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# Embedding diffeomorphisms in flows in Banach spaces

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*Abstract*. This paper concerns the problem of embedding, in the flow of an autonomous vector field, a local diffeomorphism near a hyperbolic fixed point in a Banach space. To solve the problem, we first extend Floquet theory to Banach spaces, and then prove that two $C^\infty$ hyperbolic diffeomorphisms are formally equivalent if and only if they are $C^\infty$-equivalent. The latter result is a version, in the Banach space context, of a classical theorem by Chen.

## 1. *Introduction and statement of the main results*

Let $\mathbb{X}$ be a real Banach space with norm $\|\cdot\|$. Assume that $F(x)$ has a hyperbolic fixed point; without loss of generality, we suppose this fixed point is at $x = 0$ and write

$$F(x) = Lx + f(x), \tag{1.1}$$

where $f(0) = 0$, $Df(0) = 0$ and $L, L^{-1} \in \mathcal{L}(\mathbb{X})$, the linear space of bounded linear operators on $\mathbb{X}$.

We say that $F(x)$ can be *embedded in the flow* $\phi_t$ of the differentiable equations

$$\dot{x} = Tx + g(x)$$

if $F(x) = \phi_1(x)$, with $T \in \mathcal{L}(\mathbb{X})$ and $g(x) = o(\|x\|)$. We also say that $F(x)$ admits an *embedding vector field* $Tx + g(x)$.

We know that if $F(x)$ and $G(x)$ are two conjugate diffeomorphisms (i.e. there exists a diffeomorphism $H$ such that $H \circ G = F \circ H$) and $G$ admits an embedding vector field $V$, then $F$ admits the embedding vector field $H_*V$, where $H_*$ denotes the tangent operator of $H$. So, to solve the embedding problem, one approach is to find a conjugate diffeomorphism which has an embedding flow.

In one-dimensional spaces, the embedding problem has been intensively studied; see, for instance, [**1, 11–13**]. In higher-dimensional spaces, Palis [**18**] proved that

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diffeomorphisms which can be embedded in flows are rare in the sense of Baire. In [**15**] some sufficient conditions were provided for a given local diffeomorphism to admit an embedding flow, and in [**20**] it was proved that any analytic integrable volume-preserving diffeomorphism can be embedded in an analytic differential vector field. In 1992, Kuksin and Pöschel [**10**] solved the problem of embedding a local analytic symplectic map in analytic Hamiltonian flow which depends periodically on time.

The embedding problems mentioned above are all posed in finite-dimensional spaces. A natural question that arises is: what kinds of local diffeomorphisms can be embedded in a flow in a Banach space? The present paper will deal with this problem, and will give a class of local diffeomorphisms which can be embedded in flows in Banach spaces.

We say that the spectrum of the operator $T$, denoted by $\sigma(T)$, is

- *weakly non-resonant* if
$$\sum_{n=1}^{m} \lambda_n - \lambda_s \neq 2l\pi\sqrt{-1} \text{ for all } m \in \mathbb{N} \setminus \{1\},\ s \in \mathbb{N},\ l \in \mathbb{Z} \setminus \{0\} \text{ and } \lambda_n, \lambda_s \in \sigma(T);$$
- *non-resonant* if
$$\sum_{n=1}^{m} \lambda_n - \lambda_s \neq 2l\pi\sqrt{-1} \text{ for all } m \in \mathbb{N} \setminus \{1\},\ s \in \mathbb{N},\ l \in \mathbb{Z} \text{ and } \lambda_n, \lambda_s \in \sigma(T);$$
- *non-resonant of order $k$* if
$$\sum_{n=1}^{m} \lambda_n - \lambda_s \neq 2l\pi\sqrt{-1} \text{ for all } m \in \{2, \ldots, k\},\ s \in \mathbb{N},\ l \in \mathbb{Z} \text{ and } \lambda_n, \lambda_s \in \sigma(T).$$

The operator $L$ is said to be *hyperbolic* if $\operatorname{Re}\alpha \neq 0$ for all $\alpha \in \sigma(T)$. An operator $\Lambda \in \mathcal{L}(\mathbb{X})$ is a *logarithm* of the operator $L$ if $e^{\Lambda} = L$. As usual, we denote by $(\mathbb{X}, 0)$ a suitable small neighborhood of $\mathbb{X}$ around 0.

Our main results are stated in the following theorem.

THEOREM 1.1. *For the $C^\infty$ local diffeomorphism $F(x) = Lx + f(x)$ in $(\mathbb{X}, 0)$, assume that:*

(a) *the bounded linear operator $L$ is hyperbolic and has a bounded inverse;*

(b) *$L$ has a logarithm $T \in \mathcal{L}(\mathbb{X})$;*

(c) *$f(x) = O(\|x\|^2)$ is $C^\infty$-smooth.*

*If the spectrum $\sigma(T)$ of the operator $T$ is weakly non-resonant, then $F(x)$ can be embedded in the flow of a $C^\infty$ autonomous system in $(\mathbb{X}, 0)$.*

We remark that conditions (b) and (c) are necessary for a local $C^\infty$ diffeomorphism to admit an embedding flow. Theorem 1.1 is an extension to Banach spaces of [**15**, Theorem 4] for finite-dimensional spaces; if we restrict the Banach space $\mathbb{X}$ to a finite-dimensional space, then our theorem is exactly [**15**, Theorem 4]. To prove our theorem, we need to establish, in Banach space, generalizations of Floquet theory for nonlinear periodic systems and Chen's conjugate theorem for $C^\infty$ diffeomorphisms. The proof of [**15**, Theorem 4] depends heavily on the structure for finite-dimensional spaces. As we shall see, certain parts of the proof of [**15**, Theorem 4] cannot be applied to Banach spaces, so we need new approaches.

First, let us recall some results which will be useful later on.

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THEOREM A. *For $T \in \mathcal{L}(\mathbb{X})$, assume that the spectrum can be expressed as $\sigma(T) = \bigcup_{i=1}^{k} \sigma_i$, where $k \in \mathbb{N}$ and $\sigma_i \subset \{\xi \in \mathbb{C} : a_i \leq |\xi| \leq b_i\}$, with $a_1 \geq 0$ and $b_i < a_{i+1}$ for $i = 1, \ldots, k-1$. Then $\mathbb{X}$ can be written in the form $\mathbb{X} = \mathbb{X}_1 \times \cdots \times \mathbb{X}_k$, with coordinates $x = (x_1, \ldots, x_k)$, such that $T = \text{blockdiag}(T_1, \ldots, T_k)$ and the map $F(x) = Tx + f(x)$ on $\mathbb{X}$ can be expressed as $F(x) = (T_1x_1 + f_1(x), \ldots, T_kx_k + f_k(x))$, where $\sigma(T_i) = \sigma_i$ for $i = 1, \ldots, k$. With this notation, we write $T = T_1 \oplus \cdots \oplus T_k$.*

This result follows from the spectral decomposition theorem, which can be found in [**7**, p. 111].

THEOREM B. *Let $T \in \mathcal{L}(\mathbb{X})$. We denote by $\mathcal{F}(T)$ the family of all functions which are analytic on some neighborhood of $\sigma(T)$. Then $f(\sigma(T)) = \sigma(f(T))$ for any $f \in \mathcal{F}(T)$.*

This result is called the spectral mapping theorem; it can be found in, e.g., [**6**, Theorem 1.20].

THEOREM C. *If $T_1, T_2 \in \mathcal{L}(\mathbb{X})$ and $T_1T_2 = T_2T_1$, then*

$$\sigma(T_1 + T_2) \subset \sigma(T_1) + \sigma(T_2) \quad \text{and} \quad \sigma(T_1T_2) \subset \sigma(T_1)\sigma(T_2).$$

This result is a special case of [**19**, Theorem 11.23], where the argument was presented for Banach algebras. We note that $\mathcal{L}(\mathbb{X})$ is a Banach algebra under the norm of bounded linear operators on $\mathbb{X}$.

## 2. *The embedding flows of a linear map*

In this section we obtain some sufficient conditions for a linear map on a Banach space to have an embedding flow.

THEOREM 2.1. *Let $A \in \mathcal{L}(\mathbb{X})$. The following properties hold.*

(i) *If $A$ is an isomorphism on $\mathbb{X}$, then it has a logarithm $B$; the logarithm may be complex.*

(ii) *Assume that $\sigma(A) \subset \{c \in \mathbb{C} \mid c \notin \mathbb{R}_-\}$, where $\mathbb{R}_-$ denotes the set of non-positive real numbers; then the operator $A$ has a real logarithm.*

(iii) *Let $A = M^2$ with $M \in \mathcal{L}(\mathbb{X})$; if $\sigma(M) = \sigma_1 \cup \sigma_2$, where $\sigma_1 \subset \mathbb{R}$ and* cl $\sigma_2 \cap \mathbb{R} = \emptyset$, *then the operator $A$ has a real logarithm. Here* 'cl' *denotes the closure of a set.*

This theorem extends [**15**, Theorem 2] to Banach spaces. The main tools in the proof of [**15**, Theorem 2] are from matrix theory, and the proof is quite long. Clearly, the method used in [**15**] is not applicable to Theorem 2.1. Here we shall use the functional calculus of operators to prove Theorem 2.1; our proof is much shorter than that of [**15**, Theorem 2].

*Proof of Theorem 2.1.* Recall from [**6**, pp. 10–15] that if $T \in \mathcal{L}(\mathbb{X})$ and $f \in \mathcal{F}(T)$, the space of analytic functions in some neighborhood $U$ of $\sigma(T)$ whose boundary $\Gamma$ comprises a finite number of rectifiable Jordan curves, then the operator $f(T)$ defined by

$$f(T) = \frac{1}{2\pi i} \oint_{\Gamma} f(\lambda)(\lambda I - T)^{-1}\, d\lambda,$$

where $i = \sqrt{-1}$ and $I$ is the identity operator on $\mathbb{X}$, is analytic.

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(i) Since $A$, $A^{-1} \in \mathcal{L}(\mathbb{X})$, we have $0 \notin \sigma(A)$. By [**6**, Theorem 1.6], the spectrum $\sigma(A)$ is a bounded closed set, so there is an open subset $U$ of $\mathbb{C}$ such that $\sigma(A) \subset U$, $0 \notin \operatorname{cl} U$, and $\Gamma := \partial U$ has at most two rectifiable Jordan curves. Then $\log z$ is analytic on $U \cup \Gamma$. We fix a branch of the complex logarithm on $\Gamma$, and define

$$B := \log A = \frac{1}{2\pi i} \oint_{\Gamma} \log z (zI - A)^{-1}\, dt. \tag{2.1}$$

Then $B$ is a logarithm of $A$. We note that the operator $B$ may be complex.

(ii) Choose $\Gamma$ to be a contour which is symmetric with respect to the real axis in $\mathbb{C}$ and intersects the positive axis only. Then

$$\begin{aligned} \overline{B} = \overline{\log A} &= -\frac{1}{2\pi i} \oint_{\Gamma} \overline{\log z} (\overline{z} I - A)^{-1}\, d\overline{z} \\ &= \frac{1}{2\pi i} \oint_{\Gamma} \log z (zI - A)^{-1}\, dz = B, \end{aligned}$$

where an overbar denotes the conjugate of a complex number. To establish the second equality above, we have used the symmetry of $\Gamma$ with respect to the real axis as well as the change of orientation of $\Gamma$. This proves that $B$ is a real logarithm of $A$.

(iii) By Theorem A, we can set $M := M_R \oplus M_C$ such that $\sigma(M_R) = \sigma_1$ and $\sigma(M_C) = \sigma_2$. From the proof of statement (ii), there exist $R$, $C \in \mathcal{L}(\mathbb{X})$ such that $M_R^2 = e^R$ and $M_C = e^C$. So we have

$$A = M^2 = M_R^2 \oplus M_C^2 = e^R \oplus e^{2C} = e^{R \oplus 2C}.$$

This means that $R \oplus 2C$ is a real logarithm of $A$.

Thus the proof of the theorem is complete. □

The following result, which shows the existence of an embedding flow of a linear isomorphism, is an easy consequence of Theorem 2.1.

COROLLARY 2.2. *If the operator $A \in \mathcal{L}(\mathbb{X})$ satisfies condition (ii) or (iii) of Theorem 2.1, then the isomorphism $P(x) := Ax$ can be embedded in the flow of the linear system $\dot{x} = Bx$, where $B$ is the real logarithm of $A$.*

## 3. *Embedding local diffeomorphisms in periodic differential systems*

The next result shows that a local diffeomorphism in a Banach space can be embedded in the flow of a periodic differential system. The key idea used in the proof comes from the paper [**15**].

THEOREM 3.1. *Consider the $C^k$ local diffeomorphism $F(x) = Lx + f(x)$ in $(\mathbb{X}, 0)$, where $k \in \{1, 2, \ldots, \infty\}$. If $L$ has a real logarithm $T \in \mathcal{L}(\mathbb{X})$, then $F(x)$ is the Poincaré map of a periodic system*

$$\dot{x} = Tx + v(x, t), \quad x \in (\mathbb{X}, 0), \tag{3.1}$$

*where $v(x, t) = O(\|x\|^2)$ is $C^k$ in its variables and periodic of period 1 in $t$.*

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*Proof.* Choose $0 < \delta < 1/4$ such that

$$\|e^{tT} - I\| \leq \tfrac{1}{2} \quad \text{for all } t \in [-2\delta, 2\delta].$$

Define $\phi(t) : [-1/2, 1/2] \to [0, 1]$ to be a $C^k$ function satisfying

$$\phi(t) = \begin{cases} 1 & \text{if } t \in [-\delta, \delta], \\ 0 & \text{if } 2\delta < |t| \leq \tfrac{1}{2}. \end{cases}$$

Set $E(t) = \{t + 1/2\} - 1/2$, where $\{\cdot\}$ denotes the decimal-part function. Then $\psi(t) = \phi(E(t))$ is a $C^k$ periodic function of period 1, with $\psi(t) = 1$ for $|t| < \delta$.

Define a $C^k$ map

$$\begin{array}{rccc} G: & \mathbb{X} \times \mathbb{R} & \to & \mathbb{X} \times \mathbb{R} \\ & (x, t) & \mapsto & (g(x, t), t), \end{array}$$

where

$$g(x, t) := (1 - \psi(t))x + \psi(t)e^{TE(t)}x.$$

It is easy to show that $g(x, t)$ is periodic of period 1 in $t$ and is equal to $e^{TE(t)}x$ for $|t| < \delta$. Clearly, $g|_{t=0} = g|_{t=1}$ is the identity operator. Moreover, we have

$$\|D_x g - I\| \leq \psi(t)\|I - e^{TE(t)}\| \leq \sup_{|t| \leq 2\delta} \|I - e^{Tt}\| < \frac{1}{2},$$

where $\|\cdot\|$ is the usual norm of a bounded linear operator on $\mathbb{X}$. This proves that the fibre of the map $G$ is a $C^k$ bounded linear operator in $x$, with a bounded inverse, which is periodic of period 1 in $t$. Recall that the map $G$ on $\mathbb{X}$ induces a map on the tangent bundle $T\mathbb{X}$, and the fibre of $G$ at $x$ is a tangential map from $T_x\mathbb{X}$ to $T_{G(x)}\mathbb{X}$. Furthermore, $G$ takes the line segment $\{x = \text{constant},\ t \in [-\delta, \delta]\}$ to the solution of the autonomous system

$$\dot{x} = Tx, \quad \dot{t} = 1. \tag{3.2}$$

Now define

$$W(x, t) := (w(x, t), 1) := G_*^{-1}(Tx, 1),$$

where $G_*^{-1}$ denotes the tangent operator of $G^{-1}$. Then $W(x, t)$ is periodic of period 1 in $t$, and $w(x, t) \equiv 0$ for $|t| < \delta$. Moreover, $G^{-1}$ takes the flow $e^{Tt}x$ of $\dot{x} = Tx$ to the flow

$$\frac{e^{Tt}}{(1 - \psi(t))I + \psi(t)e^{TE(t)}}x$$

of $\dot{x} = w(x, t)$. The latter has $e^T x$ as its time-1 map.

Set $H := L^{-1}F(x) = x + h(x)$; then $h(x) = O(\|x\|^2)$. Define $\alpha : \mathbb{R} \to [0, 1]$ to be a $C^{k+1}$ periodic function of period 1 that satisfies

$$\alpha(t) = \begin{cases} 0 & \text{for } t \in \left[0, \dfrac{\delta}{3}\right], \\ 1 & \text{for } t \in \left[\dfrac{2\delta}{3}, \delta\right]. \end{cases}$$

We construct a family of smooth curves $\{\gamma_x(t)\}$ in $(\mathbb{X}, 0) \times [0, \delta]$ by taking

$$\gamma_x(t) := \{(x + \alpha(t)h(x), t) \mid t \in [0, \delta]\},$$

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which connects the points $(x, 0)$ and $(L^{-1}F(x), \delta)$. In fact, this defines a homotopy of curves in $(\mathbb{X}, 0)$. We can check that any two curves $\gamma_{x_1}(t)$ and $\gamma_{x_2}(t)$ which pass through two different points $x_1$ and $x_2$ in $(\mathbb{X}, 0)$ will never intersect.

The maps $H_t : x \mapsto x + \alpha(t)h(x)$ form a family of $C^k$ bounded operators, with inverses being $C^k$ bounded operators in $(\mathbb{X}, 0)$. The tangent vector to the curve $\gamma_x$ at $(x, t)$ is

$$(\alpha'(t)h(x), 1).$$

Thus, we get a $C^k$-smooth vector field in $(\mathbb{X}, 0) \times [0, \delta]$ which has the integral curves $\gamma_x(t)$. Define the vector field

$$Q(x, t) = \begin{cases} w(x, t) & \text{for } t \notin [0, \delta] \ (\mathrm{mod}\ \mathbb{Z}), \\ \alpha'(t)h(H_t^{-1}(x)) & \text{for } t \in [0, \delta] \ (\mathrm{mod}\ \mathbb{Z}). \end{cases}$$

Then $Q$ is periodic of period 1 in $t$, and $Q(0, t) \equiv 0$. Therefore, the Poincaré map of the system

$$\dot{x} = Q(x, t)$$

is $F(x)$, because it is the composition of the map $x \mapsto x + \alpha(\delta)h(x) = L^{-1}F(x)$ with the map $x \mapsto Lx$.

Set $V(x, t) = (v(x, t), 1) := G_*(Q(x, t), 1)$. Since $g|_{t=0} = g|_{t=1}$ is the identity map, $F(x)$ is also the Poincaré map of the system

$$\dot{x} = v(x, t).$$

Moreover, we have

$$D_x v|_{x=0} = T.$$

Indeed, since $H_t^{-1}(0) = 0$ and $h(0) = 0$, we have

$$\begin{aligned} Q(0, t) &= \begin{cases} w(0, t) & \text{for } t \notin [0, \delta] \ (\mathrm{mod}\ \mathbb{Z}), \\ \alpha'(t)h(H_t^{-1}(0)) & \text{for } t \in [0, \delta] \ (\mathrm{mod}\ \mathbb{Z}) \end{cases} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} D_x Q|_{x=0} &= \begin{cases} D_x w|_{x=0} & \text{for } t \notin [0, \delta] \ (\mathrm{mod}\mathbb{Z}), \\ \alpha'(t) D_x h(H_t^{-1}(x))|_{x=0} & \text{for } t \in [0, \delta] \ (\mathrm{mod}\mathbb{Z}) \end{cases} \\ &= D_x w|_{x=0}, \end{aligned}$$

where we have used the fact that $w(x, t) \equiv 0$ for $|t| < \delta$.

From $(Tx, 1) = G_*(w(x, t), 1)$, $(v(x, t), 1) = G_*(Q(x, t), 1)$ and the fact that $G$ is a linear operator in $x$, we get that the linear part of $v(x, t)$ is $Tx$. This shows that $F(x)$ is the Poincaré map of the $C^k$ periodic system

$$\dot{x} = v(x, t),$$

and completes the proof of the theorem. □

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4. *Formal normal forms of periodic systems*

Consider a $C^k$-smooth ($k \in \{1, 2, \ldots, \infty\}$) periodic system of period 1 in $t$,

$$\dot{x} = Tx + g(x, t), \tag{4.1}$$

where $g(x, t) = O(\|x\|^2) \in C^k((\mathbb{X}, 0) \times \mathbb{R}, (\mathbb{X}, 0))$ is a periodic operator of period 1 in $t$. We say that two systems are *formally equivalent* if there exists a formal change of variables $y = x + \phi(x)$ that transforms one system to another. A system is in *formal normal form* if its nonlinear terms are all resonant.

THEOREM 4.1. *If the linear operator $T$ is weakly non-resonant and $g(x, t) = O(\|x\|^2)$ is $C^\infty$, then system (4.1) is formally equivalent to an autonomous system in* $(\mathbb{X}, 0)$.

To prove this theorem, we need the following result.

LEMMA 4.2. *Denote by $\mathcal{F}(\mathbb{X}^n, \mathbb{X})$ the Banach space of symmetric $n$-multilinear bounded operators from $\mathbb{X}^n$ to $\mathbb{X}$. Define a linear operator $\mathcal{P}$ on $\mathcal{F}(\mathbb{X}^n, \mathbb{X})$ by*

$$\begin{aligned} \mathcal{P}: \ & \mathcal{F}(\mathbb{X}^n, \mathbb{X}) \longrightarrow \mathcal{F}(\mathbb{X}^n, \mathbb{X}), \\ (\mathcal{P}u)(x_1, \ldots, x_k) \ &= \ \sum_{s=1}^{k} u(x_1, \ldots, Tx_s, \ldots, x_k) - Tu(x_1, \ldots, x_n). \end{aligned}$$

*Then $\sigma(\mathcal{P}) \subset \{\alpha_1 + \cdots + \alpha_k - \alpha \mid \alpha, \alpha_i \in \sigma(T) \text{ for } i = 1, \ldots, k\}$.*

In some sense, this lemma is an extension of Bibikov's result [**2**, Lemma 1.1], which presented the spectrum of a linear partial differential operator on a space of homogeneous polynomials in finite-dimensional spaces. In [**15, 16, 20**], we used Bibikov's lemma to study normal forms of differential vector fields. We note, however, that Bibikov's original proof cannot be applied to Lemma 4.2; our idea for the proof is motivated, instead, by [**7**, Lemma 8.2].

*Proof of Lemma 4.2.* Clearly, $\mathcal{P}$ is well defined. Set

$$\begin{gathered} [\mathcal{P}_T^s u](x_1, \ldots, x_k) = u(x_1, \ldots, Tx_s, \ldots, x_k) \quad \text{for } s = 1, \ldots, k, \\ L_T u(x_1, \ldots, x_k) = Tu(x_1, \ldots, x_k). \end{gathered}$$

We claim that

$$\sigma(L_T) \subset \sigma(T), \quad \sigma(\mathcal{P}_T^s) \subset \sigma(T).$$

Indeed, if $T - \alpha I_{\mathbb{X}}$ is invertible, where $I_{\mathbb{X}}$ is the identity operator on $\mathbb{X}$, then

$$\begin{aligned} [L_T - \alpha I_{\mathcal{F}(\mathbb{X}^n, \mathbb{X})}]^{-1} &= [L_T - \alpha L_{I_{\mathbb{X}}}]^{-1} = L_{[T - \alpha I_{\mathbb{X}}]^{-1}}, \\ [\mathcal{P}_T^s - \alpha I_{\mathcal{F}(\mathbb{X}^n, \mathbb{X})}]^{-1} &= [\mathcal{P}_T^s - \alpha \mathcal{P}_{I_{\mathbb{X}}}^s]^{-1} = \mathcal{P}_{[T - \alpha I_{\mathbb{X}}]^{-1}}^s. \end{aligned}$$

This means that $\rho(T) \subset \rho(L_T)$ and $\rho(T) \subset \rho(\mathcal{P}_T^s)$, and the claim follows.

It is now easy to check that $\mathcal{P} = \sum_{s=1}^k \mathcal{P}_T^s - L_T$ and that $\mathcal{P}_T^i$, $\mathcal{P}_T^j$ and $L_T$ commute pairwise. Hence, it follows from Theorem C that

$$\sigma(\mathcal{P}) \subset \sum_{s=1}^{k} \sigma(\mathcal{P}_T^s) - \sigma(L_T) \subset \{\alpha_1 + \cdots + \alpha_k - \alpha \mid \alpha_i, \alpha \in T\}.$$

This proves the lemma. □

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*Proof of Theorem 4.1.* Assume that system (4.1) can be transformed to the formal periodic system of period 1 in $t$,

$$\dot{y} = Ty + h(y, t), \quad y \in (\mathbb{X}, 0), \tag{4.2}$$

by the formal periodic transformation of period 1 in $t$,

$$x = y + \zeta(y, t), \quad y \in (\mathbb{X}, 0), \tag{4.3}$$

where $h$ and $\zeta$ are formal series in $y$, starting from the second-order terms, with coefficients being bounded symmetric multilinear operators that are periodic of period 1 in $t$. Let $g$, $h$ and $\zeta$ have the Taylor expansions

$$g(x, t) = \sum_{k=2}^{\infty} g_k(t)x^k, \quad h(y, t) = \sum_{k=2}^{\infty} h_k(t)y^k, \quad \zeta(y, t) = \sum_{k=2}^{\infty} \zeta_k(t)y^k,$$

where $v_k(t)$ for $v \in \{g, h, \zeta\}$ are periodic symmetric $k$-multilinear bounded operators of period 1: that is, $v_k(t) \in \mathcal{F}(\mathbb{X}^k, \mathbb{X})$ for each $t$, and $v_k(t)x^k$ is understood in the sense of $v_k(t)(x_1, \ldots, x_k)$ with $x_1 = \cdots = x_k = x \in \mathbb{X}$.

After some calculation, it can be shown that $\zeta_k(t)$ satisfies the homological equation

$$\frac{d}{dt}\zeta_k(t) + \mathcal{P}\zeta_k(t) = G_k(t) - h_k(t), \tag{4.4}$$

where $\mathcal{P}$ is the linear operator

$$[\mathcal{P}\zeta_k(t)](y_1, \ldots, y_k) = \sum_{s=1}^{k} \zeta_k(t)(y_1, \ldots, Ty_s, \ldots, y_k) - T\zeta_k(t)(y_1, \ldots, y_k)$$

and $G_k(t)$ is the coefficient of $y^k$ in the Taylor expansion of

$$\sum_{j\geq 2} g_j(t)\left(y + \sum_{l\geq 2} \zeta_l(t)y^l\right)^j - \sum_{j\geq 2} C_j(t)(y),$$

with

$$\begin{aligned} C_j(t)(y) = {} & \zeta_j(t)(h(y, t), y, \ldots, y) + \zeta_j(t)(y, h(y, t), \ldots, y) \\ & + \cdots + \zeta_j(t)(y, y, \ldots, h(y, t)). \end{aligned}$$

The operator $G_k(t)$ is found inductively, and $G_k$ and $h_k$ are symmetric periodic bounded operators of period 1.

Solving the linear equation (4.4) gives (see, for instance, [**4**, pp. 13 and 14])

$$\zeta_k(t) = e^{-\mathcal{P}t}\left[\zeta_k(0) + \int_0^t e^{\mathcal{P}s}(G_k(s) - h_k(s))\, ds\right]. \tag{4.5}$$

For a periodic operator $h_k(t)$ of period 1, the operator $\zeta_k(t)$ is periodic of period 1 if and only if

$$(e^{\mathcal{P}} - I)\zeta_k(0) = \int_0^1 e^{\mathcal{P}s}(G_k(s) - h_k(s))\, ds. \tag{4.6}$$

From Lemma 4.2 and Theorem B, we have

$$\sigma(e^{\mathcal{P}}) = e^{\sigma(\mathcal{P})} \subset \{\exp(\alpha_1 + \cdots + \alpha_k - \alpha) \mid \alpha, \alpha_i \in \sigma(T), k \geq 2\}.$$

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If $T$ is non-resonant, then $1 \notin \sigma(e^{\mathcal{P}})$ and $e^{\mathcal{P}} - I$ is invertible. Thus equation (4.5) has a unique periodic solution of period 1.

If $T$ is weakly non-resonant, we can choose a constant operator $C_k$ such that for $h_k(t) = C_k$, the equation (4.4) has a solution $\zeta_k(t)$. To achieve this, set $\Lambda = \int_0^1 e^{\mathcal{P}s}\, ds$; then $\Lambda$ is a bounded linear operator which satisfies $e^{\mathcal{P}} - I = \mathcal{P}\Lambda = \Lambda\mathcal{P}$. Furthermore, since $0 \notin \sigma(\mathcal{P})$, Theorem C implies that

$$\sigma(\Lambda) \subset \{(e^{\alpha} - 1)\alpha^{-1} \mid \alpha \in \sigma(\mathcal{P})\},$$

and consequently $\Lambda$ is invertible.

For the linear operator $\mathcal{P} : \mathcal{F}(\mathbb{X}^k, \mathbb{X}) \to \mathcal{F}(\mathbb{X}^k, \mathbb{X})$, we write the decomposition

$$\mathcal{F}(\mathbb{X}^k, \mathbb{X}) = \mathcal{P}\mathcal{F}(\mathbb{X}^k, \mathbb{X}) \oplus \mathcal{C}.$$

Since each element in $\mathcal{P}\mathcal{F}(\mathbb{X}^k, \mathbb{X})$ is symmetric, all elements in $\mathcal{C}$ must be symmetric multilinear bounded operators. Because $\Lambda$ is a bounded linear operator which is invertible, so is $\Lambda^{-1}$ by the inverse mapping theorem. Set

$$\Lambda^{-1} \int_0^1 e^{\mathcal{P}s} G_k(s)\, ds = R_k \oplus \mathcal{C}_k,$$

with

$$R_k \in \mathcal{P}\mathcal{F}(\mathbb{X}^k, \mathbb{X}) \quad \text{and} \quad \mathcal{C}_k \in \mathcal{C}.$$

We can therefore choose $h_k(s)$ such that equation (4.6) is simplified to

$$\mathcal{P}\zeta_k(0) = \Lambda^{-1} \int_0^1 e^{\mathcal{P}s} G_k(s)\, ds - \mathcal{C}_k = R_k.$$

This last equation always has a solution $\zeta_k(0)$. Consequently, the homological equation (4.4) has a periodic solution for a suitable choice of the symmetric operator $h_k$. This completes the proof of the theorem. □

From Theorem 4.1 and its proof, we easily obtain the following.

COROLLARY 4.3. *The following statements hold.*

(i) *If the linear operator $T$ is non-resonant, then system (4.1) is formally equivalent to its linear part in* $(\mathbb{X}, 0)$.

(ii) *If the linear operator $T$ is weakly non-resonant of order $k$ ($k < \infty$), then system (4.1) is formally equivalent to*

$$\dot{x} = Tx + g_1(x) + g_2(x, t), \quad x \in (\mathbb{X}, 0),$$

*where $g_1$ is a polynomial of degree at most $k$ and $g_2 = O(\|x\|^{k+1})$ is a periodic operator of period* 1 *in $t$.*

## 5. *Smooth equivalence of hyperbolic diffeomorphisms*

For diffeomorphisms in Banach spaces, ElBialy [**7**] proved the existence of finitely smooth normal forms of local contractions; Rodrigues and Solá-Morales [**17**] proved the existence of $C^1$-smooth normal forms of a saddle. In this section we shall prove the following.

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THEOREM 5.1. *In the Banach space* $\mathbb{X}$, *two real* $C^\infty$ *hyperbolic diffeomorphisms in* $(\mathbb{X}, 0)$ *are formally conjugate if and only if they are* $C^\infty$*-conjugate in* $(\mathbb{X}, 0)$.

This theorem is an extension of Chen's classical theorem (see [**3**]) to Banach spaces. If $\mathbb{X}$ is finite-dimensional, then Theorem 5.1 is exactly Chen's theorem. In my opinion, the proof in [**3**] is difficult to apply in the context of Banach spaces. Therefore our approach differs from that used by Chen, and mainly employs the stable manifold theorem in Banach spaces and the homotopy method which appeared in [**14**].

Before proving this theorem, we need some preparatory work. The first preliminary result concerns the existence of a $C^\infty$ function whose Taylor expansion is equal to the given formal series.

LEMMA 5.2. *Assume that, for each* $t \in [0, 1]$, $A_m(t) \in \mathcal{F}(\mathbb{X}^m, \mathbb{X})$ *is a* $C^\infty$*-smooth periodic* $m$*-multilinear bounded operator of period 1. Then there exists a* $C^\infty$ *Fréchet-differentiable periodic operator of period 1 in* $t$ *such that its Taylor expansion in* $(\mathbb{X}, 0)$ *has the* $A_m(t)$ *as its coefficients.*

*Proof.* Define a $C^\infty$ function $\phi(r) : \mathbb{R}^+ \to [0, 1]$ by

$$\phi(r) = \begin{cases} 1 & \text{if } 0 \le r \le \frac{1}{2}, \\ 0 & \text{if } r > 1, \end{cases}$$

where $\mathbb{R}^+$ denotes the set of non-negative real numbers. Set

$$F(x, t) := \sum_{m \in \mathbb{N}^+} f_m(x, t),$$

where $\mathbb{N}^+$ denotes the set of non-negative integers and

$$f_m(x, t) = \frac{\phi(M_m \|x\|)}{m!} A_m(t) x^m$$

with

$$M_m = \sup_{t \in [0,1], 0 \le s \le m} \|D^s A_m(t)\| + m!.$$

Recall that $A_m(t)x^m$ denotes $A_m(t)(x_1, \ldots, x_m)$ for $x_1 = \cdots = x_m = x \in \mathbb{X}$, and $D^s$ the $s$th derivative of $A_m$ with respect to $t$.

Since $\phi = 0$ for $\|x\| \ge M_m^{-1}$, $F(x, t)$ is a finite summation for each given $x$. The series $F(x, t)$ and all its Fréchet derivatives are uniformly convergent. So $F$ is a $C^\infty$ periodic function of period 1 in $t$. Moreover, the coefficients of its Taylor expansion are given by $A_m(t)$. □

The following result, due to Rodrigues and Solà-Morales [**17**], is about bounding the norms of a linear operator.

LEMMA 5.3. *Let* $\mathbb{Y}$ *be a Banach space. Suppose that* $A, A^{-1} \in \mathcal{L}(\mathbb{Y})$ *and that* $|\sigma(A)| \subset (c, d)$ *with* $c > 0$, *where* $|\sigma(A)|$ *denotes the set of absolute values of the elements of* $\sigma(A)$. *Then there exists an equivalent norm* $\|\cdot\|$ *on* $\mathbb{X}$ *such that* $\|A\| < d$ *and* $\|A^{-1}\| < c^{-1}$.

From this lemma it is straightforward to obtain the following.

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COROLLARY 5.4. *Let $\mathbb{Y}$ be a Banach space, and suppose that $A$, $A^{-1} \in \mathcal{L}(\mathbb{Y})$.*

- *If $|\sigma(A)| \subset (c, d)$ with $c > 0$ and $d < 1$, then there exist a norm on $\mathbb{X}$ and a number $0 < \alpha < 1$ such that*

$$\|A^n v\| < \alpha^n \|v\| \quad \text{for all } n \in \mathbb{N} \text{ and } v \in \mathbb{X}.$$

- *If $|\sigma(A)| \subset (c, d)$ with $c > 1$, then there exist a norm on $\mathbb{X}$ and a number $\beta > 1$ such that*

$$\|A^n v\| < \beta^n \|v\| \quad \text{for all } -n \in \mathbb{N} \text{ and } v \in \mathbb{X}.$$

The next lemma gives the existence and smoothness of stable and unstable manifolds of a local hyperbolic diffeomorphism (see, e.g., [**8**, p. 262] or [**5**, Theorem 2.1]).

LEMMA 5.5. *Let $L \in \mathcal{L}(\mathbb{X})$. Assume that $|\sigma(L)| \subset (a, b) \cup (c, d)$, where $0 < a < b < 1 < c < d$, and that $F(x) = Lx + f(x)$ is a local $C^k$ diffeomorphism in $(\mathbb{X}, 0)$ with $k \in \{1, 2, \ldots, \infty\}$, $f(0) = 0$ and $Df(0) = 0$. Let $E_s$ and $E_u$ be, respectively, the stable and unstable linear subspaces of $L$. Then the local diffeomorphism $F(x)$ has local $C^k$ stable and unstable manifolds given, respectively, by*

$$W^s_{\text{loc}} := \{x = (\xi, \eta) \mid \eta = h^s(\xi),\ \xi \in E_s\}, \quad W^u_{\text{loc}} := \{x = (\xi, \eta) \mid \xi = h^u(\eta),\ \eta \in E_u\},$$

*where*

$$h^s : E_s \longrightarrow E_u, \quad h^u : E_u \longrightarrow E_s$$

*are in class $C^k$ and satisfy $h^s(0) = 0$, $Dh^s(0) = 0$, $h^u(0) = 0$ and $Dh^u(0) = 0$.*

By Theorem A, the diffeomorphism $F(x)$ can be written as

$$F(x) = \begin{pmatrix} L_s \xi \\ L_u \eta \end{pmatrix} + \overline{F}(\xi, \eta),$$

where $L_s$ and $L_u$ are linear operators on the stable and unstable manifolds, respectively. Moreover, using the above expressions for the local stable and unstable manifolds, we can make the change of variables

$$u = \eta - h^s(\xi), \quad v = \xi - h^u(\eta);$$

then the diffeomorphsim $F(x)$ is locally $C^k$-conjugate to

$$\widetilde{F}(x) = \widetilde{F}(u, v) := \begin{pmatrix} L_s u + F_s(u, v) \\ L_u v + F_u(u, v) \end{pmatrix}, \tag{5.1}$$

where $F_t(0, 0) = 0$ and $DF_t(0, 0) = 0$ for $t \in \{s, u\}$, and $F_s(0, v) = 0$, $F_u(u, 0) = 0$. Here we have used the fact that the transformation is a local $C^k$ diffeomorphism. In the coordinates $(u, v)$, the new diffeomorphism $\widetilde{F}$ has $u = 0$ and $v = 0$ as, respectively, the stable and unstable manifolds.

Let $G(x)$ be a local $C^k$ diffeomorphism in $(\mathbb{X}, 0)$ with the same linear part as $F(x)$. Then, using a similar argument to that in the last paragraph, we can rewrite $G(x)$ as

$$\widetilde{G}(x) = \widetilde{G}(u, v) := \begin{pmatrix} L_s u + G_s(u, v) \\ L_u v + G_u(u, v) \end{pmatrix}, \tag{5.2}$$

where $G_t(0, 0) = 0$ and $DG_t(0, 0) = 0$ for $t \in \{s, u\}$, and $G_s(0, v) = 0$, $G_u(u, 0) = 0$.

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Set

$$H(x) = H(u, v) := \begin{pmatrix} H_s(u, v) \\ H_u(u, v) \end{pmatrix} = \begin{pmatrix} G_s(u, v) - F_s(u, v) \\ G_u(u, v) - F_u(u, v) \end{pmatrix}.$$

Then $H_s(0, v) = 0$ and $H_u(u, 0) = 0$.

If $G(x) - F(x) = O(\|x\|^m)$, then the difference between their stable manifolds and the difference between their unstable manifolds are both $O(\|x\|^m)$. Consequently, we have $H(x) = \widetilde{G}(x) - \widetilde{F}(x) = O(\|x\|^m)$. If $G(x) - F(x)$ is $C^\infty$-flat, then $H(x) = \widetilde{G}(x) - \widetilde{F}(x)$ is $C^\infty$-flat.

In what follows, we shall denote by $\Pi_s$ and $\Pi_u$ the projections of a vector in $\mathbb{X}$ to the stable and unstable manifolds, respectively. We also denote by $\text{jet}^k_{x=0} H$ the sum of the terms of degree less than or equal to $k$ in the Taylor expansion of $H$ at $x = 0$.

LEMMA 5.6. *Assume that $H$ is as described above. Then there exists a neighborhood $U$ of the origin in which $H$ can be rewritten as $H = H_1 + H_2$, where $H_1$ and $H_2$ satisfy*

$$\Pi_s H_1(u, 0) = 0, \quad \Pi_u H_2(0, v) = 0,$$

*and are such that:*

(i) *if $H$ is $C^k$ and $\text{jet}^k_{x=0} H = 0$, then the $H_i$ $(i = 1, 2)$ are $C^{[k/2]}$, with*

$$\|H_1\| \le C_0 \|v\|^{[k/2]}, \quad \|H_2\| \le C_0 \|u\|^{[k/2]};$$

(ii) *if $H$ is $C^\infty$ and $\text{jet}^\infty_{x=0} H = 0$, then the $H_i$ $(i = 1, 2)$ are $C^\infty$, with*

$$\|H_1\| \le C_m \|v\|^m, \quad \|H_2\| \le C_m \|u\|^m \quad \text{for all } m \in \mathbb{N},$$

*where the $C_m$ are constants.*

*Proof.* (i) Expanding $H$ to order $[k/2]$ in $u$ gives

$$H(u, v) = \sum_{i=0}^{[k/2]} T_i(u, v) + R_{[k/2]}(u, v),$$

where $T_i(u, v)$ is a symmetric $i$-multilinear map in $u$, with coefficients being $C^{k-i}$ functions in $v$, and $R_{[k/2]} = o(u^{[k/2]})$. Set

$$H_1 = \sum_{i=0}^{[k/2]} T_i(u, v), \quad H_2 = R_{[k/2]}(u, v).$$

Then $H_1$ and $H_2$ are $C^{[k/2]}$ and satisfy the stated estimates.

(ii) Let

$$\phi(r) = \begin{cases} 1 & \text{if } 0 \le r \le \frac{1}{2}, \\ 0 & \text{if } r > 1 \end{cases}$$

be the $C^\infty$ cut-off function from $\mathbb{R}^+$ to $[0, 1]$. Set

$$H_1(x) = \sum_{m=0}^{\infty} \frac{1}{m!} D^m_u H(0, v) u^m \phi(M_m \|u\|) \quad \text{and} \quad H_2 = H - H_1,$$

where $D^m_u H(0, v)$ is the $m$th Fréchet derivative of $H(u, v)$ with respect to $u$ at $u = 0$ (which is a symmetric $m$-multilinear operator on $u$) and $M_m = \sup_{x \in U, 0 \le i \le m} \|D^i H(x)\| + m!$. Then $H_1$ and $H_2$ are $C^\infty$ and satisfy the claimed estimates. □

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Choose a $C^\infty$ cut-off function $\psi(s)$ from $\mathbb{R}$ to $[0, 1]$ such that

$$\psi(s) = \begin{cases} 1 & \text{if } |s| \le 1, \\ 0 & \text{if } |s| \ge 2 \end{cases}$$

and $\sup_{s\in\mathbb{R}} |\psi'(s)| \le 2$. Define

$$\overline{p}(x) = \psi(\|x\|/r)\, p(x),$$

with $p \in \{H_1, H_2, F_\delta, G_\delta (\delta = s, u)\}$, where $r$ is a positive number to be determined. Set

$$\overline{F}(x) = \overline{F}(u, v) := \begin{pmatrix} L_s u + \overline{F}_s(u, v) \\ L_u v + \overline{F}_u(u, v) \end{pmatrix}, \quad \overline{G}(x) = \overline{G}(u, v) := \begin{pmatrix} L_s u + \overline{G}_s(u, v) \\ L_u v + \overline{G}_u(u, v) \end{pmatrix}.$$

It is obvious that $\overline{F}$ and $\overline{G}$ are in class $C^k$, while $\overline{H}_1$ and $\overline{H}_2$ are in class $C^{[k/2]}$ (respectively, $C^\infty$), provided that $F$ and $G$ are in class $C^k$ (respectively, $C^\infty$). Furthermore, depending on whether the class is $C^k$ or $C^\infty$, the functions $\overline{H}_1$ and $\overline{H}_2$ satisfy one of the following estimates:

- $\|\overline{H}_1\| \le C_0 \|v\|^{[k/2]} \psi(\|v\|/r)$ and $\|\overline{H}_2\| \le C_0 \|u\|^{[k/2]} \psi(\|u\|/r)$, subject to $\text{jet}^k_{x=0} H = 0$;
- $\|\overline{H}_1\| \le C_m \|v\|^m \psi(\|v\|/r)$ and $\|\overline{H}_2\| \le C_m \|u\|^m \psi(\|u\|/r)$ for all $m \in \mathbb{N}$, subject to $\text{jet}^\infty_{x=0} H = 0$.

To prove the conjugacy between $\overline{F}$ and $\overline{G}$, we will use the so-called homotopy method which was introduced in [**9**] (see also [**13**]) to obtain finitely smooth normal forms for dynamical systems in finite-dimensional spaces. Set

$$\Phi_\tau = \overline{F}(x) + \tau \overline{H}_1, \quad \Psi_\tau = \overline{F}(x) + \overline{H}_1 + \tau \overline{H}_2 \quad \text{for } \tau \in [0, 1]. \tag{5.3}$$

We shall prove that $\Phi_0$ and $\Phi_1$, and $\Psi_0$ and $\Psi_1$, are $C^\infty$ conjugate. As a consequence, $\overline{F}(x)$ and $\overline{G}(x)$ will be $C^\infty$ conjugate.

The next lemma is a version, for diffeomorphims in Banach spaces, of the fundamental result [**9**, Lemma 5], which was formulated for vector fields in finite-dimensional spaces (see also [**13**, p. 128] for the case of diffeomorphisms in finite-dimensional space).

LEMMA 5.7. *Let*

$$F_\tau(x) = F(x) + \tau H(x), \quad \tau \in [0, 1],$$

*be a family of diffeomorphisms in* $(\mathbb{X}, 0)$ *with fixed point* $x = 0$. *If there exists a* $C^\infty$ *bounded map* $R : (\mathbb{X}, 0) \times [0, 1] \to (\mathbb{X}, 0)$ *that satisfies the homological equation*

$$DF_\tau(x) R(x, \tau) - R \circ (F_\tau, \tau) = -H(x), \tag{5.4}$$

*then the diffeomorphisms* $F(x)$ *and* $F(x) + H(x)$ *are* $C^\infty$*-conjugate in* $(\mathbb{X}, 0)$.

*Proof.* The proof is the same as that in [**9**]. Because it is short, we shall reproduce it here. By assumption, the map

$$\phi : (x, \tau) \mapsto (F_\tau(x), \tau)$$

is a diffeormorphism in $(\mathbb{X}, 0) \times [0, 1]$. Consider the differential system

$$\dot{x} = R(x, \tau), \quad \dot{\tau} = 1.$$

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Denote by $g^t$ its local flow; then it is well-defined in $(\mathbb{X}, 0)$ for $t \in [0, r)$ with $r > 1$ and, moreover, it is a $C^\infty$ diffeomorphism in $(\mathbb{X}, 0)$.

From the homological equation (5.4), we have

$$\begin{pmatrix} DF_\tau & H \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R \\ 1 \end{pmatrix} = \begin{pmatrix} R \\ 1 \end{pmatrix} \circ \phi,$$

i.e. $D\phi v = v \circ \phi$ where $v = (R,\ 1)^T$. This implies that $\phi$ and $g^t$ commute, that is,

$$\phi \circ g^t = g^t \circ \phi. \tag{5.5}$$

Write $g^1(x, 0) = (g(x), 1)$. From (5.5) and the computations

$$g^1 \circ \phi(x, 0) = g^1(F_0(x), 0) = g^1(F(x), 0) = (g \circ F(x), 1)$$
$$\text{and} \quad \phi \circ g^1(x, 0) = \phi(g(x), 1) = (F_1 \circ g(x), 1),$$

we know that $F(x)$ and $F(x) + H(x)$ are conjugate via the $C^\infty$ diffeomorphism $g$ in $(\mathbb{X}, 0)$. □

LEMMA 5.8. *The linear functional equation (5.4) has a formal series solution*

$$R(x, \tau) = -\sum_{m=1}^{\infty} (F_\tau^{-m})_* \hat{H}(x),$$

*where* $\hat{H}(x) = H \circ F_\tau^{-1}(x)$ *and* $(F_\tau^{-m})_* \hat{H} = (DF_\tau^{-m} \hat{H}) \circ F_\tau^m$.

*Proof.* The homological equation (5.4) can be written as

$$(F_\tau)_* R = R - \hat{H}.$$

Simple calculations show that

$$\begin{aligned}(F_\tau)_* R &= -\sum_{m=1}^{\infty} (F_\tau)_* (F_\tau^{-m})_* \hat{H} \\ &= -\sum_{m=1}^{\infty} (F_\tau^{-m+1})_* \hat{H} = -\hat{H} + R,\end{aligned}$$

where we have used the fact that $f_*(g_* h) = (f \circ g)_* h$. □

Now we will prove convergence of the series solution and its derivatives of arbitrary order in Lemma 5.8, with the $F_\tau$ in Lemma 5.7 replaced by $\Phi_\tau$ and $\Psi_\tau$ from (5.3).

LEMMA 5.9. *With the same notation as in (5.3), set*

$$\hat{H}_1(x) = \overline{H}_1 \circ \Phi_\tau^{-1}(x), \quad \hat{H}_2(x) = \overline{H}_2 \circ \Psi_\tau^{-1}(x).$$

*If* $\text{jet}_{x=0}^\infty H = 0$, *then*

$$R_1(x, \tau) = -\sum_{m=1}^{\infty} (\Phi_\tau^{-m})_* \hat{H}_1(x)$$

*and*

$$R_2(x, \tau) = -\sum_{m=1}^{\infty} (\Psi_\tau^{-m})_* \hat{H}_2(x)$$

*are* $C^\infty$ *and bounded in* $(\mathbb{X}, 0)$.

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*Proof.* First, note that

$$(\Phi_\tau^{-k})_* \hat{H}_1(x) = (D\Phi_\tau^{-k} \overline{H}_1) \circ \Phi_\tau^{k-1}.$$

From the discussion preceding Lemma 5.7 and from Corollary 5.4, we obtain that for any $m \in \mathbb{N}$ there exists an $\overline{\alpha}$ $(0 < \overline{\alpha} < 1)$ such that

$$\|\overline{H}_1 \circ \Phi_\tau^{k-1}\| \le C_m \|\Pi_s \Phi_\tau^{k-1}\|^m \psi(\|\Pi_s \Phi_\tau^{k-1}\|/r)\| \le C_m (\overline{\alpha}^{k-1}\|v\|)^m.$$

By the chain rule and induction, we can prove that there exist $C$, $\gamma > 0$ such that in the ball of radius 2 centered at the origin of $\mathbb{X}$, upon choosing $r$ sufficiently small we have

$$\|D\Phi_\tau^{-k}(\cdot)\psi(\cdot/r) \circ \Phi_\tau^{k-1}\| \le M_1 \gamma^k.$$

Selecting $m$ large enough so that $\gamma \overline{\alpha}^m < 1$, we get

$$\begin{aligned}
\|R_1(x, \tau)\| &\le \sum_{k=1}^{\infty} \|(\Phi_\tau^{-k})_* \hat{H}_1(x)\| \\
&\le \sum_{k=1}^{\infty} M_1 C_m \overline{\alpha}^{-m} (\gamma \overline{\alpha}^m)^k \|v\|^m \\
&\le \frac{M_1 C_m \overline{\alpha}^{-m}}{1 - \gamma \overline{\alpha}^m} \|v\|^m \le \frac{M_1 C_m \overline{\alpha}^{-m}}{1 - \gamma \overline{\alpha}^m} \|x\|^m.
\end{aligned}$$

This proves that the series $R_1(x, \tau)$ is convergent and bounded in $(\mathbb{X}, 0)$.

As for proving that $R_1$ is $C^\infty$ in the Fréchet sense, it is sufficient to prove that the series

$$\sum_{m=1}^{\infty} D^s((\Phi_\tau^{-m})_* \hat{H}_1(x))$$

is convergent for any $s \in \mathbb{N}$. This follows from the estimates

$$\|\overline{H}_1\| \le C_m \|v\|^m \psi(\|v\|/r),$$

for arbitrary $m \in \mathbb{N}$, and

$$\|D^s \Phi_\tau^{-k}(\cdot)\psi(\cdot/r) \circ \Phi_\tau^{k-1}\| \le M_s \gamma_s^k,$$

for a suitable choice of positive numbers $M_s$ and $\gamma_s$.

By working in a similar way to the proof for $R_1$, we can also show that $R_2$ is $C^\infty$ and bounded. The only difference is that, for $R_2$, one needs to use the projection $\Pi_u$ to the unstable manifold. □

Summarizing Lemmas 5.7, 5.8 and 5.9, we obtain the following.

COROLLARY 5.10. *If $F(x)$ and $G(x)$ are two $C^\infty$ local hyperbolic diffeomorphisms in $(\mathbb{X}, 0)$ and their difference is flat, i.e.* $\text{jet}_{x=0}^\infty(F - G) = 0$, *then they are $C^\infty$-conjugate in $(\mathbb{X}, 0)$.*

*Proof of Theorem 5.1.* We need only prove the 'only if' direction. Assume that $F(x)$ and $G(x)$ are two formally conjugate $C^\infty$ local hyperbolic diffeomorphisms in $(\mathbb{X}, 0)$. Then there exists a formal transformation, denoted by $\widetilde{R}$, such that

$$\text{jet}_{x=0}^\infty G = \text{jet}_{x=0}^\infty \widetilde{R}^{-1} \circ F \circ \widetilde{R}.$$

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By Lemma 5.2, there exists a local $C^\infty$ diffeomorphism $R$ such that $\mathrm{jet}^\infty_{x=0} R = \widetilde{R}$. Set

$$H(x) = R^{-1} \circ F \circ R(x).$$

Then $\mathrm{jet}^\infty_{x=0} G = \mathrm{jet}^\infty_{x=0} H$. By Corollary 5.10, $G$ and $H$ are $C^\infty$-conjugate because they are hyperbolic. Consequently, $G$ and $F$ are $C^\infty$-conjugate in $(\mathbb{X}, 0)$. This completes the proof of Theorem 5.1. □

## 6. *Equivalence of vector fields and conjugacy of the Poincaré maps*

THEOREM 6.1. *In the Banach space $\mathbb{X}$, two real $C^k$ periodic vector fields of period* 1 *are $C^k$-equivalent if and only if their Poincaré maps are $C^k$-conjugate.*

*Proof.* Let $V(x, t)$ and $W(x, t)$ be two $C^k$ periodic vector fields of period 1 in $t$, and let $\phi_t$ and $\psi_t$ be the corresponding flows. Define two maps on $\mathbb{X}$ by $\Phi(x) := \phi_1(x)$ and $\Psi(x) = \psi_1(x)$.

*Necessity (the 'only if' statement).* By assumption there exists a $C^k$ periodic smooth bounded operator $H(x, t)$ of period 1 in $t$, with a $C^k$ bounded inverse, which sends the orbits of the flow $\phi_t$ to those of $\psi_t$, i.e. for arbitrary $x_0 \in \mathbb{X}$, we have $y_0 = H_0(x_0)$ and

$$H_t \circ \phi_t(x_0) = \psi_t(y_0) = \psi_t(H_0(x_0)),$$

where we are using $H_t(x)$ to denote $H(x, t)$. Then $H_0$ is the conjugacy between $\Phi$ and $\Psi$, because for any $x_0 \in \mathbb{X}$,

$$H_0 \circ \Phi(x_0) = H_0 \circ \phi_1(x_0) = H_1 \circ \phi_1(x_0) = \psi_1(y_0) = \Psi(y_0) = \Psi \circ H_0(x_0).$$

*Sufficiency (the 'if' statement).* Assume there exists a $C^k$-smooth bounded operator $H(x)$ such that

$$H \circ \Phi = \Psi \circ H.$$

To obtain the time-dependent mapping between the flows $\phi_t$ and $\psi_t$, we set

$$H_t := \psi_t \circ H \circ \phi_t^{-1}.$$

Then $H_t$ is $C^k$-smooth, bounded and invertible in $(\mathbb{X}, 0)$. It is also periodic of period 1 because

$$H_{t+1} = \psi_{t+1} \circ H \circ \phi_{t+1}^{-1} = \psi_t \circ \Psi \circ H \circ \Phi^{-1} \circ \phi_t^{-1} = \psi_t \circ H \circ \phi_t^{-1} = H_t,$$

where we have used the facts that $\phi_{t+1} = \phi_t \circ \Phi$, $\psi_{t+1} = \psi_t \circ \Psi$ and $H \circ \Phi = \Psi \circ H$. Hence, $H_t$ is the conjugacy between the two flows $\phi_t$ and $\psi_t$. □

## 7. *Extension of Floquet theory to Banach spaces and the proof of Theorem 1.1*

7.1. *Linear periodic systems.* Consider the following linear periodic system of period 1 in $t$:

$$\dot{x} = A(t)x, \tag{7.1}$$

where $A(t) \in \mathcal{L}(\mathbb{X})$ for each $t$ and $A$ is continuous and periodic of period 1 in $t$. Let $\Phi(t, s)$ be the solution of the initial value problem in $\mathcal{L}(\mathbb{X})$, namely $\dot{U} = A(t)U$, $U(s) = I$, where

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$I$ is the identity operator on $\mathbb{X}$. Then $\Phi(t, s) = \Phi(t, \tau)\Phi(\tau, s)$ for any $t, \tau, s \in \mathbb{R}$, and $x = \Phi(t, s)x_0$ is the solution of (7.1) satisfying initial condition $x(s) = x_0$. Moreover, $\Phi(t, s)$ is a homeomorphism of $\mathbb{X}$ onto $\mathbb{X}$, and $\Phi(t, s)$ and $\Phi(t, s)^{-1} = \Phi(s, t)$ are continuous in $t$ and $s$ (for more details, see the book of Deimling [**4**, p. 13]). For simplicity, set $s = 0$ and $\Phi(t) = \Phi(t, 0)$.

The following results provide an extension of classical Floquet theory to linear periodic systems in Banach spaces.

THEOREM 7.1. *For the periodic system (7.1), with $\Phi(t)$ as defined above, set $M := \Phi(0)^{-1}\Phi(1)$. If the spectrum of $M$ satisfies condition (iii) of Theorem 2.1, then the periodic system (7.1) can be transformed into an autonomous system.*

*Proof.* First, observe that $\Phi(t + 1) = \Phi(t)\Phi(0)^{-1}\Phi(1)$, because both sides are solutions of (7.1) and satisfy the same initial condition.

By assumption, there exists an operator $R \in \mathcal{L}(\mathcal{X})$ such that $e^{2R} = M^2$. Set $Q(t) := \Phi(t)e^{-tR}$. Then the change of variables $x = Q(t)y$ transforms the periodic system (7.1) to the autonomous system $\dot{y} = Ry$. □

7.2. *Nonlinear periodic systems.* Following on from the previous subsection, we now consider the nonlinear periodic system

$$\dot{x} = Tx + v(x, t), \tag{7.2}$$

where $v(x, t) = O(\|x\|^2)$ is a $C^\infty$ function in $(\mathbb{X}, 0)$ which is periodic of period 1 in $t$.

THEOREM 7.2. *If $T \in \mathcal{L}(\mathbb{X})$ and the spectrum $\sigma(T)$ is weakly non-resonant, then the periodic system (7.2) is $C^\infty$-equivalent to an autonomous system in* $(\mathbb{X}, 0)$.

*Proof.* From Theorem 4.1, the periodic system (7.2) is formally equivalent to an autonomous system

$$\dot{y} = Ty + \widetilde{h}(y)$$

via a formal transformation

$$x = y + \widetilde{\zeta}(y, t),$$

where $\widetilde{\zeta}(y, t)$ starts from the terms of order 2 in $y$ and is periodic of period 1 in $t$. By Lemma 5.2, there exist $C^\infty$ functions $h(y)$ and $\zeta(y, t)$ such that

$$\mathrm{jet}^\infty_{y=0} h(y) = \widetilde{h}(y), \quad \mathrm{jet}^\infty_{y=0} \zeta(y, t) = \widetilde{\zeta}(y, t).$$

Then system (7.2) is $C^\infty$-equivalent to the system

$$\dot{y} = Ty + w(y, t) \tag{7.3}$$

via the change of variables

$$x = y + \zeta(y, t).$$

Obviously, we have $\mathrm{jet}^\infty_{y=0} w(y, t) = \widetilde{h}(y)$.

Consider the $C^\infty$ autonomous system

$$\dot{y} = Ty + h(y). \tag{7.4}$$

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Denote by $P_w(y)$ and $P_h(y)$ the Poincaré maps of systems (7.3) and (7.4), respectively. Since $\mathrm{jet}^{\infty}_{y=0} h(y) = \widetilde{h}(y) = \mathrm{jet}^{\infty}_{y=0} w(y, t)$, we have $\mathrm{jet}^{\infty}_{y=0} P_h(y) = \mathrm{jet}^{\infty}_{y=0} P_w(y)$. This means that $P_h$ and $P_w$ are formally conjugate; therefore, they are $C^{\infty}$ conjugate by Theorem 5.1. Moreover, Theorem 6.1 gives that systems (7.3) and (7.4) are $C^{\infty}$-equivalent. This implies that the periodic system (7.2) is $C^{\infty}$-equivalent to the autonomous system (7.4), as claimed. □

*Proof of Theorem 1.1.* By Theorem 3.1, the $C^{\infty}$ diffeomorphism $F(x) = Lx + f(x)$ can be embedded in the $C^{\infty}$ periodic system (7.2) in $(\mathbb{X}, 0)$. Theorem 7.2 says that system (7.2) is $C^{\infty}$-equivalent to the autonomous system (7.4) in $(\mathbb{X}, 0)$. Since $F(x)$ and $P_h(y)$ are the Poincaré maps of systems (7.2) and (7.4), respectively, it follows from Theorem 6.1 that $F(x)$ and $P_h(y)$ are $C^{\infty}$-conjugate in $(\mathbb{X}, 0)$. Consequently, $F(x)$ can be embedded in a $C^{\infty}$ autonomous system, because $P_h$ has this property. The proof of our main theorem is thus complete. □

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