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A CHARACTERIZATION OF SPECIAL SUBVARIETIES

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Abstract. We prove that an algebraic subvariety of a Shimura variety is weakly special if and only if analytic components of its preimage in the symmetric space are algebraic. We also prove an analogous result in the case of abelian varieties.

§1. *Introduction.* The aim of this note is to obtain a new characterization of special subvarieties of Shimura varieties. For generalities on Shimura varieties we refer to [3, 5] or [12].

Let (G, X) be a Shimura datum and X^+ a connected component of X. We let K be a compact open subgroup of $G(\mathbb{A}_f)$ and $\Gamma := G(\mathbb{Q})_+ \cap K$ where $G(\mathbb{Q})_+$ denotes the stabilizer in $G(\mathbb{Q})$ of X^+. We let $S := \Gamma\backslash X^+$, a connected component of $\mathrm{Sh}_K(G, X)$.

A special subvariety of S is a subvariety of Hodge type in the sense of [15]. In §2 we give a description of slightly more general notion of weakly special subvarieties in terms of sub-Shimura data of (G, X). In [15] Moonen proves that a subvariety of S is weakly special if and only if it is a totally geodesic submanifold of S. A special point is a special subvariety of dimension zero and a weakly special subvariety containing a special point is special.

Special subvarieties are interesting for many reasons, one of which is the following conjecture.

CONJECTURE 1.1 (André–Oort). *Let Z be an irreducible subvariety of $\mathrm{Sh}_K(G, X)$ containing a Zariski-dense set of special points. Then Z is special.*

This conjecture has recently been proved under the assumption of the generalized Riemann hypothesis for complex multiplication (CM) fields (see [10, 22]). Part of the strategy consisted in establishing a geometric characterization of special subvarieties of Shimura varieties. This criterion says roughly that subvarieties contained in their image by certain Hecke correspondences are special.

Very recently Pila came up with a new and very promising strategy for attacking the André–Oort conjecture unconditionally (see [18, 19]). A step in this strategy consists in establishing a criterion for a subvariety of S to contain special subvarieties involving certain algebraicity properties of their preimages in X^+. Let us explain this in more detail. The Borel embedding of X^+ into its compact dual $X^{+\vee}$ (see §3) gives meaning to the notion of algebraicity of subsets of X^+. Namely, a subset Y of X^+ is algebraic if there exists an algebraic subset Z

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of $X^{+\vee}$ such that $Y = Z \cap X^+$. Let $\pi : X^+ \longrightarrow S$ be the natural projection. Let Z be an irreducible subvariety of S and let $\widetilde{Z} := \pi^{-1}(Z)$. Let $\widetilde{Z}^{\mathrm{alg}}$ be the union of *maximal* algebraic subsets (in the sense explained above) of $\widetilde{Z}$. Pila conjectures that $\widetilde{Z}^{\mathrm{alg}}$ consists exactly of preimages of weakly special subvarieties of Z. He establishes this conjecture in the case where the Shimura variety is a product of modular curves. The general case, however, seems to be very hard. In this paper we establish the following criterion for a subvariety to be special which implies Pila's conjecture in the case where Z is a curve.

THEOREM 1.2. *An irreducible subvariety Z of S is weakly special if and only if some (equivalently any) analytic component of $\pi^{-1}Z$ is algebraic in the sense explained above.*

Suppose that Z is a curve. Let Y be an analytic component of $\widetilde{Z} := \pi^{-1}(Z)$. Suppose that Y is algebraic, hence Y is a component of Z^{alg}. As $\dim(Y) = 1$, we have $\pi(Y) = Z$ and our theorem shows that Z is special, as predicted by Pila's conjecture.

The main ingredient of the proof is a well-known statement about the image of the monodromy representation associated to the smooth locus of Z due to Deligne [**4**] and André [**1**].

The compact dual $X^{+\vee}$ of X^+ has a natural model over $\overline{\mathbb{Q}}$ (see §3.3). We say that a point of X^+ is algebraic if its image by the Borel embedding is in $X^{+\vee}(\overline{\mathbb{Q}})$. With this definition we check that CM points of X^+ are algebraic (Proposition 3.7).

Let Y be an irreducible algebraic subvariety of X^+. Hence $Y = X^+ \cap Z$ for an irreducible algebraic subvariety of $X^{+\vee}$. We say that Y is defined over $\overline{\mathbb{Q}}$ if Z is defined over $\overline{\mathbb{Q}}$. We will check that if Z is a special subvariety of S then any analytic component of $\pi^{-1}Z$ is defined over $\overline{\mathbb{Q}}$. Note also that S has a canonical model over $\overline{\mathbb{Q}}$ and that special subvarieties of S are defined over $\overline{\mathbb{Q}}$.

We recall the following result due to Cohen [**2**] and Shiga–Wolfart [**21**].

THEOREM 1.3. *Let $\pi : X^+ \longrightarrow S := \Gamma\backslash X^+$ be the map uniformizing the Shimura variety S. Assume that S is a Shimura variety of abelian type. Let x be a point of $X^+(\overline{\mathbb{Q}})$ such that $\pi(x) \in S(\overline{\mathbb{Q}})$. Then x is a CM point.*

In [**2**] the theorem is only given for Shimura varieties of Hodge type (special subvarieties of some moduli space of polarized abelian varieties with some level structure) but the extension to Shimura varieties of abelian type is straightforward. See [**16**, 2.10] for the definition and the group theoretic description of Shimura varieties of abelian type.

A corollary of our main result is the following generalization.

THEOREM 1.4. *Let $\pi : X^+ \longrightarrow S := \Gamma\backslash X^+$ be the map uniformizing the Shimura variety S. Assume that S is a Shimura variety of abelian type. Let $\tilde{Y}$ be an algebraic subvariety of X^+ defined over $\overline{\mathbb{Q}}$ such that $Y := \pi(\tilde{Y})$ is an algebraic subvariety of S defined over $\overline{\mathbb{Q}}$. Then Y is a special subvariety of S.*

In the last section we indicate the easy analog of the main result in the context of abelian varieties. The principle of the proof is the same as in the Shimura

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case. The monodromy theorem of Deligne and André in this case is replaced by an elementary property of the Albanese variety.

§2. *Weakly special subvarieties.*

Definition 2.1. Let (G, X) be a Shimura datum and let K be a compact open subgroup of $G(\mathbb{A}_f)$. An algebraic subvariety Z of $\mathrm{Sh}_K(G, X)$ is weakly special if there exists a sub-Shimura datum (H, X_H) of (G, X) and a decomposition $(H^{\mathrm{ad}}, X_H^{\mathrm{ad}}) = (H_1, X_1) \times (H_2, X_2)$ and a point $y_2 \in X_2$ such that Z is the image of $X_1^+ \times \{y_2\}$ in $\mathrm{Sh}_K(G, X)$.

In this situation, Z is special (or Hodge type) if and only if y_2 is a special point of X_2. In particular Z is special if and only if it contains a special point.

The image of $X_1^+ \times \{y_2\}$ in $\mathrm{Sh}_K(G, X)$ here is defined as follows. We have an inclusion $X_H \subset X_H^{\mathrm{ad}} = X_1 \times X_2$ which induces an identification $X_H^+ = X_1^+ \times X_2^+$. Via this identification we consider $X_1^+ \times \{y_2\}$ as a subset of X_H and take its image in $\mathrm{Sh}_K(G, X)$.

In [15], Moonen proves that an irreducible subvariety Z of $\mathrm{Sh}_K(G, X)$ is weakly special if and only if it is totally geodesic.

The following properties follow immediately from the definition.

PROPOSITION 2.2.

(1) *Let $K' \subset K$ be a compact open subgroup and let $f : \mathrm{Sh}_{K'}(G, X) \longrightarrow \mathrm{Sh}_K(G, X)$ be the map induced by the inclusion. A subvariety Z of $\mathrm{Sh}_K(G, X)$ is weakly special if one (equivalently any) component of its preimage in $\mathrm{Sh}_{K'}(G, X)$ is weakly special.*

(2) *Let $g \in G(\mathbb{A}_f)$ and T_g be the corresponding Hecke correspondence. A subvariety Z of $\mathrm{Sh}_K(G, X)$ is weakly special if and only if components of its image by T_g are weakly special.*

(3) *Let $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ be the adjoint Shimura datum associated to (G, X). Choose a compact open subgroup K^{ad} containing the image of K and let $\alpha : \mathrm{Sh}_K(G, X) \longrightarrow \mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ be the corresponding morphism. A subvariety Z of $\mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is weakly special if one (equivalently any) component of its preimage in $\mathrm{Sh}_K(G, X)$ is weakly special.*

For the third property, it suffices to notice that the components of X and X^{ad} are the same.

§3. *Algebraic structure of hermitian symmetric domains.*

3.1. *Borel and Harish-Chandra's embeddings.* Let (G, X) be a Shimura datum and let $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ be the associated adjoint Shimura datum. As connected components of X and X^{ad} are the same, we can and do assume that $G = G^{\mathrm{ad}}$. This does not cause any loss of generality.

Let X^+ be a connected component of X. Fix a point $x \in X^+$ and let $h_x : \mathbb{S} \to G_{\mathbb{R}}$ be the corresponding morphism. Let K_x be the associated maximal compact subgroup of $G(\mathbb{R})$. By definition K_x is the centralizer of $x(\sqrt{-1})$ in $G(\mathbb{R})$ and

$$X^+ = G(\mathbb{R})/K_\infty.$$

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We refer to Mok's book [14, 3.3] for the construction of the Borel embedding. Attached to x, there is a parabolic subgroup P_x of $G_{\mathbb{C}}$ such that

$$K_x = G(\mathbb{R}) \cap P_x(\mathbb{C}).$$

In §3.3 we'll describe P_x as the stabilizer of the filtration induced by x on a rational faithful representation of G. The compact dual $X^{+\vee}$ is the complex homogeneous space $G(\mathbb{C})/P_x$ and the Borel embedding is given by the natural inclusion

$$X = G(\mathbb{R})/K_x \hookrightarrow X^{+\vee} = G(\mathbb{C})/P_x(\mathbb{C}). \tag{1}$$

Definition 3.1. View X^+ as a subset of $X^{+\vee}$. A subset Y of X^+ is called irreducible algebraic if there exists an irreducible closed algebraic subset Z of $X^{+\vee}$ of dimension at least one such that $Y = Z \cap X^+$. An algebraic subset of X^+ is a finite union of irreducible algebraic subsets.

To clarify the picture it is useful to recall briefly the link between the Harish-Chandra and the Borel embeddings. For simplicity of notations, we will denote by X a connected component of X until the end of this section. We refer to [9, VIII-7] and [14, V] for a detailed discussion. Fix a point x_0 of X. Let $\mathfrak{p}^+$ be the holomorphic tangent bundle at x_0 in X. Then by results of Harish-Chandra X can be canonically realized as a bounded symmetric domain in $\mathfrak{p}^+ \simeq \mathbb{C}^N$. Moreover there exists a map

$$\eta : \mathfrak{p}^+ \longrightarrow X^\vee \tag{2}$$

which is an embedding onto a dense open subset of $X^\vee$. The restriction of η to $X = G(\mathbb{R})/K_\infty$ is the Borel embedding (1) or (5).

Remark 3.2. We could define algebraic subsets of positive dimension of X as intersections of closed algebraic subvarieties of $\mathfrak{p}^+$ of positive dimension with X. Using the previous discussion, we see that the two definitions coincide. Moreover the notion of algebraic subvarieties of X is independent of the choice of x_0.

The next lemma shows that algebraic subvarieties of hermitian symmetric domains have a good behaviour when we pass to a hermitian symmetric subdomain. This will be used in the proof of the Theorem 1.2.

LEMMA 3.3. *Assume that X is realized as a bounded symmetric domain in the holomorphic tangent bundle $\mathfrak{p}^+$ at some $x_0 \in X$. Let X' be a bounded symmetric subdomain of X containing x_0. Let $\mathfrak{p}'^+$ be the holomorphic tangent bundle at $x_0 \in X'$. Then an algebraic subvariety of X' in $\mathfrak{p}'^+$ is the intersection of an algebraic subvariety of X in $\mathfrak{p}^+$ with X'.*

Proof. Let $Y \subset X'$ be an algebraic subvariety. Then $Y = X' \cap Z$ for an algebraic subvariety of $\mathfrak{p}'^+$. Using [20, Chapter II, Proposition 8.1], we see that the inclusion map $X' \subset X$ is just the restriction of the inclusion map $\mathfrak{p}'^+ \subset \mathfrak{p}^+$. Therefore Z is an algebraic subvariety of $\mathfrak{p}^+$ and $Y = X' \cap (X \cap Z)$ is the intersection of an algebraic subvariety of X in $\mathfrak{p}^+$ with X'. □

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Explicit realizations of the four classical families of irreducible bounded symmetric domains have been given by É. Cartan. The reader can consult [9, Chapter X, Ex. D.1] for a summary and [14, Chapter IV] for proofs. Let us mention the two following examples.

Let p and q be two positive integers. Then

$$D^I_{p,q} := \{Z \in M_{p,q}(\mathbb{C}),\ Z^t\overline{Z} - I_q < 0\} \subset M_{p,q}(\mathbb{C}) \simeq \mathbb{C}^{pq} \tag{3}$$

is a bounded symmetric realization of $\mathrm{SU}(p, q)/\mathrm{SU}(p+q)$ and the inclusion $D^I_{p,q} \subset \mathbb{C}^{pq}$ is the Harish-Chandra embedding.

Let n be a positive integer, $S_n(\mathbb{C}) := \{Z \in M_{n,n}(\mathbb{C}),\ Z^t = Z\}$ be the set of symmetric complex matrices in $M_{n,n}(\mathbb{C})$. Then

$$D^{III}_n := \{Z \in S_n(\mathbb{C}),\ Z^t\overline{Z} - I_n < 0\} \subset S_n(\mathbb{C}) \simeq \mathbb{C}^{n(n+1)/2} \tag{4}$$

is a bounded symmetric realization of $\mathrm{Sp}(n, \mathbb{R})/U(n)$ and the inclusion $D^{III}_n \subset \mathbb{C}^{n(n+1)/2}$ is the Harish-Chandra embedding.

In addition to the four classical families of irreducible bounded symmetric domains there are two irreducible bounded symmetric domains of exceptional type and any bounded symmetric domain is a product of irreducible ones.

3.2. *Unbounded realizations.* It is sometimes useful in the theory of hermitian symmetric domains to use unbounded realizations. If X is a hermitian symmetric domain and X is realized as an open subset of some $\mathbb{C}^N$, we may define a notion of algebraic subvarieties of X by taking an intersection of an algebraic subvariety of $\mathbb{C}^N$ as in the Harish-Chandra realization of X. We would like to know that the algebraic subvarieties of X are in fact independent of the realization (bounded or unbounded). It may be possible to study this question using the results of Piatetskii-Shapiro [17] and Korányi–Wolf [11] (see also [20, Chapter III]). We will check this in the case of the generalized Siegel upper half plane which is the most relevant in the theory of Shimura varieties.

Let $X = \mathrm{Sp}(n, \mathbb{R})/U(n)$, then the bounded symmetric realization is D^{III}_n (4) and X has the unbounded realization

$$\mathbb{H}_n := \{Z \in M_{n,n}(\mathbb{C}),\ Z^t = Z,\ \Im(Z) > 0\} \subset S_n(\mathbb{C}) \simeq \mathbb{C}^{n(n+1)/2}$$

as a Siegel space.

The rational map $\Phi : \mathbb{C}^{n(n+1)/2} \to \mathbb{C}^{n(n+1)/2}$

$$Z \mapsto (I_n + \sqrt{-1}Z)(I_n - \sqrt{-1}Z)^{-1}$$

induces a biholomorphic transformation from $\mathbb{H}_n$ to D^{III}_n. If V is an algebraic subvariety of $\mathbb{C}^{n(n+1)/2}$ then

$$\Phi(V \cap \mathbb{H}_n) = \Phi(V) \cap D^{III}_n.$$

The map Φ establishes a bijection between the algebraic subvarieties of $\mathbb{H}_n$ and of D^{III}_n.

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The main Theorem 1.2 applied to the space $\mathbb{H}_n$ gives the following.

THEOREM 3.4. *Let $\mathcal{A}_n$ be the moduli space of principally polarized abelian varieties of dimension g and let $\pi : \mathbb{H}_n \longrightarrow \mathcal{A}_n$ be the uniformizing map.*

An irreducible subvariety Z of $\mathcal{A}_n$ is weakly special if and only if some (equivalently any) analytic component of $\pi^{-1}Z$ is an algebraic subvariety of $\mathbb{H}_n$.

3.3. *Hodge theoretic interpretation and rationality.* Let x be a point of X. Let $\mu_x : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$ be the cocharacter associated to x. Let M_X be the $G(\mathbb{C})$-conjugacy class of μ_x. Let V be a faithful representation of G on a $\mathbb{Q}$-vector space of finite dimension. Then μ_x defines a filtration

$$F_x^\star V_{\mathbb{C}} := \{\cdots \supset F_p V_{\mathbb{C}} \supset F_{p+1} V_{\mathbb{C}} \supset \cdots\}_{p\in\mathbb{Z}}$$

by $\mathbb{C}$-vector subspaces of $V_{\mathbb{C}}$.

Fix $x_0 \in X$ and let Q be the subgroup of $\mathrm{GL}(V_{\mathbb{C}})$ stabilizing $F_{x_0}^\star V_{\mathbb{C}}$. There exists a surjective map $M_X \to X^\vee$ sending μ_x to the associated filtration $F_x^\star V_{\mathbb{C}}$ and we can realize this way $X^\vee$ as a subvariety of the flag variety $\Theta_{\mathbb{C}} := \mathrm{GL}(V)/Q$. The Borel embedding (1) is the map $X \to X^\vee$ given by

$$x \mapsto F_x^* V_{\mathbb{C}}. \tag{5}$$

Note that $\Theta_{\mathbb{C}}$ has a natural model Θ over $\mathbb{Q}$. For any extension L of $\mathbb{Q}$ a point $z \in \Theta(L)$ defines a filtration

$$F_z^\star V_L := \{\cdots \supset F_p V_L \supset F_{p+1} V_L \supset \cdots\}_{p\in\mathbb{Z}}$$

by L-vector subspaces of V_L. In this situation the stabilizer of $F_z^\star V_L$ is a parabolic subgroup of GL_{V_L} conjugate in $\mathrm{GL}(V_{\mathbb{C}})$ to Q. Then $X^\vee$, as a subvariety of $\Theta_{\mathbb{C}}$ is defined over the reflex field $E(G, X)$ of (G, X). This is a direct consequence of the definition of the reflex field. See [**13**, III-1] for details about these constructions.

Definition 3.5. View X as a subset of $X^\vee \subset \Theta_{\mathbb{C}}$. A closed point P of X is said to be algebraic if $P \in X^\vee(\overline{\mathbb{Q}})$.

3.4. *Complex multiplication and rationality.*

Definition 3.6. Let $x \in X$; the Mumford–Tate group $MT(x)$ of x is the smallest $\mathbb{Q}$ subgroup of G such that there is a factorization of x

$$x : \mathbb{S} \to MT(x)_{\mathbb{R}} \hookrightarrow G_{\mathbb{R}}.$$

A point $x \in X$ is said to be special or CM if $MT(x)$ is a torus.

PROPOSITION 3.7. *View X as a subset of $X^\vee \subset \Theta_{\mathbb{C}}$. Then a CM point $x \in X$ is an algebraic point.*

Let x be a CM point of X. Then we have a factorization of the associated cocharacter

$$\mu_x : \mathbb{G}_{m_{\mathbb{C}}} \to MT(x)_{\mathbb{C}} \hookrightarrow G_{\mathbb{C}} \hookrightarrow \mathrm{GL}(V_{\mathbb{C}}).$$

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We can choose a $\mathbb{Q}$-torus T_x maximal in $\mathrm{GL}(V_{\mathbb{Q}})$ such that we have a factorization

$$\mu_x : \mathbb{G}_{m_{\mathbb{C}}} \to T_{x,\mathbb{C}} \hookrightarrow \mathrm{GL}(V_{\mathbb{C}}).$$

The cocharacter μ_x is defined over a number field, therefore we have a factorization

$$\mu_x : \mathbb{G}_{m_{\overline{\mathbb{Q}}}} \to T_{x,\overline{\mathbb{Q}}} \hookrightarrow \mathrm{GL}(V_{\overline{\mathbb{Q}}}).$$

Therefore the associated filtration $F_x^\star V_{\mathbb{C}}$ is fixed by $T_x(\overline{\mathbb{Q}})$. We just need to prove the following lemma.

LEMMA 3.8. *Let $y \in \Theta(\mathbb{C})$ and T_y be a maximal $\overline{\mathbb{Q}}$-torus in $\mathrm{GL}(V_{\overline{\mathbb{Q}}})$ such that the filtration $F_y^\star V_{\mathbb{C}}$ is fixed by $T_y(\overline{\mathbb{Q}})$. Then $y \in \Theta(\overline{\mathbb{Q}})$.*

Let Q_y be the stabilizer of $F_y^\star V_{\mathbb{C}}$ in $\mathrm{GL}(V_{\mathbb{C}})$. Then Q_y is a parabolic subgroup of $\mathrm{GL}(V_{\mathbb{C}})$ containing $T_{y,\mathbb{C}}$. We have a decomposition $Q_y = RM$ as an almost direct product with R the unipotent radical of Q_y and M a Levi subgroup of Q_y. Then $T_{y,\mathbb{C}}$ is contained in some conjugate of M and replacing M by this conjugate we may assume that $T_{y,\mathbb{C}} \subset M$. Let Z be the connected centre of M; then as Z commutes with $T_{y,\mathbb{C}}$ and as $T_{y,\mathbb{C}}$ is maximal in $\mathrm{GL}(V_{\mathbb{C}})$ we see that $Z \subset T_{y,\mathbb{C}}$. The subtori of $T_{\mathbb{C}}$ correspond to sub-$\mathbb{Z}$-modules of the group of character $X^*(T_{y,\mathbb{C}}) = X^*(T_{y,\overline{\mathbb{Q}}})$ and are therefore defined over $\overline{\mathbb{Q}}$. Therefore Z is defined over $\overline{\mathbb{Q}}$. As M is the centralizer in $\mathrm{GL}(V_{\mathbb{C}})$ of Z, M is defined over $\overline{\mathbb{Q}}$. As Q_y is the normalizer of M, Q_y is defined over $\overline{\mathbb{Q}}$. The associated point of the flag variety is therefore algebraic.

Remark 3.9. The Proposition 3.7 is well-known for the classical irreducible bounded symmetric domains such as $D^I_{p,q}$ or D^{III}_n. Note that in these cases the bounded symmetric domains are explicitly realized as sets matrices with complex coefficients and an algebraic point is just given by a matrix with coefficients in $\overline{\mathbb{Q}}$. It is maybe possible to check this property for the exceptional ones as well. The proof given above is independent of the classification.

§4. *Proof of the main result.* In this section we prove the Theorems 1.2 and 1.4.

Let (G, X) be a Shimura datum and K a compact open subgroup of $G(\mathbb{A}_f)$. Let X^+ be a connected component of X and let Γ be $G(\mathbb{Q})_+ \cap K$ where $G(\mathbb{Q})_+$ is the stabilizer of X^+ in $G(\mathbb{Q})$. We let S be $\Gamma\backslash X^+$. This is a connected component of $\mathrm{Sh}_K(G, X)$. We also let $\pi : X^+ \longrightarrow S$ be the natural projection. With these notations, our result is the following.

THEOREM 4.1. *Let Y be an algebraic subvariety of S. The variety Y is weakly special if and only if one (equivalently any) component of $\pi^{-1}(Y)$ is algebraic.*

Remark 4.2. As the assumptions and conclusions of the theorem are invariant under translation by Hecke operators, the result is true for any component of $\mathrm{Sh}_K(G, X)$.

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Proof. The Borel embedding $X \to X^\vee$ is $G(\mathbb{R})$-invariant and the image of an algebraic subvariety of $X^\vee$ by an element of $G(\mathbb{R})$ is an algebraic subvariety of $X^\vee$. Let $\tilde{Y}$ be an algebraic component of $\pi^{-1}(Y)$. Let Z be an algebraic subvariety of $X^\vee$ such that $\tilde{Y} = X^+ \cap Z$. Let $\gamma \in \Gamma \subset G(\mathbb{R})$; then

$$\gamma\tilde{Y} = \gamma(X^+ \cap Z) \subset X^+ \cap \gamma \cdot Z = \gamma\gamma^{-1}(X^+ \cap \gamma \cdot Z) \subset \gamma\tilde{Y}.$$

Notice that the last inclusion follows from

$$\gamma\gamma^{-1}(X^+ \cap \gamma \cdot Z) \subset \gamma(\gamma^{-1}X^+ \cap \gamma^{-1}\gamma Z) = \gamma(X^+ \cap Z).$$

It follows that $\gamma\tilde{Y} = X^+ \cap \gamma \cdot Z$ is algebraic. As the analytic components of $\pi^{-1}(Y)$ are permuted transitively by Γ, we see that if one component is algebraic then so is any other.

Next, we reduce the situation to the case where Y is Hodge generic in S. Let S_Y be the smallest special subvariety containing Y. Such a variety S_Y exists in view of the fact that components of intersections of special subvarieties are special (this is a consequence of the interpretation of special subvarieties as loci of Hodge classes). By [**22**, Lemma 2.1] and its proof, there exists a sub-Shimura datum (H, X_H) where H is the generic Mumford–Tate group on X_H such that S_Y is the image $\Gamma_H\backslash X_H^+$ in S where X_H^+ is a connected component of X_H and $\Gamma_H = \Gamma \cap H(\mathbb{Q})$. By Lemma 3.3 a component $\widetilde{Y}$ of the preimage of Y in X_H^+ is still algebraic. We replace (G, X) by (H, X_H) and S by S_Y, and hence assume that *Y is Hodge generic in S*.

Using Proposition 2.2(3) we may and do assume that the group G is semi-simple of adjoint type. Let

$$(G, X) = (G_1, X_1) \times (G_2, X_2)$$

be the decomposition of the Shimura datum (G, X) associated to Y as in [**15**, §3.6]. In this decomposition, G_1 is the Zariski closure of the algebraic monodromy group Γ_Y associated to Y. The existence of such a decomposition is a consequence of a theorem of Deligne [**4**] and André [**1**] on the monodromy groups associated to variations of (mixed) Hodge structures (see [**15**, §3] for details and explanations).

As the group K can be chosen as small as needed, we assume using Proposition 2.2(1) that $\Gamma := K \cap G(\mathbb{A}_f)$ is neat and that $\Gamma = \Gamma_1 \times \Gamma_2$ where Γ_i are arithmetic subgroups of $G_i(\mathbb{Q})_+$. The subvariety Y is now of the form

$$Y_1 \times \{y_2\} \tag{6}$$

where Y_1 is a subvariety of $\Gamma_1\backslash X_1^+$ and y_2 is a Hodge generic point of $\Gamma_2\backslash X_2$ (see [**15**, Proposition 3.7]). To show that Y is weakly special we need to show that $Y_1 = \Gamma_1\backslash X_1^+$. Replacing Y by Y_1 and (G, X) by (H_1, X_1) does not change our assumptions, hence we now assume that the algebraic monodromy group Γ_Y of Y is Zariski dense in G.

We are now in the following situation. Write $\widetilde{Y} = Z \cap X^+$ where Z is an algebraic subvariety of $X^\vee = G(\mathbb{C})/P_x(\mathbb{C})$. Fix a point $y \in \widetilde{Y}$. As Γ_Y is Zariski

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dense in $G_{\mathbb{Q}}$, it is Zariski dense in $G_{\mathbb{C}}$ and consequently the orbit $\Gamma_Y \cdot y$ is Zariski dense in $X^\vee$. As $\Gamma_Y \cdot y \subset Y \subset Z$, it follows that Z is Zariski dense in $X^\vee$. As Z is an algebraic subvariety of $X^\vee$, we find that $Z = X^\vee$ and therefore $Y = Z \cap X^+ = X^+$. □

We can now give the proof of Theorem 1.4. In the previous situation if we moreover assume that Y and $\tilde{Y}$ are defined over $\overline{\mathbb{Q}}$, then the point y_2 of equation (6) is defined over $\overline{\mathbb{Q}}$. Moreover we have a decomposition $\tilde{Y} = \tilde{Y}_1 \times \{\tilde{y}_2\}$ with $\tilde{y}_2 \in Y'$ mapping to y_2. As $\tilde{Y}$ is defined over $\overline{\mathbb{Q}}$ then $\tilde{y}_2$ is defined over $\overline{\mathbb{Q}}$. If S is a Shimura variety of abelian type then by the result of Cohen and Shiga–Wolfart (Theorem 1.3) $\tilde{y}_2$ is a CM point. The Theorem 1.4 is therefore a consequence of the last part of Definition 2.1.

§5. *The case of abelian varieties.* In this section we prove the analog of Theorem 1.2 for complex abelian varieties. Let A be an abelian variety over $\mathbb{C}$. Then A is of the form $A = V_A/H_1(A, \mathbb{Z})$ for a $\mathbb{C}$-vector space V_A of dimension $g = \dim(A)$. A subvariety Y of A is said to be weakly special if $Y = P + B$ for a point $P \in A$ and an abelian subvariety B of A. Let $\pi : V_A \to A$ be the uniformizing map.

PROPOSITION 5.1. *Let Y be a subvariety of A. Let $\tilde{Y} \subset V_A$ be an analytic component of $\pi^{-1}(Y)$. Then Y is weakly special if and only if $\tilde{Y}$ is an algebraic subvariety of $V_A \simeq \mathbb{C}^g$.*

The group $H_1(A, \mathbb{Z})$ acts transitively on the components of $\pi^{-1}(Y)$. As this action is algebraic, if some component $\tilde{Y}$ of $\pi^{-1}(Y)$ is algebraic, the same property holds for any component of $\pi^{-1}(Y)$.

Note also that Y is weakly special if and only if for any point $Q \in A(\mathbb{C})$, the translate $Q + Y$ of Y is weakly special. The components of $\pi^{-1}(Y)$ are algebraic if and only if those of $\pi^{-1}(Q + Y)$ are. Hence we may and do assume that the origin O of A is in Y. Weakly special subvarieties of A containing O are just abelian subvarieties of A and are of the form $\pi(V')$ for a $\mathbb{C}$-subvector space V' of V. We therefore get the "only if" part of the theorem and we just need to prove the other direction.

Let $\mathrm{Alb}(Y)$ be the Albanese variety of Y normalized by the choice of the point $O \in Y$ and let $a : Y \to \mathrm{Alb}(Y)$ be the associated Albanese morphism. Let $\iota : Y \to A$ be the inclusion map. Using the functorial property of the Albanese map, we know that there exists a morphism of abelian variety $\phi : \mathrm{Alb}(Y) \to A$ such that $\iota = \phi a$. Let $B := \phi(\mathrm{Alb}(Y))$, then B is an abelian subvariety of A.

By [**23**, Lemma 12.11], $a(Y)$ generates $\mathrm{Alb}(Y)$ as a group. Therefore B is in fact the smallest abelian subvariety of A containing Y. Let $\Gamma_Y \subset H_1(A, \mathbb{Z})$ be the image of $\pi_1(Y, O)$ in $H_1(A, \mathbb{Z})$. Then $\Gamma_Y = \iota_\star H_1(Y, \mathbb{Z})$. Moreover,

$$\Gamma_Y = (\phi a)_\star H_1(Y, \mathbb{Z}) = \phi_\star H_1(\mathrm{Alb}(Y), \mathbb{Z})$$

is of finite index in $H_1(B, \mathbb{Z})$.

There exists a $\mathbb{C}$-vector subspace V_B of V_A such that $H_1(B, \mathbb{Z}) = V_B \cap H_1(A, \mathbb{Z})$ and $B = V_B/H_1(B, \mathbb{Z})$. There exists a component $\tilde{Y}$ of $\pi^{-1}(Y)$

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contained in $V_B = \pi^{-1}(B)$. Let $\tilde{y}$ be a point of $\tilde{Y}$ mapping to O. Then as Γ_Y is of finite index in $H_1(B, \mathbb{Z})$, $\Gamma_Y \cdot \tilde{y} \subset \tilde{Y}$ is Zariski dense in V_B. If $\tilde{Y}$ is algebraic then $\tilde{Y} = V_B$ and $Y = B$ is weakly special.

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