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Growth rates for geometric complexities and counting functions in polygonal billiards

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Abstract. We introduce a new method for estimating the growth of various quantities arising in dynamical systems. Applying our method to polygonal billiards on surfaces of constant curvature, we obtain estimates almost everywhere on direction complexities and position complexities. As a byproduct, we recover the power bounds of Boshernitzan on the number of billiard orbits between almost all pairs of points in a planar polygon [M. Boshernitzan. Private letter to H. Masur (1986)].

0. *Introduction and overview*

The complexity of a dynamical system is measured with respect to a coding of its orbits. The coding, in turn, is determined by partitioning the phase space of the system into appropriately chosen subsets. For dynamical systems with singularities, such as polygonal billiards, connected components in the complement to the singular set yield a natural partition. The convexity of its atoms with respect to the geodesic structure in the phase space imposed by geometric optics is crucial in the study of billiard complexity [**5**].

In the present study, as well as in [**5**], P is a geodesic polygon in a surface of constant curvature. Let, for concreteness, P be a planar polygon. We denote by $f_P(n)$ the number of words of length n generated by coding billiard orbits by visited components. When P is simply connected, this coincides with the coding by sides in P. It is known that $f_P(n)$ is subexponential in n (see [**4, 8**]), and for general P no better bound is known. If P is a *rational polygon* (i.e. its angles are commensurable with π), then $f_P(n) = O(n^3)$ (see [**5**]). It is expected that for any planar polygon $f_P(n) = O(n^d)$ (see [**3**]).

In order to advance the understanding of billiard complexity, we introduce the notion of *partial complexities*. Let Ψ be the phase space, and let $\mathcal{P}$ be the defining partition. By iteration, we obtain an increasing tower $\mathcal{P}_n$ of partitions; the full complexity is $f(n) = |\mathcal{P}_n|$. If $R \subset \Phi$, let $\mathcal{P}_n(R)$ be the induced tower of its partitions. The partial

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complexity based on R is $f_R(n) = |\mathcal{P}_n(R)|$. Certain partial complexities have been studied before.

In this work we introduce a new general approach to estimating partial complexities. The setting is as follows. There is a family of subsets $R_\theta, \theta \in \Theta$, foliating the phase space. Let $f_\theta(n)$ be the partial complexity with the base R_θ. Let $g_\theta(n)$ be the *counting function for singular billiard orbits* starting from R_θ. In certain situations, $f_\theta(n)$ and $g_\theta(n)$ have the same growth, as $n \to \infty$. See §4.

Suppose that we have a bound on the *average counting function* $G(n) = \int_\Theta g_\theta(n)$. The Tchebysheff inequality and the zero–one law yield bounds for individual $g_\theta(n)$ valid for almost all θ. See §2. Combined with the preceding remarks, these yield estimates on partial complexities for almost all values of the parameter.

In this work we implement the general scheme outlined above for polygonal billiards. We now describe the contents of the paper in more detail.

In §1 we investigate counting functions and their averages. We establish the relevant framework in sufficient generality, with a view towards a broad range of geometric–dynamic applications. The main results are Propositions 1 and 2. They yield geometric formulas for the averages of counting functions, valid under mild assumptions of transversality type.

Section 2 is analytic. The setting is as follows. There is a family of positive functions, $g_\theta(p)$, of non-negative argument (p takes integer values and real values in the discrete and continuous cases, respectively), depending on the parameter $\theta \in \Theta$. Set $G(p) = \int_\Theta g_\theta(p)\, d\theta$. From upper bounds on $G(p)$ we derive estimates on individual $g_\theta(p)$; they are valid for almost all θ. Precise formulations depend on the details of a situation. See Propositions 3 and 4.

Section 3 sets the stage for applications to billiard dynamics. Our billiard table is a geodesic polygon, P, in a simply connected surface of constant curvature. There are two versions of billiard dynamics: the *billiard flow* and the *billiard map*. For our discussion of partial complexities, it is convenient to treat them separately. Accordingly, §3 consists of several subsections, each dealing with a particular partial complexity for a particular version of billiard dynamics.

We use two geometric parameters for partial complexities: the direction and the position. The *direction complexity* describes how the set of phase points starting in the same direction splits directionwise after bouncing repeatedly off of the sides of P. The direction complexity is defined for planar polygons. The *position complexity* concerns the splitting of beams of billiard orbits emanating from a point of P. It is always defined.

In each subsection we introduce a particular counting function and check the assumptions of §1; then we evaluate the integral over the parameter space, that is, we compute the average counting functions. It turns out that these have intriguing geometric meanings. The results from §3 are as follows. Let $G_P(l)$ be the average position counting function for the billiard flow in a geodesic polygon P. For planar polygons we have $G_P(l) = c_0(P)l^2$. See Corollary 2 in §3.2. This formula has been stated in† [**1**]. For spherical polygons we have $G_P(l) = c_+(P)l + c'_+(P)f(l)$ where f is a

† We thank M. Boshernitzan and S. Troubetzkoy for pointing this out to us.

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universal periodic function. See Corollary 3 in §3.4. For polygons in $\mathbb{H}^2$ we have $G_P(l) = c_-(P)\cosh l$. See Corollary 4 in §3.5. The constants in these formulas depend only on the numbers of sides and interior obstacles for the polygon in question.

In §4 we obtain general relationships between partial complexities with one-dimensional bases and counting functions. The main result of this section is Proposition 5. It says that if the base is one-dimensional, then the difference between the partial complexity and the counting function is bounded, as time goes to infinity. Other assumptions on the bases have to do with convexity in the phase space. The framework of this section is that of *piecewise convex transformations* [**5**].

In §5 we return to polygonal billiards. Combining the material of the preceding sections, we obtain bounds on the position and direction complexities for the billiard flow and the billiard map. Some of our results are as follows. Let P be a Euclidean polygon. Let $\theta \in S^1$ (respectively $z \in P$) be any direction (respectively position). Let $fd_\theta(n)$ (respectively $h_z(l)$) be the direction complexity for the billiard map (respectively position complexity for the billiard flow). Then for almost all directions θ (respectively for almost all positions z) we have $fd_\theta(n) = O(n^{1+\varepsilon})$ (respectively $h_z(l) = O(l^{2+\varepsilon})$), where $\varepsilon > 0$ is arbitrary. See Corollaries 6 and 8. Now let P be a spherical polygon, and let $h_z(l)$ be the position complexity for the billiard flow in P. Then for almost every $z \in P$ there is a $C = C(z)$ and there are arbitrarily large l such that $h_z(l) \leq Cl$. See Corollary 10. For any $\varepsilon > 0$ and almost every $z \in P$ we have $h_z(l) = O(l^{1+\varepsilon})$. See Corollary 11.

In the study of polygonal billiards the device of *unfolding billiard orbits* is indispensable [**3**]. If $P \subset M$, and β is a billiard orbit in P, its unfolding is a geodesic in M. Several arguments in §3 use the technique of lifting billiard orbits to the *universal covering space*† of P. This notion was not written up in the billiard literature. In Appendix A we expose the relevant material. Corollary A1 of Proposition A1 computes the pullbacks of Lebesgue measures by unfolding maps. These formulas are crucial for §3.

To conclude the introduction, we now briefly survey the relevant literature. The subexponential growth of (full) billiard complexity for arbitrary Euclidean polygons is established in [**4, 8**]. Both approaches are indirect, in that they do not yield explicit bounds. Explicit bounds are known only for rational Euclidean polygons; the complexity is cubic in this case [**5**]. The arguments in [**5**] rely on a theorem from [**10**]; it says that the number of billiard orbits between corners in a rational polygon grows quadratically with the length. In our terminology, it means that the position counting function satisfies $g_z(l) = O(l^2)$. Here $z \in P$ is a corner in a rational planar polygon. By comparison, our Corollary 8 and Proposition 5 yield that $g_z(l) = O(l^{2+\varepsilon})$ for any $\varepsilon > 0$ and almost all z in an arbitrary planar polygon. The directional complexity $fd_\theta(n)$ has been studied in [**6, 7**]. The work [**7**] concerns the directional complexity for the billiard in a rational, planar polygon P. Assume that P is convex. Then [**7**] derives an explicit formula for $fd_\theta(n)$, valid for *minimal directions*. (The set of non-minimal directions is countable.) By this formula, $fd_\theta(n) = O(n)$. On the other hand, in [**6**] it was shown that $fd_\theta(n) = O(n^d)$ for any $P \subset \mathbb{R}^2$ and an arbitrary θ. (The degree does not depend on θ.) Our Corollary 6 says that for any planar polygon, any $\varepsilon > 0$ and almost all directions θ, we have $fd_\theta(n) = O(n^{1+\varepsilon})$.

† Not to be confused with the concept of universal covering space in topology.

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It is plausible that the bounds from Corollaries 8, 6, etc., hold for any point, any direction, etc.

1. *Averages of counting functions*

In this section we introduce the framework of counting functions in differentiable dynamics. We apply it to the billiard dynamics later on. Our phase spaces are 'manifolds'. By this we mean compact manifolds, possibly with boundaries, corners and singular points. Our setting involves: (i) a foliation of the phase space by closed submanifolds that are fibers for a projection onto a manifold of smaller dimension; (ii) a submanifold in the phase space, transversal to the fibers; (iii) a *weight function* on the product of the phase space and the time. (See the details below.) The dynamics in question may be discrete or continuous. We consider the two cases separately. The two subsections that follow are parallel; the expositions differ in technical details.

1.1. *Discrete dynamics.*

In what follows we use $\mathbb{Z}_+$ (respectively $\mathbb{Z}_-$) to denote the set of non-negative (respectively non-positive) integers. Let $T : X \to X$, $T^{-1} : X \to X$ be piecewise diffeomorphisms such that the following conditions are satisfied.

(A 1) There is a fibration $\eta : X \to \Theta$ whose base is a compact manifold and whose fibers $R_\theta = \eta^{-1}(\theta)$ are compact submanifolds, such that $\dim(R_\theta) = \dim(X) - \dim(\Theta)$. We use the notation $X = \bigcup_{\theta\in\Theta} R_\theta$.

(A 2) There is a closed submanifold, $Y \subset X$, $\dim(Y) = \dim(\Theta)$, such that for $k \in \mathbb{Z}_+$ the manifolds $T^{-k}(Y)$ are transversal to the fibers R_θ.

(A 3) There is a *weight function*, that is, a continuous, non-negative function $w(x, t)$ on $X \times \mathbb{Z}_+$. In special cases, the weight function w depends only on time, for example, $w = \chi_n$, the indicator function of $[0, n - 1]$.

Remark 1. Condition (A 2) may be weakened, as follows.

(A 2)$'$. There is a closed submanifold, $Y \subset X$, and a set $\Theta_{\text{ex}} \subset \Theta$ of measure zero such that for $k \in \mathbb{Z}_+$ and $\theta \in \Theta \setminus \Theta_{\text{ex}}$ the manifolds $T^{-k}(Y)$ and R_θ are transversal.

All of our results remain valid if we replace condition (A 2) by the weaker condition (A 2)$'$. However, in our applications to polygonal billiards, condition (A 2) always holds, with possible exceptions for polygons on surfaces of positive curvature. See §3.4. To simplify the exposition, we assume in the following that $\Theta_{\text{ex}} = \emptyset$.

In view of condition (A 2), $\Gamma(\theta) = \{(x, k) \mid x \in R_\theta, k \in \mathbb{Z}_+, T^k(x) \in Y\}$ is a countable (at most) set. The sets $\Gamma_k(\theta) = \{(x, k) \mid x \in R_\theta, T^k(x) \in Y\}$ are finite for all $k \in \mathbb{Z}_+$, and $\Gamma(\theta) = \cup\Gamma_k(\theta)$.

We define the *weighted counting function* by

$$g(\theta; w) = \sum_{(x,k)\in\Gamma(\theta)} w(x, k). \tag{1}$$

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The *pure counting function* $g_n(\theta)$ corresponds to the weight $w = \chi_n$. We have

$$g_n(\theta) = \sum_{k=0}^{n} |\Gamma_k(\theta)|. \tag{2}$$

PROPOSITION 1. *Let $d\theta$, dy be finite, Lebesgue-class measures on Θ, Y, respectively. Then for $k \in \mathbb{Z}_+$ there are functions $r_k(\cdot) \geq 0$ on Y, determined by the data contained in (A 1) and (A 2) alone, such that*

$$\int_\Theta g(\theta; w)\, d\theta = \int_Y \left\{ \sum_{k \in \mathbb{Z}_+} w(T^{-k}(y), k) r_k(y) \right\} dy. \tag{3}$$

Proof. For any $k \geq 0$ set $f_k = \eta \circ T^{-k} : Y \to \Theta$. By conditions (A 1) and (A 2), f_k is a local diffeomorphism. Therefore, $f_k^*(d\theta) = r_k(y)\, dy$.

It suffices to establish (3) for the special case when $w(x, i) = 0$ if $i \neq k$. A point $x \in X$ contributes to the integral on the left-hand side of (3) if and only if $T^k(x) \in Y$ or, equivalently, if $\eta(x) = f_k(y)$, $y \in Y$. The claim follows by a straightforward change of variables. □

1.2. *Continuous dynamics.* Let $b^t : \Psi \to \Psi$ be a flow of piecewise diffeomorphisms on a phase space Ψ with the following data.

(B 1) There is a fibration $q : \Psi \to Z$ with a compact base and fibers $q^{-1}(z) = R_z \subset \Psi$, transversal to the flow. We use the notation $\Psi = \bigcup_{z \in Z} R_z$.

(B 2) There is a closed submanifold, $M \subset \Psi$, $\dim(M) = \dim(Z) - 1$, transversal to the flow, and such that $N = \bigcup_{t \in \mathbb{R}_+} b^{-t} \cdot M$ is transversal to the fibers R_z.

(B 3) There is a *weight function*, that is, a continuous, non-negative function $w(x, t)$ on $\Psi \times \mathbb{R}_+$. In special cases, w depends only on time, for example, $w = \chi_l$, the indicator function of $[0, l]$.

Remark 2. Condition (B 2) may be weakened, as follows.

(B 2)′. There is a closed submanifold, $M \subset \Psi$, $\dim(M) = \dim(Z) - 1$, transversal to the flow, and such that $N = \bigcup_{t \in \mathbb{R}_+} b^{-t} \cdot M$ is transversal to the fibers R_z, except for a set $Z_{\text{ex}} \subset Z$ of measure zero where the transversality fails.

All of our results remain valid if we replace condition (B 2) by the weaker condition (B 2)′. However, in our applications to polygonal billiards, condition (B 2) always holds, with possible exceptions for polygons on surfaces of positive curvature. See §3.4. To simplify the exposition, we assume in the following that $Z_{\text{ex}} = \emptyset$.

In view of condition (B 2), $G(z) = \{(x, t) \mid x \in R_z, 0 \leq t, b^t(x) \in M\}$ is a countable (at most) set. The sets $G_l(z) = \{(x, t) \mid x \in R_z, 0 \leq t \leq l, b^t(x) \in M\}$ are finite for all $l \in \mathbb{R}_+$, and $G(z) = \cup G_l(z)$.

We define the *weighted counting function* by

$$g(z; w) = \sum_{(x,t) \in G(z)} w(x, t). \tag{4}$$

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The *pure counting function* $g_l(z)$ corresponds to the weight $w = \chi_l$. We have

$$g_l(z) = |G_l(z)|. \tag{5}$$

PROPOSITION 2. *Let dz, dm be finite, Lebesgue-class measures on Z, M, respectively; let dt be the Lebesgue measure on $\mathbb{R}$. Then there exists a positive function $r(\cdot)$ on $M \times \mathbb{R}_+$, determined by the data contained in (B 1) and (B 2) alone, and such that*

$$\int_Z g(z;\, w)\, dz = \int_{M\times\mathbb{R}_+} w(b^{-t}\cdot m,\, t) r(m,\, t)\, dm\, dt. \tag{6}$$

Proof. We define the mapping $f : M \times \mathbb{R}_+ \to Z$ by $f = q \circ b^{-t}$. By conditions (B 1) and (B 2), f has full rank almost everywhere. The pull-back by f of dz is absolutely continuous with respect to $dm\, dt$, hence $f^*(dz) = r(m,\, t)\, dm\, dt$.

For $0 < l$ set $w_l(x,\, t) = w(x,\, t)\chi_l(t)$, and let $g_l(z;\, w)$ be the corresponding counting function. Set $I_l(w) = \int_Z g_l(z;\, w)\, dz$. A point, $x \in \Psi$, contributes to $I_l(w)$ if and only if $x \in \varphi(M \times [0,\, l])$. Under the change of variables $dz = d(q \circ \varphi(m,\, t)) = r(m,\, t)\, dm\, dt$, we have

$$I_l(w) = \int_{M\times[0,l]} w(b^{-t}\cdot m,\, t) r(m,\, t)\, dm\, dt.$$

In the limit $l \to \infty$, we obtain the claim. □

1.3. *Special cases.* We discuss a few special cases of Propositions 1 and 2. First, we give the discrete version. The function $g_n(\theta)$ counts the number of visits in Y of points $x \in R_\theta$ during the first n steps of their journey. Set $\rho_k = \int_Y r_k(y)\, dy$, and $R_n = \sum_{k=0}^{n-1} \rho_k$. Then ρ_k is the volume of $Y_k = T^{-k}(Y)$ with respect to the measure $\eta^*(d\theta)$. Proposition 1 yields

$$\int_\Theta g_n(\theta)\, d\theta = R_n. \tag{7}$$

In the continuous case the function $g_l(z)$ counts the number of visits in M of orbits $b^t \cdot x,\ x \in R_z$, during the period $0 \le t \le l$. Let $R(l)$ be the volume of the manifold $N_l \subset \Psi$ with respect to the measure $q^*(dz)$. Proposition 2 yields

$$\int_Z g_l(z)\, dz = R(l). \tag{8}$$

2. *Bounds on counting functions*

In this section we analyze the setting of §1 from the measure-theoretic viewpoint. This allows us to obtain pointwise upper bounds on counting functions in a broad spectrum of situations.

Let $(X,\, \mu)$ be a finite measure space. Let $f(x;\, t)$ (for $t \in \mathbb{R}_+$) be a family of non-negative L^1 functions on X. Set

$$F(t) = \int_X f(x;\, t)\, d\mu(x). \tag{9}$$

LEMMA 1. *For almost every $x \in X$ there exists $C = C(x) > 0$ such that for arbitrarily large n there is $t \ge n$ satisfying $f(x,\, t) < CF(t)$.*

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Proof. For $0 < C$ and $n \in \mathbb{Z}_+$ let

$$B_n(C) = \{x \in X \mid CF(t) < f(x; t)\ \forall t > n\},$$

and set

$$B(C) = \bigcup_{n\in\mathbb{N}} B_n(C).$$

Integrating the inequality above, we obtain $\mu(B_n(C)) \le C^{-1}$ for any n. Thus, $\mu(B(C)) \le C^{-1}$ and hence $\mu(\bigcap_{C\in\mathbb{R}_+} B(C)) = 0$. However, $\bigcap_{C\in\mathbb{R}_+} B(C) \subset X$ is the complement to the set of points $x \in X$ satisfying the hypothesis of the lemma. □

Let the setting be as in Lemma 1. In addition, we suppose that (i) the functions $f(x; t)$ are non-decreasing in t and (ii) $F(t) \to \infty$.

LEMMA 2. *Let $\varepsilon > 0$ be arbitrary. Then for almost every $x \in X$ there exists $T = T(x, \varepsilon) > 0$ such that for all $t > T$ we have*

$$f(x; t) \le F(t)(1 + \log(1 + F(t)))^{1+\varepsilon}. \tag{10}$$

Proof. Denote by $f(x; t^-)$ (respectively $F(t^-)$) the limits of $f(x; s)$ (respectively $F(s)$), as $s \to t$ from the left. For $n \in \mathbb{Z}_+$ set $t_n = \inf\{t \mid F(t) \ge 2^n\}$. Then $F(t_{(n+1)}{}^-) \le 2F(t_n)$. Let $A_n \subset X$ be the set of points satisfying the inequality

$$f(x; t_n^-) \le \tfrac{1}{2}F(t_n{}^-)(1 + \log(1 + \tfrac{1}{2}F(t_n{}^-)))^{1+\epsilon}. \tag{11}$$

It suffices to prove that the set $\bigcup_{n\in\mathbb{Z}_+} \bigcap_{k>n} A_k$ has full measure. Indeed, for $x \in A_n$ and $t \in [t_n, t_{n+1})$ we have

$$\begin{aligned} f(x; t) \le f(x; t_{n+1}^-) &\le \tfrac{1}{2}F(t_{n+1}{}^-)(1 + \log(1 + \tfrac{1}{2}F(t_{n+1}{}^-)))^{1+\varepsilon} \\ &\le F(t_n)(1 + \log(1 + F(t_n)))^{1+\varepsilon} \le F(t)(1 + \log(1 + F(t)))^{1+\varepsilon}. \end{aligned}$$

Thus, the points $x \in \bigcup_{n\in\mathbb{Z}_+} \bigcap_{k>n} A_k$ have the property (10).

If $B_n \subset X$ is any sequence of sets, we set $\limsup_{n\to\infty} B_n = \bigcap_{i\in\mathbb{Z}_+} \bigcup_{j>i} B_j$. Let B_n be the complement of A_n in X. Then $\limsup_{n\to\infty} B_n$ is the complement to $\bigcup_{n\in\mathbb{Z}_+} \bigcap_{k>n} A_k$. It remains to prove that $\mu(\limsup_{n\to\infty} B_n) = 0$.

By the Tchebysheff inequality, we have

$$\mu(B_n) \le 2(1 + \log(1 + \tfrac{1}{2}F(t_n{}^-)))^{-(1+\varepsilon)}. \tag{12}$$

Set $\mu_n = \mu(B_n)$. Suppose first that F is a continuous function. Then $F(t_n{}^-) = F(t_n) = 2^n$. By (12)

$$\mu_n \le 2(1 + \log(1 + 2^n))^{-(1+\varepsilon)},$$

hence, the series $\sum \mu_n$ converges. Since

$$\mu\left(\limsup_{n\to\infty} B_n\right) \le \sum_{n_0}^{\infty} \mu_n$$

for any n_0, the claim follows.

In general, F need not be continuous. It is thus possible that $t_n = t_{n+1}$ for some n, implying that $B_n = B_{n+1}$. From the series $\sum \mu_n$ we drop the terms μ_n such that $B_n = B_{n-1}$. By (12), the remaining terms satisfy

$$\mu_n \le 2(1 + \log(1 + 2^{n-2}))^{-(1+\varepsilon)}.$$

Now the preceding argument applies. □

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In §§3 and 5 we apply these results in the billiard setting. In §3 we estimate the integrals (9), hence the bounds provided by Lemmas 1 and 2 will be more specific. The propositions below anticipate these applications.

PROPOSITION 3. *Let the setting and the assumptions be as in Lemma 2. Let* $0 < \varepsilon$ *be arbitrary.*

(1) *Let* $F(t) = O(t^p)$ *for* $0 < p$. *Then for almost every* $x \in X$ *we have* $f(x; t) = O(t^{p+\varepsilon})$.

(2) *Let* $F(t) = O(e^{at})$ *for* $0 < a$. *Then for almost every* $x \in X$ *we have* $f(x; t) = O(e^{(a+\varepsilon)t})$.

Proof. The first claim is immediate from Lemma 2 and $(\log t)^{1+\varepsilon} = o(t^{\varepsilon})$. The second claim follows in the same way from $t^{1+\varepsilon} = o(e^{\varepsilon t})$. □

For applications to the billiard map we need a counterpart of Proposition 3 for the case of discrete time. We state this below. Its proof is analogous to the proof of Proposition 3. Moreover, the discrete time case may be formally reduced to the continuous time case. We leave the details to the reader.

Let (X, μ) be a finite measure space. Let $f(x; n)$ be a sequence of non-negative L^1 functions on X such that for every $x \in X$ the numerical sequence $f(x; n)$ is non-decreasing. Set $F(n) = \int_X f(x; n)\, d\mu$.

PROPOSITION 4. *Let* $0 < \varepsilon$ *be arbitrary. Then the following claims hold.*

(1) *Let* $F(n) = O(n^p)$ *for* $0 < p$. *Then for almost every* $x \in X$ *we have* $f(x; n) = O(n^{p+\varepsilon})$.

(2) *Let* $F(n) = O(e^{an})$ *for* $0 < a$. *Then for almost every* $x \in X$ *we have* $f(x; n) = O(e^{(a+\varepsilon)n})$.

Remark 3. All of the bounds $f(\cdot) = O(\cdot)$ in the above propositions are equivalent to the formally stronger bounds $f(\cdot) = o(\cdot)$.

3. *Counting functions for polygonal billiard*

We now apply the preceding material to billiard dynamics. Our billiard table is a geodesic polygon in the Euclidean plane $\mathbb{R}^2$, in the round sphere $\mathbb{S}^2$ or in the hyperbolic plane $\mathbb{H}^2$. We refer to [**5**] and Appendix A for the background. Throughout this section, we use the following notation. We denote by dz the Lebesgue measure on the surface in question. If P is a polygon, we denote by $K(P)$ its set of corners. If $v \in K(P)$, we denote its angle by $\alpha(v)$. We do not require that P be simply connected. If p and q are the numbers of corners and obstacles in P, we set $\kappa(P) = p + 2q - 2$. Thus, P is simply connected if and only if $q = 0$, that is, if $\kappa(P) = p - 2$.

3.1. *Direction counting functions for billiard maps in Euclidean polygons.*

Let $P \subset \mathbb{R}^2$ be a Euclidean polygon, and let $T : X(P) \to X(P)$ be the *billiard map*. Elements of the phase space $X = X(P)$ are oriented geodesic segments in $\mathbb{R}^2$ with endpoints in ∂P. A segment $x \in X$ ending in a corner of P is *singular*; the element Tx is not well defined.

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A billiard orbit $x, Tx, \ldots, T^{k-1}x$ is a *singular orbit of length* k if $T^{k-1}x$ is the first singular element in the sequence.

Assigning to $x \in X$ its *direction*, $\eta(x) \in S^1$, we obtain a fibration $\eta : X \to S^1$ with fibers $R_\theta \subset X$. See Figure 1. Let v be an arbitrary corner of P. We define $gd_\theta(n; v)$ as the counting function for phase points with direction θ whose orbits of length $k \leq n$ hit the corner v. See (13).

THEOREM 1. *Let $P \subset \mathbb{R}^2$ be an arbitrary polygon. Let $d\theta$ be the Lebesgue measure on S^1. For any $K \subset K(P)$ we have*

$$\int_{S^1} \sum_{v \in K} gd_\theta(n; v)\, d\theta = \left(\sum_{v \in K} \alpha(v) \right) n. \tag{13}$$

Proof. It suffices to prove the claim for a singleton, $K = \{v\}$. Let $Y = Y(v) \subset X$ be the set of segments $x \in X$ ending at v. Let dy be the angular measure on Y. These data fit into the setting of §1.1; $gd_\theta(n; v)$ is a pure counting function.

Let $B(z, \alpha)$ be a conical beam of light with the apex angle α, emanating from $z \in \mathbb{R}^2$. After reflecting in ∂P, it splits into a finite number of beams $B(z_i, \alpha_i)$ satisfying $\sum \alpha_i = \alpha$. (The *conservation of light volume* is a consequence of the flatness of ∂P.)

By the preceding remark, the functions $r_k(\cdot)$ of Proposition 1 satisfy $r_k(\cdot) \equiv 1$. The claim now follows from the special case of Proposition 1 considered in §1.3. □

COROLLARY 1. *Let $P \subset \mathbb{R}^2$ be an arbitrary polygon. Then*

$$\int_{S^1} gd_\theta(n)\, d\theta = \pi \kappa(P) n. \tag{14}$$

Proof. This follows from Theorem 1 via $\sum_{v \in K(P)} \alpha(v) = (p + 2q - 2)\pi$. □

3.2. *Position counting functions for billiard flows in Euclidean polygons.* Let $b^t : \Psi \to \Psi$ be the billiard flow for $P \subset \mathbb{R}^2$. For $z \in P$ and $v \in K(P)$ let $gc_z(l; v)$ be the number of billiard flow orbits that start from $z \in P$ and wind up at v by time l. Then $gc_z(l) = \sum_{v \in K(P)} gc_z(l; v)$ is the number of singular billiard orbits of length at most l starting from z. This is the *position counting function for the billiard flow* of P.

THEOREM 2. *Let $P \subset \mathbb{R}^2$ be a Euclidean polygon. For any $K \subset K(P)$ we have*

$$\int_P \sum_{v \in K} gc_z(l; v)\, dz = \frac{1}{2} \left(\sum_{v \in K} \alpha(v) \right) l^2. \tag{15}$$

Proof. It suffices to prove the claim for $K = \{v\}$. We view elements of Ψ as pairs (z, θ) where $z \in P$ is the basepoint, and θ is the direction. Let $M = \{(v, \theta) \mid (v, -\theta) \in \Psi\}$. Let $q : \Psi \to P$ be the obvious projection. Its fibers R_z are the base sets for the counting functions $gc_z(l; v)$. See Figure 1. Set $w = \chi_l$. These data satisfy the assumptions of Proposition 2; $gc_z(l; v)$ is a pure counting function.

We set dm to be the angular measure, and compute the function $r(m, t)$ in (6). By Corollary A1 in Appendix A, $r = t\chi_l$. Proposition 2 implies the claim. □

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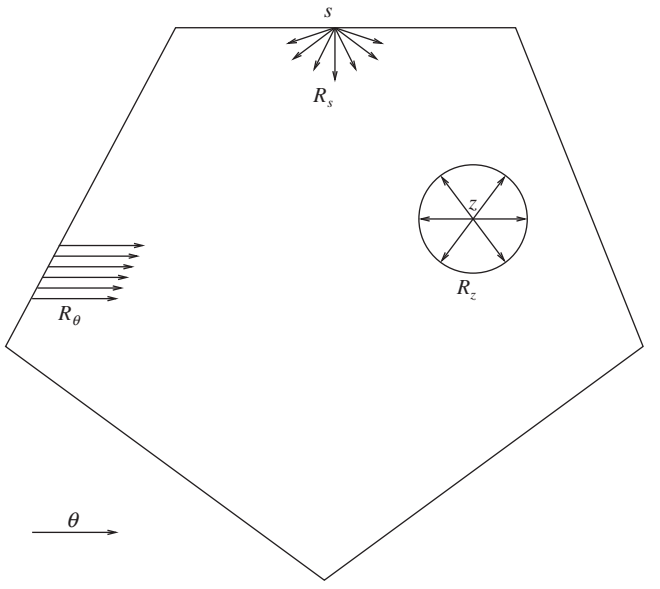


FIGURE 1. Base sets for the billiard partial complexities and counting functions.

When $K = K(P)$, the left-hand side in (15) is the average of the position counting function. The argument of Corollary 1 yields the following.

COROLLARY 2. *Let $P \subset \mathbb{R}^2$ be an arbitrary polygon. Then*

$$\int_P gc_z(l)\, dz = \frac{1}{2}\pi\kappa(P)l^2. \tag{16}$$

3.3. *Position counting functions for billiard maps in Euclidean polygons.* We now discuss two billiard map analogs of the preceding example. Let $P \subset \mathbb{R}^2$ be a Euclidean polygon, and let $T : X(P) \to X(P)$ be the billiard map. The phase space $X = X(P)$ consists of pairs (s, α) where s is the arclength parameter on ∂P, and $0 < \alpha < \pi$ is the *outgoing angle*. See [**3, 5**] and Appendix A for details. An orbit $x, Tx, \ldots, T^{k-1}x$ is singular and of (combinatorial) length k if its last segment ends at a corner of P.

Let $s \in \partial P$, $v \in K(P)$. Define $GD_s(n; v)$ to be the set of phase points $(s, \alpha) \in X$ whose orbits of length less than or equal to n end at v. Set

$$gd_s(n; v) = |GD_s(n; v)|, \quad god_s(n; v) = \sum_{(s,\alpha)\in GD_s(n;v)} \sin\alpha.$$

The expressions

$$gd_s(n) = \sum_{v\in K(P)} gd_s(n; v), \quad god_s(n) = \sum_{v\in K(P)} god_s(n; v)$$

are the *pure position counting function* and the *optical position counting function* for the billiard map in P.

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Let $z \in \mathbb{R}^2$ and let $\gamma \subset \mathbb{R}^2$ be an oriented piecewise C^1 curve. Denote by $d_z s$ the projection of the arclength form ds of γ onto the direction perpendicular to the line from z to $s \in \gamma$. The integral $\int_\gamma d_z s = |\mathrm{opt}(\gamma, z)| \le |\gamma|$ is the *optical length of* γ*, as viewed from* z.

Let $z \in P$. Unfolding k-segment billiard orbits emanating from z, we obtain a set of linear segments in $\mathbb{R}^2$ starting from z. Let $\partial_z(P; k) \subset \mathbb{R}^2$ be the curve traced by their endpoints. We say that $\partial_z(P; k) \subset \mathbb{R}^2$ is the *outer boundary of* P*, as viewed from* z*, after* k *iterates*.

THEOREM 3. *Let P be a Euclidean polygon, and let $K \subset K(P)$ be a set of corners. Then*

$$\int_{\partial P} \sum_{v \in K} gd_s(n; v)\, ds = \sum_{v \in K} \sum_{k=1}^{n} |\partial_v(P; k)|, \tag{17}$$

$$\int_{\partial P} \sum_{v \in K} god_s(n; v)\, ds = \sum_{v \in K} \sum_{k=1}^{n} |\mathrm{opt}(\partial_v(P, k))|. \tag{18}$$

Proof. It suffices to prove the claims for a singleton, $K = \{v\}$. Let $\eta : X \to \partial P$ be the natural projection. Using the arclength parametrization, we identify ∂P with the interval $[0, |\partial P|] \subset \mathbb{R}$. For $0 \le s \le |\partial P|$ let $R_s = \eta^{-1}(s) \subset X$ be the fiber. Then R_s are the base sets for the counting functions $gd_s(n; v)$, $god_s(n; v)$. See Figure 1. Let $Y = Y(v) \subset X$ be the set of phase points whose T^{-1}-orbits emanate from v. The assumptions of §1.1 are satisfied. The weight functions are $w(s, \alpha, t) = \chi_n(t)$ and $w_o(s, \alpha, t) = \sin\alpha \cdot \chi_n(t)$ for the two cases at hand. Let φ be the angle parameter on Y. The measures on ∂P and Y have densities ds and $d\varphi$, respectively.

The integrals in the right-hand side of (3) are over the curves $\partial_v(P; k)$, $0 \le k \le n-1$. The integrands are $ds(\varphi)$ and $\sin\alpha \cdot ds(\varphi) = d_v s(\varphi)$ in respective cases. □

We need estimates on lengths and optical lengths.

LEMMA 3. *For any polygon $P \subset \mathbb{R}^2$ there exist $0 < c_1 < c_2 < \infty$ such that for n sufficiently large*

$$c_1 n^2 \le \sum_{v \in K} \sum_{k=1}^{n} |\mathrm{opt}(\partial_v(P, k))| \le c_2 n^2, \quad c_1 n^2 \le \sum_{v \in K} \sum_{k=1}^{n} |(\partial_v(P, k))|. \tag{19}$$

Proof. There exist positive constants d_1, d_2 and $m_0 \in \mathbb{Z}_+$, such that for any orbit γ of the billiard map with $m > m_0$ segments, we have $d_1|\gamma| \le m \le d_2|\gamma|$ (see [**3**]).

Let $v \in K(P)$. We estimate $\sum_{k=m_0}^{n} |\mathrm{opt}(\partial_v(P, k))|$, as $n \to \infty$. Let $\theta_1 \le \theta \le \theta_2$ be the angular parameter for orbits emanating from v; let $r(\theta)$ be the geometric length of the orbit. Suppose that $r_1 \le r(\theta) \le r_2$. Then the optical length in question is sandwiched between the lengths of circular arcs of radii r_1, r_2 of angular size $\theta_2 - \theta_1$. By the preceding remarks, if k is sufficiently large, the bounds r_1, r_2 are proportional to k. The total angular size does not depend on k. Hence, for sufficiently large k we have linear upper and lower bounds on $\sum_{v \in K} |\mathrm{opt}(\partial_v(P, k))|$. The other inequality follows from $|\mathrm{opt}(\partial_v(P, k))| \le |\partial_v(P, k)|$. □

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3.4. *Position counting functions for billiard flows in spherical polygons.* We freely use the analogy to the planar case discussed in §3.2. Set

$$\zeta(x) = 1 - \cos x - \frac{2}{\pi}x. \tag{20}$$

THEOREM 4. *Let $P \subset \mathbb{S}^2$ be a geodesic polygon, and let $K \subset K(P)$. Then*

$$\int_P \sum_{v\in K} gc_z(l;\, v)\, dz = \left(\sum_{v\in K} \alpha(v)\right)\left(\frac{2}{\pi}l + \zeta(l - \pi\lfloor l/\pi \rfloor)\right). \tag{21}$$

Proof. It suffices to prove the claim when $K = \{v\}$. Let $M = M(v) \subset \Psi$ be as in §3.2, and let $d\alpha$ be the angular measure on it. Assumptions (B 1) and (B 3) of §1 are satisfied; the transversality of $b^t \cdot M$ and R_z may fail for at most a countable set of parameters $P_{\text{ex}} \subset P$. See Remark A1 in Appendix A. Hence, condition (B 2)$'$ is fulfilled and the results of §1.2 apply. The function $gc_z(l;\, v)$ is a pure counting function. The claim now follows from Proposition 2 and Corollary A1 in Appendix A. □

COROLLARY 3. *Let $P \subset \mathbb{S}^2$ be an arbitrary polygon. Then*

$$\int_P gc_z(l)\, dz = (\kappa(P)\pi + \text{area}(P))\left(\frac{2}{\pi}l + \zeta(l - \pi\lfloor l/\pi \rfloor)\right). \tag{22}$$

Proof. For a spherical polygon we have $\sum_{v\in K(P)} \alpha(v) = \text{area}(P) + \kappa(P)\pi$. Substitute this into (21). □

3.5. *Position counting functions for billiard flows in hyperbolic polygons.* To avoid repetition, we rely on the analogy with the planar case.

THEOREM 5. *Let $P \subset \mathbb{H}^2$ be a geodesic polygon, and let $K \subset K(P)$. Then*

$$\int_P \sum_{v\in K} gc_z(l;\, v)\, dz = \left(\sum_{v\in K} \alpha(v)\right)\cosh l. \tag{23}$$

Proof. We repeat verbatim the proof of Theorem 4, and use claim (2) of Corollary A1. □

COROLLARY 4. *Let $P \subset \mathbb{H}^2$ be a polygon. Then*

$$\int_P gc_z(l)\, dz = (\kappa(P)\pi - \text{area}(P))\cosh l. \tag{24}$$

Proof. Repeat the argument of Corollary 3 and use the formula

$$\sum_{v\in K(P)} \alpha(v) = \kappa(P)\pi - \text{area}(P)$$

relating the angles and the area of geodesic polygons in $\mathbb{H}^2$. □

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4. *Partial complexities and counting functions*

In this section we establish a framework that allows us to study the complexity of a wide class of dynamical systems. Our motivation comes from the billiard dynamics. In fact, the polygonal billiard is the target of applications for our results. The framework is more general, however. The following observations are our guiding principles. First, natural partitions of the billiard-type systems are geared towards the singularities. Second, the billiard dynamics satisfies a certain convexity principle that is instrumental in the study of complexity. These principles are manifest in the framework of *piecewise convex transformations* [**5**].

There are two approaches to the billiard dynamics: the *billiard flow* and the *billiard map* (see Appendix A). The framework of piecewise convex transformations is geared towards the billiard map. We begin by establishing its counterpart for flows.

4.1. *Piecewise convex transformations and piecewise convex flows.* A piecewise convex transformation is a triple (X, Γ, T), where X is a two-dimensional convex cell complex, $\Gamma \subset X$ is the graph formed by the union of one-cells and $T : X \to X$ is an invertible map, regular on the two-cells of the complex and compatible with the convex structure [**5**].

Let Ψ be a compact manifold, with boundary and corners, in general. Let $b^t : \Psi \to \Psi$ be a flow, possibly with singularities; let $X \subset \Psi$ be a cross section. We assume that the singular set of the flow is contained in X. For $z \in \Psi \setminus X$ let $\tau_+(z)$, $\tau_-(z)$ be the times when it first reaches X under b^t, b^{-t} for $0 < t$. We assume that for any $z \in \Psi \setminus X$ there is $0 < \varepsilon = \varepsilon(z)$ such that $b^t(z)$ is regular for $|t| < \varepsilon$.

A *piecewise convex flow* is determined by the following data: a flow, $b^t : \Psi \to \Psi$, a cross section, $X \subset \Psi$, and the structure of a convex cell complex on X, compatible with the Poincaré map. Billiard flows for polygons on surfaces of constant curvature are piecewise convex flows [**5**].

4.2. *Partial complexities for maps and flows.* Let (X, Γ_n, T^n) be the iterates† of a piecewise convex transformation (X, Γ, T). Let $F(\Gamma_n)$ be the finite set of open faces of Γ_n; these are the continuity regions for T^n. The function $f(n) = |F(\Gamma_n)|$ is the (full) complexity of (X, Γ, T).

Let $R \subset X$ be a closed subset. Set

$$F_R(n) = \{A \in F(\Gamma_n) \mid A \cap R \neq \emptyset\}.$$

Definition 1. The function $f_R(n) = |F_R(n)|$ is the *partial complexity of the piecewise convex transformation* (X, Γ, T) *based on the subset* R.

Let $b^t : \Psi \to \Psi$ be a piecewise convex flow and let $R \subset \Psi$ be a closed, convex set transversal to the flow. For $0 < l$ let $O_R(l)$ be the set of regular flow orbits of length l starting from R. Let $\alpha_0, \alpha_1 \in O_R(l)$. A *homotopy* is a continuous family of regular orbits $\alpha_p \in O_R(l)$, $0 \leq p \leq 1$, interpolating between α_0, α_1. If such a homotopy exists, we say, for brevity, that the orbits α_0, α_1 are R-*homotopic*. We denote by $H_R(l)$ the set of R-homotopy classes.

† They are piecewise convex transformations as well [**5**].

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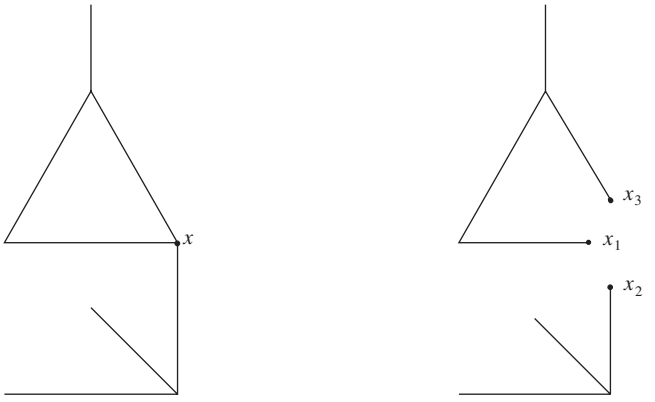


FIGURE 2. Removing a vertex in a graph.

Definition 2. The function $h_R(l) = |H_R(l)|$ is the *partial complexity (based on R) of the piecewise convex flow* $b^t : \Psi \to \Psi$.

4.3. *Partial complexities and counting functions.* Let $R \subset \Psi$ be a base set for partial complexity. We assume that R is a *convex subgraph* with respect to the cell complex structure mentioned above. We can assume, without loss of generality, that R has no isolated vertices. For $x \in R$ its *valence* $\mathrm{val}(x)$ is the number of edges of x minus one. In particular, if x is an interior point of an edge, then $\mathrm{val}(x) = 1$. Set $\mathrm{val}(R) = \max_{x \in R} \mathrm{val}(x)$. We endow $R \setminus \{x\}$ with the graph structure where x is replaced by $1 + \mathrm{val}(x)$ vertices; each of them is the endpoint of a unique edge. If $x, y, z, \ldots \in R$ are distinct points, then the inductively defined graph structure on R without $x, y, z, \ldots$ does not depend on any particular order in which we remove these points. We denote this graph by $R \setminus \{x, y, z, \ldots\}$. See Figure 2 for an illustration.

Let $E(R)$ and $V(R)$ be the sets of edges and vertices, and let $c(R)$ be the number of connected components of the graph. Let $h_i = h_i(R)$ be the Betti numbers of R, and set $\chi(R) = |V(R)| - |E(R)|$. Then $c(R) = h_0$, $\chi(R) = h_0 - h_1$.

LEMMA 4. *Let R be a finite graph, and let $x_1, \ldots, x_p \in R$ be distinct points. Then*

$$\chi(R) + \sum_{i=1}^{p} \mathrm{val}(x_i) \le c(R \setminus \{x_1, \ldots, x_p\}) \le c(R) + \sum_{i=1}^{p} \mathrm{val}(x_i). \tag{25}$$

If R is a forest, then the bound on the right in (25) becomes an equality.

Proof. It suffices to prove the claims when R is connected, and we remove a single vertex, x. Equation (25) becomes

$$\chi(R) + \mathrm{val}(x) \le c(R \setminus \{x\}) \le c(R) + \mathrm{val}(x). \tag{26}$$

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We have $|V(R \setminus \{x\})| = |V(R)| + \mathrm{val}(x)$, $|E(R \setminus \{x\})| = |E(R)|$, and $\chi(R \setminus \{x\}) = \chi(R) + \mathrm{val}(x)$. Equivalently, we have $\chi(R \setminus \{x\}) = h_0(R) + \mathrm{val}(x) - h_1(R)$ and $h_0(R \setminus \{x\}) = h_0(R) + \mathrm{val}(x) + (h_1(R \setminus \{x\}) - h_1(R))$. The former (respectively latter) identity implies the left (respectively right) inequality in (26).

When R is a tree, we have $c(R \setminus \{x\}) = c(R) + \mathrm{val}(x)$, and the remaining claim follows. □

We introduce counting functions for singular orbits of the billiard map and the billiard flow.

By definition, an orbit $\alpha = \{b^t(z), 0 \le t \le l\}$, does not pass through singular points in Ψ. It is regular if it does not contain any singular points in Ψ; it is *singular* if one of its endpoints is singular. The set $S_R(l)$ of singular orbits of length at most l, based in R, is finite. The quantities $gc_R(l) = |S_R(l)|$ and $gd_R(n) = |R \cap \Gamma_n|$ are the *counting functions of singular orbits based in* R for the flow and the map, respectively.

We now relate partial complexities and counting functions. We do this for a piecewise convex flow $b^t : \Psi \to \Psi$ and for a piecewise convex transformation (X, Γ, T). In both cases the partial complexity is based on a one-dimensional subset, say R. Recall that $gc_R(l)$, $gd_R(n)$ are the respective counting functions, and $h_R(l)$, $f_R(n)$ are the respective complexities. We refer to these situations as *the continuous case and the discrete case*, respectively.

PROPOSITION 5. *Let the setting be as above. Then the following statements hold.*

(1) *In the continuous case there exist* $h_0 \in \mathbb{N}$ *and* $l_0 \in \mathbb{R}_+$ *such that* $h_R(l) = h_0 + gc_R(l)$ *for* $l_0 \le l$.

(2) *In the discrete case there exist* $f_0, n_0 \in \mathbb{N}$ *such that for* $n_0 \le n$ *we have* $f_R(n) = f_0 + gd_R(n)$.

Proof. In both cases the graph R is equipped with an increasing tower of finite sets, say $X(l)$ and X_n, respectively. Let $X_\infty \subset R$ be their union. We compare the number of connected components of graphs $R \setminus X(l)$, $R \setminus X_n$ with the cardinalities of these sets.

We consider the discrete case, leaving the continuous case to the reader. Let $m < n$ be any pair of natural numbers. By (the proof of) Lemma 4,

$$c(R \setminus X_n) - c(R \setminus X_m) = [h_1(R \setminus X_n) - h_1(R \setminus X_m)] + \sum_{x \in X_n \setminus X_m} \mathrm{val}(x).$$

We have $h_1(R \setminus X_n) \le h_1(R \setminus X_m)$; the inequality holds if and only if $X_n \setminus X_m$ breaks cycles in $R \setminus X_m$. Since the sequence $h_1(R \setminus X_k)$ is non-increasing, it stabilizes. Thus, there exists $n_1 \in \mathbb{Z}_+$ such that for $n_1 \le m < n$ we have $h_1(R \setminus X_n) = h_1(R \setminus X_m)$.

The set of points $x \in R$ satisfying $1 < \mathrm{val}(x)$ is finite. Thus, there exists n_2, such that if $n_2 \le k$ and $x \in X_\infty \setminus X_k$, then $\mathrm{val}(x) = 1$.

Set $n_0 = \max(n_1, n_2)$. Then for $n_0 \le m < n$ the above equation yields

$$c(R \setminus X_n) - c(R \setminus X_m) = |X_n \setminus X_m|.$$

Specializing to $m = n_0$, we obtain

$$f_R(n) = (f_R(n_0) - gd_R(n_0)) + gd_R(n).$$ □

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5. *Bounds on partial complexities for the billiard*

We use the preceding material to derive bounds on partial complexities for the polygonal billiard.

5.1. *Direction complexities for billiard maps in Euclidean polygons.* We use the setting and the notation of §3.1. For a polygon P and a direction θ, we denote by $fd_\theta(n)$ the partial complexity with base R_θ. This is the *complexity in direction* θ.

COROLLARY 5. *For Lebesgue almost all directions* θ *there is* $C = C(\theta)$ *such that*

$$\liminf_{n\to\infty} \frac{fd_\theta(n)}{n} \le C.$$

Proof. Each R_θ is a convex graph in the phase space [**5**]. By Lemma 1 and Corollary 1, the counting functions $gd_\theta(n)$ have the desired properties. By Proposition 5, the directional complexities also have the desired properties. □

COROLLARY 6. *For any* $\varepsilon > 0$ *and almost every direction* θ *we have* $fd_\theta(n) = O(n^{1+\varepsilon})$.

Proof. The proof goes along the lines of the proof of Corollary 5. Instead of Lemma 1, we use Proposition 4 (the first claim). □

5.2. *Position complexities for billiard flows in Euclidean polygons.* Let P be a Euclidean polygon, and let $z \in P$ be any point. We consider the billiard flow in P, and use the setting of §3.2. Thus, $gc_z(l)$ is the position counting function for orbits emanating from z. We denote by $h_z(l)$ the corresponding partial complexity.

COROLLARY 7. *For almost every point* z *there is a positive number* $C = C(z)$ *such that*

$$\liminf_{l\to\infty} \frac{h_z(l)}{l^2} \le C.$$

Proof. The sets R_z satisfy the assumptions of §4. The claim follows from Lemma 1, Corollary 2 and the continuous case in Proposition 5. □

COROLLARY 8. *For any* $\varepsilon > 0$ *and almost every* $z \in P$ *we have* $h_z(l) = O(l^{2+\varepsilon})$.

Proof. The proof is similar to the preceding argument; we use the first claim in Proposition 3 instead of Lemma 1. □

5.3. *Position complexities for billiard maps in Euclidean polygons.* This is the billiard map analog of the preceding example. Let P be a Euclidean polygon, and let $s \in \partial P$. We use the setting of §3.3. There we have defined the counting functions $gd_s(n)$, $god_s(n)$. Let $f_s(n)$ be the partial complexity corresponding to $gd_s(n)$. This is the *position complexity for the billiard map*.

COROLLARY 9. *Let* $P \subset \mathbb{R}^2$ *be a polygon such that* $\sum_{v\in K}\sum_{k=1}^{n} |\partial_v(P;k)|$ *has a quadratic upper bound. Then for almost all* $s \in \partial P$ *we have* $f_s(n) = O(n^{2+\varepsilon})$ *for any* $0 < \varepsilon$.

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Proof. The sets $R_s \subset X$ satisfy the assumptions of §4. We use Theorem 3, Lemma 3 and apply Proposition 5. □

Remark 4. The statement of Corollary 9 is conditional because, in general, we have no efficient upper bound on $\sum_{k=1}^{n} |\partial_v(P; k)|$. The assumptions of Corollary 9 hold if P is a rational polygon [**10**].

5.4. *Position complexities for billiard flows in spherical polygons.* We use the setting of §3.4. For a spherical polygon, $P \subset \mathbb{S}^2$, and $z \in P$, let $h_z(l)$ be the position complexity.

COROLLARY 10. *For almost every point $z \in P$ there is $C = C(z)$ such that*

$$\liminf_{l\to\infty} \frac{h_z(l)}{l} \leq C.$$

Proof. The sets R_z satisfy the assumptions of §4. We use Lemma 1, Corollary 3 and Proposition 5. □

COROLLARY 11. *For any $\varepsilon > 0$ and almost every $z \in P$ we have $h_z(l) = O(l^{1+\varepsilon})$.*

Proof. See the proof of Corollary 8. □

5.5. *Position complexities for billiard flows in hyperbolic polygons.* This is the hyperbolic plane counterpart of §3.2, and we use the setting of §3.5.

COROLLARY 12. *Let $P \subset \mathbb{H}^2$ be a geodesic polygon, let $z \in P$ and let $h_z(l)$ be the position complexity. Then for almost every point $z \in P$ we have $h_z(l) = O(e^{(1+\varepsilon)l})$.*

Proof. We verify that the sets R_z satisfy the assumptions of §4, and mimic the proof of Corollary 8; we use Corollary 4, Proposition 3 and the continuous case of Proposition 5. □

A. *Appendix. Covering spaces for polygonal billiards*

Let M be a simply connected surface of constant curvature, and let $P \subset M$ be a connected geodesic polygon. We normalize the metric so that the curvature is zero ($M = \mathbb{R}^2$), one ($M = \mathbb{S}^2$) or minus one ($M = \mathbb{H}^2$).

Let A be the set of sides in P. We denote its elements by $a, b, \ldots$. For a side, say $a \in A$, let $s_a \in \mathrm{Iso}(M)$ be the corresponding geodesic reflection. We associate with P a *Coxeter system* (G, A) (see [**2**]). We denote by $\sigma_a, \sigma_b, \ldots \in G$ the elements corresponding to $a, b, \ldots \in A$. They generate G. The defining relations are $\sigma_a^2 = 1$ and $(\sigma_a\sigma_b)^{n(a,b)} = 1$; the latter arise only for the sides a, b with a common corner if the angle, $\theta(a, b)$, between them is π-rational. In this case $n(a, b)$ is the denominator of $\theta(a, b)/\pi$.

For any 'generalized polyhedron' P there is a corresponding topological space $\mathfrak{C}$ endowed with several structures, and a Coxeter system [**2**]. Our situation fits into the framework of [**2**], and we apply its results. First, $\mathfrak{C}$ is a differentiable surface. Second, $\mathfrak{C}$ is tiled by subsets P_g, $g \in G$, labelled by elements of the Coxeter group G; we call them the *tiles*, and identify P_e with P. The group G acts on $\mathfrak{C}$ properly discontinuously, preserving the tiling: $g \cdot P_h = P_{gh}$.

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Since P_e is identified with $P \subset M$, it inherits from M a Riemannian structure. The action of G is compatible with this structure, and extends it to all of $\mathfrak{C}$. This Riemannian structure generally has cone singularities at vertices† of the tiling $\mathfrak{C} = \bigcup_{g\in G} P_g$. At other points, the Riemannian structure is isometric to that of M; in particular, except for cone points, $\mathfrak{C}$ has constant curvature. The group G acts on $\mathfrak{C}$ by isometries.

Definition A1. The space $\mathfrak{C}$ endowed with the Riemannian structure, the isometric action of G and the G-invariant tiling $\mathfrak{C} = \bigcup_{g\in G} P_g$ is the *universal covering space* of the geodesic polygon $P \subset M$.

If X is a Riemannian manifold (with boundary and singularities, in general), we denote by $TX = \bigcup_{x\in X} T_x X$ its *unit tangent bundle*. The classical construct of *geodesic flow*, $G^t_X : TX \to TX$, extends to manifolds with boundaries and singularities. In particular, G^t_X makes sense when $X = M$, P or $\mathfrak{C}$. Another classical construct, the *exponential map*, also extends to our situation. For $x \in X$ as above, and $(v, t) \in T_x X \times \mathbb{R}_+$, we set $\exp_X(v, t) \in X$ to the basepoint of $G^t_X(v)$. We use the notation $\exp^x_X$ to indicate that we are exponentiating from the point x. If X is non-singular, then $\exp^x_X : T_x X \times \mathbb{R}_+ \to X$ is a differentiable mapping. For X with singularities, such as our P and $\mathfrak{C}$, the maps $\exp^x_X$ are defined on proper subsets of $T_x X \times \mathbb{R}_+$; these subsets have full Lebesgue measure. Generally, the maps do not extend by continuity to all of $T_x X \times \mathbb{R}_+$.

Let X, Y be non-singular Riemannian manifolds of the same dimension; let $\varphi : X \to Y$ be a local isometry. It induces a local diffeomorphism $\Phi : TX \to TY$ commuting with the geodesic flows: $\Phi \circ G^t_X = G^t_Y \circ \Phi$. The exponential maps also commute: $\varphi \circ \exp^x_X = \exp^{\varphi(x)}_Y \circ d_x\varphi$. These relationships hold, in particular, for coverings of non-singular Riemannian manifolds. Suitably interpreted, they extend to (branched) coverings of Riemannian manifolds with boundaries, corners and singularities. In our case $X = \mathfrak{C}$, while $Y = M$ or $Y = P$. We now define the mappings $f : \mathfrak{C} \to P$, $F : T\mathfrak{C} \to TP$ and $\varphi : \mathfrak{C} \to M$, $\Phi : T\mathfrak{C} \to TM$.

The identification $P_e = P$ defines f, φ on P_e. To extend them to all of $\mathfrak{C}$, we use the tiling $\mathfrak{C} = \bigcup_{g\in G} P_g$ and the actions of G on $\mathfrak{C}$ and M. In order to distinguish between these actions, we denote them by $g \cdot x$ and $g(x)$, respectively. Then there is a unique $x \in P_e$ such that $z = g \cdot x$. We set $f(z) = x \in P$ and $\varphi(z) = g(x) \in M$. By basic properties of Coxeter groups [**2**], the mappings f, φ are well defined. Moreover, $f : \mathfrak{C} \to P$ and $\varphi : \mathfrak{C} \to M$ are the unique G-equivariant mappings which are identical on‡ P_e. By construction, both mappings are continuous; they are diffeomorphisms in the interior of each tile, $P_g \subset \mathfrak{C}$. Under mild geometric restrictions on the polygon, φ is a diffeomorphism on the interior of the union of any pair of adjacent tiles.

The potential locus of non-differentiability for both f and φ is the set V of vertices in the tiling $\mathfrak{C} = \bigcup_{g\in G} P_g$. We have $V = f^{-1}(K(P))$ where $K(P)$ is the set of corners of P. By equivariance, $\varphi(V) = \bigcup_{g\in G} g(K(P)) \subset M$§. There are two kinds of points in V: vertices coming from the corners of P with π-rational and π-irrational angles. Their

† Each vertex, v, corresponds to a corner of P. The metric at v is regular if and only if the corner angle is π/n, $n = 2, 3, \ldots$.

‡ The action of G on P is trivial.

§ The representation $M = \bigcup_{g\in G} g(P)$ is not a tiling, in general.

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cone angles are integer multiples of 2π and are infinite, respectively. Vertices $v \in V$ with cone angle 2π are, in fact, regular points in $\mathfrak{C}$, and the mappings f, φ are both regular there. At a vertex v with the cone angle $2k\pi > 2\pi$ the mapping φ is differentiable, but not a diffeomorphism; it is locally conjugate to $z \mapsto z^k$. Near such a vertex, φ is a branched covering of degree k. At a vertex with infinite cone angle, φ has infinite branching.

Remark A1. The set $\varphi(V) \subset M$ is countable. (It is finite if and only if the group generated by geodesic reflections in the sides of P is a finite Coxeter group. Typically, $\varphi(V) \subset M$ is a dense, countable set.) Let $M = \mathbb{S}^2$, and let $z \mapsto z'$ denote the antipodal map. Set $F = P \cap (\varphi(V) \cup (\varphi(V))')$. Points of F are exceptional, in the following sense. Let $z \in P$ be such that the beam R_z of billiard orbits emanating from z contains a sub-beam focusing at a corner of P. Then $z \in F$. This follows from Proposition A1 below.

Thus, F contains all points $z \in P$ for which the transversality assumption in condition (B 2) of §1.2 fails. Since F is countable, the set of exceptional parameters has measure zero, and condition (B 2)$'$ is satisfied. See Remark 2 in §1.

Furthermore, the mappings f and φ are local isometries. They are isometries on every tile $P_g \subset \mathfrak{C}$; we have $f(P_g) = P$, $\varphi(P_g) = g(P) \subset M$. Let $g \cdot a$ be a side of P_g, let $h = \sigma_a g$ and let P_h be the adjacent tile. The maps $f : P_g \to P$, $P_h \to P$ and $\varphi : P_g \to g(P)$, $P_h \to h(P)$ are coherent around the common (open) side $g \cdot a$. The map f is never an isometry on $P_g \cup P_h$; for φ this is the case if and only if the interiors of $g(P)$, $h(P)$ are disjoint in M. The latter generally fails for non-convex P.

By coherence of f and φ across the sides separating adjacent tiles, we lift them to the tangent bundles, obtaining the mappings of unit tangent bundles $F : T\mathfrak{C} \to TP$, $\Phi : T\mathfrak{C} \to TM$, which are also defined on vectors based at the vertices of the tiling $\mathfrak{C} = \bigcup_{g \in G} P_g$. Let v be a vertex, and let α be the angle of the corner $f(v) \in K(P)$. Then $\Phi : T_v\mathfrak{C} \to T_{\varphi(v)}M$ is m-to-one if $\alpha = m\pi/n$ and ∞-to-1 if α is π-irrational. The geodesics $\gamma(t)$ in $\mathfrak{C}$ cannot be extended further (generally) once they reach a vertex. All other geodesics in $\mathfrak{C}$ are defined for $-\infty < t < \infty$.

Using the inclusion $P \subset M$, we identify TP with the subset of TM consisting of M-tangent vectors with basepoints in P, and directed inward. Any $v \in TP$ defines the billiard orbit in P, $\beta(t) = \exp_P(tv)$, $0 \le t$, and the geodesic in M, $\gamma(t) = \exp_M(tv)$, $0 \le t$. They are related by the canonical *unfolding of billiard orbits*. This is an inductive procedure which replaces the consecutive reflections about the sides of P by consecutive reflections of the 'latest billiard table' $g(P)$ about the appropriate side, yielding the next billiard table $h(P)$, and continuing the geodesic straight across the common side of $g(P)$ and $h(P)$. See [**5**, §3.1]. Let $x \in P$ and let $v \in T_xP$. We denote by β_v (respectively γ_v) the billiard orbit in P (respectively the geodesic in M) that emanates from x in the direction v. The *unfolding operator*, $U_x : \beta_v \mapsto \gamma_v$, preserves the parameterizations.

PROPOSITION A1. *Let $x \in P$, $v \in T_xP$. Identify P and $P_e \subset \mathfrak{C}$ and let $x \in P_e$, $v \in T_x\mathfrak{C}$ be the corresponding data. Then for $t \in \mathbb{R}_+$ we have*

$$U_x(\exp_P(v, t)) = \varphi(\exp_{\mathfrak{C}}(v, t)). \tag{27}$$

Proof. We freely use the preceding discussion. As $t \in \mathbb{R}_+$ goes to infinity, $\exp_P(v, t)$ runs with the unit speed along a billiard orbit in P. The curve $\exp_{\mathfrak{C}}(v, t)$ is the geodesic

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in $\mathfrak{C}$ defined by the data (x, v), and $\varphi(\exp_{\mathfrak{C}}(v, t))$ is the geodesic in M emanating from x in the direction v. The billiard orbit in P and the geodesic in M are related by the unfolding operator. □

For $x \in P$ let $E_x P = T_x P \times \mathbb{R}_+$ be the full tangent space (or the full tangent cone) at x. If $S \subset T_x P$ is a segment, let $ES_x P = S \times \mathbb{R}_+$ be the corresponding subcone. We use the analogous notation for $x \in \mathfrak{C}$ or $x \in M$. In polar coordinates (t, θ) in $\mathbb{R}^2$, the Lebesgue measure on $E_x P$ is given by the density $t\, dt\, d\theta$.

COROLLARY A1. *Let $x \in P \subset M$ be arbitrary, and let $\exp_P^x : E_x P \to P$ be the exponential mapping. The pullback by $\exp_P^x$ of the Lebesgue measure on P to $E_x P$ is the smooth measure with the density $d\nu(t, \theta)$. The following relationships hold.*

(1) *When $M = \mathbb{R}^2$, we have $d\nu = t\, dt\, d\theta$.*
(2) *When $M = \mathbb{H}^2$, we have $d\nu = \sinh t\, dt\, d\theta$.*
(3) *When $M = \mathbb{S}^2$, we have $d\nu = |\sin t|\, dt\, d\theta$.*

Proof. By Proposition A1, the measure in question coincides with the pullback to the tangent space $E_x M$ of the Riemannian measure on M by the exponential map $E_x M \to M$. The latter is well known. □

We point out that the preceding material has a billiard map version, which we now briefly discuss. Let $\beta(t) = (z(t), \theta(t))$, $t \in \mathbb{R}$, be an orbit of the billiard flow. We obtain the corresponding billiard map orbit $\beta_d(k)$, $k \in \mathbb{Z}$, by restricting $\beta(t)$ to the consecutive times t_k such that $z(t_k) \in \partial P$. The correspondence $\beta(\cdot) \mapsto \beta_d(\cdot)$ is invertible. This allows us to formulate the billiard map versions of the universal covering space, the lifting of billiard map orbits to the universal covering space and the relationship between the liftings and the unfoldings, á là Proposition A1. Since we are not directly using this material in the body of the paper, we omit the details.

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