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Homoclinic classes for generic C^1 vector fields

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Abstract. We prove that homoclinic classes for a *residual* set of C^1 vector fields X on closed n-manifolds are maximal transitive, and depend continuously on periodic orbit data. In addition, X does not exhibit cycles formed by homoclinic classes. We also prove that a homoclinic class of X is isolated if and only if it is Ω-isolated, and it is the intersection of its stable set with its unstable set. All these properties are well known for structurally stable Axiom A vector fields.

1. *Introduction*

We show some properties of homoclinic classes for generic C^1 flows on closed n-manifolds. By homoclinic class we mean the closure of the transversal homoclinic points associated to a hyperbolic periodic orbit; so, homoclinic classes are transitive and the closure of its periodic orbits [**16**, Ch. 2, §8].

For structurally stable Axiom A vector fields it is known that homoclinic classes are maximal transitive and depend continuously on the periodic orbit data. In addition, if H is a homoclinic class of X then it is saturated, that is, $H = W^s_X(H) \cap W^u_X(H)$, where $W^s_X(H)$ is the stable set of H and $W^u_X(H)$ is the unstable set of H [**2**, p. 371]. Moreover, such vector fields do not exhibit cycles formed by homoclinic classes. In this paper we shall prove these properties for generic C^1 vector fields on closed n-manifolds M (neither structural stability nor Axiom A is assumed). Furthermore, we prove that generically a homoclinic class is isolated if and only if it is isolated from the non-wandering set. In particular, all the mentioned properties hold for a dense set of C^1 vector fields on M. It is interesting to observe that neither structural stability nor Axiom A is dense in the space of C^1 vector fields on M, for all $n \geq 3$.

To state our results in a precise way we use the following notation. M is a compact boundaryless n-manifold, $\mathcal{X}^1(M)$ is the space of C^1 vector fields endowed with the

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C^1 topology. Given $X \in \mathcal{X}^1(M)$, X_t denotes the flow induced by X. The ω-limit set of p is the set $\omega_X(p)$ of accumulation points of the positive orbit of p. The α-limit set of p is $\alpha_X(p) = \omega_{-X}(p)$, where $-X$ denotes the time-reversed flow of X. The non-wandering set $\Omega(X)$ of X is the set of p such that for every neighborhood U of p and $T > 0$ there exists $t > T$ such that $X_t(U) \cap U \neq \emptyset$. Clearly $\Omega(X)$ is closed, non-empty and contains any ω-limit (α-limit) set. A compact invariant set B of X is Ω-*isolated* if $\Omega(X) \setminus B$ is closed. B is *isolated* if $B = \bigcap_{t\in\mathbb{R}} X_t(U)$ for some compact neighborhood U of B (in this case U is called the *isolating block*). We denote by $\mathrm{Per}(X)$ the union of the periodic orbits of X and $\mathrm{Crit}(X)$ the set formed by the union of $\mathrm{Per}(X)$ and the singularities of X.

A set is *transitive* for X if it is the ω-limit set of one of its orbits. A transitive set Λ of X is *maximal transitive* if it contains every transitive set T of X satisfying $\Lambda \cap T \neq \emptyset$. Note that a maximal transitive set is maximal with respect to the inclusion order. In [**4, 5**] it was asked whether every homoclinic class $H_f(p)$ of a generic diffeomorphism f satisfies the property that if T is a transitive set of f and $p \in T$, then $T \subset H_f(p)$. In [**1**], Arnaud also considered homoclinic classes for C^1 diffeomorphisms on M, and in particular she gives a positive answer to this question [**1**, Corollary 40]. On the other hand, item (1) of Theorem A below states that generically any transitive set of a C^1 vector field intersecting the homoclinic class is included in it, and thus the diffeomorphism version of it extends this result of Arnaud.

If Λ is a compact invariant set of X, we denote

$$W^s_X(\Lambda) = \{q \in M : \mathrm{dist}(X_t(q), \Lambda) \to 0, t \to \infty\}$$

and

$$W^u_X(\Lambda) = \{q \in M : \mathrm{dist}(X_t(q), \Lambda) \to 0, t \to -\infty\},$$

where dist is the metric on M. These sets are called the stable and unstable sets of Λ, respectively. We shall denote $W^s_X(p) = W^s_X(\mathcal{O}_X(p))$ and $W^u_X(p) = W^u_X(\mathcal{O}_X(p))$, where $\mathcal{O}_X(p)$ is the orbit of p. We say that Λ is *saturated* if $W^s_X(\Lambda) \cap W^u_X(\Lambda) = \Lambda$.

A *cycle* of X is a finite set of compact invariant sets $\Lambda_0, \Lambda_1, \ldots, \Lambda_n$ such that $\Lambda_n = \Lambda_0$ and $\Lambda_0, \Lambda_1, \ldots, \Lambda_{n-1}$ are disjoint, and

$$(W^u_X(\Lambda_i) \setminus \Lambda_i) \cap (W^s_X(\Lambda_{i+1}) \setminus \Lambda_{i+1}) \neq \emptyset$$

for all $i = 0, \ldots, n-1$.

A compact invariant set Λ of X is *hyperbolic* if there is a continuous tangent bundle decomposition $E^s \oplus E^X \oplus E^u$ over Λ such that E^s is contracting, E^u is expanding and E^X denotes the direction of X. We say that $p \in \mathrm{Crit}(X)$ is hyperbolic if $\mathcal{O}_X(p)$ is a hyperbolic set of X.

The Stable Manifold Theorem [**10**] asserts that $W^s_X(p)$ is an immersed manifold tangent to $E^s \oplus E^X$ for every p in a hyperbolic set Λ of X. Similarly for $W^u_X(p)$. This remark applies when $\Lambda = \mathcal{O}_X(p)$ for some $p \in \mathrm{Crit}(X)$ hyperbolic. As already defined, the homoclinic class associated to a hyperbolic periodic orbit p of X, $H_X(p)$, is the closure of the transversal intersection orbits in $W^s_X(p) \cap W^u_X(p)$.

We say that X is *Axiom A* if $\Omega(X)$ is both hyperbolic and the closure of $\mathrm{Crit}(X)$. The non-wandering set of a non-singular Axiom A flow splits in a finite disjoint union of homoclinic classes [**16**, Ch. 0, p. 3].

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Another interesting property of homoclinic classes for Axiom A vector fields is their continuous dependence on the periodic orbit data, that is, the map $p \in \mathrm{Per}(X) \mapsto H_X(p)$ is upper-semicontinuous.

In general, we say that a compact set sequence Λ_n *accumulates* on a compact set Λ if for every neighborhood U of Λ there is an $n_0 > 0$ such that $\Lambda_n \subset U$ for all $n \geq n_0$. Note that this kind of accumulation is weaker than the usual Hausdorff metric accumulation.

If $\mathcal{Y}$ denotes a metric space, then $\mathcal{R} \subset \mathcal{Y}$ is *residual* in $\mathcal{Y}$ if $\mathcal{R}$ contains a countable intersection of open-dense subsets of $\mathcal{Y}$. Clearly a countable intersection of residual subsets of $\mathcal{Y}$ is a residual subset of $\mathcal{Y}$. For example, the set of Kupka–Smale vector fields $\mathcal{KS}^1(M)$ on M is a residual subset of $\mathcal{X}^1(M)$ [**6**, Ch. 3, §3]. Recall that a vector field is Kupka–Smale if all its periodic orbits and singularities are hyperbolic and the invariant manifolds of such elements intersect transversely.

THEOREM A. *The following properties hold for a residual subset of vector fields X in $\mathcal{X}^1(M)$.*

(1) *The homoclinic classes of X are maximal transitive sets of X. In particular, different homoclinic classes of X are disjoint.*
(2) *The homoclinic classes of X are saturated.*
(3) *The homoclinic classes of X depend continuously on the periodic orbit data, that is, the map $p \in \mathrm{Per}(X) \mapsto H_X(p)$ is upper-semicontinuous.*
(4) *A homoclinic class of X is isolated if and only if it is Ω-isolated.*
(5) *The hyperbolic homoclinic classes of X are isolated.*
(6) *There is no cycle of X formed by homoclinic classes of X.*
(7) *X has finitely many homoclinic classes if and only if the union of the homoclinic classes of X is closed and every homoclinic class of X is isolated.*

When M has dimension three we obtain the following corollaries using Theorem A, [**13**], and [**14**]. Recall that an isolated set Λ of a C^r vector field X is C^r robust transitive ($r \geq 1$) if it exhibits an isolating block U such that, for every vector field Y C^r close to X, $\bigcap_{t\in\mathbb{R}} Y_t(U)$ is both transitive and non-trivial for Y.

COROLLARY 1.1. *The properties below are equivalent for a residual set of non-singular three-dimensional C^1 vector fields X and every non-trivial homoclinic class $H_X(p)$ of X:*

(1) *$H_X(p)$ is hyperbolic;*
(2) *$H_X(p)$ is isolated;*
(3) *$H_X(p)$ is C^1 robust transitive for X.*

COROLLARY 1.2. *The properties below are equivalent for a residual set of non-singular three-dimensional C^1 vector fields X:*

(1) *X is Axiom A;*
(2) *X has finitely many homoclinic classes;*
(3) *the union of the homoclinic classes of X is closed and every homoclinic class of X is isolated.*

The equivalence between items (1) and (2) of the above corollary follows from [**13, 14**]. It shows how difficult is to prove the genericity of vector fields exhibiting finitely many homoclinic classes. The equivalence between (2) and (3) follows from Theorem A(7).

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To prove Theorem A we show in §3 that homoclinic classes $H_X(p)$ for a residual set of C^1 vector fields X satisfy $H_X(p) = \Lambda^+ \cap \Lambda^-$, where Λ^+ is Lyapunov stable for X and Λ^- is Lyapunov stable for $-X$. The main technical tool to prove such a result is Lemma 3.6, a stronger version of Hayashi's C^1 Connecting Lemma [**8**], recently published in [**19**, Theorem E, p. 5214] (see also [**1, 7, 9**]).

In §2 we study compact invariant sets Λ of X satisfying $\Lambda = \Lambda^+ \cap \Lambda^-$, where $\Lambda^\pm$ is Lyapunov stable for $\pm X$. The proof of Theorem A and Corollary 1.1 will be given in the final section.

Remark 1.3. We observe that Theorem A is valid for a residual set of C^1 diffeomorphisms on any n-manifold M by the usual method of suspension.

2. *Lyapunov stability lemmas*

In this section we shall establish useful properties of Lyapunov stable sets. A reference for Lyapunov stability theory is [**3**].

Recall that we have denoted by X_t, $t \in \mathbb{R}$, the flow generated by $X \in \mathcal{X}^1(M)$. Given $A \subset M$ and $R \subset \mathbb{R}$ we set $X_R(A) = \{X_t(q) : (q,t) \in A \times R\}$. We denote by $\mathrm{Cl}(A)$ the closure of A, and by $\mathrm{Int}(A)$ the interior of A. If $\epsilon > 0$ and $q \in M$, we set $B_\epsilon(q)$ to be the ϵ-ball centered at q.

A compact subset $A \subseteq M$ is *Lyapunov stable* for X if for every open set U containing A there exists an open set V containing A such that $X_t(V) \subset U$ for every $t \geq 0$. Clearly a Lyapunov stable set is forward invariant.

The following lemma summarizes some classical properties of Lyapunov stable sets (see [**3**, Ch. V]).

LEMMA 2.1. *Let Λ^+ be a Lyapunov stable set of X. Then:*

(1) *if $x_n \in M$ and $t_n \geq 0$ satisfy $x_n \to x \in \Lambda^+$ and $X_{t_n}(x_n) \to y$, then $y \in \Lambda^+$;*
(2) *$W^u_X(\Lambda^+) \subset \Lambda^+$;*
(3) *if Γ is a transitive set of X and $\Gamma \cap \Lambda^+ \neq \emptyset$, then $\Gamma \subset \Lambda^+$.*

We are interested in invariant compact sets $\Lambda = \Lambda^+ \cap \Lambda^-$ of X, where Λ^+ is a Lyapunov stable set for X and Λ^- is a Lyapunov stable set for the reversed flow $-X$. We shall call such sets *neutral* for the sake of simplicity. As we shall see in the next section, homoclinic classes are neutral sets for generic C^1 vector fields on closed n-manifolds.

Elementary properties of neutral sets are given in the following lemma.

LEMMA 2.2. *Let Λ be a neutral set of X. Then:*

(1) *Λ is saturated;*
(2) *Λ is transitive for X if and only if Λ is maximal transitive for X. In particular, different transitive neutral sets of X are disjoint.*

Proof. Let $\Lambda = \Lambda^+ \cap \Lambda^-$ with $\Lambda^\pm$ being Lyapunov stable for $\pm X$. Clearly $W^u_X(\Lambda) \subset \Lambda^+$ by Lemma 2.1(2). Similarly, $W^s_X(\Lambda) \subset \Lambda^-$. Hence

$$W^u_X(\Lambda) \cap W^s_X(\Lambda) \subset \Lambda^+ \cap \Lambda^- = \Lambda.$$

Conversely, $\Lambda \subset W^u_X(\Lambda) \cap W^s_X(\Lambda)$ since Λ is invariant. This proves (1).

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Now, by Lemma 2.1(3), if Γ is a transitive set intersecting Λ, then $\Gamma \subset \Lambda^+$ and $\Gamma \subset \Lambda^-$. Thus, $\Gamma \subset \Lambda^+ \cap \Lambda^- = \Lambda$, and so Λ is maximal transitive. The converse is obvious. Different transitive neutral sets of X are maximal transitive, and so they are necessarily disjoint. This finishes the proof. □

Note that a Smale horseshoe with a first tangency is an example of a maximal transitive set which is not neutral, see Proposition 2.6. This example also provides a hyperbolic homoclinic class which is not neutral (compare with Theorem 3.1).

PROPOSITION 2.3. *There is no cycle of X formed by transitive neutral sets.*

Proof. By contradiction suppose that there exists a cycle $\Lambda_0, \dots, \Lambda_n$ of X such that every Λ_i is a transitive neutral set of X. Recall $\Lambda_n = \Lambda_0$.

Set $\Lambda_i = \Lambda_i^+ \cap \Lambda_i^-$ where each $\Lambda_i^\pm$ is Lyapunov stable for $\pm X$. Choose

$$x_i \in (W_X^u(\Lambda_i) \setminus \Lambda_i) \cap (W_X^s(\Lambda_{i+1}) \setminus \Lambda_{i+1})$$

according to the definition.

We claim that $x_i \in \Lambda_0^-$ for every i. Indeed, as $W_X^s(\Lambda_0) \subset \Lambda_0^-$ one has $x_{n-1} \in \Lambda_0^-$. Assume by induction that $x_i \in \Lambda_0^-$ for some i. As $x_i \in W_X^u(\Lambda_i)$, the backward invariance of Λ_0^- implies

$$\Lambda_0^- \cap \Lambda_i \supset \alpha_X(x_i) \neq \emptyset.$$

By Lemma 2.1(3) one has $\Lambda_i \subset \Lambda_0^-$ since Λ_i is transitive. In particular, $W_X^s(\Lambda_i) \subset \Lambda_0^-$ by Lemma 2.1(2) applied to $-X$. As $x_{i-1} \in W_X^s(\Lambda_i)$, one has $x_{i-1} \in \Lambda_0^-$. The claim follows by induction.

By the claim $x_0 \in \Lambda_0^-$. As $W_X^u(\Lambda_0) \subset \Lambda_0^+$ and $x_0 \in W_X^u(\Lambda_0)$ (by definition) one has $x_0 \in \Lambda_0^+ \cap \Lambda_0^- = \Lambda_0$. This contradicts $x_0 \in W_X^u(\Lambda_0) \setminus \Lambda_0$ and the proposition is proved. □

LEMMA 2.4. *If Λ is neutral for X, then for every neighborhood U of Λ there exists a neighborhood $V \subset U$ of Λ such that*

$$\Omega(X) \cap V \subset \bigcap_{t \in \mathbb{R}} X_t(U).$$

Proof. Let U be a neighborhood of a neutral set Λ of X. Choose $U' \subset \mathrm{Cl}(U') \subset U$ with U' being another neighborhood of Λ. We claim that there is a neighborhood $V \subset U'$ of Λ so that:

(1) $t \geq 0$ and $p \in V \cap X_{-t}(V)$ implies $X_{[0,t]}(p) \subseteq U'$;

(2) $t \leq 0$ and $p \in V \cap X_t(V)$ implies $X_{[-t,0]}(p) \subseteq U'$.

Indeed, if this were not true then there would exist a neighborhood U of Λ and sequences $p_n \to \Lambda$, $t_n > 0$ such that $X_{t_n}(p_n) \to \Lambda$ but $X_{[0,t_n]}(p_n) \not\subseteq U'$. Choose $q_n \in X_{[0,t_n]}(p_n) \setminus U'$. Write $q_n = X_{t_n'}(p_n)$ for some $t_n' \in [0, t_n]$ and assume that $q_n \to q$ for some $q \notin U'$. Let $\Lambda = \Lambda^+ \cap \Lambda^-$ with $\Lambda^\pm$ Lyapunov stable for $\pm X$. Since Λ^+ is Lyapunov stable for X and $t_n' > 0$, Lemma 2.1(1) implies that $q \in \Lambda^+$. On the other hand, as we can write $q_n = X_{t_n'-t_n}(X_{t_n}(p_n))$ where $t_n' - t_n > 0$ and $X_{t_n}(p_n) \to \Lambda$, again using Lemma 2.1(1) we have that $q \in \Lambda^-$. This proves that $q \in \Lambda$, a contradiction since $q \notin U'$. This proves the claim.

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Next we prove that $\Omega(X) \cap V \subseteq \bigcap_{t\in\mathbb{R}} X_t(U)$. Indeed, choose $q \in \Omega(X) \cap V$. By contradiction, we assume that there is a $t_0 > 0$ (say) such that $X_{t_0}(q) \notin U$. Then, there is a ball $B_\epsilon(q) \subseteq V$ such that $X_{t_0}(B_\epsilon(q)) \cap \mathrm{Cl}(U') = \emptyset$. As $q \in \Omega(X)$ there exists $t > t_0$ such that $B_\epsilon(q) \cap X_{-t}(B_\epsilon(q)) \neq \emptyset$. Pick $p \in B_\epsilon(q) \cap X_{-t}(B_\epsilon(q))$. By (1) above one has $X_{[0,t]}(p) \subseteq U'$ since $B_\epsilon(q) \subset V$. This contradicts $X_{t_0}(p) \in X_{t_0}(B_\epsilon(q))$ and $X_{t_0}(B_\epsilon(q)) \cap \mathrm{Cl}(U') = \emptyset$. The proof is complete. □

A first consequence of the above lemma is the following corollary. Given compact subsets $A, B \subset M$ we denote $\mathrm{dist}(A, B) = \inf\{\mathrm{dist}(a, b) : a \in A, b \in B\}$.

COROLLARY 2.5. *If Λ is a neutral set of X and Λ_n is a sequence of transitive sets of X such that $\mathrm{dist}(\Lambda_n, \Lambda) \to 0$ as $n \to \infty$, then Λ_n accumulates on Λ.*

Proof. Let Λ_n and Λ be as in the statement. Fix a neighborhood U of Λ and let $V \subset U$ be the neighborhood of Λ obtained by the previous lemma. As $\mathrm{dist}(\Lambda_n, \Lambda) \to 0$ as $n \to \infty$ we have that $\Lambda_n \cap V \neq \emptyset$ for every n large. Let q_n be the dense orbit of Λ_n. Clearly $q_n \in \Omega(X)$. We can assume that $q_n \in V$ for n large, and so $q_n \in \Omega(X) \cap V$. Then, $X_t(q_n) \in U$ for every t. In particular, $\Lambda_n = \omega_X(q_n) \subset \mathrm{Cl}(U)$. This proves the corollary since U is arbitrary. This finishes the proof. □

PROPOSITION 2.6. *A neutral set is isolated if and only if it is Ω-isolated.*

Proof. We first claim that any saturated Ω-isolated set Λ of X is isolated. Indeed, since Λ is Ω-isolated, there is a $U \supset \Lambda$ open such that $\mathrm{Cl}(U) \cap \Omega(X) = \Lambda$. This U is an isolating block for Λ. If $x \in \cap_{t\in\mathbb{R}} X_t(U)$, then $\omega_X(x) \cup \alpha_X(x) \subset \mathrm{Cl}(U) \cap \Omega(X) \subset \Lambda$. So, $x \in W^s_X(\Lambda) \cap W^u_X(\Lambda) = \Lambda$. This proves that $\bigcap_{t\in\mathbb{R}} X_t(U) \subset \Lambda$. The opposite inclusion follows since Λ is invariant. The claim follows.

To prove that invariant Ω-isolated neutral sets are isolated we use the above claim and Lemma 2.2(1). To prove that isolated neutral sets are Ω-isolated we use Lemma 2.4. □

PROPOSITION 2.7. *Transitive hyperbolic neutral sets are isolated.*

Proof. By Proposition 2.6 it suffices to show that transitive neutral hyperbolic sets Λ are Ω-isolated.

Suppose by contradiction that Λ is not Ω-isolated. Then, there is a sequence $p_n \in \Omega(X) \setminus \Lambda$ converging to $p \in \Lambda$. Fix U to be a neighborhood of Λ and let V be given in Lemma 2.4 for U. We can assume that $p_n \in V$ for every n. As p_n is non-wandering for X, for every n there are sequences $q_i \in V \to p_n$ and $t_i > 0$ such that $X_{t_i}(q_i) \to p_n$ as $i \to \infty$. By (1) in the proof of Lemma 2.4 we have $X_{[0,t_i]}(q_i) \subset U$ for every i. Therefore, we can construct a periodic pseudo-orbit of X arbitrarily close to p_n. By the Shadowing Lemma for flows [**11**, Theorem 18.1.6, p. 569] applied to the hyperbolic set Λ, such a periodic pseudo-orbit can be shadowed by a periodic orbit. This proves that $p_n \in \mathrm{Cl}(\mathrm{Per}(X))$. As the neighborhood U is arbitrary, we can assume that $p_n \in \mathrm{Per}(X)$ for every n. Note that $\mathcal{O}_X(p_n)$ converges to Λ by Corollary 2.5.

As Λ is transitive we have that if $E^s \oplus E^X \oplus E^u$ denotes the corresponding hyperbolic splitting, then $\dim(E^s) = s$ and $\dim(E^u) = u$ are constant in Λ. Clearly neither $s = 0$ nor $u = 0$ since Λ is not Ω-isolated. As $\mathcal{O}_X(p_n)$ converges to Λ both the local stable and unstable manifolds of p_n have dimension s and u, respectively. Moreover, both invariant

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manifolds have uniform size as well. This implies that $W^u_X(p_n) \cap W^s_X(p) \neq \emptyset$ and $W^s_X(p_n) \cap W^u_X(p) \neq \emptyset$ for n large. As $p_n \in \mathrm{Per}(X)$ and $p \in \Lambda$, we conclude by the Inclination Lemma [**6**] that $p_n \in \mathrm{Cl}(W^s_X(p) \cap W^u_X(p))$. As $p \in \Lambda$, $W^{s,u}_X(p) \subset W^{s,u}_X(\Lambda)$. Therefore, $p_n \in \mathrm{Cl}(W^s_X(\Lambda) \cap W^u_X(\Lambda))$. As Λ is saturated, $W^s_X(\Lambda) \cap W^u_X(\Lambda) = \Lambda$ and hence $p_n \in \mathrm{Cl}(\Lambda) = \Lambda$. But this is impossible since $p_n \in \Omega(X) \setminus \Lambda$ by assumption. This concludes the proof. □

Denote by $\mathcal{F}$ the collection of all isolated transitive neutral sets of X.

PROPOSITION 2.8. *A subcollection $\mathcal{F}'$ of $\mathcal{F}$ is finite if and only if $\bigcup_{\Lambda\in\mathcal{F}'} \Lambda$ is closed.*

Proof. Obviously $\bigcup_{\Lambda\in\mathcal{F}'} \Lambda$ is closed if $\mathcal{F}'$ is finite. Conversely, suppose that $\bigcup_{\Lambda\in\mathcal{F}'} \Lambda$ is closed. If $\mathcal{F}'$ were infinite, then there would exist a sequence $\Lambda_n \in \mathcal{F}'$ of (different) sets accumulating some $\Lambda \in \mathcal{F}'$. By Corollary 2.5 we have $\Lambda_n \subseteq U$ for some isolating block U of Λ and n large. Then we have that $\Lambda_n = \Lambda$ for n large, a contradiction which completes the proof. □

3. *Homoclinic classes*

The main result of this section is as follows.

THEOREM 3.1. *There is a residual subset $\mathcal{R}$ of $\mathcal{X}^1(M)$ such that every homoclinic class of every vector field in $\mathcal{R}$ is neutral.*

COROLLARY 3.2. *The following properties are equivalent for $X \in \mathcal{R}$ and every compact invariant set Λ of X:*

(1) *Λ is a transitive neutral set with periodic orbits of X;*
(2) *Λ is a homoclinic class of X;*
(3) *Λ is a maximal transitive set with periodic orbits of X.*

Proof. That (2) implies (1) follows from Theorem 3.1. That (1) implies (3) follows from (2) in the proof of Lemma 2.4. Let us prove that (3) implies (2). If Λ is as in (3) and $p \in \mathrm{Per}(X) \cap \Lambda$, then $\Lambda \cap H_X(p) \neq \emptyset$. By Theorem 3.1 we can assume that $H_X(p)$ is neutral, and so it is maximal transitive (using (1) implies (3)). As both Λ and $H_X(p)$ are maximal transitive we conclude that $\Lambda = H_X(p)$ and the proof follows. □

COROLLARY 3.3. *For $X \in \mathcal{R}$, a non-singular compact isolated set of X is neutral and transitive if and only if it is a homoclinic class.*

Proof. The converse follows from Theorem 3.1. To prove the direct, denote Λ to be a transitive isolated neutral set of a generic C^1 vector field X. By Proposition 2.6 it follows that Λ is also Ω-isolated. Since Λ is transitive we have $\Lambda \subset \Omega(X)$. Thus, by [**17**] it follows that $\Lambda = \mathrm{Cl}(\Lambda \cap \mathrm{Per}(X))$, and so $\Lambda \cap \mathrm{Per}(X) \neq \emptyset$. Then the conclusion follows from the previous corollary. □

The proof of Theorem 3.1 follows immediately from the following two lemmas.

LEMMA 3.4. *There exists a residual set $\mathcal{R}$ of $\mathcal{X}^\infty(M)$ such that, for every $Y \in \mathcal{R}$ and $\sigma \in \mathrm{Crit}(Y)$, $\mathrm{Cl}(W^u_X(\sigma))$ is Lyapunov stable for X and $\mathrm{Cl}(W^s_X(\sigma))$ is Lyapunov stable for $-X$.*

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LEMMA 3.5. *There exists a residual set $\mathcal{R}$ in $\mathcal{X}^\infty(M)$ such that every $X \in \mathcal{R}$ satisfies*

$$H_X(p) = \mathrm{Cl}(W^u_X(p)) \cap \mathrm{Cl}(W^s_X(p))$$

for all $p \in \mathrm{Per}(X)$.

Lemma 3.4 was proved in [**15**, Theorem 6.1, p. 372] when σ is a singularity and the same proof works when σ is a periodic orbit. We shall give another proof of this lemma in Appendix A for completeness.

Before the proof of Lemma 3.5, let us introduce some notation. Recall that M is a closed n-manifold, $n \geq 3$. We denote by 2^M_c the space of all compact subsets of M endowed with the Hausdorff topology. Recall that $\mathcal{KS}^1(M) \subset \mathcal{X}^1(M)$ denotes the set of Kupka–Smale C^1 vector fields on M.

Given $X \in \mathcal{X}^1(M)$ and $p \in \mathrm{Per}(X)$ we denote by $\Pi_X(p)$ the period of p. We set $\Pi_X(p) = 0$ if p is a singularity of X.

If $T > 0$ we denote

$$\mathrm{Crit}_T(X) = \{p \in \mathrm{Crit}(X) : \Pi_X(p) < T\}.$$

If $p \in \mathrm{Crit}(X)$ is hyperbolic, then there is a continuation $p(Y)$ of p for Y close enough to X so that $p(X) = p$.

Note that if $X \in \mathcal{KS}^1(M)$ and $T > 0$, then

$$\mathrm{Crit}_T(X) = \{p_1(X), \ldots, p_k(X)\}$$

is a finite set. Moreover,

$$\mathrm{Crit}_T(Y) = \{p_1(Y), \ldots, p_k(Y)\}$$

for every Y close enough to X.

Let $\mathcal{Y}$ be a metric space. A set-valued map

$$\Phi : \mathcal{Y} \to 2^M_c$$

is *lower semi-continuous* at $Y_0 \in \mathcal{Y}$ if for every open set $U \subset M$ one has $\Phi(Y_0) \cap U \neq \emptyset$ which implies $\Phi(Y) \cap U \neq \emptyset$ for every Y close to Y_0. Similarly, we say that Φ is *upper semi-continuous* at $Y_1 \in \mathcal{Y}$ if for every compact set $K \subset M$ one has $\Phi(Y_1) \cap K = \emptyset$ which implies $\Phi(Y) \cap K = \emptyset$ for every Y close to Y_1. We say that Φ is *lower semi-continuous* if it is lower semi-continuous at every $Y_0 \in \mathcal{Y}$. A well-known result [**12**, Corollary 1, p. 71] asserts that if $\Phi : \mathcal{X}^1(M) \to 2^M_c$ is a lower semi-continuous map, then it is upper semi-continuous at every Y in a residual subset of $\mathcal{X}^1(M)$.

The lemma below is the flow version of [**19**, Theorem E, p. 5214] (see also [**1, 7–9**]).

LEMMA 3.6. *Let $Y \in \mathcal{X}^1(M)$ and $x \notin \mathrm{Crit}(Y)$. For any C^1 neighborhood $\mathcal{U}$ of Y there are $\rho > 1$, $L > 0$ and $\epsilon_0 > 0$ such that, for any $0 < \epsilon \leq \epsilon_0$ and any two points $p, q \in M$ satisfying:*

(a) $p, q \notin B_\epsilon(Y_{[-L,0]}(x))$;
(b) $\mathcal{O}^+_Y(p) \cap B_{\epsilon/\rho}(x) \neq \emptyset$;
(c) $\mathcal{O}^-_Y(q) \cap B_{\epsilon/\rho}(x) \neq \emptyset$;

there is a $Z \in \mathcal{U}$ such that $Z = Y$ off $B_\epsilon(Y_{[-L,0]}(x))$ and $q \in \mathcal{O}^+_Z(p)$.

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Proof of Lemma 3.5. Given $X \in \mathcal{X}^1(M)$ we denote by $\mathrm{Per}_T(X)$ the set of periodic orbits of X with period $< T$.

We first prove a local version of Lemma 3.5.

LEMMA 3.7. *If $X \in \mathcal{KS}^1(M)$ and $T > 0$ then there are a neighborhood $\mathcal{V}_{X,T} \ni X$ and a residual subset $\mathcal{P}_{X,T}$ of $\mathcal{V}_{X,T}$ such that if $Y \in \mathcal{P}_{X,T}$ and $p \in \mathrm{Per}_T(Y)$ then $H_Y(p) = \mathrm{Cl}(W^u_Y(p)) \cap \mathrm{Cl}(W^s_Y(p))$.*

Proof. There is a neighborhood $\mathcal{V}_{X,T} \ni X$ such that

$$\mathrm{Per}_T(Y) = \{\sigma_1(Y), \ldots, \sigma_m(Y)\} \quad \text{for all } Y \in \mathcal{V}_{X,T}.$$

For each $1 \le i \le m$, let $\Psi_i : \mathcal{V}_{X,T} \ni Y \mapsto H_Y(\sigma_i(Y)) \in 2^M_c$. Note that Ψ_i, for all i, is lower semi-continuous by the persistence of transverse homoclinic orbits. Therefore, there is a residual subset $\mathcal{P}^i_{X,T}$ of $\mathcal{V}_{X,T}$ such that Ψ_i is upper semi-continuous in $\mathcal{P}^i_{X,T}$. Set $\mathcal{P}_{X,T} = \mathcal{KS}^1(M) \cap (\cap_i \mathcal{P}^i_{X,T}) \cap \mathcal{R}$, where $\mathcal{R}$ is the residual set given in Lemma 3.4. Then $\mathcal{P}_{X,T}$ is residual in $\mathcal{V}_{X,T}$.

Let us prove that $\mathcal{P}_{X,T}$ satisfies the conclusion of the lemma. For this, let $\sigma \in \mathrm{Per}_T(Y)$ for some $Y \in \mathcal{P}_{X,T}$. Then $\sigma = \sigma_i(Y)$ for some i, and so $\Psi_i(Y) = H_Y(\sigma)$.

Suppose, by contradiction, that $H_Y(\sigma) \neq \mathrm{Cl}(W^u_Y(\sigma)) \cap \mathrm{Cl}(W^s_Y(\sigma))$. Then there is an $x \in \mathrm{Cl}(W^u_Y(\sigma)) \cap \mathrm{Cl}(W^s_Y(\sigma)) \setminus H_Y(\sigma)$.

We have either (a) $x \notin \mathrm{Crit}(Y)$ or (b) $x \in \mathrm{Crit}(Y)$.

It is enough to prove the lemma in case (a). Indeed, suppose that case (b) holds. As Y is Kupka–Smale we have that $\mathcal{O}_Y(x)$ is hyperbolic. Clearly $\mathcal{O}_Y(x)$ is neither a sink nor a source, and so $W^s_Y(x) \setminus \mathcal{O}_Y(x) \neq \emptyset$ and $W^u_Y(x) \setminus \mathcal{O}_Y(x) \neq \emptyset$. Note that $\mathrm{Cl}(W^u_Y(\sigma))$ is Lyapunov stable since $Y \in \mathcal{R}$. As $x \in \mathrm{Cl}(W^u_Y(\sigma))$ we conclude that $W^u_Y(x) \subseteq \mathrm{Cl}(W^u_Y(\sigma))$. As $x \in \mathrm{Cl}(W^s_Y(\sigma))$, there is an $x' \in \mathrm{Cl}(W^s_Y(\sigma)) \cap (W^u_Y(x) \setminus \mathcal{O}(x))$ arbitrarily close to x (for this use the Grobman–Hartman Theorem as in the proof of Lemma 3.4 in Appendix A). Obviously $x' \notin \mathrm{Crit}(Y)$. If $x' \in H_Y(\sigma)$ we would have that $x \in H_Y(\sigma)$ since $\alpha_Y(x') = O_Y(x)$, contradicting $x \notin H_Y(\sigma)$. Henceforth $x' \in \mathrm{Cl}(W^u_Y(\sigma)) \cap \mathrm{Cl}(W^s_Y(\sigma)) \setminus H_Y(\sigma)$ and $x' \notin \mathrm{Crit}(Y)$. Then we conclude as in case (a), replacing x by x'.

Now we prove the lemma in case (a).

As $x \notin H_Y(\sigma)$, there is a compact neighborhood K of x such that $K \cap H_Y(\sigma) = \emptyset$. As Ψ_i is upper semi-continuous at Y, there is a neighborhood $\mathcal{U}$ of Y such that

$$K \cap H_Z(\sigma(Z)) = \emptyset, \tag{1}$$

for all $Z \in \mathcal{U}$.

Let ρ, L, ϵ_0 be the constants in Lemma 3.6 for $Y \in \mathcal{X}^1(M)$, x and $\mathcal{U}$ as above.

As $x \notin \mathrm{Crit}(Y)$, $Y_{[-L,0]}(x) \cap \mathcal{O}_Y(\sigma) = \emptyset$. Then, there exists $0 < \epsilon < \epsilon_0$ such that $\mathcal{O}_Y(\sigma) \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset$ and $B_\epsilon(x) \subseteq K$.

Choose an open set V containing $\mathcal{O}_Y(\sigma)$ such that $V \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset$.

As $x \in \mathrm{Cl}(W^u_Y(\sigma))$, one can choose $p \in (W^u_Y(\sigma) \setminus \{\sigma\}) \cap V$ such that

$$\mathcal{O}^+_Y(p) \cap B_{\epsilon/\rho}(x) \neq \emptyset.$$

Similarly, as $x \in \mathrm{Cl}(W^s_Y(\sigma))$, one can choose $q \in (W^s_Y(\sigma) \setminus \{\sigma\}) \cap V$ such that

$$\mathcal{O}^-_Y(q) \cap B_{\epsilon/\rho}(x) \neq \emptyset.$$

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We can assume that $\mathcal{O}_Y^-(p) \subset V$ and $\mathcal{O}_Y^+(q) \subset V$. Henceforth

$$(\mathcal{O}_Y^-(p) \cup \mathcal{O}_Y^+(q)) \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset. \tag{2}$$

Observe that $q \notin \mathcal{O}_Y^+(p)$ for, otherwise, p would be a homoclinic orbit of Y passing through K contradicting (1).

By construction ϵ, p, q satisfy (b) and (c) of Lemma 3.6. As $p, q \in V$ and $V \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset$ we have that ϵ, p, q also satisfy (a) of Lemma 3.6. Then, by Lemma 3.6, there is a $Z \in \mathcal{U}$ such that $Z = Y$ off $B_\epsilon(Y_{[-L,0]}(x))$ and $q \in \mathcal{O}_Z^+(p)$. Clearly $\sigma(Z) = \sigma$ and by (2) we have $p \in W_Z^u(\sigma)$ and $q \in W_Z^s(\sigma)$ since $Z = Y$ off $B_\epsilon(Y_{[-L,0]}(x))$. Hence $\mathcal{O} = \mathcal{O}_Z(p) = \mathcal{O}_Z(q)$ is a homoclinic orbit of σ. As $q \notin \mathcal{O}_Y^+(p)$, we have that $\mathcal{O} \cap B_\epsilon(x) \neq \emptyset$. Perturbing Z we can assume that $\mathcal{O}$ is transverse, i.e. $\mathcal{O} \subseteq H_Z(\sigma)$. As $\mathcal{O} \cap B_\epsilon(x) \neq \emptyset$ and $B_\epsilon(x) \subset K$ we would obtain $K \cap H_Z(\sigma(Z)) \neq \emptyset$ contradicting (1). This finishes the proof. □

We now continue with the proof of Lemma 3.5. Fix $T > 0$. For any $X \in \mathcal{KS}^1(M)$ consider $\mathcal{V}_{X,T}$ and $\mathcal{P}_{X,T}$ as in Lemma 3.7. Choose a sequence $X^n \in \mathcal{KS}^1(M)$ such that $\{X^n : n \in \mathbb{N}\}$ is dense in $\mathcal{X}^1(M)$ (recall that $\mathcal{X}^1(M)$ is a separable metric space). Denote $\mathcal{V}_{n,T} = \mathcal{V}_{X^n,T}$ and $\mathcal{P}_{n,T} = \mathcal{P}_{X^n,T}$.

Define

$$\mathcal{O}^T = \bigcup_n \mathcal{V}_{n,T} \quad \text{and} \quad \mathcal{P}^T = \bigcup_n \mathcal{P}_{n,T}.$$

Clearly $\mathcal{O}^T$ is open and dense in $\mathcal{X}^1(M)$.

We claim that $\mathcal{P}^T$ is residual in $\mathcal{O}^T$. Indeed, for any n there is a sequence $D_{k,n,T}$, $k \in \mathbb{N}$, such that

$$\mathcal{P}_{n,T} = \bigcap_k D_{k,n,T},$$

and $D_{k,n,T}$ is open and dense in $\mathcal{V}_{n,T}$ for any k. As

$$\mathcal{P}^T = \bigcup_n \mathcal{P}_{n,T} = \bigcup_n \left(\bigcap_k D_{k,n,T} \right) = \bigcap_k \left(\bigcup_n D_{k,n,T} \right)$$

and $\bigcup_n D_{k,n,T}$ is open and dense in $\bigcup_n \mathcal{V}_{n,T} = \mathcal{O}^T$, we conclude that $\mathcal{P}^T$ is residual in $\mathcal{O}^T$. This proves the claim.

In particular, $\mathcal{P}^T$ is residual in $\mathcal{X}^1(M)$ for every T. Set $\mathcal{P} = \bigcap_{N \in \mathbb{N}} \mathcal{P}^N$. It follows that $\mathcal{P}$ is residual in $\mathcal{X}^1(M)$. Choose $X \in \mathcal{P}$, $p \in \mathrm{Per}(X)$ and $N_0 \in \mathbb{N}$ larger than $\Pi_X(p) + 1$. By definition $X \in \mathcal{P}^{N_0}$, and so $X \in \mathcal{P}_{X^n,N_0}$ for some n. As $N_0 > \Pi_X(p)$ we have $p \in \mathrm{Per}_{N_0}(X)$. Then $H_X(p) = \mathrm{Cl}(W_X^u(p)) \cap \mathrm{Cl}(W_X^s(p))$ by Lemma 3.7 applied to X^n and $T = N_0$. This completes the proof of the lemma. □

4. *Proof of Theorem A and Corollary 1.1*

Proof of Theorem A. By Theorem 3.1 we have that homoclinic classes for a residual subset of C^1 vector fields on closed manifolds are neutral sets. This leads us to apply the results in §2. Thus, Theorem A(1) follows from Lemma 2.2(2). Theorem A(2) follows from Lemma 2.2(1). Theorem A(3) follows from Corollary 2.5. Theorem A(4) follows from

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Proposition 2.6. Similarly, Theorem A(5) follows from Proposition 2.7. Theorem A(6) follows from Proposition 2.3.

To prove Theorem A(7) we proceed as follows. If X has finitely many homoclinic classes, then the union of the homoclinic classes of X is obviously closed. By [**17**] it follows that $\Omega(X)$ is the union of the homoclinic classes of X (recall that X is C^1 generic). This implies that every homoclinic class of X is Ω-isolated, and so, they are isolated by Theorem A(4). Conversely, suppose that the union of the homoclinic classes of X is closed and that every homoclinic class of X is isolated. Let $\mathcal{F}'$ be the collection of all homoclinic classes of X. As every homoclinic class of X is isolated by assumption one has $\mathcal{F}' \subset \mathcal{F}$ (recall the notation in Proposition 2.8). We have that $\bigcup_{\Lambda\in\mathcal{F}'} \Lambda$ is closed by hypothesis. Then Proposition 2.8 implies that $\mathcal{F}'$ is finite and the proof follows. □

Proof of Corollary 1.1. That (1) implies (2) follows from Theorem A(5). To prove that (2) implies (1) we proceed as follows. If $H_X(p)$ is isolated, $H_X(p)$ is Ω-isolated by Theorem A(4). In particular, $H_X(p)$ is not in the closure of the sinks and sources of X unless it is either a sink or a source of X and we would be done. By [**13, 14**], as M is three-dimensional and X is non-singular and generic, one has that $H_X(p)$ is hyperbolic. That (1) implies (3) follows from the hyperbolic theory. Indeed, if $H_X(p)$ is hyperbolic, then $H_X(p)$ is isolated, transitive and hyperbolic. In other words, $H_X(p)$ is a basic set of X. Then, the conclusion follows from the structural stability of basic sets [**16**]. That (3) implies (1) follows from [**13, 18**] since $H_X(p)$ has no singularities (recall X has no singularities by hypothesis). This finishes the proof. □

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A. *Appendix. Proof of Lemma 3.4.*

Here we give a proof of Lemma 3.4 using Lemma 3.6. The proof given in [**15**, Theorem 6.1, p. 372] uses a different version of the C^1 Closing Lemma rather than Lemma 3.6.

Clearly, Lemma 3.4 is a consequence of the following local lemma.

LEMMA A.1. *If $X \in \mathcal{KS}^1(M)$ and $T > 0$, then there is a neighborhood $\mathcal{U}_{X,T}$ of X and a residual subset $\mathcal{R}_{X,T}$ of $\mathcal{U}_{X,T}$ such that if $Y \in \mathcal{R}_{X,T}$ and $p \in \mathrm{Crit}_T(Y)$, then $\mathrm{Cl}(W^u_Y(p))$ is Lyapunov stable for Y and $\mathrm{Cl}(W^s_Y(p))$ is Lyapunov stable for $-Y$.*

Proof. Recall that $\mathrm{Crit}(Y) = \{p_1(Y), \ldots, p_k(Y)\}$ for every Y in some neighborhood $\mathcal{U}_{X,T}$ of X, where $p_i(Y)$, $1 \le i \le k$, is either a periodic orbit or a singularity of Y.

For any $i \in \{1, \ldots, k\}$ we define $\Phi_i : \mathcal{U}_{X,T} \to 2^M_c$ by

$$\Phi_i(Y) = \mathrm{Cl}(W^u_Y(p_i(Y))).$$

By the continuous dependence of unstable manifolds we have that Φ_i is a lower semi-continuous map, and so Φ_i is also upper semi-continuous for every vector field in some

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residual subset $\mathcal{R}_i$ of $\mathcal{U}_{X,T}$. Set $\mathcal{R}_{X,T} = \mathcal{KS}^1(M) \cap \left(\bigcap_i \mathcal{R}_i\right)$. Then $\mathcal{R}_{X,T}$ is residual in $\mathcal{U}_{X,T}$. Let us prove that $\mathcal{R}_{X,T}$ satisfies the conclusion of the lemma.

Let $\sigma \in \mathrm{Crit}_T(Y)$ for some $Y \in \mathcal{R}_{X,T}$. Then, $\sigma = p_i(Y)$ for some i, and so $\Phi_i(Y) = \mathrm{Cl}(W^u_Y(\sigma))$.

Suppose by contradiction that $\mathrm{Cl}(W^u_Y(\sigma))$ is not Lyapunov stable for Y.

Then, there are an open set U containing $\mathrm{Cl}(W^u_Y(\sigma))$ and two sequences $x_n \to x \in \mathrm{Cl}(W^u_Y(\sigma))$, $t_n \geq 0$ such that

$$Y_{t_n}(x_n) \notin U.$$

As in the proof of Lemma 3.5 we have either (a) $x \notin \mathrm{Crit}(Y)$ or (b) $x \in \mathrm{Crit}(Y)$.

Again it is enough to prove the lemma in case (a). Indeed, suppose that case (b) holds. As Y is Kupka–Smale we have that $\mathcal{O}_Y(x)$ is hyperbolic. Clearly $\mathcal{O}_Y(x)$ is neither a sink or a source and so $W^s_Y(x) \setminus \mathcal{O}_Y(x) \neq \emptyset$ and $W^u_Y(x) \setminus \mathcal{O}_Y(x) \neq \emptyset$. Let $V \subset U$ be a small neighborhood of x given by the Grobman–Hartman Theorem [**6**] such that $\partial(W^u_Y(x, V)) = D^u_Y(x)$ is a fundamental domain for $W^u_Y(x)$ (here $W^u_Y(x, V)$ denotes the connected component of $V \cap W^u_Y(x)$ containing $\mathcal{O}_X(x)$). Note that $D^u_Y(x) \subset W^u_Y(x) \setminus \mathcal{O}_Y(x)$. As $x_n \to x$, we can assume that $x_n \in \mathrm{Int}(V)$ for all n. As $Y_{t_n}(x_n) \notin U$, we have that $x_n \notin W^s_Y(\sigma)$. So there is an $s_n > 0$ such that $x'_n = Y_{s_n}(x_n) \in \partial V$ and $Y_s(x_n) \in \mathrm{Int}(V)$ for $0 \leq s < s_n$. Since $Y_{t_n}(x_n) \notin U$ we have that $Y_{t_n}(x_n) \notin \mathrm{Cl}(V)$ for all n. From this we conclude that $s_n < t_n$ for all n. On the other hand, as $x_n \to x$, passing to a subsequence if necessary, we can assume that $x'_n \to x'$ for some $x' \in D^u_Y(x) \subset W^u_Y(x) \setminus \mathcal{O}_Y(x)$.

Now we have the following claim.

CLAIM A.2. $x' \in \mathrm{Cl}(W^u_Y(\sigma))$.

Proof of the Claim. As $x \in \mathrm{Cl}(W^u_Y(\sigma))$, using the Connecting Lemma [**8**], there is a Z C^1 near Y such that $W^u_Z(\sigma(Z)) \cap W^s_Z(x(Z)) \neq \emptyset$. In other words, there is a saddle connection between $\sigma(Z)$ and $x(Z)$. Breaking this saddle connection as in the proof of Lemma 2.4 in [**6**, p. 101], using the Inclination Lemma [**6**], we can find Z' C^1 close to Z so that $W^u_{Z'}(\sigma(Z'))$ passes close to x'. This contradicts the upper-semicontinuity of Φ_i at Y. Thus, $x' \in \mathrm{Cl}(W^u_Y(\sigma))$ and the claim is proved. □

Back to the proof of Lemma 3.4, as $x' \in D^u_Y(x)$ we have that $x' \notin \mathrm{Crit}(Y)$. As $x'_n \to x'$ and $Y_{t_n - s_n}(x'_n) \notin U$ with $t_n - s_n > 0$, we conclude as in case (a) replacing x by x', x_n by x'_n and t_n by $t_n - s_n$.

Now we prove the lemma in case (a).

As $\mathrm{Cl}(W^u_Y(\sigma)) \subseteq U$ and Φ_i is upper semi-continuous, there is a C^1 neighborhood $\mathcal{U} \subset \mathcal{U}_{X,T}$ of Y such that

$$\mathrm{Cl}(W^u_Z(\sigma(Z))) \subseteq U, \tag{3}$$

for all $Z \in \mathcal{U}$.

Let ρ, L, ϵ_0 be as in Lemma 3.6 for $X = Y$, x, and $\mathcal{U}$ as above. As $x \notin \mathrm{Crit}(Y)$, $Y_{[-L,0]}(x) \cap \mathcal{O}_Y(\sigma) = \emptyset$. As $x \in \mathrm{Cl}(W^u_Y(\sigma))$, $Y_{[-L,0]}(x) \subset \mathrm{Cl}(W^u_Y(\sigma))$ and so $Y_{[-L,0]}(x) \subset U$. Then, there exists $0 < \epsilon \leq \epsilon_0$ such that $B_\epsilon(Y_{[-L,0]}(x)) \cap \mathcal{O}_Y(\sigma) = \emptyset$ and $B_\epsilon(Y_{[-L,0]}(x)) \subseteq U$. Choose an open set V containing $\mathcal{O}_Y(\sigma)$, $V \subset \mathrm{Cl}(V) \subset U$, such that $V \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset$. For n large, we have $x_n \in B_{\epsilon/\rho}(x)$ and we set $q = Y_{t_n}(x_n) \notin U$. As $x_n \to x$ and $t_n > 0$ we have $\mathcal{O}^-_Y(q) \cap B_{\epsilon/\rho}(x) \neq \emptyset$. As $x \in \mathrm{Cl}(W^u_Y(\sigma))$, there is a

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$p \in (W^u_Y(\sigma) \setminus \{\sigma\}) \cap V$ such that $\mathcal{O}^+_Y(p) \cap B_{\epsilon/\rho}(x) \neq \emptyset$ and $\mathcal{O}^-_Y(p) \subseteq V$. By construction ϵ, p, q satisfy (b) and (c) of Lemma 3.6. As $V \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset$ and $q \notin U$, we have that ϵ, p, q also satisfy (a) of Lemma 3.6. Then, by Lemma 3.6, there is a $Z \in \mathcal{U}$ such that $Z = Y$ off $B_\epsilon(Y_{[-L,0]}(x))$ and $q \in \mathcal{O}^+_Z(p)$. As $V \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset$ and $\mathcal{O}^-_Y(p) \subset V$ we have

$$\mathcal{O}^-_Y(p) \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset. \tag{4}$$

Now, (4), together with $V \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset$ and $Z = Y$ off $B_\epsilon(Y_{[-L,0]}(x))$ imply that $\sigma(Z) = \sigma$ and $p \in W^u_Z(\sigma)$. As $q \notin U$ and $q \in W^u_Z(\sigma)$ (recall $p \in W^u_Z(\sigma)$ and $q \in \mathcal{O}^+_Z(p)$), we have a contradiction by (3). This finishes the proof. □

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References

[1] M. C. Arnaud. Création de connexions en topologie C^1. *Ergod. Th. & Dynam. Sys.* **21** (2001), 339–381.
[2] F. Béguin, C. Bonatti and J. L. Vieitez. Construction de flots de Smale en dimension 3. *Ann. Fac. Sci. de Toulouse Math.* **6** (1999), 369–410.
[3] N. P. Bhatia and G. P. Szego. *Stability Theory of Dynamical Systems*. Springer, Heidelberg, 1979.
[4] C. Bonatti and L. J. Díaz. Connexions heterocliniques et genericité d'une infinité de puits ou de sources. *Ann. Sci. École Norm. Sup.* **32** (1999), 135–150.
[5] C. Bonatti, L. J. Díaz and E. Pujals. A C^1-dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources. *Preprint*, 1999.
[6] W. de Melo and J. Palis. *Geometric Theory of Dynamical Systems—An Introduction*. Springer, Berlin, 1982.
[7] S. Hayashi. A C^1 make or break lemma. *Bull. Braz. Math. Soc.* **31** (2000), 337–350.
[8] S. Hayashi. Connecting invariant manifolds and the solution of the C^1 stability and Ω-stability conjectures for flows. *Ann. Math.* **145** (1997), 81–137.
[9] S. Hayashi. Hyperbolicity, stability, and the creation of homoclinic points. *Documenta Math., Extra Volume ICM* **2** (1998).
[10] M. Hirsch, C. Pugh and M. Shub. Stable manifolds and hyperbolic sets. *Proc. Symp. Pure Math.—Global Analysis* **14** (1970), 133–163.
[11] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, Cambridge, 1995.
[12] K. Kuratowski. *Topology II*. Academic Press–PWN-Polish Science Publishers, Warszawa, 1968.
[13] S. T. Liao. On hyperbolicity properties of nonwandering sets of certain 3-dimensional differential systems. *Acta Math. Sci.* **3** (1983), 361–368.
[14] R. Mañé. An ergodic closing lemma. *Ann. Math.* **116** (1982), 503–540.
[15] C. A. Morales and M. J. Pacifico. Mixing attractors for 3-flows. *Nonlinearity* **14** (2001), 359–378.
[16] J. Palis and F. Takens. *Hyperbolicity and Sensitive-Chaotic Dynamics at Homoclinic Bifurcations*. Cambridge University Press, Cambridge, 1993.
[17] C. Pugh. An improved closing lemma and a general density theorem. *Amer. J. Math.* **89** (1967), 1010–1021.
[18] L. Wen. On the C^1-stability conjecture for flows. *J. Diff. Eqns* **129** (1995), 334–357.
[19] L. Wen and Z. Xia. C^1 connecting lemmas. *Trans. Amer. Math. Soc.* **352** (2000), 5213–5230.

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