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FRACTIONAL SCHRÖDINGER–POISSON SYSTEM WITH SINGULARITY: EXISTENCE, UNIQUENESS, AND ASYMPTOTIC BEHAVIOR

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Abstract. In this paper, we consider the following fractional Schrödinger–Poisson system with singularity

$$\begin{cases} (-\Delta)^s u + V(x)u + \lambda\phi u = f(x)u^{-\gamma}, & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3, \end{cases}$$

where $0 < \gamma < 1$, $\lambda > 0$ and $0 < s \leq t < 1$ with $4s + 2t > 3$. Under certain assumptions on V and f, we show the existence, uniqueness, and monotonicity of positive solution u_λ using the variational method. We also give a convergence property of u_λ as $\lambda \to 0$, when λ is regarded as a positive parameter.

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1. Introduction. Nonlinear equations involving fractional powers of the Laplacian have attracted increasing attentions in recent years. The fractional Laplacian is the infinitesimal generator of Lévy stable diffusion process and arises in anomalous diffusion in plasma, population dynamics, geophysical fluid dynamics, flames propagation, chemical reactions in liquids, and American options in finance, see [**2**] for instance. In this paper, we study the uniqueness, monotonicity, and asymptotic behavior of positive solutions for the following fractional Schrödinger–Poisson system:

$$\begin{cases} (-\Delta)^s u + V(x)u + \lambda\phi u = f(x)u^{-\gamma}, & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3, \end{cases} \tag{SP_λ}$$

where $\lambda > 0$ is a parameter, $0 < \gamma < 1$ and $0 < s \leq t < 1$ with $4s + 2t > 3$. The fractional Laplacian operator $(-\Delta)^s$ is defined by

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$$(-\Delta)^s u(x) = C_{N,s} P.V. \int_{\mathbb{R}^3} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \mathrm{d}y, \quad u \in \mathbb{S}(\mathbb{R}^N),$$

where $P.V.$ stands for the Cauchy principal value, $C_{N,s}$ is a normalized constant, and $\mathbb{S}(\mathbb{R}^N)$ is the Schwartz space of rapidly decaying function. Throughout the paper, we suppose V and f satisfy:

(V_1) $V \in C(\mathbb{R}^3)$ satisfies $\inf_{x\in\mathbb{R}^3} V(x) > V_0 > 0$, where V_0 is a constant.
(V_2) $\mathrm{meas}\{x \in R^3 : -\infty < V(x) \le h\} < +\infty$ for all $h \in \mathbb{R}$.
(f_1) $f \in L^{\frac{2}{1+\gamma}}(\mathbb{R}^3)$ is a nonnegative function.

Recently, many scholars pay attentions to the following fractional Schrödinger–Poisson system which can be viewed as a more general case of system (SP_λ)

$$\begin{cases} (-\Delta)^s u + V(x)u + \lambda\phi u = h(x,u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = \lambda u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.1}$$

Zhang et al. [40] proved the existence and asymptotics of radial solutions for system (1.1) with $V(x) = 0$ and subcritical or critical nonlinearity. Based on the perturbation method and the mountain pass theorem, Li [20] obtained the existence of nontrivial solutions for system (1.1) with $V(x) = \lambda = 1$. Under some hypotheses, He and Jing [16] showed the existence and multiplicity of nontrivial solutions for system (1.1) by replacing λ with a suitable function $K(x)$. Gu et al. [14] showed the existence of positive solutions for system (1.1) with $\lambda = 1$ and $h(x,u) = K(x)h(u) + |u|^{2_s^*-2}u$ satisfying some suitable conditions. Guo [15], Yu et al. [36], and Teng [29, 34] obtained the existence of a positive (nontrivial) ground state solution and a sign-changing (least energy) solution for system (1.1) involving critical nonlinearities. Xiang and Wang [33] considered the existence, multiplicity, and asymptotic behavior of nonnegative solutions for a fractional Schrödinger–Poisson–Kirchhoff type system involving critical nonlinearities. Li et al. [21] studied the existence of infinitely many solutions for a fractional Schrödinger–Poisson–Kirchhoff type system in the subcritical and critical case. For more works on fractional Schrödinger–Poisson system, one can see [7, 30, 37] and the references cited therein.

If $\phi = 0$, system (SP_λ) reduces to a fractional Schrödinger equation with singularity which has been studied extensively in recent years. Some salient results have been obtained for this kind of the following semilinear problem on the bounded domains

$$\begin{cases} (-\Delta)^s u = \delta k(x) u^{-\gamma} + M u^q, \ u > 0 & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $0 < \gamma < 1$. Fang [4] showed the existence and uniqueness of classical solution for problem (1.2) when $\delta k(x) = 1$ and $M = 0$ (the purely singular problem). Mukherjee and Sreenadh [24] and Saoudi [26] both studied the existence and multiplicity of positive solutions to the critical singular problem (1.2). Wang and Zhang [32] further considered problem (1.2) with subcritical growth and relaxed the request according to $k(x)$ in [24] where authors request $k(x)$ has a uniform positive lower bound. Ghanmi and Saoudi [8] showed the existence of at least two solutions for problem (1.2) with $\delta = 1$ and u^q replaced by $f(x,u)$ satisfying some suitable conditions. Goyal [13] obtained the existence, nonexistence, and multiplicity of positive solutions for problem (1.2) with Hardy potential. Fiscella [5, 6] proved the existence of two solutions for problem (1.2) involving Kirchhoff prototype and a critical nonlinearity. Giacomoni et al. [9, 10] investigated the existence,

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multiplicity, and Hölder regularity of weak solutions for problem (1.2) with critical growth for any $\gamma > 0$. Finally, we also mention [**1**, **11**] where the existence of three solutions and bifurcation results were given for different fractional singular problems.

When $s = t = 1$ and on the bounded domain $\Omega \subset \mathbb{R}^3$, system (SP_λ) becomes to the following Schrödinger–Poisson system with singularity

$$\begin{cases} -\Delta u + \lambda\phi u = f(x)u^{-\gamma}, & x \in \Omega, \\ -\Delta\phi = u^2, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = \phi = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $0 < \gamma < 1$. When $f(x) = \mu$ is a positive parameter, Zhang [**38**] investigated problem (1.3) and obtained the existence and uniqueness results for $\lambda = 1$ and multiplicity of solutions for $\lambda = -1$. When $\lambda = 1$ and $f(x) = \frac{\mu}{|x|^\beta}$, Lei and Liao [**18**] generalized a part of results in Zhang [**38**] to the critical problem and obtained two positive solutions for problem (1.3) with critical exponent. When $\lambda = -1$ and $f(x) = \frac{\mu}{|x|^\beta}$, Wang [**31**] considered the existence and multiplicity of positive solutions for problem (1.3) with three times growth. Mukherjee and Sreenadh [**25**] investigated the existence, multiplicity, and regularity of positive weak solutions of a nonlinear Choquard equation with singular nonlinearity. Mu and Lu [**23**], Li et al. [**22**], and Zhang [**39**] studied the existence, uniqueness, and multiple results to singular Schrödinger–Kirchhoff–Poisson system. On the unbounded domain $\mathbb{R}^N$, Sun and Li [**28**] established some existence, symmetry, and uniqueness results for positive ground state solutions to a singular semilinear elliptic equation. Lei et al. [**19**] studied a Schrödinger–Newton system with singularity and critical growth terms. We [**35**] obtained the existence, uniqueness, and asymptotical behavior of solutions to a Choquard equation with singularity. For more related topics, one can see [**12**, **17**, **27**] and the references cited therein.

However, to the best of our knowledge, there are no results on the existence of positive solutions for fractional Schrödinger–Poisson system with singularity on unbounded domains. Here, we need to overcome the lack of compactness as well as the nondifferentiability of the functional I_λ on E (I_λ and E can be found in Section 2) and indirect availability of critical point theory due to the presence of singular term.

A starting point of studying system (SP_λ) is the following fact. For every $u \in H^s(\mathbb{R}^3)$, there is a unique $\phi_u^t \in \mathcal{D}^{t,2}(\mathbb{R}^3)$ such that $(-\Delta)^t\phi_u^t = u^2$, $x \in \mathbb{R}^3$, see, for example, [**7**, **21**] or Section 2 of this article. Inserting this ϕ_u^t into the first equation of the system (SP_λ) leads to

$$\begin{cases} (-\Delta)^s u + V(x)u + \lambda\phi_u^t u = f(x)u^{-\gamma}, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3. \end{cases} \tag{1.4}$$

We say that $(u, \phi_u^t) \in E \times \mathcal{D}^{t,2}(\mathbb{R}^3)$ is a solution of system (SP_λ) if u is a weak solution of equation (1.4), that is, $u > 0$ in $\mathbb{R}^3$ and for every $v \in E$,

$$(u, v)_E + \lambda \int_{\mathbb{R}^3} \phi_u^t uv \, dx - \int_{\mathbb{R}^3} f(x)u^{-\gamma} v dx = 0. \tag{1.5}$$

For the sake of simplicity, we just say u instead of (u, ϕ_u^t) is a solution of system (SP_λ). By the variational method, we obtain the following existence and uniqueness of positive solution and the asymptotic behavior of solutions with respect to the parameter λ.

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THEOREM 1.1. *Let $\lambda > 0$ and $\gamma \in (0, 1)$. Assume (V_1), (V_2), and (f_1). Then, system (SP_λ) admits a unique positive solution $u_\lambda \in E$.*

As a consequence of Theorem 1.1, we can obtain the following results.

THEOREM 1.2. *Let $\lambda > 0$ and $\gamma \in (0, 1)$. Assume (V_1) and (V_2). If $f_1(x), f_2(x) \in L^{\frac{2}{1+\gamma}}(\mathbb{R}^3)$ are two nonnegative functions, and $u_1(x)$ and $u_2(x)$ are the corresponding solutions to system (SP_λ) provided by Theorem 1.1. Then, $f_1(x) \geq f_2(x)$ implies $u_1(x) \geq u_2(x)$ for all $x \in \mathbb{R}^3$.*

THEOREM 1.3. *Let $\gamma \in (0, 1)$. Assume (V_1), (V_2), and (f_1). For $0 < \lambda_1 \leq \lambda_2$, $u_{\lambda_1}(x)$ and $u_{\lambda_2}(x)$ are two corresponding solutions of system (SP_λ) provided by Theorem 1.1. Then, $u_{\lambda_1}(x) \geq u_{\lambda_2}(x)$ for all $x \in \mathbb{R}^3$.*

THEOREM 1.4. *Let $\gamma \in (0, 1)$. Assume (V_1), (V_2), and (f_1). For every vanishing sequence $\{\lambda_n\} \subset (0, 1)$, let u_{λ_n} be the unique positive solution to system (SP_λ) provided by Theorem 1.1. Then, u_{λ_n} converge to w_0 in E, where w_0 is the unique positive solution to problem*

$$\begin{cases} (-\Delta)^s u + V(x)u = f(x)u^{-\gamma}, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3. \end{cases} \tag{P_0}$$

2. Preliminary results and variational framework. Throughout the paper, we use the following notations.

- $L^p(\mathbb{R}^3)$ is a Lebesgue space whose norm is denoted by $\|u\|_p = (\int_{\mathbb{R}^3} |u|^p \mathrm{d}x)^{\frac{1}{p}}$.
- For any $\alpha \in (0, 1)$, $2^*_\alpha = \frac{6}{3-2\alpha}$ is the fractional critical exponent in dimension 3.
- $\to$ denotes the strong convergence and $\rightharpoonup$ denotes the weak convergence.
- $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ for any function u.
- C and C_i $(i = 1, 2, \ldots)$ denote various positive constants, which may vary from line to line.

In view of the presence of potential function $V(x)$, we will work in the space

$$E = \left\{u \in \mathcal{D}^{s,2}(\mathbb{R}^3) : \|u\|_E < +\infty\right\},$$

equipped with inner product and the norm

$$(u, v)_E = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{3+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} V(x)u(x)v(x)\mathrm{d}x, \quad \|u\|_E = (u, u)_E^{1/2}.$$

Here, $\mathcal{D}^{s,2}(\mathbb{R}^3)$ is the homogeneous fractional Sobolev space as the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \mathrm{d}x\mathrm{d}y\right)^{1/2}.$$

Using conditions (V_1) and (V_2), we can obtain the following continuous or compact embedding theorem (see [**21**], Lemma 2.2).

LEMMA 2.1. *Let $0 < s < 1$ and suppose that (V_1) and (V_2) hold. If $p \in [2, 2^*_s]$, then the embeddings*

$$E \hookrightarrow \mathcal{D}^{s,2}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$$

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are continuous with $\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)} \le C\|u\|_E$ *for all* $u \in E$, *where* C *is a generic constant. In particular, for* $p \in [2, 2_s^*]$, *there exists a constant* $C_p > 0$ *such that* $\|u\|_p \le C_p\|u\|_E$ *for all* $u \in E$. *If* $p \in [2, 2_s^*)$, *then the embedding*

$$E \hookrightarrow L^p(\mathbb{R}^3)$$

is compact.

Since $0 < s \le t < 1$ and $4s + 2t > 3$, for $u \in H^s(\mathbb{R}^3)$, according to [21] and [7], there exists a unique $\phi_u^t \in \mathcal{D}^{t,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(\phi_u^t(x) - \phi_u^t(y))(v(x) - v(y))}{|x-y|^{3+2t}}\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^3} u^2 v\mathrm{d}x, \quad \forall v \in \mathcal{D}^{t,2}, \tag{2.1}$$

that is, ϕ_u^t is a weak solution of

$$(-\Delta)^t \phi_u^t = u^2, \quad x \in \mathbb{R}^3,$$

and the representation formula holds

$$\phi_u^t = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}}\mathrm{d}y, \quad x \in \mathbb{R}^3,$$

which is called the t-Riesz potential and $c_t = \frac{\Gamma(3-2t)}{\pi^{\frac{3}{2}} 2^{2t}\Gamma(t)}$. Substituting ϕ_u^t to the first equation of system (SP_λ), then system (SP_λ) transforms into equation (1.4). The energy functional corresponding to equation (1.4) is given by

$$I_\lambda(u) = \frac{1}{2}\|u\|_E^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_u^t |u|^2\mathrm{d}x - \frac{1}{1-\gamma}\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x. \tag{2.2}$$

For convenience, we recall the following lemma on the properties of the function ϕ_u^t from [7, 21, 29, 30], etc.

LEMMA 2.2. *For every* $u \in H^s(\mathbb{R}^3)$ *with* $4s + 2t > 3$, *we have*

(i) $\phi_u^t : H^s(\mathbb{R}^3) \to \mathcal{D}^{t,2}(\mathbb{R}^3)$ *is continuous and maps bounded sets into bounded sets;*

(ii) $\phi_u^t \ge 0$. *Moreover,* $\phi_u^t > 0$ *when* $u \ne 0$;

(iii) for $u \in E$, *one has* $\|\phi_u^t\|^2_{\mathcal{D}^{t,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \phi_u^t u^2\mathrm{d}x \le C\|u\|_E^4$, *where* C *is independent of* u;

(iv) $\phi_{\tau u}^t = \tau^2 \phi_u^t$, *for any* $\tau \in \mathbb{R}$;

(v) assume that $u_n \rightharpoonup u$ *in* E, *then*

$$\int_{\mathbb{R}^3} \phi_{u_n}^t u_n v\mathrm{d}x \to \int_{\mathbb{R}^3} \phi_u^t uv\mathrm{d}x \text{ for any } v \in E$$

and

$$\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2\mathrm{d}x \to \int_{\mathbb{R}^3} \phi_u^t u^2\mathrm{d}x;$$

(vi) for $u, v \in H^s(\mathbb{R}^3)$, $\int_{\mathbb{R}^3} (\phi_u^t u - \phi_v^t v)(u - v)\mathrm{d}x \ge \frac{1}{2}\|\phi_u^t - \phi_v^t\|^2_{\mathcal{D}^{t,2}(\mathbb{R}^3)}$.

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Proof. The proofs of (i)–(v) of Lemma 2.2 can be found in [**7**, **21**, **29**, **30**], etc. We only prove (vi). By the definitions of ϕ_u^t and ϕ_v^t, we have

$$(-\Delta)^t(\phi_u^t - \phi_v^t) = u^2 - v^2.$$

Hence,

$$\begin{aligned}
\|\phi_u^t - \phi_v^t\|^2_{\mathcal{D}^{t,2}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|[\phi_u^t(x) - \phi_v^t(x)] - [\phi_u^t(y) - \phi_v^t(y)]|^2}{|x-y|^{3+2s}}\,\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^3} (u^2 - v^2)(\phi_u^t - \phi_v^t)\mathrm{d}x \\
&= 2\int_{\mathbb{R}^3} \Big[\phi_u^t u^2 + \phi_v^t v^2 - (\phi_u^t + \phi_v^t)\frac{u^2+v^2}{2}\Big]\mathrm{d}x \\
&\le 2\int_{\mathbb{R}^3} \big[\phi_u^t u^2 + \phi_v^t v^2 - (\phi_u^t + \phi_v^t)uv\big]\mathrm{d}x \\
&= 2\int_{\mathbb{R}^3} (\phi_u^t u - \phi_v^t v)(u - v)\mathrm{d}x,
\end{aligned}$$

and so (vi) is proved. The proof of Lemma 2.2 is completed. □

3. Proofs of main results. In this section, we will prove the main results. First, we show that the functional I_λ attains the global minimizer in E for any $\lambda > 0$.

LEMMA 3.1. *Let $\lambda > 0$ and $\gamma \in (0, 1)$. Assume (V_1), (V_2), and (f_1). The functional I_λ defined in (2.2) attains its minimum in E, that is, there exists $u_\lambda \in E$ such that $I_\lambda(u_\lambda) = m_\lambda = \min_E I_\lambda < 0$.*

Proof. For $u \in E$, it follows from Lemma 2.2(ii), Hölder's inequality, and Lemma 2.1 that

$$\begin{aligned}
I_\lambda(u) &= \frac{1}{2}\|u\|_E^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_u^t |u|^2\mathrm{d}x - \frac{1}{1-\gamma}\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x \\
&\ge \frac{1}{2}\|u\|_E^2 - \frac{1}{1-\gamma}\|f\|_{\frac{2}{1+\gamma}}\Big[\int_{\mathbb{R}^3}|u|^2\mathrm{d}x\Big]^{\frac{1-\gamma}{2}} \\
&\ge \frac{1}{2}\|u\|_E^2 - \frac{1}{1-\gamma}\|f\|_{\frac{2}{1+\gamma}} C_2^{1-\gamma}\|u\|_E^{1-\gamma},
\end{aligned} \tag{3.1}$$

where we use $\lambda > 0$ and $0 < \gamma < 1$. Thus, I_λ is coercive and bounded from below on E. Therefore, $m_\lambda = \inf_E I_\lambda$ is well defined. For $\eta > 0$ and given $u \in E \setminus \{0\}$, by Lemma 2.2 (*iv*), one has

$$I_\lambda(\eta u) = \frac{\eta^2}{2}\|u\|_E^2 + \frac{\lambda\eta^4}{4}\int_{\mathbb{R}^3}\phi_u^t|u|^2\mathrm{d}x - \frac{\eta^{1-\gamma}}{1-\gamma}\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x,$$

so $I_\lambda(\eta u) < 0$ for $\eta > 0$ small enough, then $m_\lambda = \inf_E I_\lambda < 0$, and there exists a minimizing sequence $\{u_n\} \subset E$ such that $\lim_{n\to\infty} I_\lambda(u_n) = m_\lambda < 0$. Since $I_\lambda(|u_n|) \le I_\lambda(u_n)$, we could assume that $u_n \ge 0$. The coerciveness of I_λ on E shows that $\{u_n\}$ is bounded in E. Going if necessary to a subsequence, we can assume that

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$$\begin{aligned} &u_n \rightharpoonup u_\lambda, \quad \text{in } E, \\ &u_n \to u_\lambda, \quad \text{in } L^p(\mathbb{R}^3), \ p \in [2, 2_s^*), \\ &u_n \to u_\lambda, \quad \text{a.e. in } \mathbb{R}^3. \end{aligned} \tag{3.2}$$

Since $0 < \gamma < 1$ and $f \in L^{\frac{2}{1+\gamma}}(\mathbb{R}^3)$ is a nonnegative function, by Hölder's inequality, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^3} f(x)|u_n|^{1-\gamma} \mathrm{d}x - \int_{\mathbb{R}^3} f(x)|u_\lambda|^{1-\gamma} \mathrm{d}x \right| &\le \int_{\mathbb{R}^3} f(x) \left| |u_n|^{1-\gamma} - |u_\lambda|^{1-\gamma} \right| \mathrm{d}x \\ &\le \int_{\mathbb{R}^3} f(x) \, |u_n - u_\lambda|^{1-\gamma} \, \mathrm{d}x \\ &\le \|f\|_{\frac{2}{1+\gamma}} \Big[\int_{\mathbb{R}^3} |u_n - u_\lambda|^2 \, \mathrm{d}x \Big]^{\frac{1-\gamma}{2}}, \end{aligned}$$

which yields, by (3.2),

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} f(x)|u_n|^{1-\gamma} \mathrm{d}x = \int_{\mathbb{R}^3} f(x)|u_\lambda|^{1-\gamma} \mathrm{d}x. \tag{3.3}$$

Then by the weakly lower semicontinuity of the norm, Lemma 2.2(v) and (3.3), we have

$$\begin{aligned} I_\lambda(u_\lambda) &= \frac{1}{2}\|u_\lambda\|_E^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_{u_\lambda}^t |u_\lambda|^2 \mathrm{d}x - \frac{1}{1-\gamma}\int_{\mathbb{R}^3} f(x)|u_\lambda|^{1-\gamma} \mathrm{d}x \\ &\le \liminf_{n\to\infty} \left[\frac{1}{2}\|u_n\|_E^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_{u_n}^t |u_n|^2 \mathrm{d}x - \frac{1}{1-\gamma}\int_{\mathbb{R}^3} f(x)|u_n|^{1-\gamma} \mathrm{d}x \right] \\ &= \liminf_{n\to\infty} I_\lambda(u_n) = m_\lambda. \end{aligned}$$

On the other hand, $I_\lambda(u_\lambda) \ge m_\lambda$, so $I_\lambda(u_\lambda) = m_\lambda < 0$, and this ends the proof of Lemma 3.1. □

Proof of Theorem 1.1. The proof will be complete in three steps.

Step 1. *We show that for any $0 \le \psi \in E$, it holds*

$$(u_\lambda, \psi)_E + \lambda \int_{\mathbb{R}^3} \phi_{u_\lambda}^t u_\lambda \psi \, \mathrm{d}x - \int_{\mathbb{R}^3} f(x) u_\lambda^{-\gamma} \psi \mathrm{d}x \ge 0.$$

According to Lemma 3.1, $u_\lambda \ge 0$ and $u_\lambda \not\equiv 0$. For $0 \le \psi \in E$ and $\eta \ge 0$ satisfying $u_\lambda + \eta\psi \in E$, since $I_\lambda(u_\lambda) = m_\lambda = \min_E I_\lambda$, one has

$$\begin{aligned} 0 &\le I_\lambda(u_\lambda + \eta\psi) - I_\lambda(u_\lambda) \\ &= \frac{1}{2}\big[\|u_\lambda + \eta\psi\|_E^2 - \|u_\lambda\|_E^2\big] + \frac{\lambda}{4}\int_{\mathbb{R}^3} \big[\phi_{u_\lambda+\eta\psi}^t (u_\lambda + \eta\psi)^2 - \phi_{u_\lambda}^t u_\lambda^2\big]\mathrm{d}x \\ &\quad - \frac{1}{1-\gamma}\int_{\mathbb{R}^3} f(x)\big[(u_\lambda + \eta\psi)^{1-\gamma} - u_\lambda^{1-\gamma}\big]\mathrm{d}x. \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{1}{1-\gamma}\int_{\mathbb{R}^3} f(x)\big[(u_\lambda + \eta\psi)^{1-\gamma} - u_\lambda^{1-\gamma}\big]\mathrm{d}x \\ &\quad \le \frac{1}{2}\big[\|u_\lambda + \eta\psi\|_E^2 - \|u_\lambda\|_E^2\big] + \frac{\lambda}{4}\int_{\mathbb{R}^3} \big[\phi_{u_\lambda+\eta\psi}^t (u_\lambda + \eta\psi)^2 - \phi_{u_\lambda}^t u_\lambda^2\big]\mathrm{d}x. \end{aligned} \tag{3.4}$$

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Due to $\gamma \in (0, 1)$ and $f(x)$ is nonnegative, we can get

$$f(x)\big[(u_\lambda + \eta\psi)^{1-\gamma} - u_\lambda^{1-\gamma}\big] \geq 0, \quad \forall x \in \mathbb{R}^3,$$

then

$$\liminf_{\eta \to 0^+} \int_{\mathbb{R}^3} \frac{f(x)\big[(u_\lambda + \eta\psi)^{1-\gamma} - u_\lambda^{1-\gamma}\big]}{\eta} \mathrm{d}x$$

exists. Dividing (3.4) by $\eta > 0$ and passing to the liminf as $\eta \to 0^+$, then we can obtain from Fatou's lemma that

$$\begin{aligned} \int_{\mathbb{R}^3} f(x) u_\lambda^{-\gamma} \psi \mathrm{d}x &\leq \liminf_{\eta \to 0^+} \frac{1}{1-\gamma} \int_{\mathbb{R}^3} \frac{f(x)\big[(u_\lambda + \eta\psi)^{1-\gamma} - u_\lambda^{1-\gamma}\big]}{\eta} \mathrm{d}x \\ &\leq (u_\lambda, \psi)_E + \lambda \int_{\mathbb{R}^3} \phi_{u_\lambda}^t u_\lambda \psi \, \mathrm{d}x. \end{aligned} \tag{3.5}$$

Step 2. *We prove that $u_\lambda > 0$ in $\mathbb{R}^3$ and u_λ is a solution of system (SP_λ).*

For given $\delta > 0$, define $g : [-\delta, \delta] \to \mathbb{R}$ by $g(\eta) = I_\lambda(u_\lambda + \eta u_\lambda)$, then g attains its minimum at $\eta = 0$ by Lemma 3.1 which implies that

$$g'(0) = \|u_\lambda\|_E + \lambda \int_{\mathbb{R}^3} \phi_{u_\lambda}^t u_\lambda^2 \mathrm{d}x - \int_{\mathbb{R}^3} f(x) u_\lambda^{1-\gamma} \mathrm{d}x = 0. \tag{3.6}$$

For any $v \in E$ and $\varepsilon > 0$, set $\psi_\varepsilon = u_\lambda + \varepsilon v$, then

$$\begin{aligned} &(u_\lambda(x) - u_\lambda(y))(\psi_\varepsilon^+(x) - \psi_\varepsilon^+(y)) \\ &\quad = (u_\lambda(x) - u_\lambda(y))\big[\psi_\varepsilon(x) + \psi_\varepsilon^-(x) - \psi_\varepsilon(y) - \psi_\varepsilon^-(y)\big] \\ &\quad = |u_\lambda(x) - u_\lambda(y)|^2 + \varepsilon(u_\lambda(x) - u_\lambda(y))(v(x) - v(y)) \\ &\qquad + (u_\lambda(x) - u_\lambda(y))\big[\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y)\big]. \end{aligned} \tag{3.7}$$

According to the proof of Theorem 3.2 in [**5**],

$$\liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_\lambda(x) - u_\lambda(y))\big[\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y)\big]}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y \leq 0. \tag{3.8}$$

Set $\Omega_\varepsilon = \{x \in \mathbb{R}^3 : \psi_\varepsilon \leq 0\}$, and then using (3.6)–(3.7) and applying inequality (3.5) with $\psi = \psi_\varepsilon^+$ lead to

$$\begin{aligned} 0 \leq \frac{1}{\varepsilon}\Big\{ &\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_\lambda(x) - u_\lambda(y))(\psi_\varepsilon^+(x) - \psi_\varepsilon^+(y))}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} V(x) u_\lambda \psi_\varepsilon^+ \mathrm{d}x \\ &+ \lambda \int_{\mathbb{R}^3} \phi_{u_\lambda}^t u_\lambda \psi_\varepsilon^+ \, \mathrm{d}x - \int_{\mathbb{R}^3} f(x) u_\lambda^{-\gamma} \psi_\varepsilon^+ \, \mathrm{d}x \Big\} \\ = \frac{1}{\varepsilon} &\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_\lambda(x) - u_\lambda(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_\lambda(x) - u_\lambda(y))(v(x) - v(y))}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y \\ &+ \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_\lambda(x) - u_\lambda(y))\big[\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y)\big]}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y \\ &+ \frac{1}{\varepsilon}\Big(\int_{\mathbb{R}^3} - \int_{\Omega_\varepsilon} \Big)\Big\{ V(x) u_\lambda (u_\lambda + \varepsilon v) + \lambda \phi_{u_\lambda}^t u_\lambda (u_\lambda + \varepsilon v) - f(x) u_\lambda^{-\gamma} (u_\lambda + \varepsilon v) \Big\} \, \mathrm{d}x \end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{\varepsilon}\left\{\|u_\lambda\|_E^2 + \lambda\int_{\mathbb{R}^3}\phi_{u_\lambda}^t u_\lambda^2 \mathrm{d}x - \int_{\mathbb{R}^3} f(x)u_\lambda^{1-\gamma}\mathrm{d}x\right\}\\
&\quad + \left\{(u_\lambda, v)_E + \lambda\int_{\mathbb{R}^3}\phi_{u_\lambda}^t u_\lambda v\mathrm{d}x - \int_{\mathbb{R}^3} f(x)u_\lambda^{-\gamma}v\mathrm{d}x\right\}\\
&\quad - \frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\left\{V(x)u_\lambda(u_\lambda+\varepsilon v) + \lambda\phi_{u_\lambda}^t u_\lambda(u_\lambda+\varepsilon v) - f(x)u_\lambda^{-\gamma}(u_\lambda+\varepsilon v)\right\}\mathrm{d}x\\
&\quad + \frac{1}{\varepsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(u_\lambda(x)-u_\lambda(y))\left[\psi_\varepsilon^-(x)-\psi_\varepsilon^-(y)\right]}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\\
&= \left\{(u_\lambda, v)_E + \lambda\int_{\mathbb{R}^3}\phi_{u_\lambda}^t u_\lambda v\mathrm{d}x - \int_{\mathbb{R}^3} f(x)u_\lambda^{-\gamma}v\mathrm{d}x\right\}\\
&\quad - \frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\left[V(x)u_\lambda(u_\lambda+\varepsilon v) + \lambda\phi_{u_\lambda}^t u_\lambda(u_\lambda+\varepsilon v)\right]\mathrm{d}x\\
&\quad + \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} f(x)u_\lambda^{-\gamma}(u_\lambda+\varepsilon v)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(u_\lambda(x)-u_\lambda(y))\left[\psi_\varepsilon^-(x)-\psi_\varepsilon^-(y)\right]}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\\
&\le \left\{(u_\lambda, v)_E + \lambda\int_{\mathbb{R}^3}\phi_{u_\lambda}^t u_\lambda v\mathrm{d}x - \int_{\mathbb{R}^3} f(x)u_\lambda^{-\gamma}v\mathrm{d}x\right\} - \frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\left[V(x)u_\lambda^2 + \lambda\phi_{u_\lambda}^t u_\lambda^2\right]\mathrm{d}x\\
&\quad - \int_{\Omega_\varepsilon}\left[V(x)u_\lambda v + \lambda\phi_{u_\lambda}^t u_\lambda v\right]\mathrm{d}x + \frac{1}{\varepsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(u_\lambda(x)-u_\lambda(y))\left[\psi_\varepsilon^-(x)-\psi_\varepsilon^-(y)\right]}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\\
&\le \left\{(u_\lambda, v)_E + \lambda\int_{\mathbb{R}^3}\phi_{u_\lambda}^t u_\lambda v\mathrm{d}x - \int_{\mathbb{R}^3} f(x)u_\lambda^{-\gamma}v\mathrm{d}x\right\} - \int_{\Omega_\varepsilon}\left[V(x)u_\lambda v + \lambda\phi_{u_\lambda}^t u_\lambda v\right]\mathrm{d}x\\
&\quad + \frac{1}{\varepsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(u_\lambda(x)-u_\lambda(y))\left[\psi_\varepsilon^-(x)-\psi_\varepsilon^-(y)\right]}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Passing to the liminf as $\varepsilon \to 0^+$ to the above inequality and using (3.8) with the fact that $|\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0^+$, we have

$$(u_\lambda, v)_E + \lambda\int_{\mathbb{R}^3}\phi_{u_\lambda}^t u_\lambda v\mathrm{d}x - \int_{\mathbb{R}^3} f(x)u_\lambda^{-\gamma}v\mathrm{d}x \ge 0, \quad \forall v \in E.$$

This inequality also holds for $-v$, hence we obtain

$$(u_\lambda, v)_E + \lambda\int_{\mathbb{R}^3}\phi_{u_\lambda}^t u_\lambda v\mathrm{d}x - \int_{\mathbb{R}^3} f(x)u_\lambda^{-\gamma}v\mathrm{d}x = 0, \quad \forall v \in E. \tag{3.9}$$

From an argument similar to [**24**, Theorem 6.3], we know that $u_\lambda \in C_{loc}^\alpha(\mathbb{R}^3)$ for some $\alpha \in (0, s)$. On the other hand, the (3.9) implies that

$$(-\Delta)^s u_\lambda + V(x)u_\lambda + \lambda\phi_{u_\lambda}^t u_\lambda \ge 0. \tag{3.10}$$

Assume that there exists $x_0 \in \mathbb{R}^3$ such that $u_\lambda(x_0) = 0$, and then from (3.10), we have $(-\Delta)^s u_\lambda(x_0) \ge 0$. On the other hand, since $u_\lambda \ge 0$ and $u_\lambda \not\equiv 0$, we can obtain from Lemma 3.2 in [**3**] that

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$$(-\Delta)^s u_\lambda(x_0) = -\frac{C(s)}{2}\int_{\mathbb{R}^3}\frac{u_\lambda(x_0+y)+u_\lambda(x_0-y)-2u_\lambda(x_0)}{|y|^{3+2s}}\mathrm{d}y$$
$$= -\frac{C(s)}{2}\int_{\mathbb{R}^3}\frac{u_\lambda(x_0+y)+u_\lambda(x_0-y)}{|y|^{3+2s}}\mathrm{d}y < 0,$$

where $C(s) = \left(\int_{\mathbb{R}^3}\frac{1-\cos\xi_1}{|\xi|^{3+2s}}\mathrm{d}\xi\right)^{-1}$ and $\xi = (\xi_1, \xi_2, \xi_3)$. There is a contradiction. Hence, $u_\lambda > 0$ in $\mathbb{R}^3$ and $u_\lambda \in E$ is a solution of system (SP_λ).

Step 3. *We show that u_λ is a unique solution of system (SP_λ).*

Suppose $u_* \in E$ is also a solution of system (SP_λ), then we have

$$(u_*, v)_E + \lambda\int_{\mathbb{R}^3}\phi^t_{u_*}u_* v\,\mathrm{d}x - \int_{\mathbb{R}^3} f(x)u_*^{-\gamma}v\mathrm{d}x = 0, \quad \forall v \in E. \tag{3.11}$$

Taking $v = u_\lambda - u_*$ in both equations (3.9)–(3.11) and subtracting term by term, we obtain

$$0 \geq \int_{\mathbb{R}^3} f(x)(u_\lambda^{-\gamma} - u_*^{-\gamma})(u_\lambda - u_*)\mathrm{d}x$$
$$= \|u_\lambda - u_*\|_E^2 + \lambda\int_{\mathbb{R}^3}(\phi^t_{u_\lambda}u_\lambda - \phi^t_{u_*}u_*)(u_\lambda - u_*)\mathrm{d}x$$
$$\geq \|u_\lambda - u_*\|_E^2 + \frac{1}{2}\|\phi^t_{u_\lambda} - \phi^t_{u_*}\|^2_{\mathcal{D}^{t,2}(\mathbb{R}^3)} \geq \|u_\lambda - u_*\|_E^2 \geq 0,$$

where we use Lemma 2.2(vi). So $\|u_\lambda - u_*\|_E^2 = 0$, then $u_\lambda = u_*$ and u_λ is the unique solution of equation (1.4). This ends the proof of Theorem 1.1. □

Proof of Theorem 1.2. For any $\psi \in E$ and a.e. $x,\ y \in \mathbb{R}^3$, we have

$$(\psi(x)-\psi(y))(\psi^+(x)-\psi^+(y)) = \left[\psi^+(x)-\psi^-(x)-\psi^+(y)+\psi^-(y)\right](\psi^+(x)-\psi^+(y))$$
$$= |\psi^+(x)-\psi^+(y)|^2 + \psi^-(x)\psi^+(y) + \psi^-(y)\psi^+(x)$$
$$\geq |\psi^+(x)-\psi^+(y)|^2. \tag{3.12}$$

Since $u_1(x)$ and $u_2(x) \in E$ are two positive solutions to system (SP_λ) corresponding to $f_1(x)$ and $f_2(x)$, respectively, then for any $v \in E$, we have

$$(u_1, v)_E + \lambda\int_{\mathbb{R}^3}\phi^t_{u_1}u_1 v\,\mathrm{d}x = \int_{\mathbb{R}^3} f_1(x)u_1^{-\gamma}v\mathrm{d}x,$$
$$(u_2, v)_E + \lambda\int_{\mathbb{R}^3}\phi^t_{u_2}u_2 v\,\mathrm{d}x = \int_{\mathbb{R}^3} f_2(x)u_2^{-\gamma}v\mathrm{d}x.$$

By subtracting the above two equations with choosing $v = (u_2-u_1)^+ \in E$ and applying (3.12) with $\psi = u_2 - u_1$, we have

$$0 \geq \int_{\mathbb{R}^3}(f_2(x)u_2^{-\gamma} - f_1(x)u_1^{-\gamma})(u_2-u_1)^+\mathrm{d}x$$
$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{[(u_2-u_1)(x)-(u_2-u_1)(y)][(u_2-u_1)^+(x)-(u_2-u_1)^+(y)]}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y$$
$$+ \int_{\mathbb{R}^3} V(x)(u_2-u_1)(u_2-u_1)^+\mathrm{d}x + \lambda\int_{\mathbb{R}^3}(\phi^t_{u_2}u_2 - \phi^t_{u_1}u_1)(u_2-u_1)^+\mathrm{d}x$$

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$$\geq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{[(u_2-u_1)^+(x)-(u_2-u_1)^+(y)]^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} V(x)[(u_2-u_1)^+]^2\mathrm{d}x$$

$$+\lambda\int_{[u_2-u_1\geq 0]}(\phi^t_{u_2}u_2-\phi^t_{u_1}u_1)(u_2-u_1)\mathrm{d}x$$

$$= \|(u_2-u_1)^+\|_E^2 + \lambda\int_{[u_2-u_1\geq 0]}(\phi^t_{u_2}u_2-\phi^t_{u_1}u_1)(u_2-u_1)dx$$

$$\geq \|(u_2-u_1)^+\|_E^2 \geq 0,$$

where we use $0<\gamma<1$, Lemma 2.2(vi) and $f_1(x)\geq f_2(x)$ for all $x\in\mathbb{R}^3$. So $(u_2-u_1)^+\equiv 0$ and hence $u_1(x)\geq u_2(x)$ for all $x\in\mathbb{R}^3$. This completed the proof of Theorem 1.2. □

Proof of Theorem 1.3. Since $u_{\lambda_1}(x)$ and $u_{\lambda_2}(x)\in E$ are two positive solutions of system (SP_λ) corresponding to λ_1 and λ_2, respectively, then for any $v\in E$, we have

$$(u_{\lambda_1}, v)_E + \lambda_1\int_{\mathbb{R}^3}\phi^t_{u_{\lambda_1}}u_{\lambda_1}v\,\mathrm{d}x = \int_{\mathbb{R}^3} f(x)u_{\lambda_1}^{-\gamma}v\mathrm{d}x,$$

$$(u_{\lambda_2}, v)_E + \lambda_2\int_{\mathbb{R}^3}\phi^t_{u_{\lambda_2}}u_{\lambda_2}v\,\mathrm{d}x = \int_{\mathbb{R}^3} f(x)u_{\lambda_2}^{-\gamma}v\mathrm{d}x.$$

By subtracting the above two equations with choosing $v=(u_{\lambda_2}-u_{\lambda_1})^+\in E$ and applying (3.12) with $\psi=u_{\lambda_2}-u_{\lambda_1}$, we have

$$\begin{aligned}
0 &\geq \int_{\mathbb{R}^3}(f(x)u_{\lambda_2}^{-\gamma}-f(x)u_{\lambda_1}^{-\gamma})(u_{\lambda_2}-u_{\lambda_1})^+\mathrm{d}x\\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{[(u_{\lambda_2}-u_{\lambda_1})(x)-(u_{\lambda_2}-u_{\lambda_1})(y)][(u_{\lambda_2}-u_{\lambda_1})^+(x)-(u_{\lambda_2}-u_{\lambda_1})^+(y)]}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\\
&\quad+\int_{\mathbb{R}^3}V(x)(u_{\lambda_2}-u_{\lambda_1})(u_{\lambda_2}-u_{\lambda_1})^+\mathrm{d}x+\int_{\mathbb{R}^3}(\lambda_2\phi^t_{u_{\lambda_2}}u_{\lambda_2}-\lambda_1\phi^t_{u_{\lambda_1}}u_{\lambda_1})(u_{\lambda_2}-u_{\lambda_1})^+\mathrm{d}x\\
&\geq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{[(u_{\lambda_2}-u_{\lambda_1})^+(x)-(u_{\lambda_2}-u_{\lambda_1})^+(y)]^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^3}V(x)[(u_{\lambda_2}-u_{\lambda_1})^+]^2\mathrm{d}x\\
&\quad+\lambda_1\int_{[u_{\lambda_2}-u_{\lambda_1}\geq 0]}(\phi^t_{u_{\lambda_2}}u_{\lambda_2}-\phi^t_{u_{\lambda_1}}u_{\lambda_1})(u_{\lambda_2}-u_{\lambda_1})\mathrm{d}x\\
&= \|(u_{\lambda_2}-u_{\lambda_1})^+\|_E^2+\lambda_1\int_{[u_{\lambda_2}-u_{\lambda_1}\geq 0]}(\phi^t_{u_{\lambda_2}}u_{\lambda_2}-\phi^t_{u_{\lambda_1}}u_{\lambda_1})(u_{\lambda_2}-u_{\lambda_1})\mathrm{d}x\\
&\geq \|(u_{\lambda_2}-u_{\lambda_1})^+\|_E^2\geq 0,
\end{aligned}$$

where we use $0<\gamma<1$, Lemma 2.2(vi) and $0<\lambda_1\leq\lambda_2$. So $(u_{\lambda_2}-u_{\lambda_1})^+\equiv 0$ and hence $u_{\lambda_1}(x)\geq u_{\lambda_2}(x)$ for all $x\in\mathbb{R}^3$. This completed the proof of Theorem 1.3. □

Proof of Theorem 1.4. In the proofs of Lemma 3.1 and Theorem 1.1, $\lambda=0$ is allowed. Hence, under the assumptions of Theorem 1.4, there exists a unique positive solution $w_0\in E$ to problem (P_0), that is, for any $v\in E$, it holds

$$(w_0, v)_E = \int_{\mathbb{R}^3} f(x)w_0^{-\gamma}v\mathrm{d}x. \tag{3.13}$$

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For every vanishing sequence $\{\lambda_n\} \subset (0, 1)$, since $\{u_{\lambda_n}\}$ is a positive solution sequence to system (SP_λ) provided by Theorem 1.1, then for every $v \in E$ and $n \in \mathbb{N}$, one has

$$(u_{\lambda_n}, v)_E + \lambda_n \int_{\mathbb{R}^3} \phi^t_{u_{\lambda_n}} u_{\lambda_n} v \, \mathrm{d}x = \int_{\mathbb{R}^3} f(x) u_{\lambda_n}^{-\gamma} v \mathrm{d}x. \tag{3.14}$$

By Lemma 3.1 and the proof of Theorem 1.1, we have $I_{\lambda_n}(u_{\lambda_n}) = m_{\lambda_n} < 0$, and then $\{u_{\lambda_n}\}$ is bounded in E since I_{λ_n} is coercive according to (3.1). So, there exists a subsequence of $\{u_{\lambda_n}\}$ (still denoted by $\{u_{\lambda_n}\}$) and a nonnegative function $u_0 \in E$ such that

$$\begin{aligned} &u_{\lambda_n} \rightharpoonup u_0, \quad \text{in } E, \\ &u_{\lambda_n} \to u_0, \quad \text{in } L^p(\mathbb{R}^3), \ \ p \in [2, 2^*_s), \\ &u_{\lambda_n} \to u_0, \quad \text{a.e. in } \mathbb{R}^3. \end{aligned}$$

Choosing $v = u_{\lambda_n}$ in (3.14) and passing to the liminf as $n \to \infty$, one can get from Lemma 2.2(v), (3.3), and the weakly lower semicontinuity of the norm that

$$\|u_0\|_E^2 \le \int_{\mathbb{R}^3} f(x) u_0^{1-\gamma} \mathrm{d}x. \tag{3.15}$$

On the other hand, passing to the liminf as $n \to \infty$ in (3.14) and using Lemma 2.2(v), Fatou's lemma, for any $0 \le v \in E$, we have

$$(u_0, v)_E \ge \int_{\mathbb{R}^3} f(x) u_0^{-\gamma} v \mathrm{d}x. \tag{3.16}$$

Similar to Step 1 in the proof of Theorem 1.1, we have $u_0 > 0$ in $\mathbb{R}^3$. Then, it follows from choosing $v = u_0 \in E$ in (3.16) that

$$\|u_0\|_E^2 \ge \int_{\mathbb{R}^3} f(x) u_0^{1-\gamma} \mathrm{d}x.$$

This combined with (3.15) leads to

$$\|u_0\|_E^2 = \int_{\mathbb{R}^3} f(x) u_0^{1-\gamma} \mathrm{d}x \text{ and } u_{\lambda_n} \to u_0 \text{ in } E. \tag{3.17}$$

Using (3.17) and similar to Step 2 in the proof of Theorem 1.1, we can further obtain that $u_0 \in E$ is also a solution of problem (P_0). By the uniqueness of solution to problem (P_0), $u_0 = w_0$. Hence, $u_{\lambda_n} \to w_0$ in E, where w_0 is the unique positive solution to problem (P_0). This completed the proof of Theorem 1.4. □

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