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Non-degenerate singularities of integrable dynamical systems

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Abstract. We give a natural notion of non-degeneracy for singular points of integrable non-Hamiltonian systems, and show that such non-degenerate singularities are locally geometrically linearizable and deformation rigid in the analytic case. We conjecture that the same result also holds in the smooth case, and prove this conjecture for systems of type $(n, 0)$, i.e. n commuting smooth vector fields on an n-manifold.

1. *Introduction*

There are many natural dynamical systems that are non-Hamiltonian, for example because they have non-holonomic constraints or because they don't conserve energy, but which are still integrable in a natural sense (see, e.g., [**2, 4, 10, 14**] for some examples). It is interesting to study the topology, and in particular the singularities, of such integrable non-Hamiltonian systems. Unlike the Hamiltonian case, which has been very extensively studied, the non-Hamiltonian case is still largely open. To our knowledge, even the notion of non-degeneracy of singularities for integrable non-Hamiltonian systems has not appeared in the literature before.

The aim of this paper is to establish and study this notion of non-degeneracy. In particular, we want to extend the geometric local linearization theorems of Vey [**15**] and Eliasson [**9**] to the non-Hamiltonian case. We will show that, similar to the Hamiltonian case, non-degenerate singularities of analytic integrable dynamical systems are rigid with respect to deformations, and are geometrically linearizable (see Theorems 4.3, 4.5 and 4.8). We conjecture that the same theorems are also true for smooth non-Hamiltonian integrable systems, and prove this conjecture for the class of systems of type $(n, 0)$, i.e. n commuting smooth vector fields on a n-manifold (Theorem 6.2). This last theorem is the starting point of a very recent work by N. V. Minh and the author [**21**] on the geometry of non-degenerate $\mathbb{R}^n$-actions on n-manifolds.

In this paper, we will work in both the analytic (real or complex) and the smooth categories. The analytic part of this paper relies heavily on our theorem [**18, 19**] on the existence of convergent Poincaré–Dulac–Birkhoff normalization for analytic integrable dynamical systems.

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2. *Geometric equivalence of integrable systems*

Recall that a dynamical system given by a vector field X on an m-dimensional manifold M is called *integrable* (in the non-Hamiltonian sense) if there exist integers $p \geq 1$, $q \geq 0$, $p + q = m$, p vector fields $X_1 = X, X_2, \ldots, X_p$, and q functions $F_1, \ldots, F_q$ on M, such that the vector fields $X_1, \ldots, X_p$ commute with each other, and the functions $F_1, \ldots, F_q$ are common first integrals for these vector fields:

$$[X_i, X_j] = 0 \quad \text{for all } i, j = 1, \ldots, p \tag{2.1}$$

and

$$X_i(F_j) = 0 \quad \text{for all } i = 1, \ldots, p, \ j = 1, \ldots, q. \tag{2.2}$$

Moreover, one requires that

$$dF_1 \wedge \cdots \wedge dF_q \neq 0 \quad \text{and} \quad X_1 \wedge \cdots \wedge X_p \neq 0 \tag{2.3}$$

almost everywhere. We will also say that the m-tuple $(X_1, \ldots, X_p, F_1, \ldots, F_q)$ is an *integrable system of type* (p, q). This notion of non-Hamiltonian integrability is a very natural extension of the notion of integrability à la Liouville from the Hamiltonian case to the non-Hamiltonian case, and it retains the main dynamical features of Hamiltonian integrability, see, e.g., [**1, 2, 4, 10, 14, 18, 20**]. A Hamiltonian system with n degrees of freedom which is integrable à la Liouville is also integrable in the above sense with $p = q = n$ and $m = 2n$.

Geometrically, an integrable system $(X_1, \ldots, X_p, F_1, \ldots, F_q)$ of type (p, q) may be viewed as a singular p-dimensional foliation (given by the infinitesimal $\mathbb{K}^p$-action generated by $X_1, \ldots, X_p$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and moreover each leaf of this foliation admits a natural induced affine structure from the action), and this foliation is integrable in the sense that it admits a complete set of first integrals, i.e. the functional dimension of the algebra of first integrals of the foliation is equal to the codimension of the foliation.

Denote by $\mathcal{F}$ the algebra of common first integrals of $X_1, \ldots, X_p$. Instead of taking $F_1, \ldots, F_q$, we can choose from $\mathcal{F}$ any other family of q functionally independent functions, and they will still form with $X_1, \ldots, X_p$ an integrable system. Moreover, in general, there is no natural preferred choice of q functions in $\mathcal{F}$. So, instead of specifying q first integrals, sometimes it is better to look at the whole algebra $\mathcal{F}$ of first integrals.

Notice also that, if $f_{ij} \in \mathcal{F}$ $(i, j = 1, \ldots, p)$ such that the matrix (f_{ij}) is invertible, then by putting

$$\hat{X}_i = \sum_{ij} f_{ij} X_j \quad \text{for all } i = 1, \ldots, p, \tag{2.4}$$

we get another integrable system $(\hat{X}_1, \ldots, \hat{X}_p, F_1, \ldots, F_q)$, which, from the geometric point of view, is essentially the same as the original system, because it gives rise to the same integrable singular foliation, and the same affine structure on the leaves of the foliation.

Definition 2.1. Two integrable dynamical systems $(X_1, \ldots, X_p, F_1, \ldots, F_q)$ and $(X'_1, \ldots, X'_p, F'_1, \ldots, F'_q)$ of type (p, q) on a manifold M are said to be *geometrically equal* if they have the same algebra of first integrals (i.e. $F'_1, \ldots, F'_p$ are functionally dependent of $F_1, \ldots, F_p$ and vice versa), and there exists a matrix $(f_{ij})_{i=1,\ldots,p}^{j=1,\ldots,p}$, whose

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entries f_{ij} are first integrals of the system, and whose determinant is non-zero everywhere, such that one can write

$$X_i' = \sum_j f_{ij} X_j \quad \text{for all } i = 1, \ldots, p. \tag{2.5}$$

Two integrable systems are said to be *geometrically equivalent* if they become geometrically the same after a diffeomorphism.

In this paper, we will be mainly interested in the local structure of integrable dynamical systems, up to geometric equivalence, in the sense of the above definition. It is clear that, near a regular point, i.e. a point z such that $X_1 \wedge \cdots \wedge X_p(z) \neq 0$, any two integrable systems of the same type (p, q) will be locally geometrically equivalent, and will be equivalent to the rectified system $X_1 = \partial/\partial x_1, \ldots, X_p = \partial/\partial x_p$. The local structure becomes interesting only at singular points. Remark also that, in the definition of geometric equivalence, we don't really care about the choice of first integrals $F_1, \ldots, F_q$ and can change them to other functionally independent first integrals at will.

If $X_1 \wedge \cdots \wedge X_p(z) = 0$ but $X_{k+1} \wedge \cdots \wedge X_p(z) \neq 0$, for example, then we can simultaneously rectify $X_{k+1}, \ldots, X_p$, i.e. find a coordinate system in which

$$X_{k+1} = \frac{\partial}{\partial x_1}, \ldots, X_p = \frac{\partial}{\partial x_{p-k}}. \tag{2.6}$$

Then the system does not depend on the coordinates $x_1, \ldots, x_{p-k}$, and we can reduce the problem to that of a system of type (k, q) by forgetting about $x_1, \ldots, x_{p-k}$ and $X_{k+1}, \ldots, X_p$. After such a reduction, we may assume that z is a fixed point of the system, i.e. all the vector fields of the system vanish at z. The situation is similar to that of integrable Hamiltonian systems, where the local study of singular points can also be reduced to the study of fixed points.

3. *Linear integrable systems*

Let $(X_1, \ldots, X_p, F_1, \ldots, F_q)$ be an integrable system of type (p, q) on a manifold M, and assume that $z \in M$ is a fixed point of the system, i.e. $X_1(z) = \cdots = X_p(z) = 0$. Fix a local coordinate system around z. Denote by Y_i the linear part of X_i at z, and by G_j the homogeneous part (i.e. the non-constant terms of lowest degree in the Taylor expansion) of F_j, with respect to the above coordinate system. Then, the first terms of the Taylor expansion of the identities $[X_i, X_k] = 0$ and $X_i(F_j) = 0$ show that the vector fields $Y_1, \ldots, Y_p$ commute with each other and have $G_1, \ldots, G_q$ as common first integrals. Hence, $(Y_1, \ldots, Y_p, G_1, \ldots, G_q)$ is again an integrable system of type (p, q), provided that the independence conditions $Y_1 \wedge \cdots \wedge Y_p \neq 0$ and $dG_1 \wedge \cdots \wedge dG_q \neq 0$ (almost everywhere) still hold.

The above observations lead to the following definition.

Definition 3.1. An integrable system $(Y_1, \ldots, Y_p, G_1, \ldots, G_q)$ of type (p, q) is called *linear* with respect to a given coordinate system if the vector fields $Y_1, \ldots, Y_p$ are linear and the functions $G_1, \ldots, G_q$ are homogeneous. If, moreover, it is obtained from another integrable system $(X_1, \ldots, X_p, F_1, \ldots, F_q)$ by the above construction,

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then we will say that $(Y_1, \dots, Y_p, G_1, \dots, G_q)$ is the *linear part* of the system $(X_1, \dots, X_p, F_1, \dots, F_q)$. If all the vector fields $Y_1, \dots, Y_p$ are semisimple, then we will say that $(Y_1, \dots, Y_p, G_1, \dots, G_q)$ is a *non-degenerate linear integrable system.*

Recall that the set of linear vector fields on $\mathbb{K}^m$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, is a Lie algebra which is naturally isomorphic to $gl(m, \mathbb{K})$. Any linear vector field admits a unique decomposition into the sum of its semisimple part and nilpotent part (the Jordan decomposition), and it can be diagonalized over $\mathbb{C}$ if and only if it is semisimple, i.e. its nilpotent part is zero. It is also well known that if we have a family of commuting semisimple elements of $gl(m, \mathbb{C})$, then they can be simultaneously diagonalized over $\mathbb{C}$. Thus, if $(Y_1, \dots, Y_p, G_1, \dots, G_q)$ is a non-degenerate linear integrable system, then there exists a complex coordinate system in which the vector fields $Y_1, \dots Y_p$ are diagonal.

The above notion of non-degeneracy is absolutely similar to the Hamiltonian case, where one also asks that the (Hamiltonian) vector fields Y_i be semisimple. It is already well known that in the Hamiltonian case not every integrable linear system is non-degenerate.

Example 3.2. In $\mathbb{R}^4$, take $G_1 = x_1y_1 - x_2y_2$, $G_2 = y_1y_2$, $Y_1 = x_1(\partial/\partial x_1) - y_1(\partial/\partial y_1) - x_2(\partial/\partial x_2) + y_2(\partial/\partial y_2)$, $Y_2 = y_2(\partial/\partial x_1) + y_1(\partial/\partial x_2)$. Then this is a degenerate (non-semisimple) integrable linear Hamiltonian system.

Let $(Y_1, \dots, Y_p, G_1, \dots, G_q)$ be a non-degenerate linear integrable system. We will work over $\mathbb{C}$, and assume that the coordinate system is already chosen so that the vector fields $Y_1, \dots, Y_p$ are linear:

$$Y_i = \sum_{i=j}^{m} c_{ij} x_j \frac{\partial}{\partial x_j}. \tag{3.1}$$

The independence condition $Y_1 \wedge \cdots \wedge Y_p \neq 0$ means that the matrix $(c_{ij})_{j=1,\dots,m}^{i=1,\dots,p}$ is of rank p. The set of polynomial common first integrals of $Y_1, \dots, Y_p$ is the vector space spanned by the monomial functions $\prod_{j=1}^{m} x_j^{\alpha_j}$ such that

$$\sum_{j=1}^{m} \alpha_j c_{ij} = 0 \quad \text{for all } i = 1, \dots, p. \tag{3.2}$$

This linear equation is called the *resonance equation* of the vector fields $Y_1, \dots, Y_p$.

The set of non-negative integer solutions of the resonance equation (3.2) is the intersection

$$S \cap \mathbb{Z}_+^m, \tag{3.3}$$

where

$$S = \left\{ (\alpha_i) \in \mathbb{R}^m \;\middle|\; \sum_{j=1}^{m} \alpha_j c_{ij} = 0 \text{ for all } i = 1, \dots, p \right\} \tag{3.4}$$

is the q-dimensional space of all real solutions of (3.2), and $\mathbb{Z}_+^m$ is the set of non-negative m-tuples of integers. The functional independence of $G_1, \dots, G_q$ implies that this set $S \cap \mathbb{Z}_+^m$ must have dimension q over $\mathbb{Z}$. In particular, the set $S \cap \mathbb{R}_+^m$ has dimension q over $\mathbb{R}$, and the resonance equation (3.2) is equivalent to a linear system of equations with

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integer coefficients. In other words, using a linear transformation to replace Y_i by new vector fields

$$\tilde{Y}_i = \sum_j a_{ij} Y_j \tag{3.5}$$

with an appropriate invertible matrix (a_{ij}) with constant coefficients, we may assume that

$$\tilde{Y}_i = \sum_{i=j}^{m} \tilde{c}_{ij} x_j \frac{\partial}{\partial x_j}, \tag{3.6}$$

where

$$\tilde{c}_{ij} = \sum_k a_{ik} c_{kj} \in \mathbb{Z} \quad \text{for all } i,\ j. \tag{3.7}$$

Of course, if $(Y_1, \ldots, Y_p, G_1, \ldots, G_q)$ is an integrable system, and $\tilde{Y}_i = \sum_j a_{ij} Y_j$ is an invertible linear transformation of the vector fields Y_i, then $(\tilde{Y}_1, \ldots, \tilde{Y}_p, G_1, \ldots, G_q)$ is again an integrable system which, from the geometric point of view, is the same as the system $(Y_1, \ldots, Y_p, G_1, \ldots, G_q)$.

Conversely, if the first integrals are not yet given, but the coefficients c_{ij} are integers, and the set of non-negative solutions to the resonance equation (3.2) has dimension q, then we can choose q linearly independent non-negative integer solutions of (3.2), and the q corresponding monomial functions will be functionally independent common first integrals of $Y_1, \ldots, Y_p$, and we get an integrable system.

Notice that, given a set of linear vector fields as above, the choice of common first integrals in order to turn it into an integrable system is far from unique. Moreover, the algebra of polynomial first integrals does not admit a set of q generators in general, even though its functional dimension is equal to q. The following simple example illustrates the situation: consider a linear integrable four-dimensional system of type (1, 3), i.e. with one vector field and three functions. The vector field is $Y = x_1(\partial/\partial x_1) + x_2(\partial/\partial x_2) - x_3(\partial/\partial x_3) - x_4(\partial/\partial x_4)$. The corresponding resonance equation is: $\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = 0$. The algebra of algebraic first integrals is generated by the functions x_1x_3, x_1x_4, x_2, x_3, x_2x_4; it has functional dimension three but cannot be generated by just three functions.

Remark 3.3. If z is an isolated singular point of X_1 in an integrable system $(X_1, \ldots, X_p, F_1, \ldots, F_q)$, then it will be automatically a fixed point of the system. Indeed, if $X_i(z) \neq 0$ for some i, then due to the commutativity of X_1 with X_i, X_1 will vanish not only at z, but on the whole local trajectory of X_i which goes through z, and so z will be a non-isolated singular point of X_1. In the definition of non-degeneracy of linear systems, we don't require the origin to be an isolated singular point. For example, the system $(x_1(\partial/\partial x_1), x_2)$ is a non-degenerate linear system of type (1, 1), for which the origin is a non-isolated singular point.

The independent vector fields $\sqrt{-1}\tilde{Y}_i = \sqrt{-1}\sum_{i=j}^{m} \tilde{c}_{ij} x_j (\partial/\partial x_j)$ with integer coefficients $\tilde{c}_{ij}$ generate an effective linear torus action on $\mathbb{C}^m$. Thus, up to geometric equivalence, the classification of complex non-degenerate linear integrable systems of type (p, q) is nothing but the classification of effective linear actions of the torus $\mathbb{T}^p$ on $\mathbb{C}^m$,

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i.e. complex linear m-dimensional representations of $\mathbb{T}^p$. The classification in the real case is more complicated: two real linear systems may be non-equivalent but have the same complexification.

4. *Linearization and rigidity of non-degenerate singularities*

Definition 4.1. A fixed point of an integrable system $(X_1, \ldots, X_p, F_1, \ldots, F_q)$ of type (p, q) is called *non-degenerate* if its linear part is a non-degenerate linear integrable system. A singular point of an integrable system is called *non-degenerate* if it becomes a non-degenerate fixed point after a reduction.

Remark 4.2. Though the choice of first integrals is not important in Definition 2.1 of geometric equivalence, the q-tuple $F_1, \ldots, F_q$ of first integrals in the above definition of non-degeneracy is assumed to be chosen so that not only are they functionally independent, but their homogeneous parts are also functionally independent. (According to a simple analogue of Ziglin's lemma [**16**], in the analytic case, such a choice is always possible.)

THEOREM 4.3. (Geometric linearization) *An analytic (real or complex) integrable system near a non-degenerate fixed point is locally geometrically equivalent to a non-degenerate linear integrable system, namely its linear part.*

Proof. The proof is a consequence of the main results of [**18, 19**], which say that if a system is analytically integrable, then in a neighborhood of any singular point it admits a local analytic effective torus action (the torus is a real torus but it acts in complex space), whose dimension is equal to the so-called toric degree of the system at that point, and the linearization of this torus action is equivalent to the Poincaré–Dulac normalization of the system. (This torus action is intrinsic to the system and is defined as a kind of double commutator, i.e. any vector field which commutes with the system also commutes with this torus action.) It remains to prove that, in the non-degenerate case, the Poincaré–Dulac normalization is actually a geometric linearization of the system.

Indeed, in the non-degenerate complex analytic case, it follows directly from the definition of the toric degree (see [**18**] or [**20**]), that the toric degree at the isolated singular point is equal to p, and so there is an effective analytic torus action of dimension p around the singular point which preserves the system. By a local diffeomorphism, we may assume that this torus action is linear and is generated by p vector fields $\sqrt{-1}\tilde{Y}_1, \ldots, \sqrt{-1}\tilde{Y}_p$, where each $\tilde{Y}_i$ is a linear diagonal with integer coefficients: $\tilde{Y}_i = \sum_{i=j}^{m} \tilde{c}_{ij} x_j (\partial/\partial x_j)$, $c_{ij} \in \mathbb{Z}$ for all i, j. (The Poincaré–Dulac normalization amounts to the linearization of this torus action, see [**18**]).

Moreover, from the construction of this torus action, we have that $\tilde{Y}_i \wedge X_1 \wedge \cdots \wedge X_p = 0$ for all $i = 1, \ldots, p$ (because the torus action also preserves the first integrals, so its generators must be tangent to the complex common level sets of the first integrals). Since $\tilde{Y}_1, \ldots, \tilde{Y}_p$ are independent, by dimensional consideration, the inverse is also true: $X_i \wedge \tilde{Y}_1 \wedge \cdots \wedge \tilde{Y}_p = 0$ for all $i = 1, \ldots, p$. Lemma 4.4 below says that we can write $X_i = \sum_j f_{ij} \tilde{Y}_i$ in a unique way, where f_{ij} are local analytic functions, which are also first integrals of the system. The fact that the matrix (f_{ij}) is invertible, i.e. has non-zero

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determinant at z, is also clear, because the set $(\tilde{Y}_1, \ldots, \tilde{Y}_p)$ is nothing but a linear transformation of the linear part of $(X_1, \ldots, X_p)$.

What we have proved is that, near a non-degenerate fixed point, an integrable system is geometrically equivalent to its linear part, at least in the complex analytic case. In the real analytic case, the vector fields $(\tilde{Y}_1, \ldots, \tilde{Y}_p)$ are not real in general, but the proof will remain the same after a complexification, because the Poincaré–Dulac normalization in the real case can be chosen to be real (see [**18, 19**]). □

LEMMA 4.4. (Division lemma) *If $(Y_1, \ldots, Y_p, G_1, \ldots, G_q)$ is a non-degenerate linear integrable system, and X is a local analytic vector field which commutes with $Y_1, \ldots, Y_p$ such that $X \wedge Y_1 \wedge \cdots \wedge Y_p = 0$, then we can write $X = \sum f_i Y_i$ in a unique way, where f_i are local analytic functions which are common first integrals of $Y_1, \ldots, Y_p$.*

Proof. Without loss of generality, we may assume that $Y_i = \sum_j c_{ij} Z_j$, where c_{ij} are integers and $Z_i = x_i(\partial/\partial x_i)$ in some coordinate system $(x_1, \ldots, x_m)$. We will write $X = \sum_i g_i Z_i$, where $x_i g_i$ are analytic functions. The main point is to prove that the g_i are analytic functions, and the rest of the lemma will follow easily. Let $\prod_i x_i^{\alpha_i}$ be a polynomial first integral of the linear system. Then we also have $X(\prod_i x_i^{\alpha_i}) = 0$, which implies that $\sum_i \alpha_i g_i = 0$. If $\alpha_1 \neq 1$ then $x_1 g_1 = (-\sum_{i=2}^m x_1 g_i)/\alpha_1$ vanishes when $x_1 = 0$, and so $x_1 g_1$ is divisible by x_1, which means that g_1 is analytic. Thus, for each i, if we can choose a monomial first integral $\prod_i x_i^{\alpha_i}$ such that $\alpha_i \neq 0$, then g_i is analytic. Assume now that all monomial first integrals $\prod_i x_i^{\alpha_i}$ must have $\alpha_1 = 0$. It means that all the first integrals are also invariant with respect to the vector field $Z_1 = x_1(\partial/\partial x_1)$. Then Z_1 must be a linear combination of $Y_1, \ldots, Y_p$ (because the system is already 'complete' and one cannot add another independent commuting vector field to it), and we have $[Z_1, X] = 0$. From this relation it follows easily that g_1 is also analytic in this case. Thus, all functions g_i are analytic. □

Theorem 4.3 can be extended to the case of non-fixed non-degenerate singular points in an obvious way, with the same proof, using our results [**18, 19**] on the toric characterization of local normalizations of vector fields.

THEOREM 4.5. *Any analytic integrable dynamical system near a non-degenerate singular point is locally geometrically equivalent to a direct product of a linear non-degenerate integrable system and a constant (regular) integrable system.*

We also have an extension of Ito's theorem [**11**] to the non-Hamiltonian case. Ito's theorem says that an analytic integrable Hamiltonian system at a non-resonant singular point (without the requirement of non-degeneracy of the momentum map at that point) can also be locally geometrically linearized (i.e. locally one can choose the momentum map so that the system becomes non-degenerate and geometrically linearizable). For Hamiltonian vector fields, there are many auto-resonances due to their Hamiltonian nature, which are not counted as resonance in the Hamiltonian case. So, in the non-Hamiltonian case, we have to replace the adjective 'non-resonant' by 'minimally-resonant'.

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Definition 4.6. A vector field X in a integrable dynamical system $(X_1 = X, \ldots, X_p, F_1, \ldots, F_q)$ of type (p, q) is called *minimally resonant* at a singular point z if its toric degree at z is equal to p (maximal possible).

THEOREM 4.7. *Minimally-resonant singular points of analytic integrable systems are also locally geometrically linearizable in the sense that one can change the auxiliary commuting vector fields (keeping the first vector field and the functions intact) in order to obtain a new integrable system which is locally geometrically linearizable.*

Proof. The proof is similar to the proof of Theorem 4.3 and is also a direct consequence of the main results of [**18**]. □

In order to give another justification for our notion of non-degeneracy of singular points of integrable non-Hamiltonian systems, we will also show that such singularities are deformation rigid.

THEOREM 4.8. (Rigidity of non-degenerate singularities) *Let*

$$(X_{1,\theta}, \ldots, X_{p,\theta}, F_{1,\theta}, \ldots, F_{q,\theta}) \tag{4.1}$$

be an analytic family of integrable systems of type (p, q) depending on a parameter θ which can be multi-dimensional: $\theta = (\theta_1, \ldots, \theta_s)$, and assume that z_0 is a non-degenerate fixed point when $\theta = 0$. Then there exists a local analytic family of fixed points z_θ, such that z_θ is a fixed point of $(X_{1,\theta}, \ldots, X_{p,\theta}, F_{1,\theta}, \ldots, F_{q,\theta})$ for each θ, and moreover, up to geometric equivalence, the local structure of $(X_{1,\theta}, \ldots, X_{p,\theta}, F_{1,\theta}, \ldots, F_{q,\theta})$ at z_θ does not depend on θ.

Proof. We can put the integrable systems in this family together to get one 'big' integrable system of type $(p, q + s)$, with the last coordinates $x_{m+1}, \ldots, x_{m+s}$ as additional first integrals. Then z_0 is still a non-degenerate fixed point for this big integrable system, and we can apply Theorem 4.3 to get the desired result. □

5. *Linearization of smooth integrable systems*

In the smooth case, we still have the same definitions of linear part, geometric equivalence, non-degeneracy and geometric linearization as in the analytic case. We have the following conjecture, which is the smooth version of Theorem 4.5.

CONJECTURE 5.1. *Any smooth integrable dynamical system near a non-degenerate singular point is locally geometrically smoothly equivalent to a direct product of a linear non-degenerate integrable system and a constant system.*

We believe that the above conjecture is true, but don't have a full proof of it in the general case. We will prove it for the case of systems of type $(n, 0)$ in the next section. As a rule, normal form results for smooth systems require more elaborate work than for analytical systems, because of the lack of complex analytic tools. We have already seen this for Hamiltonian systems, where the proof of Eliasson's local linearization theorem [**9**], which is the smooth counterpart of Vey's theorem [**15**] (see also [**19**]) is much longer than the proof of Vey's theorem.

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Let us indicate here why we believe that the above conjecture is true, and some methods which could be used to prove it.

(1) By geometric arguments similar to those used in [**17–19**], we can show the existence of a smooth torus $\mathbb{T}^d$-action which preserves the system, where d is the *real toric degree* of the system (i.e. part of the toric degree whose corresponding action is real). Up to geometric equivalence, we can also assume that the vector fields which generate this torus action are part of our system. The remaining vector fields of the system are hyperbolic and invariant with respect to this smooth torus action.

(2) Theorem 4.3 is also true in the formal case with the same proof, because the results of [**18, 19**] are also true in the formal category. So we can apply a formal linearization to our smooth system. Together with Borel's theorem, it means that there is a local smooth coordinate system in which our system is already geometrically linear up to a flat term.

(3) After Step 2 above, one can try to use results and techniques on finite determinacy of mappings à la Mather [**12**] to find a matrix whose entries are smooth first integrals, such that when multiplying this matrix with our vector fields, we obtain a new geometrically equivalent system whose vectors are linear + flat terms.

(4) One can now try to invoke an equivariant version of the Sternberg–Chen theorem [**6, 13**], due to Belitskii and Kopanskii [**3**], which says that smooth equivariant hyperbolic vector fields which are formally linearizable are also smoothly equivariantly linearizable. Of course, we will have to do it simultaneously for all commuting hyperbolic vector fields. So we need an extension of the result of Belitskii and Kopanskii to the situation of a smooth $\mathbb{R}^k$-action with some hyperbolicity property which is formally linear. Maybe we would also need a version of Belitskii–Kopanskii–Sternberg–Chen for vector fields which have first integrals. Techniques of [**5, 7–9**] may also be useful here.

6. *Smooth systems of type* $(n, 0)$

In this section, we consider a smooth integrable system of type $(n, 0)$, consisting of n commuting vector fields $X_1, \ldots, X_n$ on an n-dimensional manifold M^n. (There is no function, just vector fields.) In this case, a geometric linearization means a true linearization of the vector fields, because there is no function. We will denote by

$$\rho : \mathbb{R}^n \times M^n \to M^n \tag{6.1}$$

the (local) action of $\mathbb{R}^n$ on M^n generated by these vector fields. Moreover, for each vector $v = (v^i) \in \mathbb{R}^n$, we will define

$$X_v = \sum_{i=1}^{n} v^i X_i \tag{6.2}$$

as the *generator of the action associated to* v.

First of all, we have the following classification of non-degenerate real linear systems of type $(n, 0)$, or in other words, non-degenerate linear actions of $\mathbb{R}^n$ on $\mathbb{R}^n$. Such actions are generated by Cartan subalgebras of the Lie algebra of linear vector fields on $\mathbb{R}^n$. This Lie algebra is naturally isomorphic to $gl(n, \mathbb{R})$, and so the classification of non-degenerate linear actions of $\mathbb{R}^n$ on $\mathbb{R}^n$ corresponds to a classical classification up to conjugation of Cartan subalgebras of $gl(n, \mathbb{R})$.

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THEOREM 6.1. *Let $\rho^{(1)} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a non-degenerate linear action of $\mathbb{R}^n$ on $\mathbb{R}^n$. Then there exist non-negative integers $h, e \geq 0$ such that $2h + e = n$, a linear coordinate system $x_1, \ldots, x_n$ on $\mathbb{R}^n$, and a linear basis $(v_1, \ldots, v_n)$ of $\mathbb{R}^n$ such that the generators $Y_i = X_{v_i} = \sum_j v_i^j X_j$ of the action $\rho^{(1)}$ with respect to the basis $(v_1, \ldots, v_n)$ can be written as follows:*

$$\begin{cases} Y_i = x_i \dfrac{\partial}{\partial x_i} \quad \text{for all } i = 1, \ldots, h, \\ Y_{h+2j-1} = x_{h+2j-1} \dfrac{\partial}{\partial x_{h+2j-1}} + x_{h+2j} \dfrac{\partial}{\partial x_{h+2j}}, \\ Y_{h+2j} = x_{h+2j-1} \dfrac{\partial}{\partial x_{h+2j}} - x_{h+2j} \dfrac{\partial}{\partial x_{h+2j-1}} \quad \text{for all } j = 1, \ldots, e. \end{cases} \tag{6.3}$$

The proof of the above theorem is a simple exercise of linear algebra: since the linear vector fields X_i commute, they are simultaneously diagonalizable over $\mathbb{C}$. Their joint one-dimensional real eigenspaces correspond to *hyperbolic* components Y_i, while joint complex eigenspaces correspond to components (Y_{h+2j-1}, Y_{h+2j}), which are called *elbolic* components. (Elbolic means elliptic + hyperbolic; an elbolic component has two sub-components, one of which is elliptic and the other one is hyperbolic).

Let p be a singular point of a smooth integrable system $(X_1, \ldots, X_n)$ of type $(n, 0)$, i.e. $\dim \mathrm{Span}_{\mathbb{R}}(X_1(p), \ldots, X_n(p)) < n$. We do not require p to be a fixed point, i.e. $\mathrm{rank}\, p := \dim \mathrm{Span}_{\mathbb{R}}(X_1(p), \ldots, X_n(p))$ may be 0 or positive. Recall that, if $\mathrm{rank}\, p = k > 0$, then without loss of generality, we may assume that $X_{n-k+1}(p) \wedge \cdots \wedge X_n(p) \neq 0$, and p will be called a *non-degenerate singular point* if it becomes a non-degenerate fixed point of a system of type $(n - k, 0)$ which is obtained from the original system of type $(n, 0)$ by a local reduction with respect to the free local $\mathbb{R}^k$-action generated by $X_{n-k+1}, \ldots, X_n$ (see Definition 4.1). The main result of this section is the following local normal form theorem, which is the smooth version of Theorem 4.5 for systems of type $(n, 0)$.

THEOREM 6.2. *Let p be a non-degenerate singular point of a smooth integrable system $(X_1, \ldots, X_n)$ of type $(n, 0)$. Denote by*

$$m = n - \dim \mathrm{Span}_{\mathbb{R}}(X_1(p), \ldots, X_n(p)) \tag{6.4}$$

the corank of the system at p. Then there exists a smooth local coordinate system $(x_1, x_2, \ldots, x_n)$ in a neighborhood of p, non-negative integers $h, e \geq 0$ such that $h + 2e = m$, and a basis $(v_1, \ldots, v_n)$ of $\mathbb{R}^n$ such that the corresponding generators $Y_i = X_{v_i}$ $(i = 1, \ldots, n)$ of ρ have the following form:

$$\begin{cases} Y_i = x_i \dfrac{\partial}{\partial x_i} \quad \text{for all } i = 1, \ldots, h, \\ Y_{h+2j-1} = x_{h+2j-1} \dfrac{\partial}{\partial x_{h+2j-1}} + x_{h+2j} \dfrac{\partial}{\partial x_{h+2j}}, \\ Y_{h+2j} = x_{h+2j-1} \dfrac{\partial}{\partial x_{h+2j}} - x_{h+2j} \dfrac{\partial}{\partial x_{h+2j-1}} \quad \text{for all } j = 1, \ldots, e, \\ Y_k = \dfrac{\partial}{\partial x_k} \quad \text{for all } k = m + 1, \ldots, n. \end{cases} \tag{6.5}$$

The numbers (h, e) do not depend on the choice of local coordinates.

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Proof. [See Remark 6.3 for a different, simple proof proposed by the referee of this paper] The fact that the numbers (h, e) in the above theorem do not depend on the choice of coordinates is clear, because they are invariant of the Cartan subalgebra of the corresponding reduced system at p. We will call h the number of hyperbolic components, and e the number of elbolic components of the system at p. We will prove the above theorem by induction on the couple (h, e), and will divide the proof into several steps.

Step 1. The case when $(h, e) = (1, 0)$.
In this step, we assume that the corank of the system at p is 1. Without loss of generality, we may assume that $X_2(p) \wedge \cdots \wedge X_n(p) \neq 0$. Since the vector fields $X_1, \ldots, X_n$ commute, by applying the classical Frobenius theorem we can find a local coordinate system $(y_1, \ldots, y_n)$ in which $X_i = \partial/\partial y_i$ for $i = 2, \ldots, n$. In this coordinate system, the first vector field X_1 will have the form:

$$X_1 = f_1(y_1)\frac{\partial}{\partial y_1} + \cdots + f_n(y_1)\frac{\partial}{\partial y_n} \tag{6.6}$$

(where the functions $f_1, \ldots, f_n$ depend only on the coordinate y_1, due to the fact that X_1 commutes with the other vector fields). Moreover, we have $f_1(0) = 0$ and $f_1'(0) \neq 0$, because p is a non-degenerate singular point, so we can write $f_1(y_1) = g(y_1) \cdot y_1$, with $g(0) \neq 0$. Write $f_i(y_1) = f_i(0) + g_i(y_1) \cdot y_1$ for $i = 2, \ldots, n$ also.

Replacing X_1 by another generator $Z_1 = X_1 - \sum_{i=2}^n f_i(0)X_i$ of the system, we can write

$$Z_1 = y_1 \sum_{i=1}^n g_i(y_1)\frac{\partial}{\partial y_1} = y_1 \hat{Z}_1, \tag{6.7}$$

with $g_1(0) \neq 0$. Notice that Z_1 is a regular vector field. The regular integral curve Γ of $\hat{Z}_1$ through p is also an integral curve for Z_1, and on Γ the vector field Z_1 can be linearized, i.e. there is a coordinate function x_1 on Γ, such that the restriction of Z_1 to Γ has the form $Z_1 = ax_1(\partial/\partial x_1)$, where a is a non-zero constant.

Define new coordinates $(x_1, \ldots, x_n)$ by the following formulas: for each point q in a small neighborhood of q, $x_2(q), \ldots x_n(q)$ are the unique numbers such that $q' = \phi_{X_2}^{-x_2(q)} \circ \cdots \circ \phi_{X_n}^{-x_n(q)}(q)$ belongs to Γ, where ϕ_X denotes the flow of the vector field X, and put $x_1(q) = x_1(q')$. One can then easily verify that $(x_1, \ldots, x_n)$, together with $Y_1 = Z_1/a$ and $Y_i = X_i$ for all $i \geq 2$ satisfy equations (6.5).

Step 2. The case when $e = 0$ *and* $h > 1$ *arbitrary.*
The proof will be by induction on h, so let us assume that the theorem is already proved when there are $h - 1$ hyperbolic components and zero elbolic components. Consider now the case with h hyperbolic components and zero elbolic components.

Invoking the formal version of Theorem 4.5, we can assume, without loss of generality, that the system is already linearized up to flat terms. In other words, we can assume that:

$$\begin{cases} Y_i = x_i \dfrac{\partial}{\partial x_i} + \textit{flat} \quad \text{for all } i = 1, \ldots, h, \\ Y_k = \dfrac{\partial}{\partial x_k} + \textit{flat} \quad \text{for all } k = h+1, \ldots, n, \end{cases}$$

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where *flat* means a term which is flat at p. Since the vector fields Y_k $(k \geq h+1)$ are regular and commute with each other, by the classical Frobenius theorem we can rectify our coordinate system a bit more to kill the flat terms in the expression of Y_k, $k \geq h+1$, and get:

$$\begin{cases} Y_i = x_i \dfrac{\partial}{\partial x_i} + \textit{flat} \quad \text{for all } i = 1, \ldots, h, \\ Y_k = \dfrac{\partial}{\partial x_k} \quad \text{for all } k = h+1, \ldots, n. \end{cases}$$

Consider the vector field

$$Z_1 = Y_1 - \sum_{i=2}^{h} Y_i. \tag{6.8}$$

This vector field is not hyperbolic at p if $h < n$ (it has $n-h$ eigenvalues equal to 0), but it is hyperbolic for the reduced h-dimensional system (the local reduction is done by forgetting about the coordinates $x_{h+1}, \ldots, x_n$, or in other words, by taking the quotient of the neighborhood of p by the flows of the vector fields $Y_{h+1}, \ldots, Y_n$). So, according to the classical stable manifold theorem, we have a smooth $(h-1)$-dimensional stable manifold with respect to Z_1 on the reduced h-dimensional manifold, which, when pulled back to a neighborhood of p in M^n, becomes a smooth center-stable $(n-1)$-dimensional manifold of Z_1, which we will denote by Σ_1.

Note that Σ_1 is invariant with respect to our system, which means that all the vector fields $Y_1, \ldots, Y_n$ are tangent to Σ_1, which in turn implies that the points of Σ_1 are singular with respect to our system (the rank of the system at each point is at most $n-1$). But if we forget about Y_1, then $(Y_2, \ldots, Y_n)$ form an integrable system on Σ_1 of type $(n-1, 0)$ which admits p as a singular point with $(h-1)$ hyperbolic components, so this sub-system can be linearized on Σ_1 according to our induction hypothesis. For the moment, we don't need this linearization, just a consequence of it which says that for any point $q \in \Sigma_1$, the closure of the orbit through q of the sub-system (i.e. of the infinitesimal $\mathbb{R}^{n-1}$-action generated by $(Y_2, \ldots, Y_n)$) contains p. With this, we can show that

$$Y_1(q) = 0 \quad \text{for all } q \in \Sigma_1. \tag{6.9}$$

Indeed, if $Y_1(q) \neq 0$ then we can write $Y_1(q) = \sum_{i \geq 2} a_i Y_i(q)$, where a_i are numbers and at least one of them is different from 0. By commutativity, for any other point q' on the orbit of the system through q, we also have $Y_1(q') = \sum_{i \geq 2} a_i Y_i(q') = \sum_{i=2}^{h} a_i x_i (\partial/\partial x_i) + \sum_{k=h+1}^{n} a_k (\partial/\partial x_k) + \cdots$. But when q' is very close to q, this expression contradicts the expression $Y_1 = x_1(\partial/\partial x_i) + \textit{flat}$. So we must have $Y_1(q) = 0$.

It is now easy to see that we can write

$$\Sigma_1 = \{q \in \mathcal{U} \mid Y_1(q) = 0\}, \tag{6.10}$$

where $\mathcal{U}$ denotes a small neighborhood of p. Moreover, by construction, Σ_1 is tangent to $\{x_1 = 0\}$ at p. By a smooth change of coordinates, we can assume that $\Sigma_1 = \{x_1 = 0\}$. Do the same thing for every $i = 1, \ldots, h$. We can now assume that for every $i = 1, \ldots, h$ we have

$$\Sigma_i = \{q \in \mathcal{U} \mid Y_i(q) = 0\} = \{x_i = 0\}. \tag{6.11}$$

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Then we can write

$$Y_i = x_i \hat{Y}_i, \tag{6.12}$$

where $\hat{Y}_i$ is a regular vector field for each $i = 1, \ldots, h$.

Construct a new coordinate system $(y_1, \ldots, y_n)$ as follows.

(i) For each $i = 1, \ldots, h$.

On the regular integral curve Γ_i of the vector field $\hat{Y}_i$ through p, let y_i be a coordinate function which linearizes Y_i: the restriction of Y_i to Γ_i has the form $Y_i = y_i(\partial/\partial y_i)$. The vector fields $\hat{Y}_j$, $j \neq i$ and $Y_{h+1}, \ldots, Y_n$ satisfy the integrability condition of Frobenius and generate a regular foliation of codimension one, which we will denote by $\mathcal{F}_1$. For each point q in a small neighborhood $\mathcal{U}$ of p, define $y_i(q) = y_i(q')$, where q' is the intersection of the leaf of $\mathcal{F}_i$ through q with Γ.

(ii) For the other indices.

The vector fields $\hat{Y}_1, \ldots, \hat{Y}_h$ generate a regular h-dimensional foliation. Denote by Γ the leaf of that foliation through p. The functions $y_{h+1}(q), \ldots, y_n(q)$ are defined by the condition:

$$\phi_{X_{h+1}}^{-y_{h+1}(q)} \circ \cdots \circ \phi_{X_n}^{-y_n(q)}(q) \in \Gamma.$$

One then verifies easily that the vector fields $Y_1, \ldots, Y_n$ satisfy equations (6.5) with respect to the new coordinate system $(y_1, \ldots, y_n)$.

Step 3. The case when $(h, e) = (0, 1)$.

In this case, using formal linearization, we obtain a local smooth coordinate system $(x_1, \ldots, x_n)$ in which we have:

$$\begin{cases} Y_1 = x_1 \dfrac{\partial}{\partial x_1} + x_2 \dfrac{\partial}{\partial x_2} + \textit{flat}, \\ Y_2 = x_1 \dfrac{\partial}{\partial x_2} - x_2 \dfrac{\partial}{\partial x_1} + \textit{flat}, \\ Y_k = \dfrac{\partial}{\partial x_k} + \textit{flat} \quad \text{for all } k = 3, \ldots, n. \end{cases}$$

Using geometric arguments similar to those in [**17–19**] for constructing torus actions, we can assume that Y_2 generates an action of $\mathbb{T}^1$. Invoking Bochner's linearization theorem, we can assume that Y_2 is already linear, i.e. the flat term in its expression is actually 0:

$$\begin{cases} Y_1 = x_1 \dfrac{\partial}{\partial x_1} + x_2 \dfrac{\partial}{\partial x_2} + \textit{flat}, \\ Y_2 = x_1 \dfrac{\partial}{\partial x_2} - x_2 \dfrac{\partial}{\partial x_1}, \\ Y_k = \dfrac{\partial}{\partial x_k} + \textit{flat} \quad \text{for all } k = 3, \ldots, n. \end{cases}$$

Using arguments similar to those in Step 2, one can show that the center manifold Σ of Y_1 is a smooth submanifold of dimension $n - 2$, and Y_1 vanishes on it, i.e. we can write $\Sigma = \{q \in \mathcal{U} \mid Y_1(q) = 0\}$, where $\mathcal{U}$ denotes a small neighborhood of p. Note that Σ is also the set of fixed points of the $\mathbb{T}^1$-action generated by Y_2, and so we have

$$\Sigma = \{q \in \mathcal{U} \mid Y_1(q) = 0\} = \{q \in \mathcal{U} \mid Y_2(q) = 0\} = \{x_1 = x_2 = 0\}.$$

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One then proves easily that there is a unique local two-dimensional surface Γ which contains q and which is invariant with respect to Y_1 and Y_2. On Γ, there is a coordinate system (y_1, y_2) with respect to which the restrictions of Y_1 and Y_2 to Γ are linear. One then proceeds as in Step 1 to construct a new coordinate system $(y_1, \ldots, y_n)$ in which the vector fields $Y_1, \ldots, Y_n$ satisfy equations (6.5).

Step 4. The general case, with arbitrary (h, e).
This is just a combination of the arguments presented in the previous three steps. In fact, one can treat elbolic components in almost the same way as hyperbolic components, except that instead of integral curves one has to use integral two-dimensional disks, and instead of codimension-1 manifolds on which the corresponding vector fields vanish one has to use codimension-2 submanifolds for elbolic components. □

Remark 6.3. Another, simpler proof of Theorem 6.2 along the following lines was communicated to us by the referee.

(i) In the case when $m = n$, the linear part of an appropriate linear combination $E = \sum a_i X_i$ of the vector fields $X_1, \ldots, X_n$ is a radial vector field, i.e. has the form $E^{(1)} = \sum_{i=1}^{n} x_i(\partial/\partial x_i)$. By Sternberg's theorem, E is smoothly linearizable, i.e. we can assume that $E = \sum_{i=1}^{n} x_i(\partial/\partial x_i)$ after a smooth change of the coordinate system. Since the vector fields X_i commute with the radial vector field $E = \sum_{i=1}^{n} x_i(\partial/\partial x_i)$, they are automatically linear in the new coordinate system.

(ii) The case with $m < n$ can be reduced to the above case, by considering the m-dimensional isotropy algebra of the infinitesimal $\mathbb{R}^n$-action at the singular point, and showing the existence of m-dimensional invariant submanifolds of the subaction of this isotropy algebra, which are transverse to the local orbit through the singular point of the $\mathbb{R}^n$-action.

This new proof also works for finitely differentiable systems.

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