to denote an edge. Also we are using q_e rather than t_e which was used in (1).

We shall also make use of the *universal group* $F(\mathscr{G}, Y)$ of a graph of groups. This is defined to be the group with presentation

$$\langle e, G_y | \text{rel}\, G_y,\ e a^e e^{-1} = a^{\bar{e}},\ e\bar{e} = 1 \rangle,$$

where e, y and a run through the same sets as before. It is important to note that $F(\mathscr{G}, Y)$ is an HNN-group, with the edges e as stable letters and their inverses, and with base the free product $*_{y \in V(Y)} G_y$.

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If $y_0 \in V(Y)$, we recall that $\pi(\mathscr{G}, Y, y_0)$ is the subgroup of $F(\mathscr{G}, Y)$ consisting of all elements which can be represented by a word in the generators of the form

$$r_0, e_1, r_1, e_2, r_2, \ldots, e_n, r_n$$

with $n \geqslant 0$, where $e_1, e_2, \ldots, e_n$ is a closed path in Y from y_0 to y_0 (trivial if $n = 0$), $r_0 \in G_{y_0}$, and, for $1 \leqslant i \leqslant n$, $r_i \in G_{t(e_i)}$. There is an obvious homomorphism α from $F(\mathscr{G}, Y)$ onto $\pi(\mathscr{G}, Y, T)$, given by sending e to q_e for $e \in E(Y)$, and g to g, where $g \in G_y$, $y \in V(Y)$.

LEMMA 1. *The homomorphism α, restricted to $\pi(\mathscr{G}, Y, y_0)$, is an isomorphism onto $\pi(\mathscr{G}, Y, T)$.*

Proof. This is proposition 20 of ch. I in (8).

Given a graph of groups $(\mathscr{G}, Y)$ and a maximal tree T of Y, one can construct a tree $\tilde{Y} = \tilde{Y}(\mathscr{G}, Y, T)$ on which $\pi = \pi(\mathscr{G}, Y, T)$ acts without inversions, in such a way that the quotient graph is Y. We give a definition of $\tilde{Y}$ which is slightly different from that in (1) and (8). The two definitions are easily seen to be equivalent, and the present one makes it clear that $\tilde{Y}$ does not depend on a choice of orientation for Y. For $e \in E(Y)$, let $\tilde{G}_e = G_{\bar{e}}$, that is, $\tilde{G}_e = \{a^{\bar{e}} | a \in G_e\}$, so that $\tilde{G}_e$ is a subgroup of $G_{o(e)}$. We define

$$V(\tilde{Y}) = \{[g, y] | g \in \pi,\ y \in V(Y)\},$$

where $[g, y]$ is the ordered pair (gG_y, y),

and

$$E(\tilde{Y}) = \{[g, e] | g \in \pi, e \in E(Y)\},$$

where $[g, e]$ is the ordered pair $(g\tilde{G}_e, e)$. We then set

$$o[g, e] = [g, o(e)], \quad t[g, e] = [gq_e, t(e)]$$

and

$$\overline{[g, e]} = [gq_e, \bar{e}]$$

for $g \in \pi$, $e \in E(Y)$. It is readily verified that $o[g, e]$, $t[g, e]$ and $\overline{[g, e]}$ are well-defined, and that these definitions make $\tilde{Y}$ into a graph.

Suppose now that G is a group acting on a tree X, without inversions, that is, $ge \neq \bar{e}$ for any $g \in G$, $e \in E(X)$. Let Y be the quotient graph, let $p: X \to Y$ be the projection map, and let T be a maximal tree of Y. Then it is possible to construct an associated graph of groups $(\mathscr{G}, Y)$, the groups G_y, for $y \in V(Y)$, being the stabilizers of certain vertices of X, and the groups G_e, for $e \in E(Y)$, are the stabilizers of certain edges of X (see (1)). The construction of $(\mathscr{G}, Y)$ is not canonical; in particular, it depends on a choice of lifting $j: Y \to X$. If x_0 is a given vertex of X, and $y_0 = p(x_0)$, it is easy to see that j can be so chosen that $j(y_0) = x_0$. Let $\tilde{Y} = \tilde{Y}(\mathscr{G}, Y, T)$. For $y \in V(Y)$, we define $\tilde{y} = [1, y]$. (This is consistent with the notation of (1), but note that we do *not* define $\tilde{e} = [1, e]$ for $e \in E(Y)$. To define $\tilde{e}$ consistently, we need an orientation of Y.) The Bass–Serre theorem ((8), ch. I, théorème 13) can now be stated as follows.

THEOREM 1. *With the above notation, there is an isomorphism $\phi: \pi(\mathscr{G}, Y, T) \to G$ and an isomorphism $\psi: \tilde{Y}(\mathscr{G}, Y, T) \to X$, such that $\phi(\tilde{y}_0) = x_0$, and $\psi(gu) = \phi(g)\psi(u)$ for all $g \in \pi(\mathscr{G}, Y, T)$ and all $u \in Y$.*

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In other words, when considering the action of G on X, we may identify $(\tilde{Y}, \tilde{y}_0)$ with (X, x_0) and $\pi(\mathscr{G}, Y, T)$ with G. To ensure that $\phi(\tilde{y}_0) = x_0$, we need only choose j as above, with $j(y_0) = x_0$.

(2) We now consider the structure of groups with a length function satisfying N 1*. This has been studied extensively by Hurley (4). In Lemma 6 below, we state a reduced form theorem for such length functions, somewhat cruder than that in (4), but which makes clear that such length functions arise from pregroups in the sense of Stallings (9).

Let G be a group with length function l. For x and y in G, define $\langle x, y\rangle$ to be $d(x, y^{-1})$.

LEMMA 2. *For all x, y and z in G, $\langle xy, z\rangle + \langle x, y\rangle = \langle x, yz\rangle + \langle y, z\rangle$.*

Proof. This is merely a restatement of lemma 1 (*c*) in (2). It can be proved by repeated application of the equation $l(ab) = l(a) + l(b) - 2\langle a, b\rangle$ to the products $(xy)z$ and $x(yz)$.

LEMMA 3. *If $\langle x, y\rangle = 0$ and $l(y) \geqslant l(z)$, then $\langle xy, z\rangle = \langle y, z\rangle$.*

Proof. Note that $d(y^{-1}, (yz)^{-1}) = \frac{1}{2}(l(y) + l(yz) - l(z)) \geqslant \frac{1}{2}l(yz)$. If $l(yz) > 0$, then $d(y^{-1}, (yz)^{-1}) > d(x, y^{-1})$, for $d(x, y^{-1}) = \langle x, y\rangle = 0$ by assumption. By A 4,

$$d(x, (yz)^{-1}) = d(x, y^{-1}) = 0.$$

If $l(yz) = 0$, then by A 2 and the remarks following Lemma 1 of (2), $d(x, (yz)^{-1}) \leqslant l(yz)$, so again $d(x, (yz)^{-1}) = 0$. Hence $\langle x, yz\rangle = \langle x, y\rangle = 0$, and the result follows from Lemma 2.

By a *word* in this context we mean a finite sequence $(u_1, u_2, \ldots, u_n)$ of elements of G, with $n \geqslant 1$ and $l(u_i) = 1$ for $1 \leqslant i \leqslant n$. Such a sequence is *reduced* if $l(u_1 \ldots u_n) = n$. By A 3, if $(u_1, \ldots, u_n)$ is a word, then $\langle u_1 \ldots u_k, u_{k+1}\rangle \geqslant 0$, that is

$$\begin{aligned} l(u_1 \ldots u_{k+1}) &\leqslant l(u_1 \ldots u_k) + l(u_{k+1}) \\ &= l(u_1 \ldots u_k) + 1, \quad \text{for} \quad 1 \leqslant k \leqslant n-1. \end{aligned}$$

Hence $(u_1, \ldots, u_n)$ is reduced if and only if $l(u_1 \ldots u_k) = k$ for $1 \leqslant k \leqslant n$.

LEMMA 4. *A word $(u_1, \ldots, u_n)$ is reduced if and only if $d(u_k, u_{k+1}^{-1}) = 0$ for*

$$1 \leqslant k \leqslant n-1.$$

Proof. By the remarks just made, $(u_1, \ldots, u_n)$ is reduced if and only if

$$l(u_1 \ldots u_{k+1}) = l(u_1 \ldots u_k) + l(u_{k+1}) \quad \text{for} \quad 1 \leqslant k \leqslant n-1,$$

that is, if and only if $\langle u_1 \ldots u_k, u_{k+1}\rangle = 0$ for $1 \leqslant k \leqslant n-1$. Using Lemma 3 and induction on k, this happens if and only if $\langle u_k, u_{k+1}\rangle = 0$ for $1 \leqslant k \leqslant n-1$, which is the desired result. For by Lemma 3, if $\langle u_1 \ldots u_k, u_{k+1}\rangle = 0$, then

$$\langle u_1 \ldots u_{k+1}, u_{k+2}\rangle = \langle u_{k+1}, u_{k+2}\rangle, \quad \text{where} \quad 1 \leqslant k \leqslant n-2.$$

We remark that Lemmas 3 and 4 are related to proposition (2·3) and lemma (6·1) in (6). If $(u_1, u_2, \ldots, u_n)$ is a word, we define the element of G which it represents to be $u_1 u_2 \ldots u_n$.

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LEMMA 5. *If l satisfies* N 1*, *then every element of G of positive length is represented by a reduced word.*

Proof. Assume $g \in G$, $l(g) > 0$. By A 2 and N 1*, we can write $g = g_1 g_2 \dots g_n$, with $g_i \in G$ and $l(g_i) \leqslant 1$ for $1 \leqslant i \leqslant n$. Choose such an expression for g with n minimal. Now if x and y belong to G and $l(x) = 0$, then $l(xy) = l(yx) = l(y)$ (See (4), lemma 3·3, corollary). Thus, if some g_i has length 0, it follows that there is a consecutive pair g_j, g_{j+1} with $l(g_j g_{j+1}) \leqslant 1$. But then $g = g_1' g_2' \dots g_{n-1}'$, where $g_1' = g_1, \dots, g_{j-1}' = g_{j-1}$, $g_j' = g_j g_{j+1}, g_{j+1}' = g_{j+2}, \dots, g_{n-1}' = g_n$, contradicting the minimality of n. Hence $l(g_i) = 1$ for $1 \leqslant i \leqslant n$, and $(g_1, g_2, \dots, g_n)$ is a word representing g. If it is not reduced, then by Lemma 4, $\langle g_j, g_{j+1} \rangle$ is greater than 0 for some j, hence $l(g_j g_{j+1}) < 2$, that is, $l(g_j g_{j+1}) \leqslant 1$, and we obtain the same contradiction as before, therefore $(g_1, \dots, g_n)$ is reduced.

We now give a criterion for two reduced words to represent the same element of G. Let $B = \{g \in G \mid l(g) = 0\}$.

LEMMA 6. *Suppose $(u_1, \dots, u_m)$ and $(v_1, \dots, v_n)$ are reduced words with*

$$u_1 \dots u_m = v_1 \dots v_n.$$

Then $m = n$, and there exist elements $b_i \in B$ for $1 \leqslant i \leqslant n-1$, such that

$$u_1 = v_1 b_1, u_2 = b_1^{-1} v_2 b_2, u_3 = b_2^{-1} v_3 b_3, \dots, u_{n-1} = b_{n-2}^{-1} v_{n-1} b_{n-1}, u_n = b_{n-1}^{-1} v_n.$$

Although a direct proof can be given, this is omitted since Lemma 6 follows easily from the results in (4). We shall make no use of it in proving our embedding theorems. It is included only to establish a connexion between length functions and pregroups.

Suppose that l satisfies N 1*. Let $P = \{x \in G \mid l(x) \leqslant 1\}$, and let

$$D = \{(x, y) \in P \times P \mid l(xy) \leqslant 1\}.$$

Then we obtain a pregroup ((9), section 3·A·1) by restricting the multiplication of G to D. The only thing which is not quite clear is that Axiom (5) for a pregroup is satisfied. Bearing in mind the corollary to lemma 3·3 in (4), it suffices to prove the following lemma. Let $X = \{x \in G \mid l(x) = 1\}$.

LEMMA 7. *Suppose that w, x, y, z are in G, and that w, z, wx, xy, yz all belong to X. Then either $wxy \in P$ or $xyz \in P$.*

Proof. Suppose on the contrary that $wxy \notin P$ and $xyz \notin P$. By A 3, $\langle w, xy \rangle \geqslant 0$, that is, $l(wxy) \leqslant l(w) + l(xy) = 2$, and since $wxy \notin P$, $l(wxy) = 2$. Hence $\langle w, xy \rangle = 0$, and similarly $\langle xy, z \rangle = 0$. By Lemma 4, (w, xy, z) is a reduced word, so $l(wxyz) = 3$. Therefore $\langle wx, yz \rangle = \frac{1}{2}(l(wx) + l(yz) - l(wxyz)) = -\frac{1}{2}$, contradicting A 3.

It follows from Lemma 4 that our use of the term 'reduced word' is consistent with its use in (9), and it follows from Lemma 6 that G is the universal group $U(P)$ of the pregroup P, as defined in (9).

We finish this section with an alternative formulation of N 1*. It follows from the definition of reduced word and A 2 that, if $(u_1, \dots, u_n)$ is a reduced word, then so is $(u_n^{-1}, u_{n-1}^{-1}, \dots, u_1^{-1})$. Also, it follows from Lemma 4 that any subword of a reduced word is also reduced.

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LEMMA 8. *Axiom* N 1* *holds if and only if*:

given $x \in G$ *with* $l(x) > 0$, *there exist elements* g_i *of* G, *for* $1 \leqslant i \leqslant l(x)$, *such that* $d(x^{-1}, g_i^{-1}) = l(g_i) = i$.

Proof. Let $n = l(x) \geqslant 1$. Suppose that there exist elements $g_i \in G$ with

$$d(x^{-1}, g_i^{-1}) = l(g_i) = i \text{ for } 1 \leqslant i \leqslant n,$$

and define $g_0 = 1$. Then for $1 \leqslant i \leqslant n$, $d(x^{-1}, g_{i-1}^{-1}) = i-1$ and $d(x^{-1}, g_i^{-1}) = i$, so by A 4, $d(g_{i-1}^{-1}, g_i^{-1}) = i-1$. Define $u_i = g_{i-1}^{-1} g_i$ for $1 \leqslant i \leqslant n$. Then

$$\begin{aligned} l(u_i) &= l(g_{i-1}) + l(g_i) - 2d(g_{i-1}^{-1}, g_i^{-1}) \\ &= (i-1) + i - 2(i-1) = 1, \quad \text{for} \quad 1 \leqslant i \leqslant n. \end{aligned}$$

Now $u_1 \ldots u_n = g_0^{-1} g_n = g_n$, so g_n is in the subgroup of G generated by P, where again $P = \{g \in G | l(g) \leqslant 1\}$. But $d(x^{-1}, g_n^{-1}) = n$, and $l(x) = l(g_n) = n$, whence

$$l(x^{-1} g_n) = l(g_n^{-1} x) = 0$$

(using A 2), so $x = g_n(g_n^{-1} x)$ also belongs to the subgroup generated by P, hence N 1* holds.

Conversely, assume N 1* holds, and again suppose $l(x) = n \geqslant 1$. By Lemma 5, x is represented by a reduced word, say $(u_1, \ldots, u_n)$. For $1 \leqslant i \leqslant n$, define $g_i = u_1 \ldots u_i$. By the remarks preceding Lemma 4, $l(g_i) = i$. Moreover, $l(x^{-1} g_i) = n - i$. For this is clear if $i = n$, and if $i < n$, $x^{-1} g_i = u_n^{-1} u_{n-1}^{-1} \ldots u_{i+1}^{-1}$, and the word $(u_n^{-1}, u_{n-1}^{-1}, \ldots, u_{i+1}^{-1})$ is reduced by the remarks preceding Lemma 8. Therefore,

$$d(x^{-1}, g_i^{-1}) = \tfrac{1}{2}(n + i - (n-i)) = i,$$

and the lemma is proved.

(3) The connexion between length functions and group actions on trees, established in (2), is as follows. Let (X, x_0) be a rooted tree (i.e. X is a tree and x_0 is a vertex of X), and let G be a group acting on X. For $g \in G$, define $l(g)$ to be the length of the reduced path from x_0 to gx_0 in X. Then l is a length function on G satisfying C 0. Conversely, given a length function l on a group G satisfying C 0, there exists a rooted tree (X, x_0), with G acting on X, such that l is obtained in the manner above. We remark that the construction in (2) defined X as a one-dimensional simplicial complex, but such a complex gives rise to a graph in the present sense. (See (8), p. 26.) Moreover, our use of the term 'without inversions' is consistent with the usage of (2). In the proof of the following lemma, we use the notation of (2).

LEMMA 9. *Let* l *be a length function on a group* G *satisfying* C 0, *and let* T *be the tree constructed in theorem* 1 *of* (2), *on which* G *acts. Then*

(1) G *acts without inversions on* T *if and only if* l *satisfies* C 3.

(2) G *acts transitively on the vertices of* T *if and only if* l *satisfies* N 1*.

Proof. (1) Assume that an element g of G inverts the edge $\{[x, m], [x, m+1]\}$ of T. Then by ((2), lemma 6), $d(x^{-1}, g) \geqslant m+1$, $l(g) = 2m+1$ and $d(x^{-1}, g^{-1}) \geqslant m+1$. In particular $l(g)$ is odd. By A 4, $d(g, g^{-1}) \geqslant m+1$, that is, $l(g) - \frac{1}{2} l(g^2) \geqslant m+1 = \frac{1}{2}(l(g)+1)$, hence $l(g) \geqslant l(g^2) + 1 > l(g^2)$, so C 3 does not hold.

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Conversely, suppose C3 is false, so there exists $g \in G$ such that $l(g)$ is odd and

$$l(g^2) < l(g).$$

We can write $l(g) = 2m+1$, where m is a non-negative integer. As in the last paragraph of the proof of theorem 3 in (2), g then inverts the edge $\{[g, m], [g, m+1]\}$ of T, so G acts with inversions on T.

(2) The action of G on vertices is transitive if and only if every vertex is in the orbit of the vertex $[1, 0]$. Since $[x, 0] = [1, 0]$ for all $x \in G$, this is equivalent to: given a vertex $[x, m]$ with $m > 0$, there exists $g \in G$ such that $g[1, 0] = [x, m]$. Since $g[1, 0] = [g, l(g)]$, this is equivalent to:

(a) Given $x \in G$ and an integer m with $1 \leqslant m \leqslant l(x)$, there exists $g \in G$ such that $l(g) = m$ and $d(x^{-1}, g^{-1}) \geqslant m$.

But if $l(g) = m$, then $d(x^{-1}, g^{-1}) \leqslant l(g^{-1}) = m$, by the remarks following lemma 1 in (2). Hence, (a) is equivalent to

(b) Given $x \in G$ and an integer m with $1 \leqslant m \leqslant l(x)$, there exists $g \in G$ such that $l(g) = m = d(x^{-1}, g^{-1})$, and the equivalence of (b) and N1* is precisely the content of Lemma 8.

COROLLARY. *Let (X, x_0) be a rooted tree, let G be a group acting on X, and let l be the corresponding length function on G. Then G acts without inversions on X if and only if l satisfies* C3.

Proof. Since l satisfies C0, we may construct the tree T of theorem 1 in (2), on which G acts. By the remarks following theorem 1 in (2), T may be viewed as a G-invariant subtree of X, containing x_0, and so if G acts without inversions on X, it does so on T, and l satisfies C3 by Lemma 9. To obtain the converse, it clearly suffices to show the following lemma.

LEMMA. *Let G be a group acting on a tree X, and let T be a G-invariant subtree of X on which G acts without inversions. Then G acts without inversions on X.*

Proof. Corresponding to the subtree T of X there is a level function $\lambda: V(X) \to \mathbb{Z}$, where $\lambda(x)$ is the length of the shortest path from x to a vertex of T (see (5), lemma 1). Since T is G-invariant, $\lambda(gx) = \lambda(x)$ for all $x \in V(X)$, all $g \in G$. Let e be an edge not in T, with endpoints x and y. Since $\lambda(x)$ and $\lambda(y)$ differ by 1, it is impossible that $gx = y$ for some $g \in G$, so no element of G inverts e. This proves the lemma, and so completes the proof of the corollary.

Now let $(\mathscr{G}, Y)$ be a graph of groups. As we have noted, the universal group $F(\mathscr{G}, Y)$ has the structure of an HNN-group, and we let l_0 be the natural length corresponding to this HNN structure. Choose a vertex y_0 of Y, and let l_1 be the restriction of l_0 to $\pi(\mathscr{G}, Y, y_0)$. Let T be a maximal tree of Y, and let $\alpha: \pi(\mathscr{G}, Y, y_0) \to \pi(\mathscr{G}, Y, T)$ be the isomorphism of Lemma 1. Let $\tilde{Y} = \tilde{Y}(\mathscr{G}, Y, T)$, so that $\pi(\mathscr{G}, Y, T)$ acts without inversions on $\tilde{Y}$, whence $\pi(\mathscr{G}, Y, y_0)$ acts without inversions on $\tilde{Y}$, via α. Let l_2 be the length function on $\pi(\mathscr{G}, Y, y_0)$ corresponding to the rooted tree $(\tilde{Y}, \tilde{y}_0)$. That is, for $g \in \pi(\mathscr{G}, Y, y_0)$, $l_2(g)$ = the length of the reduced path in $\tilde{Y}$ from $\tilde{y}_0$ to $\alpha(g)\tilde{y}_0$.

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By the corollary to Lemma 9, l_2 is a length function on $\pi(\mathscr{G}, Y, y_0)$ satisfying C0 and C3.

LEMMA 10. *The mappings l_1 and l_2 are equal.*

Proof. An element $g \in \pi(\mathscr{G}, Y, y_0)$ can be represented in the form

$$g = r_0 e_1 r_1 e_2 r_2 \dots e_n r_n,$$

with $n \geqslant 0$, where $e_1, \dots, e_n$ is a closed path in Y at y_0, $r_0 \in G_{y_0}$, and $r_i \in G_{t(e_i)}$ for $1 \leqslant i \leqslant n$. Choosing such a representation with n as small as possible, we can assume that

$$(r_0, e_1, r_1, e_2 \dots e_n, r_n)$$

is a Britton-reduced word relative to the decomposition of $F(\mathscr{G}, Y)$ as HNN-group. That is, if $e_i = \bar{e}_{i+1}$ for some i, then $r_i \notin G_{e_i}^{e_i}$. Then $l_1(g) = n$, and we must show that the reduced path from $\tilde{y}_0$ to $\alpha(g)\tilde{y}_0$ has length n. We do this by an explicit construction. The result is clear if $n = 0$, so assume $n > 0$. Let $q_i = q_{e_i}$, and define

$$u_i = [r_0 q_1 r_1 q_2 \dots q_{i-1} r_{i-1}, e_i] \quad \text{for} \quad 1 \leqslant i \leqslant n,$$

using the notation for edges of $\tilde{Y}$ in Section 1. It is easily checked that $u_1, u_2, \dots, u_n$ is a path in $\tilde{Y}$ from $\tilde{y}_0$ to $\alpha(g)\tilde{y}_0$, and we have only to show it is reduced. Suppose on the contrary that $u_i = \bar{u}_{i+1}$ for some i with $1 \leqslant i \leqslant n-1$. That is,

$$[g, e_i] = [g q_i r_i, e_{i+1}], \quad \text{where} \quad g = r_0 q_1 r_1 q_2 \dots q_{i-1} r_{i-1}.$$

Translating by g^{-1}, we obtain

$$[1, e_i] = \overline{[q_i r_i, e_{i+1}]} = [q_i r_i q_{i+1}, \bar{e}_{i+1}].$$

Explicitly, this is $(\tilde{G}_{e_i}, e_i) = (q_i r_i q_{i+1} \tilde{G}_{\bar{e}_{i+1}}, \bar{e}_{i+1})$. Hence $e_i = \bar{e}_{i+1}$, and so $q_{i+1} = q_i^{-1}$, and it follows that $q_i r_i q_i^{-1} \in \tilde{G}_{e_i} = G_{\bar{e}_i}^{\bar{e}_i}$. That is, $q_{e_i} r_i q_{e_i}^{-1} = a^{\bar{e}_i}$ for some $a \in G_{e_i}$, hence, using the defining relations in $\pi(\mathscr{G}, Y, T)$, $r_i = a^{e_i} \in G_{e_i}^{e_i}$, and the subword e_i, r_i, e_{i+1} contradicts the fact that the word $(r_0, e_1, r_1, \dots, e_n, r_n)$ is Britton-reduced.

COROLLARY. *Let $G = \langle t_i, A \mid t_i B_i t_i^{-1} = C_i \rangle$, $i \in I$, be an HNN-group with stable letters t_i, base A and associated pairs B_i, C_i. Then the natural length l on G corresponding to this decomposition is a length function on G, and satisfies* C0, C3 *and* N1*.

Proof. G can be identified with $F(\mathscr{G}, Y)$, where Y has just one vertex, with the group A associated with it, and one unoriented edge for each $i \in I$, with C_i associated. Since Y has one vertex, $G = F(\mathscr{G}, Y) = \pi(\mathscr{G}, Y, y_0)$, y_0 being the unique vertex of Y, and the natural length l on G is equal to the mapping l_2 of Lemma 10, hence l is a length function satisfying C0 and C3. If $g \in G = \pi(\mathscr{G}, Y, y_0)$ and $l(g) > 0$, write g as a Britton-reduced product

$$g = r_0 e_1 r_1 \dots e_n r_n$$

as in the proof of Lemma 10. It is clear from the definition of l that

$$l(r_0 e_1 r_1) = l(e_2 r_2) = l(e_3 r_3) = \dots = l(e_n r_n) = 1,$$

so l satisfies N1*.

We are now in a position to prove our embedding theorems.

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THEOREM 2. *Let G be a group with a length function l satisfying* C0 *and* C3. *Then G can be embedded as a retract in an HNN-group F, such that l is the restriction to G of the natural length on F. If, in addition, l satisfies* N 1*, *we may take $F = G$.*

Proof. Let (X, x_0) denote the rooted tree (T, P_0) of ((2), theorem 1) on which G acts, corresponding to the length function l. Then, for all $g \in G$, $l(g)$ is the length of the reduced path from x_0 to gx_0. By Lemma 9, G acts without inversions on X, and we can form an associated graph of groups $(\mathscr{G}, Y)$, where $Y = X/G$, and if T is a maximal tree of Y, we can by Theorem 1 identify G with $\pi(\mathscr{G}, Y, T)$, and (X, x_0) with $(\tilde{Y}, \tilde{y}_0)$, where $y_0 = p(x_0)$ and $p: X \to Y$ is the quotient map. Let $\alpha: F(\mathscr{G}, Y) \to \pi(\mathscr{G}, Y, T)$ be the homomorphism of Lemma 1, and let β be the inverse of α restricted to $\pi(\mathscr{G}, Y, y_0)$, in accordance with Lemma 1. It follows at once from Lemma 10 that β is the desired embedding (into $F(\mathscr{G}, Y)$). Further, there is a split exact sequence of groups

$$1 \to K \to F(\mathscr{G}, Y) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} G \to 1,$$

where $K = \operatorname{Ker} \alpha$, hence $\beta(G)$ is a retract of $F(\mathscr{G}, Y)$. Finally, if l satisfies N 1*, then by Lemma 9, G is transitive on the vertices of X, so Y has one vertex. But then α is an isomorphism, hence so is β.

Note that, in Theorem 2, if l satisfies A 0, then by theorem 3 of (2), G acts freely on X, so all vertex groups of $(\mathscr{G}, Y)$ are trivial. Then $F(\mathscr{G}, Y)$ is a free group, with the edges of Y as a symmetrized basis, and the natural length on $F(\mathscr{G}, Y)$ is the usual length function relative to this basis, so we obtain Lyndon's first embedding theorem.

We remark that, if l is a length function on a group G, satisfying C0 and C3, then a stronger form of C3 holds, namely

$$l(g) \text{ odd}, \quad g \in G \quad \text{implies that} \quad l(g^2) > l(g).$$

For by C3, $l(g^2) \geqslant l(g)$, and if $l(g^2) = l(g)$ then $d(g, g^{-1}) = \frac{1}{2} l(g)$ which is a half-integer, contradicting C0.

THEOREM 3. *Let G be a group with a length function l. Then there is an embedding of G into a group F, with length function $\bar{l}$, such that $\bar{l}$ satisfies* N 1*, *and $\bar{l}$ restricted to G is l.*

Proof. Let $l_1 = 2l$, so that l_1 is also a length function on G, and l_1 clearly satisfies C0 and C3. By Theorem 2, and the corollary to Lemma 10, G can be embedded in a group F_0 with length function l_0 extending l_1, such that l_0 satisfies C0, C3, and N 1*. Let $F = \{x \in F_0 \mid l_0(x) \text{ is even}\}$.

Then F is a subgroup of F_0, for $1 \in F$, and if x, y are in F, then $l_0(x) + l_0(y) - l_0(xy^{-1})$ is even by C0, hence $xy^{-1} \in F$.

For $x \in F$, define $\bar{l}(x) = \frac{1}{2} l_0(x)$. Clearly $\bar{l}$ is a length function on F, G is contained in F and $\bar{l}$ restricted to G is l. To see that $\bar{l}$ satisfies N 1*, take $x \in F$ with $l_0(x) > 0$, say $l_0(x) = 2n$, with $n > 0$. Since l_0 satisfies N 1*, x is represented by a reduced word $(u_1, u_2, \ldots, u_{2n})$ relative to l_0, by Lemma 5. Let $g_1 = u_1 u_2$, $g_2 = u_3 u_4$, $\ldots$, $g_n = u_{2n-1} u_{2n}$. By Lemma 4, $l_0(g_1) = l_0(g_2) = \ldots = l_0(g_n) = 2$, hence $g_i \in F$ and $\bar{l}(g_i) = 1$ for $1 \leqslant i \leqslant n$. Since $x = g_1 g_2 \ldots g_n$, $\bar{l}$ satisfies N 1*.

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The group F_0, obtained by the construction of Theorem 2, is $F(\mathscr{G}, Y)$, for some graph of groups $(\mathscr{G}, Y)$, and is an HNN-group. The group F defined in the proof of Theorem 3 is the kernel of the mapping from F_0 onto C_2, the cyclic group of order 2, given by sending all elements of the base group of F_0 to 1, and all stable letters to the generator of C_2.

(4) We now show that Lyndon's second embedding theorem can be obtained by the present methods. Let l be a length function on a group G satisfying A 5, C 1 and C 2. Let $l' = 2l$, so that l' is a length function on G satisfying C 0, and also A 5, C 2 and C 3. Correspondingly, we define $d'(x, y) = \frac{1}{2}(l'(x) + l'(y) - l'(xy^{-1}))$. Since l' satisfies C 0, we may construct the tree T of ((2), theorem 1), on which G acts, corresponding to l'. We change the notation, and write X instead of T. Since l' satisfies C 3, G acts without inversions on X, by Lemma 9 or its corollary, and we may form the quotient graph Y. Define a vertex $[x, m]$ of X to be *even* if m is even, and *odd* if m is odd. Since $l'(g)$ is even for all $g \in G$, the action of G on X preserves parity of vertices, so we may speak of even and odd vertices of Y. (See the definition of the action of G in (2).) Moreover, Y, like X, has a bipartite structure, with even (odd) vertices adjacent only to odd (even) vertices.

LEMMA 11. *Under these circumstances*:

(1) *if* $[x, m]$ *is even, it has trivial stabilizer for the action of* G;

(2) *if* $[x, m]$ *is odd, then* $[x, m]$ *and* $[x, m+2]$ *are in different* G*-orbits, where*

$$m + 2 \leqslant l(x);$$

(3) *if* $d'(x^{-1}, y^{-1}) = m$, *and* m *is even, then* $[x, m+1]$ *and* $[y, m+1]$ *are in different* G*-orbits (assuming, of course, that* $l(x) \geqslant m+1$ *and* $l(y) \geqslant m+1$).

Proof. (1) Assume $[x, m]$ is even. By ((2), lemma 7), an element $g \in G$ stabilizes $[x, m]$ if and only if

either (a) $d'(x^{-1}, g) \geqslant m$, $l'(g) = 2m$ and $d'(x^{-1}, g^{-1}) \geqslant m$

or (b) $d'(x^{-1}, g) < m$, $d'((gx)^{-1}, x^{-1}) \geqslant m$ and $l'(x) = l'(gx)$.

In case (a), the proof of theorem 3 in (2) shows that either

(i) $l'(g^2) = l'(g) = 2m$

or (ii) $d'(g, g^{-1}) + d'(g^{-1}, g) > l'(g) = l'(g^{-1}) = 2m$.

In the second case, since l' satisfies A 5, $g^2 = 1$, so in either case, $l'(g^2) \leqslant l'(g) = 2m$, and dividing by 2, $l(g^2) \leqslant l(g) = m$, and m is even, hence $g = 1$ since l satisfies C 1.

In case (b), again the proof of theorem 3 in (2) shows that

$$d'(gx, x) + d'((gx)^{-1}, x^{-1}) > l'(gx) = l'(x)$$

and since l' satisfies A 5, $gx = x$, so $g = 1$.

(2) Suppose $g[x, m] = [x, m+2]$ with m odd, $g \in G$. Then it is easy to see, by the methods used to prove lemma 5 in (2) that either

$$(a)\ \ d'(x^{-1}, g) < m, \quad d'((gx)^{-1}, x^{-1}) \geqslant m+2 \quad \text{and} \quad l'(g) = 2 + 2d'(x^{-1}, g)$$

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or

$$(b)\ d'(x^{-1},g) \geqslant m, \quad d'(x^{-1},g^{-1}) \geqslant m+2 \quad \text{and} \quad l'(g) = 2m+2.$$

In Case (a), after dividing by 2, we obtain

(i) $d(x^{-1},g) < m/2$, (ii) $d((gx)^{-1},x^{-1}) \geqslant m/2+1$ and (iii) $l(g) = 1+2d(x^{-1},g)$.

Now $d((gx)^{-1},g^{-1}) = l(g)-d(x^{-1},g)$ by ((2), lemma 1(a)), hence by (iii):

(iv) $d((gx)^{-1},g^{-1}) = 1+d(x^{-1},g)$.

By (i) and (iv), $d((gx)^{-1},g^{-1}) < m/2+1$, so by (ii) and A 4, $d((gx)^{-1},g^{-1}) = d(g^{-1},x^{-1})$ and by (iv), $d(g^{-1},x^{-1}) = 1+d(x^{-1},g) > d(x^{-1},g)$.

Again by A 4, $d(x^{-1},g) = d(g,g^{-1})$, and then by (iii), $l(g) = 1+2d(g,g^{-1})$, that is, $l(g^2) = 1+l(g)$, contradicting C 2.

Case (b), (i) $d(x^{-1},g) \geqslant \frac{1}{2}m$, (ii) $d(x^{-1},g^{-1}) \geqslant \frac{1}{2}m+1$, (iii) $l(g) = m+1$.

Then (i) and (ii), together with A 4, imply that $d(g,g^{-1}) \geqslant m/2$. If $d(g,g^{-1}) = m/2$, then $l(g)-\frac{1}{2}l(g^2) = m/2$, and using (iii), $l(g^2) = m+2 = l(g)+1$, contradicting C 2. If $d(g,g^{-1}) > m/2$, then similarly we obtain $l(g^2) < m+2$, so $l(g^2) \leqslant m+1 = l(g)$, contradicting C 1 since $m+1$ is even.

(3) Suppose that $d'(x^{-1},y^{-1}) = m$, $g[x,m+1] = [y,m+1]$ with m even and $g \in G$. Then again by the methods of lemma 5 in (2), either

$(a)\ d'(x^{-1},g) \geqslant m+1, \quad l'(g) = 2m+2 \quad \text{and} \quad d'(g^{-1},y^{-1}) \geqslant m+1$

or $(b)\ d'(x^{-1},g) < m+1, \quad d'((gx)^{-1},y^{-1}) \geqslant m+1 \quad \text{and} \quad l'(x) = l'(gx)$.

In Case (a), two applications of A 4 show first that $d'(x^{-1},g^{-1}) = m$, then that

$$d'(g,g^{-1}) = m.$$

Hence $l'(g)-\frac{1}{2}l'(g^2) = m$, and since $l'(g) = 2m+2$, this implies that

$$l'(g^2) = 2m+4 = l'(g)+2,$$

and dividing by 2, $l(g^2) = l(g)+1$, contradicting C 2.

In Case (b), we obtain by A 4 that $d'((gx)^{-1},x^{-1}) = m$. Hence the following conditions are satisfied:

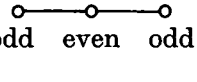
$$d'(x^{-1},g) \leqslant m, \quad d'((gx)^{-1},x^{-1}) \geqslant m, \quad \text{and} \quad l'(x) = l'(gx)$$

(the first condition holds because l' satisfies C 0).

It follows from ((2), lemma 7), taking account of the remark just before lemma 3 in (2) to dispose of the case $d'(x^{-1},g) = m$, that g stabilizes $[x,m]$. Since m is even, $g = 1$ by Part (1) of the lemma, hence $[x,m+1] = [y,m+1]$, contradicting $d'(x^{-1},y^{-1}) = m$.

COROLLARY. *The quotient graph Y contains no subgraph* , *where the lines depict unoriented edges.*

Proof. If it did, one of the vertices would be even, the other odd. Now the quotient map $X \to Y$ is locally onto, and applying this fact to the even vertex we obtain a subgraph of X of the form



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where the two odd vertices are in the same G-orbit. This, however, is ruled out by Parts (2) and (3) of Lemma 11.

Now let $p: X \to Y$ be the quotient map (recall that $Y = X/G$), and let $y_0 = p([1, 0])$. Form an associated graph of groups $(\mathscr{G}, Y)$ for the action of G on X, and let

$$F = F(\mathscr{G}, Y).$$

As in the proof of Theorem 2, G may be identified with $\pi(\mathscr{G}, Y, y_0)$, in such a way that l' is the restriction to G of the natural length on the HNN-group F. Since l' satisfies A 5, by ((2), theorem 2) the stabilizers of all edges of X for the action of G are trivial; therefore, if A is an orientation of Y,

$$F = \left(\underset{e \in A}{*} \langle e \rangle \right) * \left(\underset{y \in V(Y)}{*} G_y \right)$$

where $\langle e \rangle$ denotes the infinite cyclic subgroup of F generated by the stable letter e. Let

$$H_y = G_y * \left(\underset{e \in A,\ t(e) = y}{*} \langle e \rangle \right)$$

so that $F = \underset{y \in V(Y)}{*} H_y$. Lyndon's second embedding theorem follows at once from:

LEMMA 12. *The length function l is the restriction to G of the natural length function on F corresponding to the free product decomposition $F = \underset{y \in V(Y)}{*} H_y$.*

Proof. If g is an element of $G = \pi(\mathscr{G}, Y, y_0)$ with $l(g) = n$, so that $l'(g) = 2n$, then as in the proof of Lemma 10, g can be represented by a Britton-reduced word

$$g = r_0 e_1 r_1 e_2 \dots e_{2n} r_{2n},$$

where $e_1, \dots, e_{2n}$ is a closed path in Y from y_0 to y_0, $n \geqslant 0$, $r_0 \in G_{y_0}$, and for $1 \leqslant i \leqslant 2n$, $r_i \in G_{t(e_i)}$. Call this word w.

Now $y_0 = p([1, 0])$, so y_0 is even, hence so are $t(e_2), t(e_4), \dots, t(e_{2n}) = y_0$, and by Part (1) of Lemma 11, $r_0 = r_2 = r_4 = \dots = r_{2n} = 1$. Thus, if $n = 0$, $g = r_0 = 1$, and we can assume $n > 0$, so that

$$g = (e_1 r_1 e_2)(e_3 r_3 e_4) \dots (e_{2n-1} r_{2n-1} e_{2n}).$$

Define g_i to be $e_{2i-1} r_{2i-1} e_{2i}$ for $1 \leqslant i \leqslant n$. Then $g_i \in H_{t(e_{2i-1})}$ (for in F, $e_{2i} = (\bar{e}_{2i})^{-1}$, and $t(\bar{e}_{2i}) = t(e_{2i-1})$). Moreover, no g_i is equal to 1, because the word w is Britton-reduced. Now if g_i and g_{i+1} belong to the same free factor H_y of F, we must have

$$y = t(e_{2i-1}) = t(e_{2i+1}), \quad \text{whence} \quad o(e_{2i}) = t(e_{2i+1}) \quad \text{and} \quad t(e_{2i}) = o(e_{2i+1})$$

(bearing in mind that $e_1, e_2, \dots, e_{2n}$ is a path). By the corollary to Lemma 11, this implies that $e_{2i} = \bar{e}_{2i+1}$. But then the subword $(e_{2i}, r_{2i}, e_{2i+1}) = (e_{2i}, 1, e_{2i+1})$ of w contradicts the fact that w is Britton-reduced.

Hence, $(g_1, g_2, \dots, g_n)$ is a reduced word relative to the free product decomposition $F = \underset{y \in V(Y)}{*} H_y$, and since $g = g_1 g_2 \dots g_n$ and $l(g) = n$, this proves Lemma 12.

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