On Summable Trigonometric Integrals. By S. VERBLUNSKY, Ph.D., Magdalene College.

[*Received* 8 November 1930, *read* 26 January 1931.]

1. The analogue for trigonometric integrals of de la Vallée Poussin's classical theorem concerning the uniqueness of the trigonometrical development has recently been obtained by M. Jacob* in the following form.

Consider the integral

$$\int_0^\infty \{A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\}\,d\lambda, \quad \ldots\ldots\ldots\ldots(1)$$

where

$$\lim_{\mu\to\infty}\int_\mu^{\mu+1}|A(\lambda)|\,d\lambda = 0, \qquad \lim_{\mu\to\infty}\int_\mu^{\mu+1}|B(\lambda)|\,d\lambda = 0. \quad \ldots\ldots(2)$$

Suppose that the upper and lower limits as $h\to 0$ of

$$\int_0^\infty \left(\frac{\sin\lambda h}{\lambda h}\right)^2 \{A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\}\,d\lambda,$$

which we denote by $\overline{R}(x)$, $\underline{R}(x)$, are finite except at the points of a set which contains no perfect component, and satisfy the conditions

$$\overline{R}(x) \geqslant f(x) \geqslant \underline{R}(x)$$

for all x, where $f(x)$ is absolutely integrable in $(-\infty, \infty)$. Then (1) is the Fourier integral of $f(x)$; i.e. for almost all λ,

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\lambda x\,dx; \qquad B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx.$$

This enunciation is somewhat more than that given by Jacob, whose proof however establishes the result.

The theorem of de la Vallée Poussin has been generalised in various directions by Zygmund† and by the present writer‡, and this suggests the problem of obtaining analogous generalisations of Jacob's theorem. In this connection, a paper by M. E. Grimshaw§ is relevant.

* M. Jacob, *Math. Annalen*, 97 (1927), 663–674.
† Zygmund, *Math. Zeit.* 25 (1926), 274–290.
‡ Verblunsky, *Proc. Lond. Math. Soc.* 31 (1930), 370–386, referred to as T.
§ M. E. Grimshaw, *Proc. Lond. Math. Soc.* 29 (1928), 243–256.

https://doi.org/10.1017/S0305004100010288 Published online by Cambridge University Press

We define the upper and lower Poisson sums of the integral (1) as the upper and lower limits as $r \to 1\,(0 < r < 1)$ of

$$\int_0^\infty r^\lambda \{A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\}\,d\lambda$$

or, what is the same, the upper and lower limits as $\eta \to 0\,(\eta > 0)$ of

$$\int_0^\infty e^{-\eta\lambda}\{A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\}\,d\lambda.$$

We denote these sums by $\overline{P}(x)$, $\underline{P}(x)$ respectively. The result of Miss Grimshaw may be enunciated as follows.

Let the integral (1) be such that $\overline{P}(x)$, $\underline{P}(x)$ are absolutely integrable in every finite interval, and finite everywhere. Suppose that there exists a function $F(x)$ which is continuous at every finite point, and absolutely integrable in $(-\infty, \infty)$, and for which

$$\left.\begin{aligned} \frac{A(\lambda)}{\lambda^2} &= -\frac{1}{\pi}\int_{-\infty}^{\infty} F(x)\cos\lambda x\,dx; \\ \frac{B(\lambda)}{\lambda^2} &= -\frac{1}{\pi}\int_{-\infty}^{\infty} F(x)\sin\lambda x\,dx. \end{aligned}\right\} \quad \ldots\ldots\ldots(3)$$

Then there is a function $f(x)$ which is finite and absolutely integrable in every finite interval and such that

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\lambda x\,dx \quad (C, 2);$$

$$B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx \quad (C, 2).$$

We observe that the absolute integrability of $F(x)$ implies the continuity of $A(\lambda)$ and $B(\lambda)$. Further, the conditions (3) imply that, as $\lambda \to 0$,

$$A(\lambda) = O(\lambda^2); \qquad B(\lambda) = O(\lambda^2).$$

The theorem could not therefore be directly applied except to integrals of a most restricted type.

The object of this note is to prove the analogue of the uniqueness theorem for trigonometric series given by the writer in the paper referred to. The analogue may be enunciated as follows.

THEOREM I. *Let the coefficients in the integral* (1) *satisfy the conditions that* $A(\lambda)$ *and* $B(\lambda)$ *are absolutely integrable in every finite interval* $(0, X)$, *and that*

$$\int_\mu^{\mu+1} \{A(\lambda)\}^2 d\lambda = o(\mu^2); \qquad \int_\mu^{\mu+1} \{B(\lambda)\}^2 d\lambda = o(\mu^2) \ldots(4)$$

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as $\mu \to \infty$. *Let the upper and lower Poisson sums be finite everywhere and satisfy either of the conditions*

(i) $\overline{P}(x)$, $\underline{P}(x)$ *are absolutely integrable in* $(-\infty, \infty)$,

or (ii) $\overline{P}(x)$, $\underline{P}(x)$ *are absolutely integrable in every finite interval. Then*

$$\overline{P}(x) = \underline{P}(x) = f(x)$$

almost everywhere, and in case (i)

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\lambda x\,dx; \qquad B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx,$$

almost everywhere, while in case (ii) *these equations hold* $(C, 2)$ *almost everywhere.*

2. It will save complication if we confine our attention to trigonometric integrals of the form

$$\int_0^{\infty} A(\lambda)\cos\lambda x\,d\lambda. \qquad \text{.....(5)}$$

Integrals of the form

$$\int_0^{\infty} B(\lambda)\sin\lambda x\,d\lambda \qquad \text{.....(6)}$$

may be treated in a similar manner.

LEMMA 1. *Let* $g(x)$ *be integrable in every finite interval* $(1, X)$, *and let*

$$\int_{\mu}^{\mu+1} \{g(x)\}^2 dx = o(\mu^2) \qquad \text{.....(7)}$$

as $\mu \to \infty$. *If*

$$G(x) = \int_1^{\infty} \frac{g(\lambda)}{\lambda^2}\cos\lambda x\,d\lambda$$

is convergent in a neighbourhood of the origin, then

$$\overline{\lim_{h\to 0}}\int_1^{\infty} g(\lambda)\left(\frac{\sin\lambda h}{\lambda h}\right)^2 d\lambda \geqslant \overline{\lim_{\eta\to 0}}\int_1^{\infty} e^{-\eta\lambda} g(\lambda)\,d\lambda, \qquad \text{.....(8)}$$

$$\overline{\lim_{\eta\to 0}}\int_1^{\infty} e^{-\eta\lambda} g(\lambda)\,d\lambda \geqslant \overline{\lim_{h\to 0}}\int_1^{\infty} g(\lambda)\left(\frac{\sin\lambda h}{\lambda h}\right)^2 d\lambda. \qquad \text{.....(9)}$$

The proof depends on the following two lemmas due to Miss Grimshaw.

LEMMA α. *If* $\phi(t)$ *is absolutely integrable in* $(0, X)$ *for every finite* X, *and* $\phi(t)/t$ *is absolutely integrable at* ∞, *and*

$$\Psi(\lambda) = \frac{1}{\pi}\int_0^{\infty} \phi(t)\frac{\sin\lambda t}{t}\,dt,$$

then

$$\overline{\lim_{t\to 0}}\frac{\phi(t)}{t^2} \leqslant \overline{\lim_{\eta\to 0}}\int_0^{\infty} e^{-\eta\lambda}\,2\lambda\Psi(\lambda)\,d\lambda$$

with an inverse relation between the upper limits.

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LEMMA β. *If* $\psi(\lambda)$ *is absolutely integrable in* $(0, X)$ *for every finite* X, *and*

$$\Psi(\lambda) = \int_0^\lambda \psi(t)\, dt$$

then, provided that

$$\int_0^\infty \psi(t)\, dt = 0,$$

$$\lim_{\eta \to 0} \int_0^\infty - e^{-\eta\lambda}\lambda^2 \psi(\lambda)\, d\lambda \leqslant \lim_{\eta \to 0} \int_0^\infty e^{-\eta\lambda} 2\lambda \Psi(\lambda)\, d\lambda$$

with an inverse relation between the upper limits.

Turning to the proof of Lemma 1 we may suppose that (7) holds for all $\mu \geqslant 1$. Further, we may suppose that

$$\int_1^\infty \frac{g(\lambda)}{\lambda^2}\, d\lambda = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Since

$$\int_1^\infty \frac{\{g(\lambda)\}^2}{\lambda^4}\, d\lambda < \infty$$

the theory of Fourier transforms* enables us to assert that

$$\int_0^\infty \{G(x)\}^2\, dx < \infty. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Put

$$g(t) = G(t).$$

Then $g(t)$ satisfies the conditions of Lemma α. Further

$$\frac{g(t)}{t^2} = -\frac{1}{2}\int_1^\infty g(\lambda) \left(\frac{\sin \lambda t/2}{\lambda t/2}\right)^2 d\lambda. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

We write, in accordance with Lemma α,

$$\Psi(\lambda) = \frac{1}{\pi}\int_0^\infty g(t) \frac{\sin \lambda t}{t}\, dt$$

$$= \frac{1}{\pi}\int_0^\infty G(t) \left(\frac{\sin \lambda t}{t}\right) dt.$$

By the theory of Fourier transforms, we have, for almost all λ,

$$\frac{d}{d\lambda}\int_0^\infty G(t) \frac{\sin \lambda t}{t}\, dt = \frac{\pi}{2}\frac{g(\lambda)}{\lambda^2} \quad (\lambda > 1)$$

$$= 0 \quad (0 < \lambda < 1).$$

Hence

$$\Psi(\lambda) = \frac{1}{2}\int_1^\lambda \frac{g(\lambda)}{\lambda^2}\, d\lambda,$$

$$\psi(\lambda) = \frac{1}{2}\frac{g(\lambda)}{\lambda^2}.$$

* Cf. Hobson, *Theory of Functions of a Real Variable*, II (1926), 742–752.

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In virtue of (10) the conditions of Lemma β are satisfied, and the result then follows.

3. Our next object is to prove that, under the conditions of the theorem,

$$F(x) = \int_0^1 A(\lambda) \frac{1-\cos \lambda x}{\lambda^2} d\lambda - \int_1^\infty A(\lambda) \frac{\cos \lambda x}{\lambda^2} d\lambda \quad \ldots(13)$$

is a continuous function. We observe in passing that, when considering integrals of the form (6), the function which corresponds to $F(x)$ is

$$\int_0^1 B(\lambda) \frac{\lambda x - \sin \lambda x}{\lambda^2} d\lambda - \int_1^\infty B(\lambda) \frac{\sin \lambda x}{\lambda^2} d\lambda.$$

The first term in (13) represents a continuous function. It remains to consider the second term. By hypothesis, the upper and lower limits as $r \to 1$ of

$$\int_1^\infty r^\lambda A(\lambda) \cos \lambda x \, d\lambda$$

are finite. We have

$$\int_1^\infty \frac{r^\lambda}{\lambda} f(\lambda) \cos \lambda x \, d\lambda = \int_1^\infty A(\lambda) \cos \lambda x \, d\lambda \int_0^r \rho^{\lambda-1} d\rho$$

$$= \int_0^r d\rho \int_1^\infty \rho^{\lambda-1} A(\lambda) \cos \lambda x \, d\lambda.$$

Hence

$$\left| \lim_{r \to 1} \int_1^\infty \frac{r^\lambda}{\lambda} A(\lambda) \cos \lambda x \, d\lambda - \int_1^\infty \frac{r^\lambda}{\lambda} A(\lambda) \cos \lambda x \, d\lambda \right|$$

$$\leqslant \int_r^1 d\rho \left| \int_1^\infty \rho^{\lambda-1} A(\lambda) \cos \lambda x \, d\lambda \right|$$

$$\leqslant (1-r) \operatorname*{Max}_{r \leqslant \rho < 1} \left| \int_1^\infty \rho^{\lambda-1} A(\lambda) \cos \lambda x \, d\lambda \right|.$$

It follows that

$$\int_1^\infty \frac{r^\lambda}{\lambda} A(\lambda) \cos \lambda x \, d\lambda$$

converges relatively uniformly to its limit. By repeating the argument, we infer that

$$\int_1^\infty \frac{r^\lambda}{\lambda^2} A(\lambda) \cos \lambda x \, d\lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

tends relatively uniformly to its limit. Now the condition (4) implies that

$$\int_\mu^{\mu+1} |A(\lambda)| \, d\lambda = o(\mu)$$

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as $\mu \to \infty$. Further by the method used for proving Tauber's theorem we can show that if

$$\int_1^\infty h(x)\,dx \quad \text{.....(15)}$$

is summable by Poisson's method and

$$\int_\mu^{\mu+1} h(x)\,dx = o\left(\frac{1}{\mu}\right)$$

as $\mu \to \infty$, then (15) is convergent. It follows that

$$\int_1^\infty A(\lambda)\frac{\cos \lambda x}{\lambda^2}\,d\lambda \quad \text{.....(16)}$$

is everywhere convergent and possesses the property that its points of discontinuity with respect to any perfect set are non-dense on that set. Hence $F(x)$ possesses the same property.

Suppose that (α, β) is an interval in which $F(x)$ is continuous. Let x_0 be a point of this interval. If we write

$$G(x) = \int_1^\infty A(\lambda)\frac{\cos \lambda x_0}{\lambda^2}\cos \lambda x\,d\lambda,$$

then we can apply Lemma 1 with

$$g(\lambda) = A(\lambda)\cos \lambda x_0.$$

It follows that

$$\overline{\lim_{h\to 0}} \int_1^\infty A(\lambda)\cos \lambda x_0 \left(\frac{\sin \lambda h}{\lambda h}\right)^2 d\lambda \geqslant \overline{\lim_{r\to 1}} \int_1^\infty r^\lambda A(\lambda)\cos \lambda x_0\,d\lambda.$$

But

$$\lim_{h\to 0} \int_0^\infty A(\lambda)\cos \lambda x_0 \left(\frac{\sin \lambda h}{\lambda h}\right)^2 d\lambda = \lim_{r\to 1} \int_0^\infty r^\lambda A(\lambda)\cos \lambda x_0\,d\lambda.$$

Hence $$\bar{D}^2 F(x_0) \geqslant \underline{P}(x_0).$$

Similarly we can show that

$$\bar{P}(x_0) \geqslant \underline{D}^2 F(x_0).$$

If now we write

$$f(x_0) = \text{Min}\,\{\bar{D}^2 F(x_0), \bar{P}(x_0)\},$$

then $$\bar{P}(x_0) \geqslant f(x_0) \geqslant \underline{P}(x_0),$$

so that, under either of the conditions (i), (ii) of the theorem, $f(x)$ is absolutely integrable in every finite interval. Further

$$\bar{D}^2 F(x_0) \geqslant f(x_0) \geqslant \underline{D}^2 F(x_0),$$

so that since $F(x)$ is continuous in (α, β) we have

$$F(x) = \int_a^x dy \int_a^y f(t)\,dt + px + q \qquad (\alpha < x < \beta).$$

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In order to employ the argument of T. § 5 to infer that $F(x)$ is everywhere continuous, it remains to prove that it is a differential coefficient.

4. For this purpose we require two simple lemmas.

LEMMA 2. *If*

$$\int_{\mu}^{\mu+1} |k(x)|\,dx = o\left(\frac{1}{\mu}\right)$$

as $\mu \to \infty$, *and*

$$\int_{1}^{\infty} k(\lambda)\,d\lambda$$

is convergent, then

$$\lim_{h\to 0}\int_{1}^{\infty} k(\lambda)\left(\frac{\sin \lambda h}{\lambda h}\right)^2 d\lambda = \int_{1}^{\infty} k(\lambda)\,d\lambda.$$

LEMMA 3. *If*

$$\int_{\mu}^{\mu+1} |k(x)|\,dx = o(1)$$

as $\mu \to \infty$, *then*

$$\lim_{h\to 0} h\int_{1}^{\infty} k(\lambda)\left(\frac{\sin \lambda h}{\lambda h}\right)^2 d\lambda = 0.$$

If in Lemma 2 we put

$$h(\lambda) = \frac{A(\lambda)\cos \lambda x}{\lambda^2}$$

it follows that the function

$$\Phi(x) = \int_{1}^{\infty} \frac{A(\lambda)\sin \lambda x}{\lambda^3}\,d\lambda$$

satisfies the relation

$$\lim_{h\to 0}\frac{\Phi(x+h)-\Phi(x-h)}{2h} = \int_{1}^{\infty}\frac{A(\lambda)}{\lambda^2}\cos \lambda x\,d\lambda. \quad \ldots(17)$$

Further,

$$\frac{\Phi(x+2h)+\Phi(x-2h)-2\Phi(x)}{4h} = -h\int_{1}^{\infty}\frac{A(\lambda)}{\lambda}\sin \lambda x\left(\frac{\sin \lambda h}{\lambda h}\right)^2 d\lambda.$$

Since

$$\int_{\mu}^{\mu+1}\frac{|A(\lambda)|}{\lambda}\,d\lambda = o(1),$$

it follows by Lemma 3 that

$$\lim_{h\to 0}\frac{\Phi(x+2h)+\Phi(x-2h)-2\Phi(x)}{4h} = 0.$$

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This combined with (17) shows that

$$\int_1^\infty \frac{A(\lambda)}{\lambda^2} \cos \lambda x \, d\lambda$$

is the differential coefficient of $\Phi(x)$. Since

$$\int_0^1 A(\lambda) \frac{1-\cos \lambda x}{\lambda^2} \, d\lambda$$

is a differential coefficient, so is $F(x)$.

Hence $F(x)$ is continuous in every finite interval, and in any such interval (α, β) we have an equation of the form

$$F(x) = \int_a^x dy \int_a^y f(t) \, dt + px + q. \quad \ldots\ldots\ldots\ldots (18)$$

5. Suppose now that condition (i) of the theorem is satisfied. Then $f(t)$ is absolutely integrable in $(-\infty, \infty)$. Writing

$$D^2F(x, h) = \frac{F(x+2h) + F(x-2h) - 2F(x)}{4h^2}$$

we have, from (18),

$$D^2F(x, h) = \tfrac{1}{4}h^2 \int_0^{2h} dt \int_{-t}^{t} f(x+u) \, du. \quad \ldots\ldots\ldots (19)$$

On the other hand

$$D^2F(x, h) = \int_0^\infty A(\lambda) \left(\frac{\sin \lambda h}{\lambda h}\right)^2 \cos \lambda x \, d\lambda.$$

Let X be chosen so that in every finite interval on the right of X, $A(\lambda)$ is quadratically integrable. Then, by the theory of Fourier's integral,

$$\lim_{\mu \to \infty} \frac{1}{\pi\mu} \int_0^\mu dz \int_{-z}^{z} K(x, h) \cos \lambda x \, dx = A(\lambda) \left(\frac{\sin \lambda h}{\lambda h}\right)^2 \quad (0 < \lambda < X)$$

$$= 0 \qquad (X < \lambda < \infty)$$

for almost all λ, where

$$K(x, h) = \int_0^X A(\lambda) \left(\frac{\sin \lambda h}{\lambda h}\right)^2 \cos \lambda x \, d\lambda.$$

Further, since

$$A(\lambda) \left(\frac{\sin \lambda h}{\lambda h}\right)^2$$

is quadratically integrable in (X, ∞), we have by the theory of Fourier transforms

$$\lim_{\mu \to 0} \frac{1}{\pi\mu} \int_0^\mu dz \int_{-z}^{z} [D^2F(x, h) - K(x, h)] \cos \lambda x \, dx = 0 \quad (0 < \lambda < X)$$

$$= A(\lambda) \left(\frac{\sin \lambda h}{\lambda h}\right)^2 \qquad (X < \lambda < \infty)$$

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for almost all λ. Hence, for almost al λ,

$$\lim_{\mu\to\infty}\frac{1}{\pi\mu}\int_0^{\mu}dz\int_{-z}^{z}D^2F(x,h)\cos\lambda x\,dx=A(\lambda)\left(\frac{\sin\lambda h}{\lambda h}\right)^2. \qquad \ldots\ldots(20)$$

But (19) gives

$$\frac{1}{\pi}\int_{-z}^{z}D^2F(x,h)\cos\lambda x\,dx=\frac{1}{4\pi h^2}\int_0^{2h}dt\int_{-t}^{t}du$$
$$\times\left\{\int_{-z-u}^{z-u}f(x)\cos\lambda x\,dx\cos\lambda u+\int_{-z-u}^{z-u}f(x)\sin\lambda x\,dx\sin\lambda u\right\}.$$

Bearing in mind the absolute integrability of $f(x)$ in $(-\infty,\infty)$, we infer that

$$\lim_{z\to\infty}\frac{1}{\pi}\int_{-z}^{z}D^2F(x,h)\cos\lambda x\,dx=\left(\frac{\sin\lambda h}{\lambda h}\right)^2\frac{1}{\pi}\int_{-\infty}^{\infty}f(x)\cos\lambda x\,dx. \qquad \ldots\ldots(21)$$

From (20) and (21) we deduce

$$A(\lambda)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(x)\cos\lambda x\,dx$$

almost everywhere, and this proves the theorem.

6. Suppose now that condition (ii) holds. We divide the given trigonometric integral into two parts

$$j(x)=\int_0^1 A(\lambda)\cos\lambda x\,d\lambda, \qquad \ldots\ldots\ldots\ldots\ldots\ldots(22)$$

$$J(x)=\int_1^{\infty}A(\lambda)\cos\lambda x\,d\lambda. \qquad \ldots\ldots\ldots\ldots\ldots\ldots(23)$$

Then in virtue of the absolute integrability of $A(\lambda)$ in every finite interval, $j(x)$ is a continuous function. Hence $J(x)$ satisfies the conditions of the theorem in case (ii), with $f(x)-j(x)$ taking the place of $f(x)$, and

$$\mathfrak{F}(x)=-\int_1^{\infty}\frac{A(\lambda)}{\lambda^2}\cos\lambda x\,d\lambda \qquad \ldots\ldots\ldots\ldots\ldots\ldots(24)$$

taking the place of $F(x)$. We have

$$\mathfrak{F}(x)=\int_a^x dy\int_a^y\{f(t)-j(t)\}\,dt+px+q \quad (a<x<b). \qquad \ldots\ldots(25)$$

If we write

$$\mathfrak{F}(x)=-\int_1^X\frac{A(\lambda)}{\lambda^2}\cos\lambda x\,d\lambda=\int_X^{\infty}\frac{A(\lambda)}{\lambda^2}\cos\lambda x\,d\lambda,$$

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then the first term tends to zero as $x \to \infty$, and the second term represents a function which is quadratically integrable in $(0, \infty)$. It follows that

$$\frac{1}{u}\int_0^u |\mathfrak{F}(x)|\,dx \to 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

as $u \to \infty$.

We now apply the following lemma, due to Miss Grimshaw.

LEMMA γ. *If the even function $\mathfrak{F}(x)$ be continuous for finite x and satisfy* (26); *if further there exist a function $k(x)$ which is integrable in every finite interval (a, b) such that*

$$\mathfrak{F}(x) = \int_a^x dy \int_a^y k(t)\,dt + px + q \quad (a < x < b),$$

then

$$\int_{-\infty}^{\infty} k(x) \cos sx\,dx = -s^2 \int_{-\infty}^{\infty} \mathfrak{F}(x) \cos sx\,dx \quad (C, 2)$$

provided that one of the integrals exists $(C, 2)$.

It follows that the sum $(C, 2)$ of

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \{f(x) - j(x)\} \cos sx\,dx$$

is

$$\frac{2s^2}{\pi}\int_0^{\infty} \cos sx\,dx \int_1^{\infty} \frac{A(\lambda)}{\lambda^2} \cos \lambda x\,d\lambda \quad (C, 2)$$

which, by the theory of Fourier transforms is, for almost all s,

$$A(s) \text{ in } 1 < s < \infty; \quad 0 \text{ in } 0 < s < 1.$$

On the other hand, from (22), by a lemma due to Pollard*, the sum $(C, 1)$ of

$$\frac{1}{\pi}\int_{-\infty}^{\infty} j(x) \cos sx\,dx$$

is, for almost all s,

$$0 \text{ in } 1 < s < \infty; \quad A(s) \text{ in } 0 < s < 1.$$

Hence for almost all λ,

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \cos \lambda x\,dx \quad (C, 2)$$

and this proves the theorem.

* Pollard, *Proc. Lond. Math. Soc.* 25 (1926), 451–468.

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