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On periodic points of λ-graph systems

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Abstract. In a previous paper (Presentations of subshifts and their topological conjugacy invariants. *Doc. Math.* **4** (1999), 285–340), the notion of λ-graph system has been introduced. The λ-graph systems are generalizations of finite directed labeled graphs. In this paper, we study periodic points of λ-graph systems. We introduce some invariants for a λ-graph system $\mathfrak{L}$ to count the cardinal number of p-periodic points of $\mathfrak{L}$. They are invariant under strong shift equivalence of λ-graph systems. We then consider the zeta functions of λ-graph systems, which are also invariant under strong shift equivalence of λ-graph systems. Some examples are also presented.

1. *Introduction*

Let Σ be a finite set with discrete topology, called an alphabet. Let $\Sigma^{\mathbb{Z}}$ be the set of all bi-infinite sequences of Σ. It is a compact Hausdorff space by the product topology and it has a natural homeomorphism σ defined by left-shift: $\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$, $(x_i)_{i\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$. Let Λ be a closed σ-invariant subset, i.e. $\sigma(\Lambda) = \Lambda$ of $\Sigma^{\mathbb{Z}}$. The restriction of σ to Λ is denoted by σ_Λ. The topological dynamical system $(\Lambda, \sigma_\Lambda)$ is called a subshift and for reasons of brevity we denote it by Λ. Throughout this paper we denote by $\mathbb{Z}_+$ and $\mathbb{N}$ the set of all non-negative integers and the set of all positive integers, respectively. In [**Ma1**], the notion of a λ-graph system has been introduced as a generalization of finite directed labeled graphs. A λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ is a labeled directed Bratteli diagram [**Br**] with an additional structure. Both the vertex set V and the edge set E of $\mathfrak{L}$ are disjoint unions

$$V = \bigcup_{n\in\mathbb{Z}_+} V_l, \quad E = \bigcup_{l\in\mathbb{Z}_+} E_{l,l+1}$$

of sequences of finite sets V_l and $E_{l,l+1}, l \in \mathbb{Z}_+$, respectively. All edges in $E_{l,l+1}$ start at some vertex of V_l and end at some vertex of $V_{l+1}, l \in \mathbb{Z}_+$, which are labeled by the map $\lambda : E \longrightarrow \Sigma$. Here it is required that every vertex has out-going edges and that every

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vertex except the vertices in V_0 has in-coming edges. The additional structure that turns the labeled directed Bratteli diagram into a λ-graph system is a map

$$\iota : \bigcup_{l \in \mathbb{N}} V_l \to V$$

such that

$$\iota(V_{l+1}) = V_l, \quad l \in \mathbb{Z}_+.$$

The map l is compatible with the labeling, in the sense that if u is the initial vertex of an edge with label α and final vertex v, then $\iota(u)$ is the initial vertex of an edge with label α and final vertex $\iota(v)$.

A symbolic matrix system over Σ consists of two sequences of rectangular matrices $(\mathcal{M}_{l,l+1}, I_{l,l+1}), l \in \mathbb{Z}_+$. The matrices $\mathcal{M}_{l,l+1}$ have their entries in formal sums of Σ and the matrices $I_{l,l+1}$ have their entries in $\{0, 1\}$. They satisfy the commutation relations

$$I_{l,l+1}\mathcal{M}_{l+1,l+2} = \mathcal{M}_{l,l+1}I_{l+1,l+2}, \quad l \in \mathbb{Z}_+.$$

The matrices $I_{l,l+1}, l \in \mathbb{Z}_+$ have one 1 in each column and at least one 1 in each row. We denote by $(\mathcal{M}, I)$ the symbolic matrix system $(\mathcal{M}_{l,l+1}, I_{l,l+1}), l \in \mathbb{Z}_+$. A λ-graph system naturally arises from a symbolic matrix system $(\mathcal{M}, I)$. The edges from a vertex $v_j^{l+1} \in V_{l+1}$ to a vertex $v_i^l \in V_l$ are given by the (i, j)-component $\mathcal{M}_{l,l+1}(i, j)$ of $\mathcal{M}_{l,l+1}$. The matrix $I_{l,l+1}$ defines a surjection ι from V_{l+1} to V_l for each $l \in \mathbb{Z}_+$. This sets up a one-to-one correspondence between λ-graph systems and symbolic matrix systems.

Every λ-graph system $\mathfrak{L}$ gives rise to a subshift $\Lambda_{\mathfrak{L}}$ by gathering the all label sequences appearing in the labeled Bratteli diagram. Conversely there is a canonical method to construct a λ-graph system from an arbitrary subshift. The constructed λ-graph system is called the canonical λ-graph system for the subshift. Strong shift equivalence between two symbolic matrix systems has been formulated in [**Ma1**] as a generalization of strong shift equivalence for non-negative matrices [**Wi**] and strong shift equivalence for symbolic matrices [**N**]. Two λ-graph systems are said to be strong shift equivalent if the corresponding symbolic matrix systems are strong shift equivalent. Strong shift equivalent λ-graph systems give rise to conjugate subshifts. In [**Ma1**], it was proved that two subshifts are topologically conjugate if and only if their canonical λ-graph systems are strong shift equivalent. Hence the strong shift equivalence of λ-graph systems is a crucial equivalence relation for studying the topological conjugacy of subshifts. We note that strong shift equivalence λ-graph systems give rise to outer conjugate gauge actions on the associated C^*-algebras [**Ma3**].

The sequence of numbers of periodic points of a subshift is a crucial topological conjugacy invariant. They have a large amount of information with regards to the conjugacy classes of subshifts rather than the topological entropy. In this paper, we study periodic points of λ-graph systems. We will first construct a topological dynamical system $(Z_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$ of a compact Hausdorff space $Z_{\mathfrak{L}}$ with a homeomorphism $\sigma_{\mathfrak{L}}$ from a λ-graph system $\mathfrak{L}$. It comes from a topological Markov chain of the Shannon graph associated with $\mathfrak{L}$ (cf. [**KM1**]), and is regarded as a cover of the presented subshift $(\Lambda_{\mathfrak{L}}, \sigma_{\Lambda_{\mathfrak{L}}})$. We then introduce two directed sequences $\triangle_{\mathfrak{L}}(p), \widehat{\triangle}_{\mathfrak{L}}(p), p \in \mathbb{N}$ of abelian groups. The former $\triangle_{\mathfrak{L}}(p), p \in \mathbb{N}$ are the K_0-groups of the space of the p-periodic points of the

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dynamical system $(Z_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$, and the latter $\widehat{\triangle}_{\mathfrak{L}}(p)$, $p \in \mathbb{N}$ are the quotient groups of the former by the K_0-groups of the space of the p-periodic points of the presented subshift $(\Lambda_{\mathfrak{L}}, \sigma_{\Lambda_{\mathfrak{L}}})$. The pair $\triangle_{\mathfrak{L}}(p), \widehat{\triangle}_{\mathfrak{L}}(p)$, $p \in \mathbb{N}$ with the natural homomorphisms $\iota^{\triangle}_{q,p} : \triangle_{\mathfrak{L}}(q) \to \triangle_{\mathfrak{L}}(p)$ and $\iota^{\widehat{\triangle}}_{q,p} : \widehat{\triangle}_{\mathfrak{L}}(q) \to \widehat{\triangle}_{\mathfrak{L}}(p)$ for $q|p$ are called the periodic dimension group for $\mathfrak{L}$ and is written as $\mathfrak{P}_{\mathfrak{L}}$. We will then prove that it is invariant under strong shift equivalence of λ-graph systems. The periodic dimension groups for λ-graph systems give us information for the cardinal numbers of periodic points both for the λ-graph systems and the presented subshifts. We will next consider two kinds of zeta functions $\zeta_{\mathfrak{L}}(z)$ and $\zeta^{\infty}_{\mathfrak{L}}(z)$ for the λ-graph system $\mathfrak{L}$. One is the zeta function $\zeta_{\mathfrak{L}}(z)$ of Artin–Mazur–Ruelle-type defined by the cardinal numbers of finite periodic points of the dynamical system $(Z_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$ (cf. [**AM, Ru**]). The other one is the zeta function $\zeta^{\infty}_{\mathfrak{L}}(z)$ defined by the cardinal numbers of periodic words with infinite periodic points of the dynamical system $(Z_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$. The former zeta function $\zeta_{\mathfrak{L}}(z)$ is a natural generalization of the ordinary zeta function of the topological Markov shift. The latter one $\zeta^{\infty}_{\mathfrak{L}}(z)$ is an obstruction such that the λ-graph system $\mathfrak{L}$ has the same periodic structure as the presented subshift. Both of the zeta functions are invariant under strong shift equivalence of λ-graph systems. The invariants presented in this paper for a λ-graph system $\mathfrak{L}$, the periodic dimension group $\mathfrak{P}_{\mathfrak{L}}$ and the zeta functions $\zeta_{\mathfrak{L}}(z)$, $\zeta^{\infty}_{\mathfrak{L}}(z)$ are easier to compute than the invariants defined in [**Ma1**], the dimension triple, the K-groups and the Bowen–Franks groups. In §6 some examples will be presented. In particular, we see that for the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_2)}$ of the Dyck shift D_2, the product formula

$$\zeta_{D_2}(z) = \zeta_{\mathfrak{L}^{Ch(D_2)}}(z) \cdot \zeta^{\infty}_{\mathfrak{L}^{Ch(D_2)}}(z)$$

holds, where $\zeta_{D_2}(z)$ means the ordinary zeta function of the Dyck shift.

2. *Dynamical systems constructed from λ-graph systems*

We say that a λ-graph system is left-resolving if for every symbol $\alpha \in \Sigma$ every vertex has at most one in-coming edge labeled α. In what follows, we assume that a λ-graph system is left-resolving. In [**Ma2**], Deaconu's continuous graphs [**De1, De2**] have been constructed from λ-graph systems. We will briefly review the construction. Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a λ-graph system over Σ. For $u \in V_{l-1}$ and $v \in V_{l+1}$, we put

$$E^{\iota}(u, v) = \{e \in E_{l,l+1} \mid t(e) = v, \iota(s(e)) = u\},$$
$$E_{\iota}(u, v) = \{e \in E_{l-1,l} \mid s(e) = u, t(e) = \iota(v)\}.$$

Then there exists a bijective correspondence between $E^{\iota}(u, v)$ and $E_{\iota}(u, v)$ that preserves labels for each pair u, v of vertices. We call this property the local property of $\mathfrak{L}$. Let $\Omega_{\mathfrak{L}}$ be the projective limit of the system $\iota_{l,l+1}(= \iota|_{V_{l+1}}) : V_{l+1} \to V_l, l \in \mathbb{Z}_+$, which is defined by

$$\Omega_{\mathfrak{L}} = \left\{ (v^l)_{l \in \mathbb{Z}_+} \in \prod_{l \in \mathbb{Z}_+} V_l \;\middle|\; \iota_{l,l+1}(v^{l+1}) = v^l, l \in \mathbb{Z}_+ \right\}.$$

We endow $\Omega_{\mathfrak{L}}$ with the projective limit topology so that it is a compact Hausdorff space. An element v in $\Omega_{\mathfrak{L}}$ is called an ι-orbit or also a vertex. Let $E_{\mathfrak{L}}$ be the set of all triplets

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$(u, \alpha, v) \in \Omega_{\mathfrak{L}} \times \Sigma \times \Omega_{\mathfrak{L}}$ such that for each $l \in \mathbb{Z}_+$, there exists $e_{l,l+1} \in E_{l,l+1}$ satisfying

$$u^l = s(e_{l,l+1}), \quad v^{l+1} = t(e_{l,l+1}), \quad \alpha = \lambda(e_{l,l+1}),$$

where $u = (u^l)_{l\in\mathbb{Z}_+}$, $v = (v^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$. Then the set $E_{\mathfrak{L}} \subset \Omega_{\mathfrak{L}} \times \Sigma \times \Omega_{\mathfrak{L}}$ is a topological Markov chain [**Kr4**] and also [**Ma2**, Proposition 2.1] called a continuous graph in the sense of Deaconu [**De1, De2**]. It is also a left-resolving Shannon graph [**KM1**].

Consider the set $Z_{\mathfrak{L}}$ of all two-sided sequences of paths of $E_{\mathfrak{L}}$,

$$Z_{\mathfrak{L}} = \left\{ (\alpha_i, u_i)_{i\in\mathbb{Z}} \in \prod_{i\in\mathbb{Z}} (\Sigma \times \Omega_{\mathfrak{L}}) \;\middle|\; (u_{i-1}, \alpha_i, u_i) \in E_{\mathfrak{L}} \text{ for all } i \in \mathbb{Z} \right\}.$$

The set $Z_{\mathfrak{L}}$ has the relative topology from the infinite product topology of $\prod_{i\in\mathbb{Z}}(\Sigma \times \Omega_{\mathfrak{L}})$ and is a zero-dimensional compact Hausdorff space. The shift map $\sigma_{\mathfrak{L}} : (\alpha_i, u_i)_{i\in\mathbb{Z}} \in Z_{\mathfrak{L}} \to (\alpha_{i+1}, u_{i+1})_{i\in\mathbb{Z}} \in Z_{\mathfrak{L}}$ is a homeomorphism and hence we have a topological dynamical system $(Z_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$. We note that it is not generally expansive for a metric defining the topology of $Z_{\mathfrak{L}}$. Let $\Lambda_{\mathfrak{L}}$ be the subshift presented by $\mathfrak{L}$ and $\sigma_{\Lambda_{\mathfrak{L}}}$ the left-shift on $\Lambda_{\mathfrak{L}}$. For $k \in \mathbb{N}$, we denote by $B_k(\Lambda_{\mathfrak{L}})$ the set of admissible words of length k in $\Lambda_{\mathfrak{L}}$.

We will now provide some basic lemmas on λ-graph systems.

LEMMA 2.1. *For an ι-orbit $\omega = (\omega^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$:*

(i) *there exist a symbol $\alpha \in \Sigma$ and an ι-orbit $u = (u^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$ such that $(\omega, \alpha, u) \in E_{\mathfrak{L}}$, and*

(ii) *there exist a symbol $\beta \in \Sigma$ and an ι-orbit $v = (v^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$ such that $(v, \beta, \omega) \in E_{\mathfrak{L}}$.*

Proof. (i) For an ι-orbit $\omega = (\omega^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$, put

$$S_\alpha(\omega) = \{e \in E \mid \exists l \in \mathbb{Z}_+; s(e) = \omega^l, \lambda(e) = \alpha\}, \quad \alpha \in \Sigma.$$

As the cardinality of the set $\bigcup_{\alpha\in\Sigma} S_\alpha(\omega)$ is infinite, we may find a symbol $\alpha_\circ \in \Sigma$ such that the cardinality of $S_{\alpha_\circ}(\omega)$ is infinite. For $n \in \mathbb{N}$ put

$$F_n(\omega, \alpha_\circ) = \{u \in \Omega_{\mathfrak{L}} \mid \exists e_n \in E_{n-1,n}; \lambda(e_n) = \alpha_\circ, s(e_n) = \omega^{n-1}, t(e_n) = u^n\}.$$

By the local property of $\mathfrak{L}$, a vertex $u \in \Omega_{\mathfrak{L}}$ belongs to $F_n(\omega, \alpha_\circ)$ if and only if, for every $k = 1, 2, \ldots, n$, there exists $e_k \in E_{k-1,k}$ such that $\lambda(e_k) = \alpha_\circ$, $s(e_k) = \omega^{k-1}$, $t(e_k) = u^k$. Since $F_n(\omega, \alpha_\circ)$ is closed in $\Omega_{\mathfrak{L}}$ for $n \in \mathbb{N}$ and the family $F_n(\omega, \alpha_\circ)$, $n \in \mathbb{N}$ has finite intersection property, we may find $u \in \bigcap_{n\in\mathbb{N}} F_n(\omega, \alpha_\circ)$ so that $(\omega, \alpha_\circ, u) \in E_{\mathfrak{L}}$.

(ii) We similarly set

$$S^\beta(\omega) = \{e \in E \mid \exists l \in \mathbb{Z}_+; t(e) = \omega^l, \lambda(e) = \beta\}, \quad \beta \in \Sigma$$

so that the cardinality of $S^{\beta_\circ}(\omega)$ is infinite for some $\beta_\circ \in \Sigma$. For $n \in \mathbb{N}$ put

$$F_n(\beta_\circ, \omega) = \{v \in \Omega_{\mathfrak{L}} \mid \exists e_n \in E_{n-1,n}; \lambda(e_n) = \beta_\circ, t(e_n) = \omega^n, s(e_n) = v^{n-1}\}.$$

Similarly to (i), we may find $v \in \bigcap_{n\in\mathbb{N}} F_n(\beta_\circ, \omega)$ so that $(v, \beta_\circ, \omega) \in E_{\mathfrak{L}}$. □

LEMMA 2.2. *For a symbol $\alpha \in B_1(\Lambda_{\mathfrak{L}})$ and an ι-orbit $u = (u^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$ with $t(e_0) = u_1, \lambda(e_0) = \alpha$ for some $e_0 \in E_{0,1}$ there exist an ι-orbit $\omega = (\omega^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$ and a sequence of edges $e_l \in E_{l,l+1}$, $l \in \mathbb{N}$, such that $s(e_l) = \omega^l$, $t(e_l) = u^{l+1}$, $\lambda(e_l) = \alpha$. Hence $(\omega, \alpha, u) \in E_{\mathfrak{L}}$.*

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Proof. By the local property of $\mathfrak{L}$, we may find a vertex $\omega^1 \in V_1$ and an edge $e_1 \in E_{1,2}$ such that $\iota(\omega^1) = \omega^0, s(e_1) = \omega^1, t(e_1) = u^2, \lambda(e_1) = \alpha$. By induction, the assertion holds. □

LEMMA 2.3. *For $\mu = (\mu_1, \ldots, \mu_k) \in B_k(\Lambda_{\mathfrak{L}})$, there exist ι-orbits $u_i = (u_i^l)_{l\in\mathbb{Z}_+}$, $i = 0, 1, \ldots, k$, such that $(u_{i-1}, \mu_i, u_i) \in E_{\mathfrak{L}}, i = 1, \ldots, k$.*

Proof. For $(\mu_1, \ldots, \mu_k) \in B_k(\Lambda_{\mathfrak{L}})$, take $e_i \in E_{i-1,i}, i = 1, \ldots, k$, such that $t(e_i) = s(e_{i+1}), \lambda(e_i) = \mu_i$. Take $u_k = (u_k^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$ such that $t(e_k) = u_k^k$. By the preceding lemma, for the edge e_k and the ι-orbit u_k, we may take $u_{k-1} \in \Omega_{\mathfrak{L}}$ such that $(u_{k-1}, \mu_k, u_k) \in E_{\mathfrak{L}}$. One sees the assertion by repeating this procedure. □

LEMMA 2.4. *The map $\pi_{\mathfrak{L}} : Z_{\mathfrak{L}} \longrightarrow \Lambda_{\mathfrak{L}}$ defined by $\pi_{\mathfrak{L}}((\alpha_i, u_i)_{i\in\mathbb{Z}}) = (\alpha_i)_{i\in\mathbb{Z}}$ is continuous and surjective.*

Proof. It suffices to show its surjectivity. For $(\alpha_i)_{i\in\mathbb{Z}} \in \Lambda_{\mathfrak{L}}$ and $n \in \mathbb{N}$, put

$$Z_{\mathfrak{L}}((\alpha_i)_{i=-n}^{n}) = \{(\beta_i, u_i)_{i\in\mathbb{Z}} \in Z_{\mathfrak{L}} \mid \beta_i = \alpha_i, i = 0, \pm 1, \ldots, \pm n\}.$$

By Lemmas 2.1 and 2.3 the above set is not empty. As it is closed and the family $Z_{\mathfrak{L}}((\alpha_i)_{i=-n}^{n}), n \in \mathbb{N}$ has finite intersection property, we may find $x \in \bigcap_{i\in\mathbb{N}} Z_{\mathfrak{L}}((\alpha_i)_{i=-n}^{n})$ that satisfies $\pi_{\mathfrak{L}}(x) = (\alpha_i)_{i\in\mathbb{Z}}$. □

Let $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ be two symbolic matrix systems over the alphabets Σ and Σ', respectively. Let $m(l)$ and $m'(l)$ be the sequences for which $\mathcal{M}_{l,l+1}, I_{l,l+1}$ are $m(l) \times m(l+1)$ matrices and $\mathcal{M}'_{l,l+1}, I'_{l,l+1}$ are $m'(l) \times m'(l+1)$ matrices, respectively. Recall that $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are said to be strong shift equivalent in 1-step [**Ma1**] if there exist alphabets C, D and specifications $\kappa : \Sigma \to C \cdot D$, $\kappa' : \Sigma' \to D \cdot C$ such that, for each $l \in \mathbb{N}$, there exist an $m(l-1) \times m'(l)$ matrix $\mathcal{H}_l$ over C and an $m'(l-1) \times m(l)$ matrix $\mathcal{K}_l$ over D satisfying the following equations:

$$I_{l-1,l}\mathcal{M}_{l,l+1} \overset{\kappa}{\simeq} \mathcal{H}_l\mathcal{K}_{l+1}, \quad I'_{l-1,l}\mathcal{M}'_{l,l+1} \overset{\kappa'}{\simeq} \mathcal{K}_l\mathcal{H}_{l+1} \tag{2.1}$$

and

$$\mathcal{H}_l I'_{l,l+1} = I_{l-1,l}\mathcal{H}_{l+1}, \quad \mathcal{K}_l I_{l,l+1} = I'_{l-1,l}\mathcal{K}_{l+1}. \tag{2.2}$$

Here a specification $\kappa : \Sigma_1 \to \Sigma_2$ between finite alphabets Σ_1, Σ_2 means a bijection from a subset of Σ_1 onto a subset of Σ_2, and the notation $\mathcal{M}_1 \overset{\kappa}{\simeq} \mathcal{M}_2$ for symbolic matrices $\mathcal{M}_1$ over Σ_1 and $\mathcal{M}_2$ over Σ_2 and a specification $\kappa : \Sigma_1 \to \Sigma_2$ means the equality $\kappa(\mathcal{M}_1(i, j)) = \mathcal{M}_2(i, j)$ for each component. We write this situation as $(\mathcal{M}, I) \underset{1-st}{\approx} (\mathcal{M}', I')$. Two symbolic matrix systems $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are said to be strong shift equivalent if there exists a finite sequence of symbolic matrix systems $(\mathcal{M}^{(i)}, I^{(i)}), i = 1, 2, \ldots, N-1$, such that

$$(\mathcal{M}, I) \underset{1-st}{\approx} (\mathcal{M}^{(1)}, I^{(1)}) \underset{1-st}{\approx} \cdots \underset{1-st}{\approx} (\mathcal{M}^{(N-1)}, I^{(N-1)}) \underset{1-st}{\approx} (\mathcal{M}', I').$$

Two λ-graph systems $\mathfrak{L}$ and $\mathfrak{L}'$ are said to be strong shift equivalent if their respective symbolic matrix systems $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are strong shift equivalent.

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THEOREM 2.5. *Let $\mathfrak{L}$ and $\mathfrak{L}'$ be λ-graph systems. If they are strong shift equivalent, there exist homeomorphisms $\Phi : Z_{\mathfrak{L}} \longrightarrow Z_{\mathfrak{L}'}$ and $\phi : \Lambda_{\mathfrak{L}} \longrightarrow \Lambda_{\mathfrak{L}'}$ such that*

$$\Phi \circ \sigma_{\mathfrak{L}} = \sigma_{\mathfrak{L}'} \circ \Phi, \quad \phi \circ \sigma_{\Lambda_{\mathfrak{L}}} = \sigma_{\Lambda_{\mathfrak{L}'}} \circ \phi, \quad \phi \circ \pi_{\mathfrak{L}} = \pi_{\mathfrak{L}'} \circ \Phi.$$

Hence the conjugacy class of the factor map

$$\pi_{\mathfrak{L}} : (Z_{\mathfrak{L}}, \sigma_{\mathfrak{L}}) \longrightarrow (\Lambda_{\mathfrak{L}}, \sigma_{\Lambda_{\mathfrak{L}}})$$

is invariant under the strong shift equivalence of λ-graph systems.

Proof. Let $\mathfrak{L} = (V, E, \lambda, \iota)$ and $\mathfrak{L}' = (V', E', \lambda', \iota')$ be λ-graph systems over Σ and Σ', respectively. We may assume that their respect symbolic matrix systems $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are strong shift equivalent in 1-step. For $(\alpha_i, u_i)_{i\in\mathbb{Z}} \in Z_{\mathfrak{L}}$, let $u_i = (u_i^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$, $i \in \mathbb{Z}$. We arbitrarily fix $i \in \mathbb{Z}$ and $l \in \mathbb{Z}_+$. Take $e^i_{l,l+1} \in E_{l,l+1}$ such that $s(e^i_{l,l+1}) = u^l_{i-1}, t(e^i_{l,l+1}) = u^{l+1}_i, \lambda(e^i_{l,l+1}) = \alpha_i$. By the first equality of (2.1), the edge $e^i_{l,l+1}$ and the vertex $u^{l-1}_{i-1} = \iota(u^l_{i-1})$ determine two edges $f^{i-1}_{l-1,l}, g^i_{l,l+l}$ such that

$$\begin{aligned} s(f^{i-1}_{l-1,l}) = u^{l-1}_{i-1}, \quad t(f^{i-1}_{l-1,l}) = u'^{l}_{i-1}, \quad \lambda_{\mathcal{H}}(f^{i-1}_{l-1,l}) = c_{\alpha_i}, \\ s(g^i_{l,l+l}) = u'^{l}_{i-1}, \quad t(g^i_{l,l+1}) = u^{l+1}_i, \quad \lambda_{\mathcal{K}}(g^i_{l,l+1}) = d_{\alpha_i} \end{aligned}$$

for some vertex $u'^{l}_{i-1} \in V'^{l}$ and symbols $c_{\alpha_i} \in C, d_{\alpha_i} \in D$, where $\lambda_{\mathcal{H}}, \lambda_{\mathcal{K}}$ denote the labelings in the symbolic matrices $\mathcal{H}_l, \mathcal{K}_{l+1}$, respectively. As $l \in \mathbb{Z}_+$ is arbitrary, we have a sequence $u'^{l}_{i-1} \in V'^{l}, l \in \mathbb{Z}_+$ of vertices in $\mathfrak{L}'$. By the second equality of (2.2), one sees that

$$\iota'(u'^{l}_{i-1}) = u'^{l-1}_{i-1}, \quad l \in \mathbb{N}.$$

This implies that $u'^{l}_{i-1} \in V'^{l}, l \in \mathbb{Z}_+$ defines an ι-orbit u'_{i-1} of V' for all $i \in \mathbb{Z}$. We note that the edges $g^i_{l-1,l}$ and $f^i_{l,l+1}$ satisfy

$$t(g^i_{l-1,l}) = s(f^i_{l,l+1}) = u^l_i,$$

by the second equality of (2.1). There uniquely exists an edge $e'^{i}_{l,l+1} \in E'^{l,l+1}$ such that $s(e'^{i}_{l,l+1}) = u'^{l}_{i-1}, t(e'^{i}_{l,l+1}) = u'^{l+1}_i, \lambda'(e'^{i}_{l,l+1}) = \kappa'^{-1}(d_{\alpha_i} c_{\alpha_i})$. Put $\alpha'_i = \kappa'^{-1}(d_{\alpha_i} c_{\alpha_i}) \in \Sigma'$. Since $l \in \mathbb{Z}_+$ is arbitrary, one obtains

$$(u'_{i-1}, \alpha'_i, u'_i) \in E_{\mathfrak{L}}.$$

Therefore, $(\alpha'_i, u'_i)_{i\in\mathbb{Z}}$ defines an element of $Z_{\mathfrak{L}'}$. It is straightforward to see that the correspondences

$$\begin{aligned} \Phi : (\alpha_i, u_i)_{i\in\mathbb{Z}} \in Z_{\mathfrak{L}} \longrightarrow (\alpha'_i, u'_i)_{i\in\mathbb{Z}} \in Z_{\mathfrak{L}'}, \\ \phi : (\alpha_i)_{i\in\mathbb{Z}} \in \Lambda_{\mathfrak{L}} \longrightarrow (\alpha'_i)_{i\in\mathbb{Z}} \in \Lambda_{\mathfrak{L}'} \end{aligned}$$

give rise to homeomorphisms and satisfy

$$\Phi \circ \sigma_{\mathfrak{L}} = \sigma_{\mathfrak{L}'} \circ \Phi, \quad \phi \circ \sigma_{\Lambda_{\mathfrak{L}}} = \sigma_{\Lambda'_{\mathfrak{L}}} \circ \phi, \quad \phi \circ \pi_{\mathfrak{L}} = \pi_{\mathfrak{L}'} \circ \Phi. \qquad \square$$

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3. *Periodic points of λ-graph systems*

In this section, we study periodic points of the dynamical system $(Z_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$. We will now provide some notation. For $p \in \mathbb{N}$, we set

$$\Lambda_{\mathfrak{L}}^p = \{(\alpha_i)_{i\in\mathbb{Z}} \in \Lambda_{\mathfrak{L}} \mid \alpha_i = \alpha_{p+i},\ i \in \mathbb{Z}\},$$

the set of all p-periodic points of $\Lambda_{\mathfrak{L}}$ under $\sigma_{\Lambda_{\mathfrak{L}}}$, and

$$Z_{\mathfrak{L}}^p = \{(\alpha_i, u_i)_{i\in\mathbb{Z}} \in Z_{\mathfrak{L}} \mid \alpha_i = \alpha_{p+i}, u_i = u_{p+i},\ i \in \mathbb{Z}\},$$

the set of all p-periodic points of $Z_{\mathfrak{L}}$ under $\sigma_{\mathfrak{L}}$. We denote by $\pi_{\mathfrak{L}}^p : Z_{\mathfrak{L}}^p \to \Lambda_{\mathfrak{L}}^p$ the restriction of $\pi_{\mathfrak{L}} : Z_{\mathfrak{L}} \to \Lambda_{\mathfrak{L}}$ to $Z_{\mathfrak{L}}^p$. For $\mu = (\mu_1, \dots, \mu_p) \in B_p(\Lambda_{\mathfrak{L}})$ we put $\mu_{np+i} = \mu_i, n \in \mathbb{Z}$, $i = 1, 2, \dots, p$, and $\mu^\infty = (\mu_i)_{i\in\mathbb{Z}}$. If $\mu^\infty \in \Lambda_{\mathfrak{L}}$, it belongs to $\Lambda_{\mathfrak{L}}^p$. Put

$$W(\Lambda_{\mathfrak{L}}^p) = \{\mu \in B_p(\Lambda_{\mathfrak{L}}) \mid \mu^\infty \in \Lambda_{\mathfrak{L}}\}.$$

We call a word of $W(\Lambda_{\mathfrak{L}}^p)$ a periodic word. For $\mu \in W(\Lambda_{\mathfrak{L}}^p)$, we set

$$\begin{aligned}\Omega_{\mathfrak{L}}^p[\mu] = \{(u^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}} \mid \exists \text{ a path } \gamma_n \text{ of } \mathfrak{L} \text{ of length } p;\\ s(\gamma_n) = u^n, t(\gamma_n) = u^{n+p}, \lambda(\gamma_n) = \mu, n \in \mathbb{Z}_+\},\\ Z_{\mathfrak{L}}^p[\mu] = \{x \in Z_{\mathfrak{L}}^p \mid \pi_{\mathfrak{L}}(x) = \mu^\infty\},\end{aligned}$$

where, for a path $\gamma_n = (e_1, \dots, e_p)$ of $\mathfrak{L}$, $\lambda(\gamma_n)$ denotes $(\lambda(e_1), \dots, \lambda(e_p))$. Hence we have a disjoint union

$$Z_{\mathfrak{L}}^p = \coprod_{\mu\in W(\Lambda_{\mathfrak{L}}^p)} Z_{\mathfrak{L}}^p[\mu].$$

LEMMA 3.1. *For $\mu \in W(\Lambda_{\mathfrak{L}}^p)$, the correspondence $\pi_{\mathfrak{L}}^p[\mu] : Z_{\mathfrak{L}}^p[\mu] \longrightarrow \Omega_{\mathfrak{L}}^p[\mu]$ defined by $\pi_{\mathfrak{L}}^p[\mu]((\alpha_i, u_i)_{i\in\mathbb{Z}}) = u_0$ gives rise to a homeomorphism.*

Proof. It suffices to show that $\pi_{\mathfrak{L}}^p[\mu]$ is surjective. For $u_0 = (u_0^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}^p[\mu]$ there exists a path γ_n of $\mathfrak{L}$ of length p such that $s(\gamma_n) = u_0^{np}, t(\gamma_n) = u_0^{(n+1)p}, \lambda(\gamma_n) = \mu = (\mu_1, \dots, \mu_p), n \in \mathbb{Z}_+$. Let $\gamma_n = (e_0^n, e_1^n, \dots, e_{p-1}^n)$ where $e_k^n \in E_{n+k,n+k+1}$. We put $u_k^{n+k} = s(e_k^n)$ for $1 \le k \le p-1$, $n \in \mathbb{Z}_+$ and $u_k^{k-i} = \iota^i(u_k^k)$ for $i = 1, 2, \dots, k$. As $\mathfrak{L}$ is left-resolving, by the local property of $\mathfrak{L}$, one has $\iota(u_k^{l+1}) = u_k^l$ for $0 \le k \le p-1, l \in \mathbb{Z}_+$. Hence $u_k = (u_k^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}, 0 \le k \le p-1$. For $i \in \mathbb{Z}_+$, let $p(i)$ satisfy $i = np + p(i)$, $0 \le p(i) \le p-1$, for some $n \in \mathbb{Z}_+$. Put $\alpha_i = \mu_{p(i)}$ and $u_i = u_{p(i)}$ for all $i \in \mathbb{Z}_+$. Then the sequence $(\alpha_i, u_i)_{i\in\mathbb{Z}}$ belongs to $Z_{\mathfrak{L}}^p$ and yields a point of $Z_{\mathfrak{L}}^p[\mu]$. It goes to u_0 by $\pi_{\mathfrak{L}}^p[\mu]$. □

For $\mu \in W(\Lambda_{\mathfrak{L}}^p)$ with $\mu = \mu_1\mu_2\cdots\mu_p$ we set $\sigma(\mu) = \mu_2\mu_3\cdots\mu_p\mu_1$.

LEMMA 3.2.

(i) $\sigma(\mu) \in W(\Lambda_{\mathfrak{L}}^p)$.

(ii) $\Omega_{\mathfrak{L}}^p[\mu]$ *is homeomorphic to* $\Omega_{\mathfrak{L}}^p[\sigma(\mu)]$.

Proof. (i) The assertion is clear.

(ii) By the preceding lemma, for $u_0 \in \Omega_{\mathfrak{L}}^p[\mu]$ there exists $(\alpha_i, u_i)_{i\in\mathbb{Z}} \in Z_{\mathfrak{L}}^p[\mu]$. As $u_1 \in \Omega_{\mathfrak{L}}^p[\sigma(\mu)]$, the correspondence $s : u_0 \in \Omega_{\mathfrak{L}}^p[\mu] \longrightarrow u_1 \in \Omega_{\mathfrak{L}}^p[\sigma(\mu)]$ gives rise to a homeomorphism. □

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Let $\mathfrak{L}$ and $\mathfrak{L}'$ be λ-graph systems. If they are strong shift equivalent, there exist shift-commuting homeomorphisms $\Phi : Z_{\mathfrak{L}} \longrightarrow Z_{\mathfrak{L}'}$ and $\phi : \Lambda_{\mathfrak{L}} \longrightarrow \Lambda_{\mathfrak{L}'}$ by Theorem 2.5. One can then make the following observation.

PROPOSITION 3.3. *For $p \in \mathbb{N}$, the respective restrictions of Φ and ϕ to $Z^p_{\mathfrak{L}}$ and $\Lambda^p_{\mathfrak{L}}$,*

$$\Phi : Z^p_{\mathfrak{L}} \longrightarrow Z^p_{\mathfrak{L}'}, \quad \phi : \Lambda^p_{\mathfrak{L}} \longrightarrow \Lambda^p_{\mathfrak{L}'}$$

are homeomorphic. In particular,

$$|W(\Lambda^p_{\mathfrak{L}})| = |W(\Lambda^p_{\mathfrak{L}'})|,$$

where $|W(\Lambda^p_{\mathfrak{L}})|, |W(\Lambda^p_{\mathfrak{L}'})|$ denote the cardinality of $W(\Lambda^p_{\mathfrak{L}})$, $W(\Lambda^p_{\mathfrak{L}'})$, respectively.

For $p \in \mathbb{N}$, let $\pi^{p*}_{\mathfrak{L}} : C(\Lambda^p_{\mathfrak{L}}) \longrightarrow C(Z^p_{\mathfrak{L}})$ be the homomorphism from the commutative C^*-algebra $C(\Lambda^p_{\mathfrak{L}})$ of all complex-valued continuous functions on $\Lambda^p_{\mathfrak{L}}$ to the commutative C^*-algebra $C(Z^p_{\mathfrak{L}})$ of all complex-valued continuous functions on $Z^p_{\mathfrak{L}}$ induced by the map $\pi^p_{\mathfrak{L}} : Z^p_{\mathfrak{L}} \longrightarrow \Lambda^p_{\mathfrak{L}}$. Define $\Pi^p_{\mathfrak{L}}$ by the homomorphism

$$\pi^{p*}_{\mathfrak{L}*} : K_0(C(\Lambda^p_{\mathfrak{L}})) \longrightarrow K_0(C(Z^p_{\mathfrak{L}}))$$

on the K_0-groups induced from the homomorphism $\pi^{p*}_{\mathfrak{L}} : C(\Lambda^p_{\mathfrak{L}}) \longrightarrow C(Z^p_{\mathfrak{L}})$. The algebras $C(\Lambda^p_{\mathfrak{L}})$ and $C(Z^p_{\mathfrak{L}})$ are naturally decomposed as the direct sums

$$C(\Lambda^p_{\mathfrak{L}}) = \bigoplus_{\mu \in W(\Lambda^p_{\mathfrak{L}})} \mathbb{C} \quad \text{and} \quad C(Z^p_{\mathfrak{L}}) = \bigoplus_{\mu \in W(\Lambda^p_{\mathfrak{L}})} C(\Omega^p_{\mathfrak{L}}[\mu]).$$

For $p, q \in \mathbb{N}$ with $q|p$, one has $W(\Lambda^q_{\mathfrak{L}}) \subset W(\Lambda^p_{\mathfrak{L}})$ and the set $Z^q_{\mathfrak{L}}$ is a clopen subset of $Z^p_{\mathfrak{L}}$ so that there exist natural embeddings

$$C(\Lambda^q_{\mathfrak{L}}) \hookrightarrow C(\Lambda^p_{\mathfrak{L}}) \quad \text{and} \quad C(Z^q_{\mathfrak{L}}) \hookrightarrow C(Z^p_{\mathfrak{L}})$$

denoted by $i^{\Lambda}_{q,p}$ and $i^{Z}_{q,p}$, respectively. They are compatible to the homomorphisms $\pi^{q*}_{\mathfrak{L}} : C(\Lambda^q_{\mathfrak{L}}) \longrightarrow C(Z^q_{\mathfrak{L}})$ and $\pi^{p*}_{\mathfrak{L}} : C(\Lambda^p_{\mathfrak{L}}) \longrightarrow C(Z^p_{\mathfrak{L}})$. This means the diagram

$$\begin{array}{ccc} C(\Lambda^q_{\mathfrak{L}}) & \xrightarrow{\pi^{q*}_{\mathfrak{L}}} & C(Z^q_{\mathfrak{L}}) \\ {\scriptstyle i^{\Lambda}_{q,p}}\downarrow & & \downarrow{\scriptstyle i^{Z}_{q,p}} \\ C(\Lambda^p_{\mathfrak{L}}) & \xrightarrow{\pi^{p*}_{\mathfrak{L}}} & C(Z^p_{\mathfrak{L}}) \end{array}$$

is commutative. Hence the K-theory

$$\begin{array}{ccc} K_0(C(\Lambda^q_{\mathfrak{L}})) & \xrightarrow{\Pi^q_{\mathfrak{L}}} & K_0(C(Z^q_{\mathfrak{L}})) \\ {\scriptstyle i^{\Lambda}_{q,p*}}\downarrow & & \downarrow{\scriptstyle i^{Z}_{q,p*}} \\ K_0(C(\Lambda^p_{\mathfrak{L}})) & \xrightarrow{\Pi^p_{\mathfrak{L}}} & K_0(C(Z^p_{\mathfrak{L}})) \end{array}$$

is also commutative. The K-theory diagram is denoted by $\mathfrak{S}_{q,p}(\mathfrak{L})$ for $p, q \in \mathbb{N}$ with $q|p$. The system $\mathfrak{S}(\mathfrak{L}) = \{\mathfrak{S}_{q,p}(\mathfrak{L})\}_{q,p}$ is called the *periodic dimension diagrams* for $\mathfrak{L}$. We denote

$$\triangle_{\mathfrak{L}}(p) = K_0(C(Z^p_{\mathfrak{L}})) \quad \text{and} \quad \widehat{\triangle}_{\mathfrak{L}}(p) = K_0(C(Z^p_{\mathfrak{L}}))/\Pi^p_{\mathfrak{L}}(K_0(C(\Lambda^p_{\mathfrak{L}})))$$

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for $p \in \mathbb{N}$. As the homomorphism $i^{Z}_{q,p*} : K_0(C(Z^q_{\mathfrak{L}})) \to K_0(C(Z^p_{\mathfrak{L}}))$ for $q|p$ is compatible with the homomorphisms $\Pi^q_{\mathfrak{L}} : K_0(C(\Lambda^q_{\mathfrak{L}})) \longrightarrow K_0(C(Z^q_{\mathfrak{L}}))$ and $\Pi^p_{\mathfrak{L}} : K_0(C(\Lambda^p_{\mathfrak{L}})) \longrightarrow K_0(C(Z^p_{\mathfrak{L}}))$, the map $i^{Z}_{q,p*}$ denoted by $\iota^{\triangle}_{q,p}$ induces a homomorphism $\widehat{\triangle}_{\mathfrak{L}}(q) \to \widehat{\triangle}_{\mathfrak{L}}(p)$ that is denoted by $\iota^{\widehat{\triangle}}_{q,p}$. We put the directed sequences of the groups

$$\triangle_{\mathfrak{L}} = \{\iota^{\triangle}_{q,p} : \triangle_{\mathfrak{L}}(q) \to \triangle_{\mathfrak{L}}(p) \text{ for } p, q \in \mathbb{N} \text{ with } q < p,\ q|p\},$$
$$\widehat{\triangle}_{\mathfrak{L}} = \{\iota^{\widehat{\triangle}}_{q,p} : \widehat{\triangle}_{\mathfrak{L}}(q) \to \widehat{\triangle}_{\mathfrak{L}}(p) \text{ for } p, q \in \mathbb{N} \text{ with } q < p,\ q|p\}.$$

The pair $\mathfrak{P}_{\mathfrak{L}} = (\triangle_{\mathfrak{L}}, \widehat{\triangle}_{\mathfrak{L}})$ is called the *periodic dimension group* of a λ-graph system $\mathfrak{L}$.

Let $v^l_1, \dots, v^l_{m(l)}$ be the vertex set V_l of $\mathfrak{L}$ at level l. For $k \in \mathbb{N}$ and $\mu \in B_k(\Lambda_{\mathfrak{L}})$, define $m(l) \times m(l+k)$ matrices $A_{l,l+k}[\mu]$ and $I_{l,l+k}$ with entries in $\{0, 1\}$ by setting

$$A_{l,l+k}[\mu](i,j) = \begin{cases} 1 & \text{if } \exists \text{ a path } \gamma;\ s(\gamma) = v^l_i,\ t(\gamma) = v^{l+k}_j,\ \lambda(\gamma) = \mu, \\ 0 & \text{otherwise,} \end{cases}$$
$$I_{l,l+k}(i,j) = \begin{cases} 1 & \text{if } \iota^k(v^{l+k}_j) = v^l_i, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \dots, m(l)$, $j = 1, 2, \dots, m(l+k)$. We denote by $A^t_{l,l+k}[\mu]$ and $I^t_{l,l+k}$ the transposed matrices of $A_{l,l+k}[\mu]$ and $I_{l,l+k}$, respectively.

LEMMA 3.4. *For $\mu \in W(\Lambda^p_{\mathfrak{L}})$, we have*

$$K_0(C(\Omega^p_{\mathfrak{L}}[\mu])) = \varinjlim_{n\to\infty} \{A^t_{np,(n+1)p}[\mu] * I^t_{np,(n+1)p} : \mathbb{Z}^{m(np)} \longrightarrow \mathbb{Z}^{m((n+1)p)}\}$$

where

$$A^t_{np,(n+1)p}[\mu] * I^t_{np,(n+1)p}(j,i) = A^t_{np,(n+1)p}[\mu](j,i) I^t_{np,(n+1)p}(j,i)$$

for $i = 1, 2, \dots, m(np)$, $j = 1, 2, \dots, m((n+1)p)$.

Proof. The vertex set $\Omega_{\mathfrak{L}}$ is written as the projective limit

$$\Omega_{\mathfrak{L}} = \varprojlim\{\iota_{l,l+1} : V_{l+1} \to V_l\}.$$

An element $(v^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$ belongs to $\Omega_{\mathfrak{L}}[\mu]$ for $\mu \in W(\Lambda^p_{\mathfrak{L}})$ if and only if there exists a sequence $\gamma_{l,l+p}$ of paths from v^l to v^{l+p} labeled μ for $l \in \mathbb{Z}_+$. By the local property of a λ-graph system the latter condition is equivalent to the condition that there exists a sequence $\gamma_{np,(n+1)p}$ of paths from v^{np} to $v^{(n+1)p}$ labeled μ for $n \in \mathbb{Z}_+$. Hence $(v^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$ belongs to $\Omega_{\mathfrak{L}}[\mu]$ if and only if it belongs to the following set:

$$\{(v^l)_{l\in\mathbb{Z}_+} \in \Pi_{l\in\mathbb{Z}_+} V_l \mid \iota^p(v^{(n+1)p}) = v^{np} \text{ and } \exists \text{ a path } \gamma_{np,(n+1)p};$$
$$s(\gamma_{np,(n+1)p}) = v^{np}, t(\gamma_{np,(n+1)p}) = v^{(n+1)p}, \lambda(\gamma_{np,(n+1)p}) = \mu\}.$$

Hence the algebra $C(\Omega^p_{\mathfrak{L}}[\mu])$ is isomorphic to the algebra of the inductive limit

$$C(\Omega^p_{\mathfrak{L}}[\mu]) = \varinjlim_{n\to\infty} \{A^t_{np,(n+1)p}[\mu] * I^t_{np,(n+1)p} : \mathbb{C}^{m(np)} \longrightarrow \mathbb{C}^{m((n+1)p)}\}$$

so that we have the desired K-theory formula. □

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For $\mu \in W(\Lambda_{\mathfrak{L}}^p)$ we put

$$\widehat{V}_0 = V_0,\ \widehat{V}_{np} = \{v_j^{np} \in V_{np} \mid \exists \text{ a vertex } v_i^{(n-1)p} \in \widehat{V}_{(n-1)p} \text{ and a path } \gamma_{(n-1)p,np};$$
$$\iota^p(v_j^{np}) = v_i^{(n-1)p},\ s(\gamma_{(n-1)p,np}) = v_i^{(n-1)p},\ t(\gamma_{(n-1)p,np}) = v_j^{np},\ \lambda(\gamma_{(n-1)p,np}) = \mu\}$$

for $n \in \mathbb{N}$. We denote by $\hat{m}(np)$ the cardinal number $|\widehat{V}_{np}|$. Then the restriction of the matrix $A^t_{np,(n+1)p}[\mu] * I^t_{np,(n+1)p}$ to the corresponding coordinates for the vertices $\widehat{V}_{np}$ naturally defines a homomorphism from $\mathbb{Z}^{\hat{m}(np)}$ to $\mathbb{Z}^{\hat{m}((n+1)p)}$. The restricted matrix is also denoted by $A^t_{np,(n+1)p}[\mu] * I^t_{np,(n+1)p}$. The inductive limit

$$\lim_{n\to\infty}\{A^t_{np,(n+1)p}[\mu] * I^t_{np,(n+1)p} : \mathbb{Z}^{\hat{m}(np)} \longrightarrow \mathbb{Z}^{\hat{m}((n+1)p)}\}$$

is still isomorphic to the group $K_0(C(\Omega_{\mathfrak{L}}^p[\mu]))$. Consider the quotient group $\mathbb{Z}^{\hat{m}(np)}/\mathbb{Z}$ of the group $\mathbb{Z}^{\hat{m}(np)}$ by the image of the map $k \in \mathbb{Z} \to (k, \dots, k) \in \mathbb{Z}^{\hat{m}(np)}$. As the quotient maps are compatible to the homomorphisms

$$A^t_{np,(n+1)p}[\mu] * I^t_{np,(n+1)p} : \mathbb{Z}^{\hat{m}(np)} \longrightarrow \mathbb{Z}^{\hat{m}((n+1)p)}, \quad n \in \mathbb{Z}_+,$$

we may consider the induced homomorphism $\mathbb{Z}^{\hat{m}(np)}/\mathbb{Z} \longrightarrow \mathbb{Z}^{\hat{m}((n+1)p)}/\mathbb{Z}$ that is also denoted by $A^t_{np,(n+1)p}[\mu] * I^t_{np,(n+1)p}$. It is easy to see that the direct sum of the inductive limits

$$\lim_{n\to\infty}\{A^t_{np,(n+1)p}[\mu] * I^t_{np,(n+1)p} : \mathbb{Z}^{\hat{m}(np)}/\mathbb{Z} \longrightarrow \mathbb{Z}^{\hat{m}((n+1)p)}/\mathbb{Z}\}$$

over $\mu \in W(\Lambda_{\mathfrak{L}}^p)$ is isomorphic to the group $\widehat{\triangle}_{\mathfrak{L}}(p)$. We then reach the following results.

THEOREM 3.5. *The periodic dimension diagram $\mathfrak{S}_{\mathfrak{L}}$ is invariant under the strong shift equivalence of a λ-graph system. In particular, the periodic dimension group $\mathfrak{P}_{\mathfrak{L}} = (\triangle_{\mathfrak{L}}, \widehat{\triangle}_{\mathfrak{L}})$ is invariant under the strong shift equivalence of a λ-graph system. They are calculated as*

$$\triangle_{\mathfrak{L}}(p) = \bigoplus_{\mu\in W(\Lambda_{\mathfrak{L}}^p)} \lim_{n\to\infty}\{A^t_{np,(n+1)p}[\mu] * I^t_{np,(n+1)p} : \mathbb{Z}^{m(np)} \longrightarrow \mathbb{Z}^{m((n+1)p)}\},$$
$$\widehat{\triangle}_{\mathfrak{L}}(p) = \bigoplus_{\mu\in W(\Lambda_{\mathfrak{L}}^p)} \lim_{n\to\infty}\{A^t_{np,(n+1)p}[\mu] * I^t_{np,(n+1)p} : \mathbb{Z}^{\hat{m}(np)}/\mathbb{Z} \longrightarrow \mathbb{Z}^{\hat{m}((n+1)p)}/\mathbb{Z}\},$$

where $\bigoplus_{\mu\in W(\Lambda_{\mathfrak{L}}^p)}$ denotes the direct sum of abelian groups over $W(\Lambda_{\mathfrak{L}}^p)$.

The matrices $A^t_{np,(n+1)p}[\mu] * I^t_{np,(n+1)p}$ pick up the p-periodic diagonal elements. The above formulae of $\triangle_{\mathfrak{L}}(p)$ and $\widehat{\triangle}_{\mathfrak{L}}(p)$ are regarded as trace formulae for the groups. In §6 we will compute the groups $\triangle_{\mathfrak{L}}(p)$, $\widehat{\triangle}_{\mathfrak{L}}(p)$, $p \in \mathbb{N}$ for some examples of λ-graph systems.

We will next study the least periodic points of the dynamical system $(Z_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$. For $p \in \mathbb{N}$, we set

$$\Lambda_{\mathfrak{L}}^{[p]} = \{(\alpha_i)_{i\in\mathbb{Z}} \in \Lambda_{\mathfrak{L}}^p \mid (\alpha_i)_{i\in\mathbb{Z}} \notin \Lambda_{\mathfrak{L}}^k \text{ for all } k \text{ with } 0 < k < p\},$$
$$Z_{\mathfrak{L}}^{[p]} = \{(\alpha_i, u_i)_{i\in\mathbb{Z}} \in Z_{\mathfrak{L}}^p \mid (\alpha_i, u_i)_{i\in\mathbb{Z}} \notin Z_{\mathfrak{L}}^k \text{ for all } k \text{ with } 0 < k < p\}.$$

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They are called the p-least periodic points of $(\Lambda_{\mathfrak{L}}, \sigma_{\Lambda_{\mathfrak{L}}})$ and $(Z_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$, respectively. The p-least periodic words $W(\Lambda_{\mathfrak{L}}^{[p]})$ are defined by the set of all words $\mu \in W(\Lambda_{\mathfrak{L}}^{p})$ such that μ does not belong to $W(\Lambda_{\mathfrak{L}}^{k})$ for all k with $0 < k < p$. Denote by $\pi_{\mathfrak{L}}^{[p]}$ the restriction of $\pi_{\mathfrak{L}}^{p}$ to the subset $Z_{\mathfrak{L}}^{[p]}$ of $Z_{\mathfrak{L}}^{p}$. The map $\pi_{\mathfrak{L}}^{[p]} : Z_{\mathfrak{L}}^{[p]} \longrightarrow \Lambda_{\mathfrak{L}}^{[p]}$ induces a homomorphism $\pi_{\mathfrak{L}}^{[p]*} : C(\Lambda_{\mathfrak{L}}^{[p]}) \longrightarrow C(Z_{\mathfrak{L}}^{[p]})$. Similarly to the preceding discussions, define $\Pi_{\mathfrak{L}}^{[p]}$ by the homomorphism

$$\pi_{\mathfrak{L}}^{[p]*}{}_{*} : K_0(C(\Lambda_{\mathfrak{L}}^{[p]})) \longrightarrow K_0(C(Z_{\mathfrak{L}}^{[p]}))$$

on the K_0-groups. For $p, q \in \mathbb{N}$ with $q < p$, $q|p$ there exist canonical embeddings

$$C(\Lambda_{\mathfrak{L}}^{[q]}) \hookrightarrow C(\Lambda_{\mathfrak{L}}^{[p]}) \quad \text{and} \quad C(Z_{\mathfrak{L}}^{[q]}) \hookrightarrow C(Z_{\mathfrak{L}}^{[p]})$$

denoted by $i_{[q],[p]}^{\Lambda}$ and $i_{[q],[p]}^{Z}$, respectively. We have a commutative diagram:

$$\begin{array}{ccc} K_0(C(\Lambda_{\mathfrak{L}}^{[q]})) & \xrightarrow{\Pi_{\mathfrak{L}}^{[q]}} & K_0(C(Z_{\mathfrak{L}}^{[q]})) \\ \Big\downarrow{\scriptstyle i_{[q],[p]*}^{\Lambda}} & & \Big\downarrow{\scriptstyle i_{[q],[p]*}^{Z}} \\ K_0(C(\Lambda_{\mathfrak{L}}^{[p]})) & \xrightarrow{\Pi_{\mathfrak{L}}^{[p]}} & K_0(C(Z_{\mathfrak{L}}^{[p]})) \end{array}$$

that is denoted by $\mathfrak{S}_{[q],[p]}(\mathfrak{L})$ for $p, q \in \mathbb{N}$ with $q|p$. The system $\mathfrak{S}([\mathfrak{L}]) = \{\mathfrak{S}_{[q],[p]}\}_{q,p}$ is called the *least periodic dimension diagrams* for $\mathfrak{L}$. We denote

$$\triangle_{\mathfrak{L}}([p]) = K_0(C(Z_{\mathfrak{L}}^{[p]})) \quad \text{and} \quad \widehat{\triangle}_{\mathfrak{L}}([p]) = K_0(C(Z_{\mathfrak{L}}^{[p]}))/\Phi_{\mathfrak{L}}^{[p]}(K_0(C(\Lambda_{\mathfrak{L}}^{[p]})))$$

for $p \in \mathbb{N}$. The map $i_{[q],[p]*}^{Z}$ denoted by $\iota_{[q],[p]}^{\triangle}$ induces a homomorphism $\widehat{\triangle}_{\mathfrak{L}}([q]) \to \widehat{\triangle}_{\mathfrak{L}}([p])$ for $q|p$, which is denoted by $\iota_{[q],[p]}^{\widehat{\triangle}}$. Similarly we put the directed sequences of the groups

$$\triangle_{[\mathfrak{L}]} = \{\iota_{[q],[p]}^{\triangle} : \triangle_{\mathfrak{L}}([q]) \to \triangle_{\mathfrak{L}}([p]) \text{ for } q|p\},$$
$$\widehat{\triangle}_{[\mathfrak{L}]} = \{\iota_{[q],[p]}^{\widehat{\triangle}} : \widehat{\triangle}_{\mathfrak{L}}([q]) \to \widehat{\triangle}_{\mathfrak{L}}([p]) \text{ for } q|p\}.$$

The pair $\mathfrak{P}_{[\mathfrak{L}]} = (\triangle_{[\mathfrak{L}]}, \widehat{\triangle}_{[\mathfrak{L}]})$ is called the *least periodic dimension group* of a λ-graph system $\mathfrak{L}$. The least periodic dimension group $\mathfrak{P}_{[\mathfrak{L}]}$ also is invariant under the strong shift equivalence of λ-graph systems.

PROPOSITION 3.6. *The least periodic dimension group $\mathfrak{P}_{[\mathfrak{L}]}$ determines the periodic dimension group $\mathfrak{P}_{\mathfrak{L}}$. Conversely, the periodic dimension group $\mathfrak{P}_{\mathfrak{L}}$ determines the least periodic dimension group $\mathfrak{P}_{[\mathfrak{L}]}$.*

Proof. Since the set $W(\Lambda_{\mathfrak{L}}^{p})$ is decomposed into the disjoint union $\coprod_{q|p} W(\Lambda_{\mathfrak{L}}^{[q]})$, one has $\triangle_{\mathfrak{L}}(p) = \bigoplus_{q|p} \triangle_{\mathfrak{L}}([q])$. This decomposition of $\triangle_{\mathfrak{L}}(p)$ is compatible to the map $\Pi_{\mathfrak{L}}^{p}$ so that one sees the decomposition $\widehat{\triangle}_{\mathfrak{L}}(p) = \bigoplus_{q|p} \widehat{\triangle}_{\mathfrak{L}}([q])$. Hence $\mathfrak{P}_{[\mathfrak{L}]}$ determines $\mathfrak{P}_{\mathfrak{L}}$.

Conversely, the set $W(\Lambda_{\mathfrak{L}}^{[p]})$ is written as $W(\Lambda_{\mathfrak{L}}^{p}) \setminus \coprod_{\substack{q|p \\ q<p}} W(\Lambda_{\mathfrak{L}}^{[q]})$. We have a natural exact sequence

$$0 \longrightarrow \bigoplus_{\substack{q|p \\ q<p}} C(Z_{\mathfrak{L}}^{[q]}) \longrightarrow C(Z_{\mathfrak{L}}^{p}) \longrightarrow C(Z_{\mathfrak{L}}^{[p]}) \longrightarrow 0.$$

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Therefore, we have

$$K_0(C(Z_{\mathfrak{L}}^{[p]})) = K_0(C(Z_{\mathfrak{L}}^{p})) \Big/ \bigoplus_{\substack{q|p \\ q<p}} K_0(C(Z_{\mathfrak{L}}^{[q]})).$$

As $K_0(C(Z_{\mathfrak{L}}^{[1]})) = K_0(C(Z_{\mathfrak{L}}^{1}))$, by induction, $\triangle_{\mathfrak{L}}([p])$ is determined by $\mathfrak{P}_{\mathfrak{L}}$. The maps $\iota_{q,p}^{\triangle} : \triangle_{\mathfrak{L}}(q) \to \triangle_{\mathfrak{L}}(p)$ for $q|p$ induce the maps $\iota_{[q],[p]}^{\triangle} : \triangle_{\mathfrak{L}}([q]) \to \triangle_{\mathfrak{L}}([p])$. Similarly, $\widehat{\triangle}_{\mathfrak{L}}([p])$ is determined by $\mathfrak{P}_{\mathfrak{L}}$ and the maps $\iota_{q,p}^{\widehat{\triangle}} : \widehat{\triangle}_{\mathfrak{L}}(q) \to \widehat{\triangle}_{\mathfrak{L}}(p)$ for $q|p$ induce the maps $\iota_{[q],[p]}^{\widehat{\triangle}} : \widehat{\triangle}_{\mathfrak{L}}([q]) \to \widehat{\triangle}_{\mathfrak{L}}([p])$. Hence the periodic dimension group $\mathfrak{P}_{\mathfrak{L}}$ determines the least periodic dimension group $\mathfrak{P}_{[\mathfrak{L}]}$. □

4. *Zeta functions of λ-graph systems*

In what follows, we denote by $|S|$ the cardinality of a set S.

We say that a word $\mu \in W(\Lambda_{\mathfrak{L}}^{p})$ has *finite periodic ι-orbits* if $|\Omega_{\mathfrak{L}}^{p}[\mu]|$ is finite, otherwise $\mu \in W(\Lambda_{\mathfrak{L}}^{p})$ has *infinite periodic ι-orbits*. For $\mu \in W(\Lambda_{\mathfrak{L}}^{p})$, we put

$$N_{\mathfrak{L}}[\mu] = \begin{cases} |\Omega_{\mathfrak{L}}^{p}[\mu]| & \text{if } \mu \text{ has finite periodic } \iota\text{-orbits,} \\ 0 & \text{otherwise.} \end{cases}$$

We set

$$N_{\mathfrak{L}}(p) = \sum_{\mu \in W(\Lambda_{\mathfrak{L}}^{p})} N_{\mathfrak{L}}[\mu],$$

$$N_{\mathfrak{L}}^{\infty}(p) = |\{\mu \in W(\Lambda_{\mathfrak{L}}^{p}) \mid |\Omega_{\mathfrak{L}}^{p}[\mu]| = \infty\}|.$$

Both of the numbers $N_{\mathfrak{L}}(p)$, $N_{\mathfrak{L}}^{\infty}(p)$, $p \in \mathbb{N}$ are always finite, and $N_{\mathfrak{L}}(p)$ counts the number of finite periodic ι-orbits, and $N_{\mathfrak{L}}^{\infty}(p)$ counts the number of periodic words with infinite periodic ι-orbits. We define two kinds of zeta functions for a λ-graph system that give us complete information for the sequences $N_{\mathfrak{L}}(p)$, $N_{\mathfrak{L}}^{\infty}(p)$, $p \in \mathbb{N}$.

Definition. Let $\mathfrak{L}$ be a λ-graph system. We define the zeta function $\zeta_{\mathfrak{L}}(z)$ of $\mathfrak{L}$ by setting

$$\zeta_{\mathfrak{L}}(z) = \exp\left[\sum_{n=1}^{\infty} \frac{N_{\mathfrak{L}}(n)}{n} z^n\right], \quad z \in \mathbb{C},$$

which counts the cardinal numbers of finite periodic ι-orbits, and the zeta function $\zeta_{\mathfrak{L}}^{\infty}(z)$ of $\mathfrak{L}$ by setting

$$\zeta_{\mathfrak{L}}^{\infty}(z) = \exp\left[\sum_{n=1}^{\infty} \frac{N_{\mathfrak{L}}^{\infty}(n)}{n} z^n\right], \quad z \in \mathbb{C},$$

which counts the cardinal numbers of periodic words with infinite periodic ι-orbits. They are considered as formal power series of z.

THEOREM 4.1. *If two λ-graph systems have the same periodic dimension diagrams, they have the same zeta functions.*

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Proof. Suppose that two λ-graph systems $\mathfrak{L}$ and $\mathfrak{L}'$ have the same periodic dimension diagrams. As the homomorphisms $\Pi^p_{\mathfrak{L}} : K_0(C(\Lambda^p_{\mathfrak{L}})) \longrightarrow K_0(C(Z^p_{\mathfrak{L}}))$ and $\Pi^p_{\mathfrak{L}'} : K_0(C(\Lambda^p_{\mathfrak{L}'})) \longrightarrow K_0(C(Z^p_{\mathfrak{L}'}))$ are conjugate for $p \in \mathbb{N}$, we may take a bijection

$$\xi^p : W(\Lambda^p_{\mathfrak{L}}) \longrightarrow W(\Lambda^p_{\mathfrak{L}'})$$

such that

$$K_0(C(\Omega^p_{\mathfrak{L}}[\mu])) = K_0(C(\Omega^p_{\mathfrak{L}'}[\xi^p(\mu)])), \quad \mu \in W(\Lambda^p_{\mathfrak{L}}).$$

Hence the set $\Omega^p_{\mathfrak{L}}[\mu]$ has finite cardinality if and only if $\Omega^p_{\mathfrak{L}}[\xi^p(\mu)]$ does so. In this case, one sees that

$$|\Omega^p_{\mathfrak{L}}[\mu]| = |\Omega^p_{\mathfrak{L}}[\xi^p(\mu)]|$$

so that $N_{\mathfrak{L}}(p) = N_{\mathfrak{L}'}(p)$ and $\zeta_{\mathfrak{L}}(z) = \zeta_{\mathfrak{L}'}(z)$. Furthermore, one also obtains that $\zeta^\infty_{\mathfrak{L}}(z) = \zeta^\infty_{\mathfrak{L}'}(z)$. □

Either by Theorem 3.5 or by direct inspection we can make the following corollary.

COROLLARY 4.2. *Let $\mathfrak{L}$ and $\mathfrak{L}'$ be λ-graph systems. Suppose that $\mathfrak{L}$ is strong shift equivalent to $\mathfrak{L}'$. Then we have*

$$\zeta_{\mathfrak{L}}(z) = \zeta_{\mathfrak{L}'}(z), \quad \zeta^\infty_{\mathfrak{L}}(z) = \zeta^\infty_{\mathfrak{L}'}(z).$$

Hence the zeta functions of a λ-graph system are invariant under strong shift equivalence.

Let Λ be a subshift. We denote by $N_\Lambda(p)$ the cardinal number of the set of p-periodic points of Λ. We say that $\mathfrak{L}$ is *periodically faithful* for Λ if $\Lambda_{\mathfrak{L}} = \Lambda$ and $N_{\mathfrak{L}}(p) = N_\Lambda(p)$, $N^\infty_{\mathfrak{L}}(p) = 0$, $p \in \mathbb{N}$.

PROPOSITION 4.3. *Let Λ be a subshift presented by a λ-graph system $\mathfrak{L}$. Then the following four conditions are equivalent to each other:*

(i) *$\mathfrak{L}$ is periodically faithful for the subshift Λ,*
(ii) *$\widehat{\Delta}(p) = 0$ for all $p \in \mathbb{N}$ and $\Lambda_{\mathfrak{L}} = \Lambda$,*
(iii) *$\widehat{\Delta}([p]) = 0$ for all $p \in \mathbb{N}$ and $\Lambda_{\mathfrak{L}} = \Lambda$,*
(iv) *$\zeta_{\mathfrak{L}}(z) = \zeta_\Lambda(z)$ and $\zeta^\infty_{\mathfrak{L}}(z) = 1$,*

where $\zeta_\Lambda(z)$ is the zeta function of the subshift Λ defined by

$$\zeta_\Lambda(z) = \exp\left[\sum_{n=1}^{\infty} \frac{N_\Lambda(n)}{n} z^n\right].$$

Proof. It is clear that condition (i) is equivalent to condition (iv), and (ii) is equivalent to (iii). The condition $\widehat{\Delta}(p) = 0$ is equivalent to the condition that the map $\Pi^p_{\mathfrak{L}} : K_0(\Lambda^p_{\mathfrak{L}}) \to K_0(Z^p_{\mathfrak{L}})$ is an isomorphism because $K_0(\Lambda^p_{\mathfrak{L}})$ is isomorphic to the direct sum $\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}$ of $|W(\Lambda^p_{\mathfrak{L}})|$-copies of $\mathbb{Z}$ and $\Pi^p_{\mathfrak{L}}$ is always injective. This condition is equivalent to the condition that

$$|\Omega^p_{\mathfrak{L}}[\mu]| = 1 \quad \text{for all } \mu \in W(\Lambda^p_{\mathfrak{L}}). \tag{$*$}$$

It is immediate that both conditions (i) and (ii) are equivalent to the above condition ($*$). □

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It is easy to see that, if Λ is a topological Markov shift, the canonical λ-graph system $\mathfrak{L}^{\Lambda}$ for Λ is periodically faithful for Λ.

The zeta function $\zeta_{\mathfrak{L}}(z)$ is a natural generalization of the ordinary zeta function of the topological Markov shift, whereas the zeta function $\zeta_{\mathfrak{L}}^{\infty}(z)$ is an obstruction such that the λ-graph system $\mathfrak{L}$ has the same periodic structure as the presented subshift $\Lambda_{\mathfrak{L}}$.

PROPOSITION 4.4. *Suppose that $\mathfrak{L}$ is strong shift equivalent to $\mathfrak{L}'$. Then $\mathfrak{L}$ is periodically faithful for $\Lambda_{\mathfrak{L}}$ if and only if $\mathfrak{L}'$ is periodically faithful for $\Lambda_{\mathfrak{L}'}$.*

Proof. As $\mathfrak{L}$ is strong shift equivalent to $\mathfrak{L}'$, the subshifts $\Lambda_{\mathfrak{L}}$ and $\Lambda_{\mathfrak{L}'}$ are topologically conjugate so that $N_{\Lambda_{\mathfrak{L}}}(p) = N_{\Lambda_{\mathfrak{L}'}}(p)$, $p \in \mathbb{N}$. Since the zeta functions are invariant under strong shift equivalence, one sees that the assertion holds. □

5. *Zeta functions of λ-graph systems associated with labeled graphs*

The formula $\zeta_{\Lambda^A}(z) = 1/\det(1 - zA)$ for the zeta function $\zeta_{\Lambda^A}(z)$ of an edge shift Λ^A defined by a non-negative matrix A can be found in the work of Bowen and Lanford [**BL1, BL2**]. The formula has been generalized to sofic shifts by Manning [**Man**] and Bowen [**Bo**]. In this section we will study our zeta functions for λ-graph systems coming from labeled graphs.

Let $\mathcal{G} = (G, \lambda)$ be a left-resolving labeled graph over Σ. Its underlying directed graph $G = (V, E)$ has the vertex set V and the edge set E. We will construct a labeled graph $\mathcal{G}_{[2]} = (G_{[2]}, \lambda_{[2]})$ over Σ from $\mathcal{G}$, called the complete in-split labeled graph of $\mathcal{G}$ (cf. [**LM**]). We set $V_{[2]} = \{(e, v) \in E \times V \mid t(e) = v\}$. For two vertices (f, u), (e, v) in $V_{[2]}$, define a labeled edge from (f, u) to (e, v) with label $\lambda(e)$ if $s(e) = u$. Denote by $E_{[2]}$ and $\lambda_{[2]}$ the labeled edges and its labeling map defined in this way. Then we have a labeled graph $\mathcal{G}_{[2]} = (G_{[2]}, \lambda_{[2]})$ with vertex set $V_{[2]}$ and edge set $E_{[2]}$. We may also construct a λ-graph system $\mathfrak{L}_{\mathcal{G}}$ associated with $\mathcal{G}$ by setting $V_l = V$, $E_{l,l+1} = E$ for all $l \in \mathbb{Z}_+$, and $\iota = \mathrm{id}$. The labeled map λ on $\mathfrak{L}_{\mathcal{G}}$ is the same as the map λ on $\mathcal{G}$. As the set of all ι-orbits of $\mathfrak{L}_{\mathcal{G}}$ is identified with the vertex set V of $\mathcal{G}$, the set $\Omega^p_{\mathfrak{L}_{\mathcal{G}}}[\mu]$ is finite for every $\mu \in W(\Lambda^p_{\mathfrak{L}_{\mathcal{G}}})$. Hence there is no word having infinite periodic ι-orbits, so that $N^{\infty}_{\mathfrak{L}_{\mathcal{G}}}(p) = 0$ and $\zeta^{\infty}_{\mathfrak{L}_{\mathcal{G}}}(z) = 1$.

For a left-resolving λ-graph system $\mathfrak{L}$, we put

$$V_{\mathfrak{L}} = \{(\alpha, u) \in \Sigma \times \Omega_{\mathfrak{L}} \mid \exists v \in \Omega_{\mathfrak{L}};\ (v, \alpha, u) \in E_{\mathfrak{L}}\}.$$

For $(\alpha, u), (\beta, v) \in V_{\mathfrak{L}}$, define a labeled edge from (β, v) to (α, u) with label α if $(v, \alpha, u) \in E_{\mathfrak{L}}$. Let us denote by $\lambda_{\mathfrak{L}}$ the labeling map. Then we have a left-resolving labeled graph $(V_{\mathfrak{L}}, E_{\mathfrak{L}}, \lambda_{\mathfrak{L}})$ over Σ. We denote it by $\mathcal{G}_{\mathfrak{L}}$. Put

$$Z_{\mathcal{G}_{\mathfrak{L}}} = \{(\alpha_i, u_i)_{i \in \mathbb{Z}} \mid (\alpha_i, u_i) \in V_{\mathfrak{L}}, (u_{i-1}, \alpha_i, u_i) \in E_{\mathfrak{L}}, i \in \mathbb{Z}\},$$

the sequences of vertices concatenating in the graph $\mathcal{G}_{\mathfrak{L}}$. We denote the left-shift on $Z_{\mathcal{G}_{\mathfrak{L}}}$ by $\sigma_{\mathcal{G}_{\mathfrak{L}}}$. One may naturally identify the topological dynamical systems $(Z_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$ and $(Z_{\mathcal{G}_{\mathfrak{L}}}, \sigma_{\mathcal{G}_{\mathfrak{L}}})$. Hence the space $Z^p_{\mathfrak{L}}$ is identified with the p-periodic points of $(Z_{\mathcal{G}_{\mathfrak{L}}}, \sigma_{\mathcal{G}_{\mathfrak{L}}})$.

LEMMA 5.1. *Let $\mathcal{G} = (G, \lambda)$ be a left-resolving finite labeled graph and $\mathfrak{L}_{\mathcal{G}}$ be the associated λ-graph system for $\mathcal{G}$. Then we have $V_{\mathfrak{L}_{\mathcal{G}}} = V_{[2]}$ and $E_{\mathfrak{L}_{\mathcal{G}}} = E_{[2]}$ in a natural way. Hence we have $\mathcal{G}_{\mathfrak{L}_{\mathcal{G}}} = \mathcal{G}_{[2]}$.*

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The zeta function $\zeta_G(z)$ for a finite directed graph G is the zeta function of the edge shift associated with G (cf. [**LM**, §6.4]). As the graph $G_{[2]}$ is strong shift equivalent to G as the vertex shifts (cf. [**LM**, Ch. 2]), one has $\zeta_{G_{[2]}}(z) = \zeta_G(z)$. By the preceding lemma, one sees that $N_{\mathfrak{L}_\mathcal{G}}(p) = N_{G_{[2]}}(p)$, $p \in \mathbb{N}$, and hence $\zeta_{\mathfrak{L}_\mathcal{G}}(z) = \zeta_{G_{[2]}}(z)$ for a left-resolving finite labeled graph $\mathcal{G} = (G, \lambda)$. Therefore, we make the following proposition.

PROPOSITION 5.2. *Let $\mathcal{G} = (G, \lambda)$ be a left-resolving finite labeled graph with underlying directed graph G. Let $\mathfrak{L}_\mathcal{G}$ be the associated λ-graph system for $\mathcal{G}$. Then we have*

$$N_{\mathfrak{L}_\mathcal{G}}(p) = N_G(p), \quad p \in \mathbb{N},$$

where $N_G(p)$ denotes the cardinal number of the p-periodic points of the vertices of the directed graph G. Hence we have

$$\zeta_{\mathfrak{L}_\mathcal{G}}(z) = \zeta_G(z) = \frac{1}{\det(I - zA_G)}, \quad \zeta^\infty_{\mathfrak{L}_\mathcal{G}}(z) = 1,$$

where A_G denotes the adjacency matrix of the graph G.

The proposition means that the pair of the zeta functions $\zeta_\mathfrak{L}(z)$ and $\zeta^\infty_\mathfrak{L}(z)$ for a λ-graph system naturally generalizes the ordinary zeta function $\zeta_G(z)$ of the labeled graphs.

6. *Examples of zeta functions of λ-graph systems*

6.1. *λ-graph systems presenting the full 2-shift Λ_2 over $\Sigma = \{\alpha, \beta\}$.*

6.1.1. Let $\mathcal{G}_{\Lambda_2}$ be the labeled graph with one vertex and two self-loops labeled α and β. Let $\mathfrak{L}_0$ be the associated λ-graph system $\mathfrak{L}_{\mathcal{G}_{\Lambda_2}}$ defined in the preceding section, which is the canonical λ-graph system $\mathfrak{L}^{\Lambda_2}$ of Λ_2. As the set $\Omega^p_{\mathfrak{L}_0}[\nu]$ is one point for every word ν of Σ, their p-periodic dimension groups and the p-least periodic dimension groups are

$$\triangle_{\mathfrak{L}_0}(p) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{2^p}, \quad \widehat{\triangle}_{\mathfrak{L}_0}(p) = 0,$$

$$\triangle_{\mathfrak{L}_0}([p]) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\sum_{q|p} \mu(p/q)2^q}, \quad \widehat{\triangle}_{\mathfrak{L}_0}([p]) = 0,$$

for all $p \in \mathbb{N}$, where $\mu(n)$, $n \in \mathbb{N}$ is the Möbius function, so that $\mathfrak{L}_0$ is periodically faithful for Λ_2. As $N_{\mathfrak{L}_0}[\nu] = 1$ for every word ν, one has $N_{\mathfrak{L}_0}(p) = 2^p$, $N^\infty_{\mathfrak{L}_0}(p) = 0$ so that the zeta functions are

$$\zeta_{\mathfrak{L}_0}(z) = \exp\left[\sum_{n=1}^\infty \frac{2^n}{n} z^n\right] = \exp[-\log(1 - 2z)] = \frac{1}{1 - 2z} = \zeta_{\Lambda_2}(z), \quad \zeta^\infty_{\mathfrak{L}_0}(z) = 1,$$

where $\zeta_{\Lambda_2}(z)$ is the zeta function of the full 2-shift (cf. [**LM**, Example 6.4.3]).

6.1.2. Let $\mathfrak{L}_1$ be the λ-graph system whose symbolic matrix system $(\mathcal{M}^1_{l,l+1}, I^1_{l,l+1})$, $l \in \mathbb{Z}_+$ is defined by

$$\mathcal{M}^1_{0,1} = [\alpha + \beta], \quad I^1_{0,1} = [1],$$

$$\mathcal{M}^1_{1,2} = [\alpha + \beta, \alpha + \beta], \quad I^1_{1,2} = [1, 1],$$

$$\mathcal{M}^1_{l,l+1} = \begin{bmatrix} \alpha + \beta & & 0 \\ & \ddots & \\ 0 & & \alpha + \beta \end{bmatrix}, \quad I^1_{l,l+1} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}, \quad 2 \le l \in \mathbb{N}.$$

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There exist two ι-orbits each of which is the same as the preceding canonical λ-graph system $\mathfrak{L}_0$. Hence we have

$$\triangle_{\mathfrak{L}_1}(p) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{2^{p+1}}, \quad \widehat{\triangle}_{\mathfrak{L}_1}(p) = \mathbb{Z},$$

$$\triangle_{\mathfrak{L}_1}([p]) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\sum_{q|p} \mu(p/q)2^{q+1}}, \quad \widehat{\triangle}_{\mathfrak{L}_1}([p]) = \mathbb{Z},$$

for all $p \in \mathbb{N}$, so that $\mathfrak{L}_1$ is not periodically faithful for Λ_2. As $N_{\mathfrak{L}_1}[\nu] = 2$ for every word ν, one has $N_{\mathfrak{L}_1}(p) = 2^{p+1}$, $N^{\infty}_{\mathfrak{L}_1}(p) = 0$ so that the zeta functions are

$$\zeta_{\mathfrak{L}_1}(z) = \exp\left[\sum_{n=1}^{\infty} \frac{2^{n+1}}{n} z^n\right] = \exp[-2\log(1-2z)] = \frac{1}{(1-2z)^2},$$

$$\zeta^{\infty}_{\mathfrak{L}_1}(z) = 1.$$

6.1.3. Let $\mathfrak{L}_2$ be the λ-graph system whose symbolic matrix system $(\mathcal{M}^2_{l,l+1}, I^2_{l,l+1})$, $l \in \mathbb{Z}_+$ is defined by

$$\mathcal{M}^2_{0,1} = [\alpha + \beta, \alpha + \beta], \quad I^2_{0,1} = [1, 1]$$

and

$$\mathcal{M}^2_{l,l+1}(i,j) = \begin{cases} \alpha + \beta & \text{for } j = 2i-1, 2i, \\ 0 & \text{elsewhere,} \end{cases}$$

$$I^2_{l,l+1}(i,j) = \begin{cases} 1 & \text{for } j = 2i-1, 2i, \\ 0 & \text{elsewhere,} \end{cases}$$

for $i = 1, 2, \ldots, 2^l$, $j = 1, 2, \ldots, 2^{l+1}$, $l \in \mathbb{N}$. Hence the vertex set $V^2_l, l \in \mathbb{Z}_+$, of $\mathfrak{L}_2$ has 2^l vertices. Each of the vertices of V_l has two out-going edges labeled α, two out-going edges labeled β, one in-coming edge labeled α and one in-coming edge labeled β. Put a $2^{p^{l+1}} \times 2^{p^l}$ matrix

$$B_{l,l+1}(i,j) = \begin{cases} 1 & \text{for } i = (j-1)2^p, (j-1)2^p + 1, \ldots, j2^p, \\ 0 & \text{elsewhere,} \end{cases}$$

for $i = 1, 2, \ldots, 2^{p^{l+1}}$, $j = 1, 2, \ldots, 2^{p^l}$. Then by Theorem 3.5 we have

$$\triangle_{\mathfrak{L}_2}(p) = \varprojlim_{l \to \infty} \{B_{l,l+1} : \mathbb{Z}^{2^{p^l}} \longrightarrow \mathbb{Z}^{2^{p^{l+1}}}\}.$$

Hence $\triangle_{\mathfrak{L}_2}(p)$ is isomorphic to the abelian group $C(\mathcal{C}, \mathbb{Z})$ of all $\mathbb{Z}$-valued continuous functions on a Cantor discontinuum $\mathcal{C}$. Similarly, the groups $\widehat{\triangle}_{\mathfrak{L}_2}(p)$, $\triangle_{\mathfrak{L}_2}([p])$ and $\widehat{\triangle}_{\mathfrak{L}_2}([p])$ are all isomorphic to $C(\mathcal{C}, \mathbb{Z})$. As every word ν of Σ does not have finite periodic ι-orbits, we have $N_{\mathfrak{L}_2}[\nu] = 0$ and $N^{\infty}_{\mathfrak{L}_2}(p) = 2^p$ so that

$$\zeta_{\mathfrak{L}_2}(z) = 1, \quad \zeta^{\infty}_{\mathfrak{L}_2}(z) = \frac{1}{1-2z} = \zeta_{\Lambda_2}(z).$$

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6.1.4. Let $\mathfrak{L}_3$ be the λ-graph system whose symbolic matrix system $(\mathcal{M}^3_{l,l+1}, I^3_{l,l+1})$, $l \in \mathbb{Z}_+$ is defined by

$$\mathcal{M}^3_{0,1} = [\alpha + \beta, \alpha + \beta], \quad I^3_{0,1} = [1, 1]$$

and

$$\mathcal{M}^3_{l,l+1}(i, j) = \begin{cases} \alpha + \beta & \text{for } i = j, \text{ and } i = l+1, \ j = l+2 \\ 0 & \text{elsewhere,} \end{cases}$$

$$I^3_{l,l+1}(i, j) = \begin{cases} 1 & \text{for } i = j, \text{ and } i = l+1, \ j = l+2 \\ 0 & \text{elsewhere,} \end{cases}$$

for $i = 1, 2, \ldots, l+1$, $j = 1, 2, \ldots, l+2$, $l \in \mathbb{N}$. Hence the vertex set V^3_l of $\mathfrak{L}_3$ has $l+1$ vertices for $l \in \mathbb{Z}_+$. Each of the vertices of V_l except v^l_{l+1} has one out-going edge labeled α and one out-going edge labeled β. The vertex v^l_{l+1} has two out-going edges labeled α and two out-going edges labeled β. Each of the vertices of V_l has one in-coming edge labeled α and one in-coming edge labeled β. By Theorem 3.5 one sees that the groups $\triangle_{\mathfrak{L}_3}(p)$, $\widehat{\triangle}_{\mathfrak{L}_3}(p)$, $\triangle_{\mathfrak{L}_3}([p])$ and $\widehat{\triangle}_{\mathfrak{L}_3}([p])$ are all isomorphic to the group $\bigoplus_1^\infty \mathbb{Z} = \mathbb{Z}^\infty$ of the direct sums of the countably infinite copies of $\mathbb{Z}$. As every word ν of Σ does not have finite periodic ι-orbits, we have $N_{\mathfrak{L}_3}[\nu] = 0$ and $N^\infty_{\mathfrak{L}_3}(p) = 2^p$ so that

$$\zeta_{\mathfrak{L}_3}(z) = 1, \quad \zeta^\infty_{\mathfrak{L}_3}(z) = \frac{1}{1-2z} = \zeta_{\Lambda_3}(z).$$

We therefore conclude that these four λ-graph systems $\mathfrak{L}_0, \mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_3$ are not strong shift equivalent to each other.

6.2. *λ-graph systems presenting the even shift Λ^{ev} over $\Sigma = \{\alpha, \beta\}$.*

6.2.1. Let $\mathfrak{L}^{ev}_0$ be the λ-graph system associated with its left-Krieger cover of the even shift Λ^{ev} [**Kr2, Kr3**], which is the canonical λ-graph system of Λ^{ev}. It is the λ-graph system associated with the symbolic matrix

$$\begin{bmatrix} \alpha & \beta & 0 \\ \beta & 0 & \alpha \\ 0 & 0 & \beta \end{bmatrix}.$$

By Proposition 5.2 the zeta functions of $\mathfrak{L}^{ev}_0$ are

$$\zeta_{\mathfrak{L}^{ev}_0}(z) = \left[\det \left(1 - z \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right) \right]^{-1} = [(1-z)(1-z-z^2)]^{-1}, \quad \zeta^\infty_{\mathfrak{L}^{ev}_0}(z) = 1.$$

6.2.2. Let $\mathfrak{L}^{ev}_1$ be the λ-graph system associated with the left-Fischer cover of Λ_{ev} [**Fi**]. It is the λ-graph system associated with the symbolic matrix

$$\begin{bmatrix} \alpha & \beta \\ \beta & 0 \end{bmatrix}.$$

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By Proposition 5.2 the zeta functions of $\mathfrak{L}_1^{ev}$ are

$$\zeta_{\mathfrak{L}_1^{ev}}(z) = \left[\det\left(1 - z\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\right)\right]^{-1} = [1 - z - z^2]^{-1}, \quad \zeta^{\infty}_{\mathfrak{L}_1^{ev}}(z) = 1.$$

Both the zeta functions $\zeta_{\mathfrak{L}_0^{ev}}(z)$ and $\zeta_{\mathfrak{L}_1^{ev}}(z)$ are different from the ordinary zeta function $\zeta_{\Lambda^{ev}}(z) = (1+z)/(1-z-z^2)$ of the even shift (cf. [**Man, Bo**]).

6.3. *λ-graph systems presenting the Dyck shift D_2 with four brackets.* The Dyck shift can be defined by the Dyck inverse monoid ([**Kr1, Kr5**], cf. [**ChS**]). The Dyck inverse monoid is the inverse monoid (with zero) with generators $\alpha_1, \alpha_2, \beta_1, \beta_2$ and relations

$$\alpha_1\beta_1 = \alpha_2\beta_2 = 1, \quad \alpha_1\beta_2 = \alpha_2\beta_1 = 0.$$

The Dyck shift D_2 is defined as the subshift $D_2 \subset \{\alpha_1, \alpha_2, \beta_1, \beta_2\}^{\mathbb{Z}}$, whose admissible words $(\gamma_i)_{0\le i\le I}$, $\gamma_i \in \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ satisfy the condition

$$\prod_{0\le i\le I} \gamma_i \neq 0.$$

6.3.1. The Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_2)}$ for the Dyck shift D_2 is defined in [**KM2**]. Its vertices at level l are the words of $\{\beta_1, \beta_2\}$ of length l. Hence the cardinal number of the vertex set at level l is 2^l. The mapping ι deletes the rightmost symbol of a word. There is an edge labeled α_j from the vertex $(\beta_{i(n)})_{2\le n\le l}$, to the vertex $(\beta_{i(n)})_{1\le n\le l}$ precisely if $i(1) = j$. There is an edge labeled β_j from the vertex $(\beta_{i(n)})_{1\le n\le l-2}$ to the vertex $(\beta_{i(n)})_{2\le n\le l}$ precisely if $i(1) = j$. It is straightforward to see that for a word $\mu \in W(\Lambda^p_{\mathfrak{L}^{Ch(D_2)}})$ the set $\Omega^p_{\mathfrak{L}^{Ch(D_2)}}[\mu]$ is homeomorphic to a Cantor discontinuum $\mathcal{C}$ if and only if p is even and the reduced form $\mathrm{red}(\mu)$ of μ is of the form $\beta_{i_1}\cdots\beta_{i_m}\alpha_{i_m}\cdots\alpha_{i_1}$ for some $m \le p/2$; otherwise $\Omega^p_{\mathfrak{L}^{Ch(D_2)}}[\mu]$ is one point. Hence we have

$$\triangle_{\mathfrak{L}^{Ch(D_2)}}(p) = \underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{N_{\mathfrak{L}^{Ch(D_2)}}(p)} \oplus \underbrace{C(\mathcal{C},\mathbb{Z})\oplus\cdots\oplus C(\mathcal{C},\mathbb{Z})}_{N^{\infty}_{\mathfrak{L}^{Ch(D_2)}}(p)},$$

$$\widehat{\triangle}_{\mathfrak{L}^{Ch(D_2)}}(p) = \underbrace{C(\mathcal{C},\mathbb{Z})\oplus\cdots\oplus C(\mathcal{C},\mathbb{Z})}_{N^{\infty}_{\mathfrak{L}^{Ch(D_2)}}(p)}.$$

By [**HI**] (and also [**I**]), the cardinal number of the words in $W(\Lambda^n_{\mathfrak{L}^{Ch(D_2)}})$ whose reduced form is of the form $\beta_{i_1}\cdots\beta_{i_m}\alpha_{i_m}\cdots\alpha_{i_1}$ for some $m \le n/2$ is calculated to be $\binom{n}{n/2}2^{n/2}$. Also [**HI**] shows that

$$|W(\Lambda^n_{\mathfrak{L}^{Ch(D_2)}})| = \begin{cases} 2\left\{3^n - \displaystyle\sum_{i=0}^{(n-1)/2}\binom{n}{i}2^i\right\} & \text{for } n \text{ odd},\\ 2\left\{3^n - \displaystyle\sum_{i=0}^{n/2}\binom{n}{i}2^i\right\} + \binom{n}{n/2}2^{n/2} & \text{for } n \text{ even}. \end{cases}$$

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Hence we have

$$N_{\mathfrak{L}^{Ch(D_2)}}(n) = \begin{cases} 2\left\{3^n - \sum_{i=0}^{(n-1)/2} \binom{n}{i} 2^i\right\} & \text{for } n \text{ odd}, \\ 2\left\{3^n - \sum_{i=0}^{n/2} \binom{n}{i} 2^i\right\} & \text{for } n \text{ even}, \end{cases}$$

and

$$N^{\infty}_{\mathfrak{L}^{Ch(D_2)}}(n) = \begin{cases} 0 & \text{for } n \text{ odd}, \\ \binom{n}{n/2} 2^{n/2} & \text{for } n \text{ even}. \end{cases}$$

In [**HI**] (and also [**I**]), the following equalities have been shown:

$$\sum_{n\,\text{odd}}^{\infty} \frac{z^n}{n} \sum_{i=0}^{(n-1)/2} \binom{n}{i} 2^i = \frac{1}{4} \log \frac{3 + 8z + \sqrt{1 - 8z^2}}{3 - 8z + \sqrt{1 - 8z^2}}, \tag{6.1}$$

$$\begin{aligned} \sum_{n\,\text{even}}^{\infty} \frac{z^n}{n} \sum_{i=0}^{n/2} \binom{n}{i} 2^i = &-\frac{1}{2} \log(1 + \sqrt{1 - 8z^2}) \\ &- \frac{1}{4} \log(3 + 8z + \sqrt{1 - 8z^2}) \\ &- \frac{1}{4} \log(3 - 8z + \sqrt{1 - 8z^2}) + \frac{3}{2} \log 2, \end{aligned} \tag{6.2}$$

$$\sum_{n\,\text{even}}^{\infty} \frac{z^n}{n} \binom{n}{n/2} 2^{n/2} = -\log(1 + \sqrt{1 - 8z^2}) + \log 2. \tag{6.3}$$

As $\sum_{n=1}^{\infty} 3^n/nz^n = -\log(1 - 3z)$, as a result of the above equalities, we have

$$\begin{aligned} \zeta_{\mathfrak{L}^{Ch(D_2)}}(z) &= \frac{(1 - 2z)\sqrt{1 - 8z^2} + 1 - 2z - 2z^2}{2(1 - 3z)^2}, \\ \zeta^{\infty}_{\mathfrak{L}^{Ch(D_2)}}(z) &= \frac{1 - \sqrt{1 - 8z^2}}{4z^2}. \end{aligned} \tag{6.4}$$

The ordinary zeta function $\zeta_{D_2}(z)$ for the Dyck shift is proved to be

$$\zeta_{D_2}(z) = \frac{3 - 8z + \sqrt{1 - 8z^2}}{4(1 - 3z)^2}$$

by [**Ke**]. Hence we know that the product formula

$$\zeta_{D_2}(z) = \zeta_{\mathfrak{L}^{Ch(D_2)}}(z) \cdot \zeta^{\infty}_{\mathfrak{L}^{Ch(D_2)}}(z) \tag{6.5}$$

holds.

6.3.2. The canonical λ-graph system $\mathfrak{L}^{C(D_2)}$ for the Dyck shift D_2 has been studied in [**Ma4**]. Its vertices at level l are the words of $\{\beta_1, \beta_2\}$ of length less than or equal to l. Hence the cardinal number of the vertex set at level l is $2^{l+1} - 1$. The mapping ι deletes the rightmost symbol of a word. There is an edge labeled α_j from the vertex

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$(\beta_{i(n)})_{2\le n\le k}, k \le l$ to the vertex $(\beta_{i(n)})_{1\le n\le k}$ precisely if $i(1) = j$. There is an edge labeled β_j from the vertex $(\beta_{i(n)})_{1\le n\le k-2}, k \le l$ to the vertex $(\beta_{i(n)})_{2\le n\le k}$ precisely if $i(1) = j$. The λ-graph system $\mathfrak{L}^{C(D_2)}$ includes the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_2)}$ as a λ-graph subsystem. It is straightforward to see that for a word $\mu \in W(\Lambda^p_{\mathfrak{L}^{C(D_2)}})$

$$|\Omega^p_{\mathfrak{L}^{C(D_2)}}[\mu]| = \begin{cases} 1 & \text{if } \mathrm{red}(\mu) = \beta_{i_1}\cdots\beta_{i_m} \text{ for some } m, \\ 2 & \text{if } \mathrm{red}(\mu) = \alpha_{i_1}\cdots\alpha_{i_m} \text{ for some } m, \\ \infty & \text{if } \mathrm{red}(\mu) = \beta_{i_1}\cdots\beta_{i_m}\alpha_{i_m}\cdots\alpha_{i_1} \text{ for some } m. \end{cases}$$

Let $J_{l,l+1}$ be a $2^l \times 2^{l+1}$ matrix for $l \in \mathbb{Z}_+$ defined by

$$J_{l,l+1}(i,j) = \begin{cases} 1 & \text{if } j = i,\ i = 1, 2, \ldots, 2^l, \\ 1 & \text{if } j = 2i-1, 2i,\ i = 2^{l-1}+1, \ldots, 2^l, \\ 0 & \text{otherwise.} \end{cases}$$

Let S_l denote the discrete set consisting of 2^l-points. Let $\widetilde{\mathcal{C}}$ be a zero-dimensional compact Hausdorff space defined by the projective limit

$$\widetilde{\mathcal{C}} = \varprojlim_{l}\{J_{l,l+1} : S_{l+1} \longrightarrow S_l\}.$$

The set $\Omega^p_{\mathfrak{L}^{C(D_2)}}[\mu]$ is homeomorphic to $\widetilde{\mathcal{C}}$ if and only if p is even and the reduced form of μ is of the form $\beta_{i_1}\cdots\beta_{i_m}\alpha_{i_m}\cdots\alpha_{i_1}$ for some $m \le p/2$; otherwise $\Omega^p_{\mathfrak{L}^{C(D_2)}}[\mu]$ is one point or two points. Hence we have

$$\triangle_{\mathfrak{L}^{C(D_2)}}(p) = \underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{N_{\mathfrak{L}^{C(D_2)}}(p)} \oplus \underbrace{C(\widetilde{\mathcal{C}},\mathbb{Z})\oplus\cdots\oplus C(\widetilde{\mathcal{C}},\mathbb{Z})}_{N^{\infty}_{\mathfrak{L}^{C(D_2)}}(p)}.$$

We notice that the space $\widetilde{\mathcal{C}}$ is not perfect, so that it is not homeomorphic to the Cantor discontinuum. By Theorem 3.5, we know that

$$\widehat{\triangle}_{\mathfrak{L}^{C(D_2)}}(p) = \underbrace{C(\widetilde{\mathcal{C}},\mathbb{Z})\oplus\cdots\oplus C(\widetilde{\mathcal{C}},\mathbb{Z})}_{N^{\infty}_{\mathfrak{L}^{C(D_2)}}(p)}.$$

By [**HI**] (and also [**I**]), the cardinal number of the set of words in $W(\Lambda^n_{\mathfrak{L}^{C(D_2)}})$ whose reduced form is of the form $\beta_{i_1}\cdots\beta_{i_m}$ for some m is calculated to be

$$\begin{cases} 3^n - \displaystyle\sum_{i=0}^{(n-1)/2}\binom{n}{i}2^i & \text{for } n \text{ odd}, \\ 3^n - \displaystyle\sum_{i=0}^{n/2}\binom{n}{i}2^i - \binom{n}{n/2}2^{n/2} & \text{for } n \text{ even}, \end{cases}$$

that is, the same as the cardinal number of the set of words in $W(\Lambda^n_{\mathfrak{L}^{C(D_2)}})$ whose reduced form is of the form $\alpha_{i_1}\cdots\alpha_{i_m}$ for some $m \le n/2$. Hence one sees

$$N_{\mathfrak{L}^{C(D_2)}}(n) = \begin{cases} 3\left\{3^n - \displaystyle\sum_{i=0}^{(n-1)/2}\binom{n}{i}2^i\right\} & \text{for } n \text{ odd}, \\ 3\left\{3^n - \displaystyle\sum_{i=0}^{n/2}\binom{n}{i}2^i - \binom{n}{n/2}2^{n/2}\right\} & \text{for } n \text{ even}, \end{cases}$$

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and

$$N^{\infty}_{\mathfrak{L}^{C(D_2)}}(n) = N^{\infty}_{\mathfrak{L}^{Ch(D_2)}}(n) = \begin{cases} 0 & \text{for } n \text{ odd}, \\ \binom{n}{n/2} 2^{n/2} & \text{for } n \text{ even}. \end{cases}$$

Therefore, one obtains by (6.1)–(6.4),

$$\zeta_{\mathfrak{L}^{C(D_2)}}(z) = \frac{\{(1-2z)\sqrt{1-8z^2}+1-2z-2z^2\}^{3/2}}{2^{9/2}(1-3z)^3},$$

$$\zeta^{\infty}_{\mathfrak{L}^{C(D_2)}}(z) = \frac{1-\sqrt{1-8z^2}}{4z^2}.$$

Since the equalities

$$\zeta_{\mathfrak{L}^{Ch(D_2)}}(z) = 4\zeta_{\mathfrak{L}^{C(D_2)}}(z)^{2/3}, \quad \zeta^{\infty}_{\mathfrak{L}^{Ch(D_2)}}(z) = \zeta^{\infty}_{\mathfrak{L}^{C(D_2)}}(z),$$

hold, the equality (6.5) implies the formula

$$\zeta_{D_2}(z) = 4\zeta_{\mathfrak{L}^{C(D_2)}}(z)^{2/3} \cdot \zeta^{\infty}_{\mathfrak{L}^{C(D_2)}}(z).$$

The zeta function $\zeta_{\mathfrak{L}^{C(D_2)}}(z)$ of $\mathfrak{L}^{C(D_2)}$ is different from the zeta function $\zeta_{\mathfrak{L}^{Ch(D_2)}}(z)$ of $\mathfrak{L}^{Ch(D_2)}$. This implies that $\mathfrak{L}^{C(D_2)}$ is not strong shift equivalent to $\mathfrak{L}^{Ch(D_2)}$.

6.4. *A λ-graph system presenting a subsystem of the Dyck shift D_2.* Let Z_2 be the subsystem of the Dyck shift D_2 consisting of all bi-infinite sequences in D_2 that do not include the symbol α_1. It is the intersection between D_2 and the full 3-shift with the symbols $\{\alpha_2, \beta_1, \beta_2\}$. The periodic points of the subshift Z_2 and its ordinary zeta function have been studied by Hamachi and Inoue [**HI**] (cf. [**I**]). Let $\mathfrak{L}^{Z_2}$ be the λ-graph system over $\{\alpha_2, \beta_1, \beta_2\}$ whose vertex set with ι-map is the same as that of $\mathfrak{L}^{Ch(D_2)}$ and whose edge set is the subset of the edge set of $\mathfrak{L}^{Ch(D_2)}$ obtained by deleting the edges labeled α_1. It is evident to see that the λ-graph system $\mathfrak{L}^{Z_2}$ presents the subshift Z_2. Since $\mathfrak{L}^{Z_2}$ is a subsystem of $\mathfrak{L}^{Ch(D_2)}$, the preceding discussions for $\mathfrak{L}^{Ch(D_2)}$ are useful for studying the periodic points of $\mathfrak{L}^{Z_2}$. We then see that for a word $\mu \in W(\Lambda^p_{\mathfrak{L}^{Z_2}})$ the set $\Omega^p_{\mathfrak{L}^{Z_2}}[\mu]$ is homeomorphic to a Cantor discontinuum $\mathcal{C}$ if and only if p is even and the reduced form $\mathrm{red}(\mu)$ of μ is of the form

$$\underbrace{\beta_2 \cdots \beta_2}_{m}\underbrace{\alpha_2 \cdots \alpha_2}_{m}$$

for some $m \leq p/2$; otherwise $\Omega^p_{\mathfrak{L}^{Z_2}}[\mu]$ is one point. Hence we have

$$\triangle_{\mathfrak{L}^{Z_2}}(p) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{N_{\mathfrak{L}^{Z_2}}(p)} \oplus \underbrace{C(\mathcal{C}, \mathbb{Z}) \oplus \cdots \oplus C(\mathcal{C}, \mathbb{Z})}_{N^{\infty}_{\mathfrak{L}^{Z_2}}(p)},$$

$$\widehat{\triangle}_{\mathfrak{L}^{Z_2}}(p) = \underbrace{C(\mathcal{C}, \mathbb{Z}) \oplus \cdots \oplus C(\mathcal{C}, \mathbb{Z})}_{N^{\infty}_{\mathfrak{L}^{Z_2}}(p)}.$$

By [**HI**], the periodic points of the subshift Z_2 are calculated as

$$|W(\Lambda^n_{\mathfrak{L}^{Z_2}})| = \begin{cases} \displaystyle\sum_{i=0}^{(n-1)/2} \binom{n}{i} + \sum_{i=0}^{(n-1)/2} \binom{n}{i} 2^{n-2i} & \text{for } n \text{ odd}, \\ \displaystyle\sum_{i=0}^{n/2-1} \binom{n}{i} + \sum_{i=0}^{n/2-1} \binom{n}{i} 2^{n-2i} + \binom{n}{n/2} & \text{for } n \text{ even}. \end{cases}$$

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Since the cardinal number of the words in $W(\Lambda^n_{\mathfrak{L}_{Z_2}})$ whose reduced form is of the form $\underbrace{\beta_2\cdots\beta_2}_{m}\underbrace{\alpha_2\cdots\alpha_2}_{m}$ for some $m \le n/2$ is $\binom{n}{n/2}$, we have

$$N_{\mathfrak{L}_{Z_2}}(n) = \begin{cases} \sum_{i=0}^{(n-1)/2}\binom{n}{i} + \sum_{i=0}^{(n-1)/2}\binom{n}{i}2^{n-2i} & \text{for } n \text{ odd}, \\ \sum_{i=0}^{n/2-1}\binom{n}{i} + \sum_{i=0}^{n/2-1}\binom{n}{i}2^{n-2i} & \text{for } n \text{ even} \end{cases}$$

and

$$N^{\infty}_{\mathfrak{L}_{Z_2}}(n) = \begin{cases} 0 & \text{for } n \text{ odd}, \\ \binom{n}{n/2} & \text{for } n \text{ even}. \end{cases}$$

In [**HI**] (and also [**I**]), the following equalities have been shown:

$$\sum_{n\,\text{odd}}^{\infty}\frac{z^n}{n}\sum_{i=0}^{(n-1)/2}\binom{n}{i} = \frac{1}{4}\log\frac{1+2z}{1-2z},$$

$$\sum_{n\,\text{odd}}^{\infty}\frac{z^n}{n}\sum_{i=0}^{(n-1)/2}\binom{n}{i}2^{n-2i} = \frac{1}{2}\log\frac{2+5z}{2-5z} + \frac{1}{4}\log\frac{5-8z+3\sqrt{1-4z^2}}{5+8z+3\sqrt{1-4z^2}},$$

$$\sum_{\substack{n\,\text{even}\\ n\ge 2}}^{\infty}\frac{z^n}{n}\sum_{i=0}^{n/2}\binom{n}{i} = \frac{1}{2}\log\frac{1-z^2}{1+\sqrt{1-4z^2}} - \frac{1}{4}\log(1-4z^2) + \frac{1}{2}\log 2,$$

$$\sum_{\substack{n\,\text{even}\\ n\ge 2}}^{\infty}\frac{z^n}{n}\sum_{i=0}^{n/2}\binom{n}{i}2^{n-2i} = \frac{1}{2}\log\frac{25z^2-4}{1+\sqrt{1-4z^2}} + \frac{1}{4}\log\{(5-8z+3\sqrt{1-4z^2})(5+8z+3\sqrt{1-4z^2})\}.$$

As

$$\sum_{\substack{n\,\text{even}\\ n\ge 2}}^{\infty}\frac{z^n}{n}\binom{n}{n/2} = \log\frac{2}{1+\sqrt{1-4z^2}},$$

as a result of the above equalities, we have

$$\zeta_{\mathfrak{L}_{Z_2}}(z) = \frac{-6z+3+\sqrt{1-4z^2}}{2(2-5z)(1-2z)}, \quad \zeta^{\infty}_{\mathfrak{L}_{Z_2}}(z) = \frac{1-\sqrt{1-4z^2}}{2z^2}.$$

The ordinary zeta function $\zeta_{Z_2}(z)$ for the subshift Z_2 is proved to be

$$\zeta_{Z_2}(z) = \frac{2z^2-3z+1+(3z-1)\sqrt{1-4z^2}}{2z^2(2-5z)(1-2z)}$$

by [**HI**]. Hence the product formula

$$\zeta_{Z_2}(z) = \zeta_{\mathfrak{L}_{Z_2}}(z)\cdot\zeta^{\infty}_{\mathfrak{L}_{Z_2}}(z)$$

holds.

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