Double generating spirographic curves

STEPHAN BERENDONK

We can use a spirograph in different ways to draw curves. Do these different modes of drawing necessarily yield different curves, or is it also possible to get the same curve by using two different modes of drawing? We address this question geometrically, not using the parametric equations of the curves.

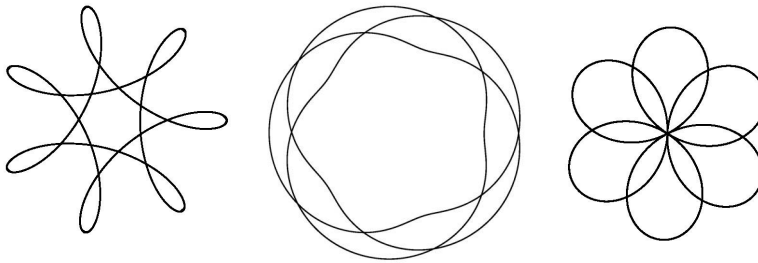


FIGURE 1: Diagrams representing the three types of curves treated in this paper: a hypotrochoid (left), an epitrochoid (middle) and a rose curve (right).

Threegraphs are twographs

Have you ever played with a spirograph? It is a drawing toy consisting of a set of plastic gears of different sizes. Some of the gears are ring-shaped other have the form of a disc (see Figure 2).

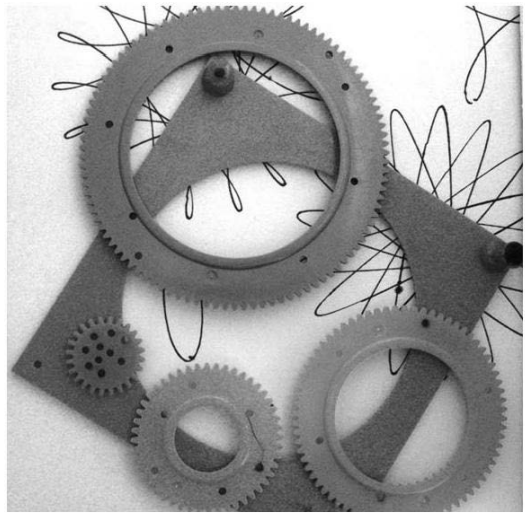


FIGURE 2

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To draw a curve with a spirograph take two gears, fix the greater one and put the smaller one inside it such that the two gears touch each other. Then put a pen in one of the holes of the smaller gear and move it carefully in such a way that the two gears keep in touch during the motion. Meanwhile the pen traces a curve and eventually, after some revolutions of the small gear, it will be back in its starting position so that the curve closes. This drawing process is illustrated in Figure 3.

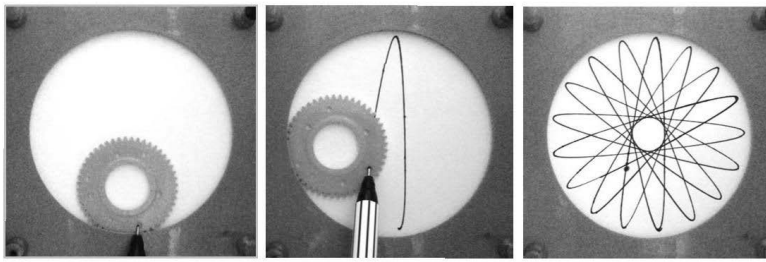


FIGURE 3

After playing with a spirograph for a while I realised that it is also possible to draw curves by putting three gears inside each other instead of only two. In that case, the small gear is not directly rolling on the inside of the fixed gear, but there is an intermediate annulus in between (see Figure 4).

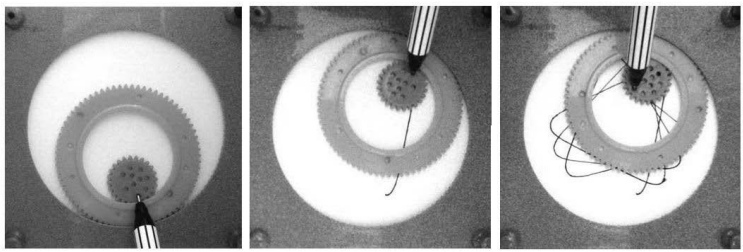


FIGURE 4

So the small gear rolls around the inside of the intermediate annulus, while simultaneously the intermediate annulus is rolling with its outside around the inside of the fixed gear. During the motion the centres of the three gears stay collinear (see Figure 5). The curves I got in this way were new ones, in the sense that I could not produce them with my spirograph set in the ordinary way by using only two gears. However, this was due to the fact that I had only a limited number of gears available. Indeed, we will show that the curves generated with three gears can also be generated with only two gears of suitable sizes.

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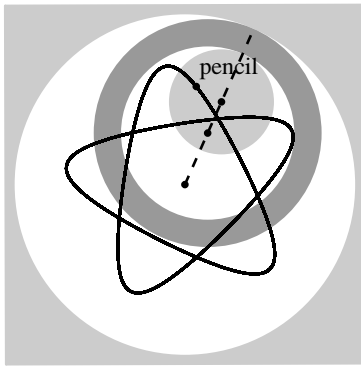


FIGURE 5

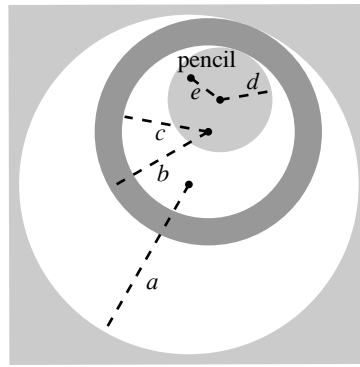


FIGURE 6

Let us agree to call a curve produced with two gears a *twograph* and a curve produced with three gears a *threegraph*. Now assume that we have drawn a threegraph by using a fixed gear of radius a, an annulus with outer circle of radius b and inner circle of radius c, and a small gear of radius d. The pencil was placed in a hole at distance e from the centre of the small gear (see Figure 6). Suppose that when we started drawing, the pencil had maximal distance to the centre O of the fixed gear and that after a short while the annulus touched the fixed gear at C. At that instant M was the centre of the small gear, P the position of the pencil, D the point where the small gear touched the annulus, $\hat{A}$ and A the points of the annulus and the fixed gear that coincided in the starting position, and E and $\hat{E}$ the points of the annulus and the small gear that coincided in the starting position (see Figure 7).

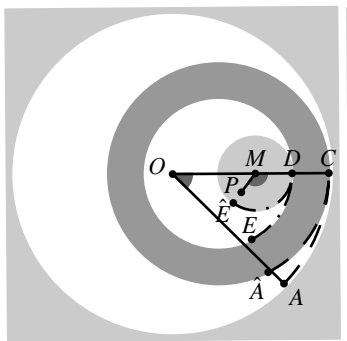


FIGURE 7

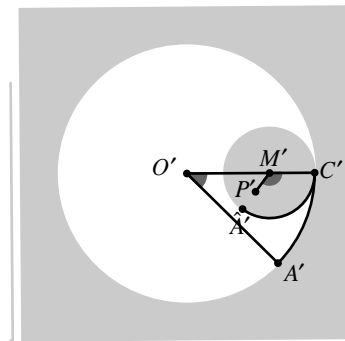


FIGURE 8

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Then we have

$$|\overline{OM}| = a - (b - c) - d \text{ and } |\overline{MP}| = e.$$

The cogs of the gears ensure that the gears roll without slipping so that the corresponding circular arcs have the same length. Thus

$$|\text{arc}\,\hat{E}D| = |\text{arc}\,ED| \text{ and } |\text{arc}\,\hat{A}C| = |\text{arc}\,AC|.$$

Now arc ED and arc $\hat{A}C$ subtend the same angle, so that

$$\frac{|\text{arc}\,ED|}{c} = \frac{|\text{arc}\,\hat{A}C|}{b}.$$

Thus we have

$$\angle PMD \cdot d = \left|\text{arc}\,\hat{E}D\right| = \left|\text{arc}\,ED\right| = \left|\text{arc}\,\hat{A}C\right| \cdot \frac{c}{b} = \left|\text{arc}\,AC\right| \cdot \frac{c}{b} = \angle AOC \cdot \frac{ac}{b}$$

Hence

$$\frac{\angle PMD}{\angle AOC} = \frac{ac}{bd}.$$

Assume that we have also drawn a twograph by using a fixed gear of radius R and a rolling gear of radius r and by placing the pencil at distance f from the centre of the rolling gear. Suppose again that when we started drawing, the pencil had maximal distance to the centre O' of the fixed gear and that after a short while the rolling gear touched the fixed gear at C'. At that instant M' was the centre of the rolling gear, P' the position of the pencil, and $\hat{A}'$ and A' the points of the rolling and the fixed gear that coincided in the starting position (see Figure 8). Then we have

$$|\overline{O'M'}| = R - r \text{ and } |\overline{M'P'}| = f$$

and

$$|\text{arc}\,\hat{A}'C'| = |\text{arc}\,A'C'|.$$

Thus

$$\frac{\angle P'M'C'}{\angle AOC} = \frac{R}{r}.$$

Now when $|\overline{OM}| = |\overline{O'M'}|$, $|\overline{MP}| = |\overline{M'P'}|$ and

$$\frac{\angle PMD}{\angle AOC} = \frac{\angle P'M'C'}{\angle A'O'C'}$$

the twograph and the threegraph will be congruent. We can arrange this by choosing f equal to e and R and r such that

$$R - r = a - (b - c) - d$$

and

$$\frac{R}{r} = \frac{ac}{bd}$$

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hold. This is the case for

$$R = \frac{ac\,[(a + c) - (b + d)]}{ac - bd} \tag{1}$$

and

$$r = \frac{bd\,[(a + c) - (b + d)]}{ac - bd}. \tag{2}$$

A threegraph given by the parameters a, b, c, d and e can thus be generated as a twograph by chosing the parameters R and r as in (1) and (2), and by putting the pencil at a distance e from the centre of the fixed gear. Note that conversely every twograph can also be generated as a threegraph, even in infinitely many different ways.

Epitrochoids are peritrochoids

Instead of rolling a gear around the inside of a fixed gear, you could also roll it around the outside (see Figure 9). Curves produced by rolling around the inside are called *hypotrochoids*. Curves produced by rolling around the outside are called *epitrochoids* (see Figure 1). Still another way to draw a curve with the spirograph is by rotating an annulus around a fixed gear, such that the inner circle of the annulus rolls around the outside of the fixed gear (see Figure 10). Curves produced in this way are called *peritrochoids*.

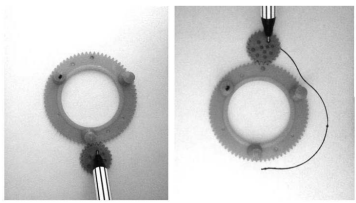


FIGURE 9

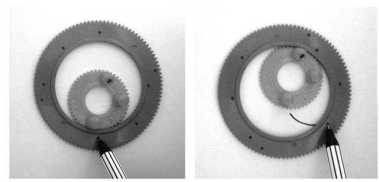


FIGURE 10

In [1] Cundy showed that epitrochoids can also be generated as peritrochoids and vice versa. Unlike Cundy, who compared the parametric equations of the curves, we determine the suitable sizes of the gears by contemplating the geometric figures. We only do the analysis. The synthesis is left as an exercise.

Suppose we were able to draw a curve simultaneously as an epitrochoid and as a peritrochoid using only one pencil P. Let R and r be the radii of the fixed gear G_1 and the rolling gear G_2 used to draw the curve as an epitrochoid and let R' and r' be the radii of the fixed gear G_1' and the inner circle of the rolling annulus G_2' used to draw the curve as a peritrochoid. Further, let e and e' be the distances between the pencil P and the centres M and M' of G_2 and G_2'. From the symmetry of the curve we can deduce that the fixed gears G_1 and G_1' had the same centre; call it O.

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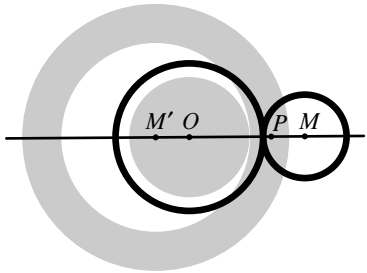


FIGURE 11

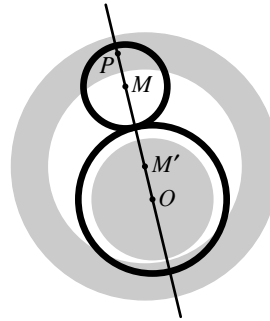


FIGURE 12

At some moment, while we were drawing the curve, the pencil P had minimal distance to O. At that instant M', O, M and P were collinear, with P between O and M and O between M' and P (see Figure 11; G_1 and G_2 are represented by the black circles and G_1' and G_2' by the grey gears). This yields

$$R + r - e = |\overline{OM}| - |\overline{MP}| = |\overline{OP}| = |\overline{M'P}| - |\overline{M'O}| = e' - (r' - R'). \qquad (3)$$

At another moment the pencil P had maximal distance to O. At that instant M', O, M and P were collinear too, with M and M' between P and O (see Figure 12). From this we obtain

$$R + r + e = |\overline{OM}| + |\overline{MP}| = |\overline{OP}| = |\overline{OM'}| + |\overline{M'P}| = r' - R' + e'. \qquad (4)$$

By adding equations (3) and (4) we get

$$e' = R + r. \qquad (5)$$

Subtracting equation (3) from (4) gives

$$e = r' - R'. \qquad (6)$$

Now consider an arbitrary moment and let C respectively C' be the points where G_1 and G_2 respectively G_1' and G_2' touched each other at that instant. Let A on G_1 and $\hat{A}$ on G_2 respectively E on G_1' and $\hat{E}$ on G_2' be the points that coincided, when P had minimal distance to O for the last time before the moment under consideration (see Figure 13).

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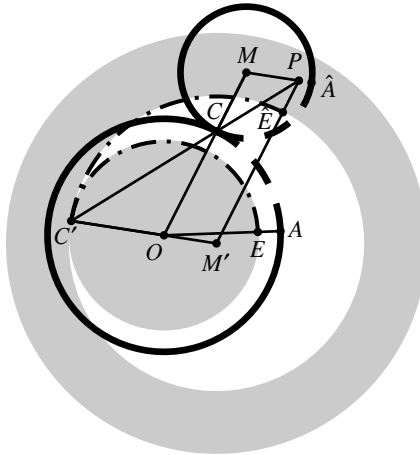


FIGURE 13

Then we have

$$\angle AOC \cdot R \;=\; \left|\operatorname{arc} AC\right| \;=\; \left|\operatorname{arc} \hat{A}C\right| \;=\; \angle CM\hat{A} \cdot r \;=\; \angle OMP \cdot r.$$

Equations (5) and (6) show, that $OM'PM$ is a parallelogram and therefore

$$\angle OMP \;=\; \angle PM'O.$$

Thus

$$\angle EOC' \cdot R' = \left|\operatorname{arc} EC'\right| = \left|\operatorname{arc} \hat{E}C'\right| = \angle \hat{E}M'C' \cdot r' = \angle PM'O.r' = \angle OMP.r'.$$

So

$$\angle OMP \cdot \frac{r'}{R'} \;=\; \angle EOC' \;=\; \angle AOC \;+\; \angle COC' \;=\; \angle OMP \cdot \frac{r}{R} \;+\; \angle OMP.$$

Dividing this equation by $\angle OMP$ gives

$$\frac{r'}{R'} \;=\; \frac{r}{R} \;+\; 1. \tag{7}$$

By using equations (6) and (7) we can express R' and r' in terms of R, r and e. We get

$$R' \;=\; e \cdot \frac{R}{r} \tag{8}$$

and

$$r' \;=\; e \cdot \left(\frac{R}{r} \;+\; 1\right). \tag{9}$$

Finally note that

$$\frac{\angle EOC'}{\angle AOC} \;=\; \frac{\angle AOC \;+\; \angle COC'}{\angle AOC} \;=\; \frac{\angle AOC \;+\; \angle OMP}{\angle AOC} \;=\; 1 \;+\; \frac{R}{r}.$$

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This completes the analysis. Thus an epitrochoid given by the parameters R, r and e can simultaneously be generated as a peritrochoid by choosing the parameters R', r' and e' as in equations (8), (9) and (5) and by rotating the annulus of the peritrochoid $1 + \frac{R}{r}$ times as fast as the rolling gear of the epitrochoid.

Rose curves are rose-like curves

Imagine we drew a curve with the spirograph in the usual way, thus as a hypotrochoid, but instead of placing the pencil P in a hole inside the rolling gear we managed to fix it to its boundary. Curves produced in this way are called *hypocyloids*. The black curve in Figure 14 for instance is a hypocycloid, which is drawn with a fixed gear of radius 5/4 times as big as the radius of the rolling gear.

To each point P of the hypocycloid we associate a point P', namely the foot of the perpendicular dropped from the centre of the fixed gear to the tangent of the hypocycloid at P. The associated points form a new curve, the *pedal* of the hypocycloid with respect to its centre (see the white curve in Figure 14). Due to their appearance those pedals are also called *rose curves* (see also Figure 1).

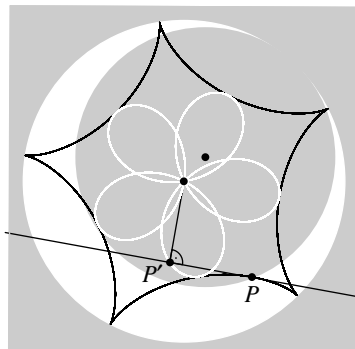


FIGURE 14

Each rose curve passes through the centre of the fixed gear that was used to draw the hypocycloid from which it descends. Some hypotrochoids drawn with a spirograph have the same property, that is, they also pass through the centre of the fixed gear used to draw it. This happens if the radius r' of the rolling gear is greater than half the radius R' of the fixed gear and the distance between the pencil and the centre of the rolling gear is equal to $R' - r'$. Hypotrochoids having this property look quite similar to rose curves. Let us therefore call them *rose-like curves*. It turns out that rose curves and rose-like curves are not just similar, but actually the same. An algebraic proof of this fact, using the parametric equations of the curves involved, can be found in [2]. We however, again, want to determine the suitable sizes for the gears geometrically.

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Suppose, for the sake of an analysis that we were able to draw a rose curve simultaneously as a pedal of a hypocycloid with respect to its centre and as a hypotrochoid. By simultaneously we mean that, if the pencil used to draw the hypocycloid is at some point P, then at the same moment the pencil of the hypotrochoid is at the associated point P'.

Let R and r be the radii of the fixed gear G_1 and the rolling gear G_2 used to draw the hypocycloid and let R' and r' be the radii of the fixed gear G_1' and the rolling gear G_2' used to draw the hypotrochoid. Further, let M and M' be the centres of the rolling gears G_1 and G_1'. From the symmetry of the curve it follows that the fixed gears G_1 and G_1' have the same centre, call it O.

We may assume that $2r > R$, since the same hypocycloid can also be drawn by rolling a gear of radius $R - r$ within the fixed gear of radius R. For a geometrical proof of this fact see [3]. Now let P be a point where the hypocycloid and its rose touch each other. Note that both pencils pass through P at the same time. Consider the moment this happens and let C be the point where G_1 and G_2 touch each other at that instant. Since P is a point on the hypocycloid with minimal distance to O, the points P, O, M and C are collinear with O between P and M, and M between O and C (see Figure 15; G_1 and G_2 are represented by the grey gears and G_1' and G_2' by the black circles). However P is also a point on the rose curve with maximal distance to O, so the points P, M' and O are also collinear with M' between P and O. This yields

$$2(R' - r') + R = (R' - r') + (R' - r') + R = |\overline{PM'}| + |\overline{M'O}| + |\overline{OC}|$$
$$= |\overline{PC}| = |\overline{PM}| + |\overline{MC}| = r + r = 2r.$$

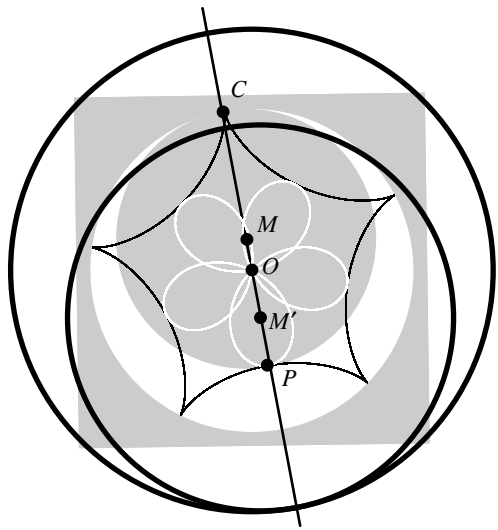


FIGURE 15

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Thus

$$R' - r' = r - \frac{R}{2}. \tag{10}$$

Now consider an arbitrary moment and let P and P' be the positions of the pencils belonging to the hypocycloid and the hypotrochoid respectively. Furthermore let C respectively C' be the points where G_1 and G_2 respectively G_1' and G_2' touched each other at that instant. Let B on G_1 be the point where the pencil P of the hypocycloid touched G_1 for the last time before the moment under consideration and let A and $\hat{A}$ be the points on G_1' respectively G_2' that coincided at that moment (see Figure 16). Note that A lies on the line BO with O between A and B.

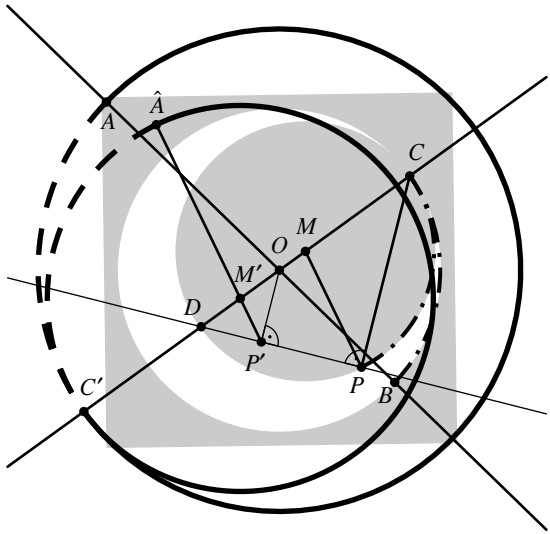


FIGURE 16

Since G_2 rolls without slipping on G_1, the point C must have zero velocity when regarded as a point on G_2. Therefore the motion of G_2 can be seen as an instantaneous rotation around C. So the pencil P moves in a direction perpendicular to CP and the tangent to the hypocycloid at P is the perpendicular to CP through P. This perpendicular meets G_2 in a second point D and, by Thales theorem, $\overline{CD}$ is a diameter of G_2.

We know that the centre M' of G_2' has to lie at the same distance from the points P' and O. Due to Thales, the midpoint of segment $\overline{DO}$ has this property. So let us assume that M' is the midpoint of $\overline{DO}$. Then M' lies on the line CO and so does C'. Note also that $OM'P'$ and CMP are similar triangles, since they are both isosceles and $\angle M'OP' = \angle MCP$. Thus we have $\angle P'M'O = \angle PMC$.

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By using these facts, we get

$$\begin{aligned}\angle BOC\cdot R' &= \angle AOC'\cdot R' = |\operatorname{arc} AC'| = |\operatorname{arc} \hat{A}C'| = \angle \hat{A}M'C'\cdot r' \\ &= \angle P'M'O\cdot r' = \angle PMC\cdot r' = \frac{|\operatorname{arc} PC|}{r}\cdot r' = \frac{|\operatorname{arc} BC|}{r}\cdot r' \\ &= \frac{\angle BOC\cdot R}{r}\cdot r'.\end{aligned}$$

Thus

$$\frac{R'}{r'} = \frac{R}{r}. \tag{11}$$

By using equations (10) and (11) we can express R' and r' in terms of R and r. We get

$$R' = \frac{R}{2}\cdot\frac{2r - R}{R - r}, \tag{12}$$

and

$$r' = \frac{r}{2}\cdot\frac{2r - R}{R - r}. \tag{13}$$

This completes the analysis. A rose curve associated to a hypocyloid with parameters R and r can thus be generated as a hypotrochoid by choosing the parameters R' and r' as in (12) and (13) and putting the pencil at a distance $R' - r'$ from the centre of the rolling gear. Note that the approach was analogous to the one we used in the case of the epi- and peritrochoids. We considered the situation at two different moments and deduced expressions for the difference and the quotient of R' and r' in terms of R and r.

References

1. H. M. Cundy, Peritrochoids, *Math. Gaz.* **52** (May 1968) pp. 161-162.
2. L. M. Hall, Trochoids, roses, and thorns – beyond the spirograph, *Coll. Math. J.* **23** (1992) pp. 20-35.
3. E. H. Lockwood, *A book of curves*, Cambridge University Press (1961).

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