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T_1-locales

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In the theory of locales (or 'pointless topology'), several authors have tried to find a suitable form of a T_1-separation axiom. Their proposals are unsatisfactory in the sense that they do not coincide for topological spaces with the T_1-axiom. Our main result is that sublocales of sober T_1-spaces are exactly locales in which primes are dual atoms. Hence, these locales should be called T_1, despite the fact that they include any locale without points. Our T_1-locales are also closed with respect to products; in fact they form the smallest epireflective subcategory of locales containing all sober T_1-spaces. Besides T_1-locales we will also consider T_2-locales and epireflective subcategories of locales in general.

All unexplained facts concerning locales can be found in Johnstone [**11**]. Recall that a *frame* is a complete lattice L in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}$$

holds for all $a \in L$, $S \subseteq L$. A frame homomorphism $K \to L$ is a map preserving finite meets and arbitrary joins. Let Frm be the category of frames. If T is a topological space, the lattice $O(T)$ of open sets of T is a frame. A continuous map $f: S \to T$ determines a frame homomorphism $O(f): O(T) \to O(S)$ sending $V \in O(T)$ to $f^{-1}(V)$. We get the functor $O: Top \to Frm^{op}$ where Top is the category of topological spaces and continuous maps. The functor O is full and faithful on the subcategory $Sob \subseteq Top$ of all sober spaces. In what follows, a subcategory always means a full subcategory. O has a right adjoint $P: Frm^{op} \to Top$ assigning to a frame L the topological space $P(L)$ of *prime* (i.e. $\wedge$-irreducible and $\neq 1$) elements of L. Open sets of $P(L)$ are

$$\hat{x} = \{a \in P(L) : x \not\leqslant a\}$$

where $x \in L$. The unit $\psi_L: L \to OP(L)$ of the adjunction maps x to $\hat{x}$. Hence ψ_L is surjective. Topological spaces $P(L)$ are sober and therefore Sob is isomorphic to a coreflective subcategory in Frm^{op}.

The above facts indicate the importance of the opposite category $Loc = Frm^{op}$. Objects of Loc are called *locales* and we keep the name 'continuous map' for their morphisms. Locales isomorphic to some $O(T)$ are called 'spatial'. Hence Sob is isomorphic to the category of spatial locales. We will often identify spatial locales with the corresponding sober spaces. Monomorphisms in Frm are injective homomorphisms and regular epimorphisms are surjective homomorphisms. This gives the description of epimorphisms and regular monomorphisms in Loc. Regular monomorphisms give sublocales, i.e. sublocales correspond to quotient frames. Of course, products of locales correspond to sums of frames.

If $\mathscr{X}$ is a subcategory of Loc then $S(\mathscr{X})$ will denote the subcategory of all sublocales of objects of $\mathscr{X}$ and $P(\mathscr{X})$ the subcategory of all products of objects of $\mathscr{X}$. Let $\hat{\mathscr{X}} = SP(\mathscr{X})$.

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Since any morphism $h\colon K \to L$ in Loc has a unique factorization $h\colon K \xrightarrow{e} H \xrightarrow{m} L$ where e is an epimorphism and m a regular monomorphism, $\hat{\mathscr{X}}$ is the smallest epireflective subcategory of Loc containing $\mathscr{X}$ (by applying [7], 37·4).

Any locale is a sublocale of a sober space (this follows from the fact that free frames are spatial and any frame is a quotient of a free frame). Similarly, any completely regular locale L is a sublocale of a completely regular space. It is enough to know that L is a sublocale of its Čech–Stone compactification $\beta(L)$ and that any compact regular locale is spatial. The last fact needs the axiom of choice. The category $\mathscr{CR}$ of completely regular locales is the epireflective hull of completely regular spaces even without the use of (AC). In fact we have $\mathscr{CR} = \widehat{\{\mathbf{R}\}}$ (it follows from [11], IV 2·9). Analogous results hold for 0-dimensional locales. We have to take the Banaschewski compactification $B(L)$ instead of $\beta(L)$ (see 1·1) and the two-element discrete space instead of $\mathbf{R}$. We do not know whether regular locales form the epireflective hull of regular spaces.

1. T_1-*locales*

We say that L is a 'T_1-locale' if its primes are dual atoms. This definition is justified by Theorem 1.2 which says that T_1-locales generalize sober T_1-spaces in the same way as locales generalize sober spaces. It is evident that spatial T_1-locales are exactly sober T_1-spaces. Let Loc_1 be the category of T_1-locales and Sob_1 the category of sober T_1-spaces.

Before proving the main result we will consider the Banaschewski compactification of locales. Let L be a frame, $Idl(L)$ the frame of all ideals of L and $B(L) = \{J \in Idl(L)$: for any $x \in J$ there is a complemented element $x \leqslant y$ in $J\}$. Put

$$b_L(J) = \bigvee J$$

for any $J \in B(L)$.

PROPOSITION 1·1. *The continuous map $b_L\colon L \to B(L)$ is the reflection of L to the category of compact 0-dimensional locales.*

Proof. It is easy to see that if $J_1 \in Idl(L)$ is the complement of $J_2 \in Idl(L)$ then there are elements $x, y \in L$ such that $J_1 = \downarrow(x)$, $J_2 = \downarrow(y)$ and x is the complement of y. It implies that the continuous map $Idl(L) \to B(L)$ corresponding to the frame inclusion

$$B(L) \subset Idl(L)$$

is the reflection of $Idl(L)$ to the category of 0-dimensional locales. $B(L)$ is compact because it is a subframe of a compact frame $Idl(L)$. Put $s_L(J) = \bigvee J$ for any $J \in Idl(L)$. Since the continuous map $s_L\colon L \to Idl(L)$ is flat, it follows by [11], III. 1·11 that $b_L\colon L \to B(L)$ is the compact 0-dimensional reflection of L.

Recall that quotient frames of L correspond to nuclei on L, i.e. to maps $j\colon L \to L$ such that (i) $a \leqslant j(a)$, (ii) $jj(a) = j(a)$ and (iii) $j(a \wedge b) = j(a) \wedge j(b)$ for all $a, b \in L$. The lattice $N(L)$ of all nuclei on L is a frame and there is a frame homomorphism $c_L\colon L \to N(L)$ given by the prescription

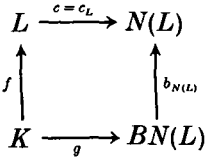
$$c_L(a)(x) = a \vee x$$

for any $x \in L$.

THEOREM 1·2. *(AC)*: $\widehat{Sob_1} = S(Sob_1) = Loc_1$.

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Proof: Let $L \in Loc_1$. Consider the following pullback of frames



Hence K consists of couples (a, J) where $a \in L$, $J \in BN(L)$ and $c(a) = \bigvee J$. It is easy to see that f is surjective and g injective. Since $BN(L)$ is compact regular, it is spatial. Hence K is spatial because it is a subframe of a spatial frame.

We will prove that $K \in Loc_1$. Let $(a, J) \in P(K)$. Consider the ideal $J_a = J \vee \downarrow u(a)$ where $u(a)$ is the open nucleus. Since $u(a)$ is the complement of $c(a)$, it holds

$$\bigvee J_a = \bigvee J \vee u(a) = c(a) \vee u(a) = 1$$

Hence $(1, J_a) \in K$. Further, $(a, \downarrow c(a)) \wedge (1, J_a) = (a, J)$ because

$$\downarrow c(a) \wedge (J \vee \downarrow u(a)) = (\downarrow c(a)) \wedge J) \vee (\downarrow c(a) \wedge \downarrow u(a)) = J.$$

It therefore holds $\quad (a, J) = (a, \downarrow c(a)) \quad$ or $\quad (a, J) = (1, J_a)$.

In the first case $a \in P(L)$ and thus a is a dual atom in L. Hence $c(a)$ is a dual atom in $N(L)$ because **2** is the corresponding quotient frame. Let $(a, \downarrow c(a)) < (x, M)$. Then $x = 1$ and since $VM = x$, there is $j \in M$ such that $c(a) \not\geqslant j$. Hence $1 = j \vee c(a) \in M$. We have proved that (a, J) is a dual atom in K.

Consider the second case If $J_1 \cap J_2 = J$ for $J_1, J_2 \in BN(L)$ then $(1, J_1), (1, J_2) \in K$ and $(1, J_1) \cap (1, J_2) = (1, J)$. The consequence is that $J \in P(BN(L))$. Since $BN(L)$ is a spatial 0-dimensional locale, J is a dual atom in $BN(L)$. Hence $(1, J)$ is a dual atom in K.

We have proved that $Loc_1 \subseteq S(Sob_1)$. It remains to verify that $\widehat{Loc_1} = Loc_1$. Recall that any frame homomorphism $f\colon L \to M$ determines a mapping

$$f_*\colon M \to L\colon f_*(a) = \bigvee_{f(x) \leqslant a} x.$$

It is easy to see that $f_*(P(M)) \subseteq P(L)$. Hence $M \in Loc_1$ whenever f is surjective and $L \in Loc_1$. Let $L_i \in Loc_1$ for $i \in I$ and $v_i\colon L_i \to L$ be injections of the sum in Frm. Let $a \in P(L)$. Then $(v_i)_*(a)$ is a dual atom in L_i for any $i \in I$. Let $x = v_{i_1}(x_1) \wedge \ldots \wedge v_{i_n}(x_n) \in L$. If $x_j \leqslant (V_{ij})_*(a)$ then $x \leqslant a$ and if $x_j \vee (v_{ij})_*(a) = 1$ then $x \vee a = 1$. Hence a is a dual atom in L and therefore $L \in Loc_1$.

The theorem justifies our definition of T_1-locales. Dowker and Strauss [2] and Banaschewski and Harting [1] proposed to take the subfit locales of Isbell [9] as T_1-locales. They were called conjunctive by Simmons [14] because the defining property is dual to Wallman's disjunctivity. Recall that a frame L is a conjunctive if for each two elements $a, b \in L$ with $a \not\leqslant b$ there is an element $c \in L$ such that $a \vee c = 1$ and $b \vee c \neq 1$. The justification for taking them as T_1-locales is that compact conjunctive locales coincide with compact sober T_1-spaces (see Isbell [9]). However, there are conjunctive sober spaces which are not T_1-spaces. Fourman [4], and more recently Dowker and Strauss [3], gave the name 'T_1' to locales which are called unordered by Isbell [10] and T_U by Johnstone [11]. In this case, there are even Hausdorff spaces which lack the 'T_1' property (see [11], III.1·5). The definition of T_1-locales is discussed by Murchiston and Stanley [13].

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2. T_2-*locales*

Up to now, no satisfactory definition of T_2-locales is known. This question was discussed by Isbell [9] who introduced strongly Hausdorff locales. They form an epireflective subcategory of *Loc* but, as the name indicates, they are for spaces more special than T_2. Dowker and Strauss [2], Simmons [14] (and Kerstan [12], as well) proposed to define T_2-locales as conjunctive locales L satisfying the condition (N_2): For any $a, b \in L$ with $a \vee b = 1, a \neq 1, b \neq 1$, there are elements $x, y \in L$ such that $x \wedge y = 0$, $x \not\leqslant a$, $y \not\leqslant b$. Spatial T_2-locales then coincide with Hausdorff spaces and any regular locale is T_2. However, it does not seem that these T_2-locales form an epireflective subcategory of *Loc*. (In a later paper [3], Dowker and Strauss proposed a definition of T_2-locales equivalent to Isbell's 'strong Hausdorffness'.)

Therefore, the problem is to describe $\widehat{Haus}$ where *Haus* denotes the category of T_2-spaces. In what follows, we present our best candidate for $\widehat{Haus}$.

We say that an element $a \in L$, $a \neq 1$ of a frame L is *semiprime* if

$$x \wedge y = 0 \Rightarrow x \leqslant a \quad \text{or} \quad y \leqslant a$$

for any $x, y \in L$. Any prime element is semiprime and $b \geqslant a$, a semiprime implies that b is semiprime. We say that L is an *S-frame* if semiprime elements of L are dual atoms. It is equivalent to the fact that the semiprime elements form an antichain. Let $\mathscr{S}$ denote the category of S-locales.

PROPOSITION 2·1. *$\mathscr{S}$ is an epireflective subcategory of Loc and $\mathscr{S} \cap Sob = Haus$.*

Proof. Let $f: L \to M$ be a frame homomorphism and $a \in M$ a semiprime element. If $x \wedge y = 0$ for $x, y \in L$ then $f(x) \wedge f(y) = 0$, i.e. $f(x) \leqslant a$ or $f(y) \leqslant a$. Hence $x \leqslant f_*(a)$ or $y \leqslant f_*(a)$. We have proved that the mapping $f_*: M \to L$ preserves semiprime elements. As in the proof of 1·2, this yields that $\mathscr{S}$ is epireflective in *Loc*.

Let T be a T_2-space and $V \in O(T)$ semiprime. Let $a, b \in T$ be distinct elements and A, B their disjoint open neighbourhoods. Then $A \subseteq V$ or $B \subseteq V$ and thus V is a dual atom. On the other hand, let $O(T)$ be a spatial S-locale and a, b distinct elements of T. Since $O(T)$ is a T_1-locale, $T - \{a, b\}$ is not semiprime. There are $A, B \in O(T)$ such that $A \cap B = \varnothing$ and $A, B \not\subseteq T - \{a, b\}$. Then A, B are disjoint neighbourhoods of a, b.

The method of the proof of 1·2 does not work for S-locales because K would be a compact spatial S-locale which must be completely regular.

LEMMA 2·2. *Any conjunctive N_2-locale is an S-locale.*

Proof. Let L be a conjunctive N_2-locale and $a, b \in L$, $a < b \neq 1$. There is $c \in L$ such that $b \vee c = 1$ and $a \vee c \neq 1$. Since $b \vee (a \vee c) = 1$, there are $x, y \in L$ such that $x \wedge y = 0$, $x \not\leqslant b$ and $y \not\leqslant a \vee c$. Hence $x, y \not\leqslant a$ and thus a is not semiprime.

EXAMPLE 2·3. Let B be a complete Boolean algebra without dual atoms (e.g. that consisting of the regular open subsets of the reals). It is easy to see that B has no semiprime elements. Adding to B a new top element, we get an S-frame L which is not conjunctive.

An element $1 \neq a$ of a frame L is called *pseudoprime* if $\bigvee J = a$ for some prime ideal J in L (see Hofman and Lawson [8]. Any prime element is pseudoprime and pseudoprimes are semiprimes. We say that L is a P-locale if pseudoprimes are dual atoms. Let $\mathscr{P}$ denote the category of P-locales. Then $\mathscr{S} \subseteq \mathscr{P} \subseteq Loc_1$.

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PROPOSITION 2·4. *$\mathscr{P}$ is an epireflective subcategory of Loc.*

Proof. Let $f: L \to M$ be a frame homomorphism and $a \in M$ pseudoprime. Then $a = \bigvee J$ for a prime ideal J and therefore $\bigvee f^{-1}(J) \leqslant f_*(a)$ and $f^{-1}(J)$ is a prime ideal. This suffices for the application of the argument from 3·1.

3. *Epireflective subcategories of locales*

Epireflective subcategories of topological spaces have been extensively studied (see Herrlich [6]). We will briefly look at epireflective subcategories of locales.

PROPOSITION 3·1. *Let $\mathscr{X}$ be epireflective in Loc. Then $\mathscr{X} \cap Sob$ is epireflective in Sob.*

Proof. It suffices to show that the epireflector $R: Loc \to \mathscr{X}$ maps spatial locales to spatial ones. Let T be a sober space and $\rho_T: T \to R(T)$ the reflection. There is a unique continuous map $f: T \to PR(T)$ such that $\psi_T . f = \rho_T$. Since $PR(T)$ is a sublocale of $R(T)$, $PR(T) \in \mathscr{X}$ and thus there is a unique continuous map $g: R(T) \to PR(T)$ such that $g . \rho_T = f$. It holds $\psi_T . g . \rho_T = \rho_T$ and therefore $\psi_T . g = 1_{R(T)}$. Hence ψ_T is an isomorphism and thus $R(T)$ is spatial.

PROPOSITION 3·2. *Let $\mathscr{Y}$ be epireflective in Sob. Then $\hat{\mathscr{Y}}$ is the smallest epireflective subcategory in Loc such that $\hat{\mathscr{Y}} \cap Sob = \mathscr{Y}$.*

Proof. It suffices to prove that $\hat{\mathscr{Y}} \cap Sob = \mathscr{Y}$. Let $T \in \hat{\mathscr{Y}} \cap Sob$. Since $T \in \hat{\mathscr{Y}}$, there are locales $T_i \in \mathscr{Y}$, $i \in I$ such that T is a sublocale of $\prod_{i \in I} T_i$. The product $\prod^*_{i \in I} T_i$ in Sob is a sublocale of $\prod_{i \in I} T_i$ (see [11], II.2·13). Since T must be a sublocale of $\prod^*_{i \in I} T_i$, $T \in \mathscr{Y}$ holds.

PROPOSITION 3·3. *Let $\mathscr{Y}$ be epireflective in Sob. Then $\check{\mathscr{Y}} = \{L \in Loc: P(L) \in \mathscr{Y}\}$ is the largest epireflective subcategory in Loc such that $\check{\mathscr{Y}} \cap Sob = \mathscr{Y}$.*

Proof. Clearly $\check{\mathscr{Y}} \cap Sob = \mathscr{Y}$. $\check{\mathscr{Y}}$ is epireflective in Loc because P is a right adjoint. Let $\mathscr{X}$ be epireflective in Loc and $\mathscr{X} \cap Sob = \mathscr{Y}$. Since $P(L)$ is a sublocale of L, it holds $\mathscr{X} \subseteq \check{\mathscr{Y}}$.

The description of $\check{\mathscr{Y}}$ is fairly explicit. Remark that $\widecheck{Sob}_1$ consists of those locales in which primes form an anti-chain. Let $\mathscr{Y}$ consist of the empty space and of the one-element space. Then $\mathscr{Y}$ is the smallest epireflective subcategory of Loc and

$$\check{\mathscr{Y}} = \{L: \text{card } P(L) \leqslant 1\}.$$

$\hat{\mathscr{Y}}$ and $\check{\mathscr{Y}}$ are not the only epireflective subcategories in Loc inducing $\mathscr{Y}$. For instance, if $\mathscr{Y}$ consists of the completely regular spaces then we have $\hat{\mathscr{Y}} = \mathscr{CR}$, $\check{\mathscr{Y}} = \{L: P(L)$ completely regular$\}$ but also $\{L: L$ regular and $P(L)$ completely regular$\}$.

The above results establish the relation between epireflective subcategories in Loc and Sob. Since epimorphisms in Sob are not onto, there are epireflective subcategories in Sob which are not epireflective in Top. Sob itself is an example (see Herrlich [5]). However, if $\mathscr{Y}$ is epireflective in Sob and $\mathscr{Y} \subseteq Sob_1$ then $\mathscr{Y}$ is epireflective in Top.

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