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Wave-front singularities for two-dimensional anisotropic elastic waves

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Abstract. Wave-front singularities for the displacement functions, associated with the radiation of linear elastic waves from a point source embedded in a finitely strained two-dimensional elastic solid, are examined in detail. It is found that generally the singularities are of order $d^{-\frac{1}{2}}$ where d measures distance away from the front. However, in certain exceptional cases singularities of order d^{-n}, where $n = \frac{1}{4}, \frac{2}{3}, \frac{3}{4}$, may be encountered.

1. *Introduction.* In an earlier set of papers (**1**, **2**, **3**), the author considered the radiation of anisotropic elastic waves from a point perturbing body force embedded in a two-dimensional solid which was initially subjected to a (finite) pure homogeneous deformation. The shape of the wave front (W) was determined in (**1**, **2**) as an envelope of plane waves. The coordinates $(\bar{\xi}, \bar{\eta})$ of W were found to satisfy the envelope equations

$$\bar{\xi}\cos\theta + \bar{\eta}\sin\theta = v(\theta) \quad \text{and} \quad -\bar{\xi}\sin\theta + \bar{\eta}\cos\theta = v'(\theta), \tag{1·1}$$

which when solved for $\bar{\xi}$ and $\bar{\eta}$ gave the following (θ) parametric representation for the two branches W_+ and W_- of W

$$\bar{\xi}_\pm(\theta) = v_\pm(\theta)\cos\theta - v'_\pm(\theta)\sin\theta \quad \text{and} \quad \bar{\eta}_\pm(\theta) = v_\pm(\theta)\sin\theta + v'_\pm(\theta)\cos\theta. \tag{1·2}$$

Here
$$v_\pm^{-1}(\theta) = R_\pm(\theta) = \left[\frac{B(\theta) \pm \sqrt{(B^2(\theta) - 4A)}}{2A(\theta)}\right]^{\frac{1}{2}}, \tag{1·3}$$

where
$$A(\theta) = \alpha\cos^4\theta + \gamma\cos^2\theta\sin^2\theta + \beta\sin^4\theta, \tag{1·4}$$

and
$$B(\theta) = (\alpha+1)\cos^2\theta + (\beta+1)\sin^2\theta. \tag{1·5}$$

In the above equations, θ is allowed to vary on the interval $0 \leqslant \theta < 2\pi$. The physical parameters of the problem α, β and γ depend on the initial deformation of the body and on its constitutive relations. Equations (1·2)–(1·5) may be combined to give explicit expressions for $\bar{\xi}_\pm$ and $\bar{\eta}_\pm$, the coordinates of $W_\pm$, as

$$\bar{\xi}_\pm(\theta) = \frac{\cos\theta}{2AR_\pm}\left[2\alpha\cos^2\theta + \gamma\sin^2\theta \pm \frac{\sin^2\theta(k_1\cos^2\theta - k_2\sin^2\theta)}{\sqrt{(B^2-4A)}}\right], \tag{1·6}$$

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and $$\overline{\eta}_{\pm}(\theta) = \frac{\sin\theta}{2AR_{\pm}}\left[2\beta\sin^2\theta + \gamma\cos^2\theta \mp \frac{\cos^2\theta(k_1\cos^2\theta - k_2\sin^2\theta)}{\sqrt{(B^2-4A)}}\right], \tag{1·7}$$

where $k_1 = 2\alpha(\beta+1) - \gamma(\alpha+1)$ and $k_2 = 2\beta(\alpha+1) - \gamma(\beta+1)$. Associated with the wave front W is a normal curve N (or slowness curve) having two branches N_+ and N_- in the (r, s) plane with coordinates

$$r_{\pm}(\theta) = R_{\pm}(\theta)\cos\theta \quad \text{and} \quad s_{\pm}(\theta) = R_{\pm}(\theta)\sin\theta. \tag{1·8}$$

In (3) attention was directed toward the more difficult problem of actually finding the displacement field when a point impulse is applied at the origin in a finitely strained solid. It was there shown that the ξ and η components of the displacement could be broken down as

$$u(\xi, \eta, \tau) = A_1 u_1(\xi, \eta, \tau) + A_2 u_2(\xi, \eta, \tau) \tag{1·9}$$

and $$v(\xi, \eta, \tau) = A_1 v_1(\xi, \eta, \tau) + A_2 v_2(\xi, \eta, \tau), \tag{1·10}$$

where A_1 and A_2 are constants. Employing a Fourier–Laplace transform inversion technique, essentially due to Burridge (4), permitted the displacement functions to be expressed as

$$u_1(\xi, \eta, \tau) = -\frac{1}{\pi\tau}\Im\sum_p\left\{\text{residues of } \frac{Q_{22}(p, 1, \overline{\xi}p + \overline{\eta})}{Q(p, 1, \overline{\xi}p + \overline{\eta})} \text{ in the upper half } p\text{-plane}\right\}, \tag{1·11}$$

$$u_2(\xi, \eta, \tau) \equiv v_1(\xi, \eta, \tau) = \frac{1}{\pi\tau}\Im\sum_p\left\{\text{residues of } \frac{Q_{21}(p, 1)}{Q(p, 1, \overline{\xi}p + \overline{\eta})} \text{ in the upper half } p\text{-plane}\right\}, \tag{1·12}$$

and

$$v_2(\xi, \eta, \tau) = -\frac{1}{\pi\tau}\Im\sum_p\left\{\text{residues of } \frac{Q_{11}(p, 1, \overline{\xi}p + \overline{\eta})}{Q(p, 1, \overline{\xi}p + \overline{\eta})} \text{ in the upper half } p\text{-plane}\right\}. \tag{1·13}$$

In the above expressions ξ and η are proportional to the space coordinates of the problem and τ to the time coordinate. The variables $\overline{\xi} \equiv \xi/\tau$ and $\overline{\eta} \equiv \eta/\tau$ are dimensionless. The Q functions are defined by

$$Q_{11}(x, y, z) \equiv x^2 + \beta y^2 - z^2, \quad Q_{22}(x, y, z) \equiv \alpha x^2 + y^2 - z^2, \quad Q_{12}(x, y) \equiv Q_{21}(x, y) \equiv \kappa xy \tag{1·14}$$

and $$Q(x, y, z) \equiv \alpha x^4 + \gamma x^2 y^2 + \beta y^4 - (\alpha+1)x^2 z^2 - (\beta+1)y^2 z^2 + z^4.$$

Although equations (1·11)–(1·13) constitute an exact solution to the above posed problem, their interpretation in a general situation is made difficult by the necessity of solving a quartic equation in p. Considerable simplification results if the displacements are just required along the $\overline{\xi}$ or $\overline{\eta}$-axes, since the quartic then reduces to a biquadratic equation in p. This case was analysed in detail (for $\alpha = \beta$) in (3). There the displacement singularities were displayed, but for this special case only.

The present paper seeks to analyse the displacement functions in the neighbourhood of the wave front. It is found that in general (there are exceptions) the displacements become unbounded as W is approached from the left-hand side when the front is transversed in the direction of increasing θ. At an ordinary point of W (to be defined

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below) where the displacement is singular, the singularity is of the order $d^{-\frac{1}{2}}$ where d measures distance away from the point in question on W. However, for some situations, i.e. for appropriate combinations of (α, β, γ), there may be cusps or point lacunas on W, in which case the singularities will be stronger than $d^{-\frac{1}{2}}$.

In the following sections of this paper, the displacement singularities at the wave front (from the theory of weak solutions to linear partial differential equations it is known that there can be no others) are found from the *exact* solution (equations (1·11)–(1·13)) to the equations of motion by using a perturbation technique (see section 2). The results of this technique can be contrasted with those given by Bazer and Yen (5) for a more general problem. Bazer and Yen determine the wave front singularities by an appeal to certain asymptotic formulae of Ludwig (6). The results of the present paper differ in detail from those of Bazer and Yen, especially at a cuspoidal point of W.

2. *Perturbation technique for determining the singularities of* W. In (3) it was shown that, for given values of $(\bar{\xi}, \bar{\eta})$ and the parameters (α, β, γ), the real p-plane roots of $Q(p, 1, \bar{\xi}p + \bar{\eta}) \equiv f(p|\bar{\xi}, \bar{\eta})$ can be found from a knowledge of the intersection points of the line L: $\bar{\xi}r + \bar{\eta}s = 1$ with the normal curve N: $Q(r, s, -1) = 0$ in the (r, s) plane. In particular if (r_0, s_0) is an intersection point of L with N then $p_0 = r_0/s_0$ is a real zero of f.

Now suppose, e.g. that $\bar{\xi} = \bar{\xi}_-(\theta)$ and $\bar{\eta} = \bar{\eta}_-(\theta)$. Then L will touch N_- (the inner branch of N) at $[r_-(\theta), s_-(\theta)]$ and also cross N_+ at two additional distinct points. The fact that L touches (is tangent to) N_- in this case clearly means that $p = p_0 = \cot\theta$ is a *double* root of $f(p|\bar{\xi}_-, \bar{\eta}_-) = 0$. If the values of $\bar{\xi}$ and $\bar{\eta}$ are now slightly perturbed away from $\bar{\xi}_-$ and $\bar{\eta}_-$, then this double p root will break up into two real roots (if $\bar{\xi}$ and $\bar{\eta}$ lie to the right of W_-) or into two complex roots (if $\bar{\xi}$ and $\bar{\eta}$ lie to the left of W_-). This perturbation, if sufficiently small, will not alter the character of the two additional p roots determined from the intersection of L with N_+. Thus *the behaviour of the displacement functions, near the point* $(\bar{\xi}_-, \bar{\eta}_-)$ *on* W_-, *can be found by locating the approximate roots of* $f(p|\bar{\xi}, \bar{\eta}) = 0$ *for small values of* $(\bar{\xi} - \bar{\xi}_-)^2 + (\bar{\eta} - \bar{\eta}_-)^2$. Furthermore, only those two roots which coalesce to p_0 when $(\bar{\xi}, \bar{\eta})$ moves to a point on W_- are needed if attention is directed toward the displacement singularity at $(\bar{\xi}_-, \bar{\eta}_-)$ when approached from the left. This is so since the real character of the two additional roots is not disturbed by the perturbation and hence these roots will not contribute to the displacement expressions.

Since $p_0 = \cot\theta$ is a double root of $f(p|\bar{\xi}_-, \bar{\eta}_-)$ then

$$f(p_0|\bar{\xi}_-, \bar{\eta}_-) = f'(p_0|\bar{\xi}_-, \bar{\eta}_-) = 0. \tag{2·1}$$

Realizing that $(\bar{\xi} - \bar{\xi}_-)$, $(\bar{\eta} - \bar{\eta}_-)$ and $(p - p_0)$ are now small (but not of the same order) a triple Taylor series expansion of $f(p|\bar{\xi}, \bar{\eta})$ about $(p_0, \bar{\xi}_-, \bar{\eta}_-)$ gives (retaining only the significant terms)

$$f(p|\bar{\xi}, \bar{\eta}) \approx H + f''(p_0|\ \bar{\xi}_-, \bar{\eta}_-)\, \tfrac{1}{2}(p - p_0)^2, \tag{2·2}$$

where

$$H = (\bar{\xi} - \bar{\xi}_-) f_{\bar{\xi}}(p_0|\bar{\xi}_-, \bar{\eta}_-) + (\bar{\eta} - \bar{\eta}_-) f_{\bar{\eta}}(p_0|\bar{\xi}_-, \bar{\eta}_-). \tag{2·3}$$

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The perturbed roots of $f(p|\bar{\xi},\bar{\eta}) = 0$ are then given by

$$p = p_0 \pm \left[-\frac{2H}{f''}\right]^{\frac{1}{2}}, \tag{2·4}$$

plus two additional real roots, which contribute nothing to the residue calculation of the displacement functions. Whence, from equation (1·11),

$$u_1 \approx -\frac{1}{\pi\tau}\Im\frac{Q_{22}(p_0,1,\bar{\xi}_-p_0+\bar{\eta}_-)}{(p-p_0)f''} = -\frac{1}{\pi\tau}\Im\frac{Q_{22}(p_0,1,\bar{\xi}_-p_0+\bar{\eta}_-)}{\pm f''[-2H/f'']^{\frac{1}{2}}}. \tag{2·5}$$

A point on $W_\pm$ for which $f''(p_0|\bar{\xi}_\pm,\bar{\eta}_\pm) \neq 0$ will be called an *ordinary point* of $W_\pm$. Clearly at an ordinary point of W_- the displacement u_1 will have a singularity like $|H|^{-\frac{1}{2}}$ as $(\bar{\xi}-\bar{\xi}_-)^2+(\bar{\eta}+\bar{\eta}_-)^2 \to 0$, provided the associated perturbed pole lies in the upper half p-plane. Obviously to treat the situation in more detail requires a thorough knowledge of the sign of $-2H/f''$ for points on W_-. This will be given below in sections 3 and 4.

With but minor alterations, the above argument also applies to the determination of the displacement singularities at ordinary points on W_+.

3. *Expressions for f'' and H.* Exploiting the two relations given in equation (2·1) (and similar relations for points on W_+) allows f'' to be written as

$$f''(p_0|\bar{\xi}_\pm,\bar{\eta}_\pm) = \frac{M_\pm}{A}\left[2\beta\sin^2\theta+\gamma\cos^2\theta \mp \frac{\cos^2\theta(k_1\cos^2\theta-k_2\sin^2\theta)}{\sqrt{(B^2-4A)}}\right]$$
$$+ 8\cos^2\theta\sin^2\theta\frac{[\gamma-(\alpha\beta+1)][\gamma-(\alpha+\beta)]}{B^2-4A}, \tag{3·1}$$

where $$M_\pm(\theta) = [2\alpha\cos^2\theta+\gamma\sin^2\theta-(\alpha+1)v_\pm^2].$$

While this expression for f'' shows its explicit dependence on θ, it is somewhat complicated and in particular the sign of f'' (which is of importance in the residue calculations leading to the wave-front singularities) is not readily available.

An alternate expression for f'' which relates its sign to geometric properties of the normal curve will now be established. Now

$$f(p|\bar{\xi},\bar{\eta}) = Q(p,1,\bar{\xi}p+\bar{\eta}) = (\bar{\xi}p+\bar{\eta})^4 Q\left(\frac{p}{\bar{\xi}p+\bar{\eta}},\frac{1}{\bar{\xi}p+\bar{\eta}},-1\right),$$

or $$f(p|\bar{\xi},\bar{\eta}) = (\bar{\xi}p+\bar{\eta})^4 F(r,s), \tag{3·2}$$

where $$F(r,s) \equiv Q(r,s,-1) \quad \text{and} \quad r(p) = \frac{p}{\bar{\xi}p+\bar{\eta}}, \quad s(p) = \frac{1}{\bar{\xi}p+\bar{\eta}}. \tag{3·3}$$

Then, at a double real root of f,

$$f''(p_0|\bar{\xi}_\pm,\bar{\eta}_\pm) = (\bar{\xi}p+\bar{\eta})^4\frac{d^2F}{dp^2}, \tag{3·4}$$

it being understood that here, and in what follows, all qualities are evaluated at $p = p_0$, $\bar{\xi} = \bar{\xi}_\pm$, $\bar{\eta} = \bar{\eta}_\pm$ after the indicated differentiation. Using the chain rule for differentiation of F gives

$$\frac{d^2F}{dp^2} = F_{rr}r_p^2 + 2F_{rs}r_ps_p + F_{ss}s_p^2. \tag{3·5}$$

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Taking d/dr and d^2/dr^2 of the implicit equation of N: $F(r,s)=0$ gives

$$\frac{ds}{dr}=-\frac{F_r}{F_s} \quad \text{and} \quad \frac{d^2s}{dr^2}=-\frac{F_{rr}F_s^2-2F_{rs}F_rF_s+F_{ss}F_r^2}{F_s^3}, \tag{3·6}$$

so that equation (3·5) may be written

$$\frac{d^2F}{dp^2}=-F_s\left(\frac{dr}{dp}\right)^2\frac{d^2s}{dr^2}, \tag{3·7}$$

which is valid at an ordinary point on N. Equations (1·3) and (1·8) allow d^2s/dr^2 on N to be written in terms of θ derivatives of v

$$\frac{d^2s}{dr^2}=-\frac{v^3}{(v\sin\theta+v'\cos\theta)^3}(v+v''). \tag{3·8}$$

After some simplification, equation (3·4) may be rewritten as

$$f''(p_0|\bar{\xi}_\pm,\bar{\eta}_\pm)=\pm 2v_\pm\sqrt{(B^2-4A)}\,(v_\pm+v''_\pm). \tag{3·9}$$

Recall (7) that the curvature of a plane curve specified parametrically by $x=x(\theta)$ and $y=y(\theta)$ is

$$k=\frac{x'y''-x''y'}{(x'^2+y'^2)^{\frac{3}{2}}}.$$

Hence, using equations (1·8) and (1·2),

normal curve curvature $$k_N=v^3\frac{v+v''}{(v^2+v'^2)^{\frac{3}{2}}}, \tag{3·10}$$

wave-front curvature $$k_W=\frac{1}{|v+v''|}. \tag{3·11}$$

Thus the sign of f'' is directly related to the sign of $(v+v'')$ on N which in turn, from equation (3·10), has the same sign as the curvature of N.

From the above remarks it is now clear that for a given ordinary point on (say) W_-, f'' will be positive if $k_{N-}<0$ and negative if $k_{N-}>0$. Since N_- is convex,

$$(v_-+v''_-)>0,$$

but $(v_++v''_+)$ will change sign at a (simple) inflection point of N_+ where $(v_++v''_+)=0$.

The expression for H given in equation (2·4) may be simplified with the help of the formulae of section 1 to

$$H=-\frac{2v_\pm}{\sin^4\theta}(\pm\sqrt{(B-4A)})\,\psi_\pm, \tag{3·12}$$

where $$\psi_\pm(\theta)=[(\bar{\xi}-\bar{\xi}_\pm)\cos\theta+(\bar{\eta}-\bar{\eta}_\pm)\sin\theta]. \tag{3·13}$$

Equation (2·3) for the approximate form of $f(p|\bar{\xi},\bar{\eta})$ when p is close to a double real zero and $(\bar{\xi},\bar{\eta})$ is close to a point on W_+ or W_- becomes,

$$f(p|\bar{\xi},\bar{\eta})\approx\pm 2v_\pm\sqrt{(B^2-4A)}\left[-\frac{\psi_\pm}{\sin^4\theta}+(v_\pm+v''_\pm)\tfrac{1}{2}(p-p_0)^2\right]. \tag{3·14}$$

It should be mentioned that equation (3·1) and the alternate expression of equation (3·9) may be combined to give an explicit formula for $(v_\pm+v''_\pm)$.

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By the examination of some typical normal curves, it is easily seen that

$$\psi_{-L}(\theta) < 0 \quad \text{and} \quad \psi_{-R}(\theta) > 0, \tag{3·15}$$

where $\psi_{-L}(\theta)$ signifies that $(\bar{\xi}, \bar{\eta})$ lies immediately to the left of W_- as W_- is traversed in the direction of increasing θ, while $\psi_{-R}(\theta)$ has the same meaning when $(\bar{\xi}, \bar{\eta})$ lies immediately to the right of W_-. Similar results for the sign of $\psi_+(\theta)$ are not so simple since $\psi_{+R}(\theta)$ and $\psi_{+L}(\theta)$ may change signs as W_+ is traversed. Fortunately any sign change of ψ_+ is coincident with that of $(v_+ + v_+'')$.

The results of this section may now be summarized as follows:

$$f(p|\bar{\xi}, \bar{\eta}) \approx \pm 2v_\pm \sqrt{(B^2 - 4A)\,(v_\pm + v_\pm'')} \left[\frac{(p-p_0)^2}{2} - \frac{\chi_\pm(\theta)}{\sin^4\theta}\right], \tag{3·16}$$

where

$$\chi_\pm(\theta) = \frac{(\bar{\xi} - \bar{\xi}_\pm)\cos\theta + (\bar{\eta} - \bar{\eta}_\pm)\sin\theta}{v_\pm + v_\pm''}. \tag{3·17}$$

This expression for $f(p|\bar{\xi}, \bar{\eta})$ is approximately true for p near a double real root $(p = p_0 = \cot\theta)$ of $f(p|\bar{\xi}_\pm, \bar{\eta}_\pm) = 0$ with $(\bar{\xi}, \bar{\eta})$ very close to $(\bar{\xi}_-, \bar{\eta}_-)$ (i.e. close to W_-) or $(\bar{\xi}, \bar{\eta})$ very close to $(\bar{\xi}_+, \bar{\eta}_+)$ (i.e. close to W_+). Furthermore

$$\chi_\pm(\theta) < 0 \quad \text{when} \quad (\bar{\xi}, \bar{\eta}) \text{ lies to the } \textit{left} \text{ of } W_+. \tag{3·18}$$

and

$$\chi_\pm(\theta) > 0 \quad \text{when} \quad (\bar{\xi}, \bar{\eta}) \text{ lies to the } \textit{right} \text{ of } W_+. \tag{3·19}$$

4. *Displacement singularity formulae at an ordinary point of W.* The viewpoint adapted in this paper is to measure the jump in the various displacement components, at any point on the wave front, as the point $(\bar{\xi}, \bar{\eta})$ *crosses the front*. This jump will be denoted by

$$[\overline{u_1}]_\pm = u_1(\bar{\xi} \text{ and } \bar{\eta} \text{ immediately to the left of } W_\pm) - u_1[\bar{\xi} = \bar{\xi}_\pm(\theta),\ \bar{\eta} = \bar{\eta}_\pm(\theta)], \tag{4·1}$$

where the second term is to be considered as a limit expression when $(\bar{\xi}, \bar{\eta})$ approaches $(\bar{\xi}_\pm, \bar{\eta}_\pm)$ from the right-hand side of $W_\pm$. This limit will always exist at an ordinary point of $W_\pm$.

In most instances it would be just as easy to give the dominating part of the displacement expression as the front is approached from the left or from the right. However, since the displacement is generally non-singular on the right side of W (for an exception to this statement see section 6 below), the two additional (unperturbed) p roots of $f = 0$ would be required in order to give the limiting value of the displacement as W_+ is approached from the right. Although the limiting form of the displacement function as W is approached from the right and left would provide some additional information, it would contribute nothing further to our knowledge of the singularities involved at W. For this reason primary attention is directed herein toward the jump in displacement across W. In this connexion it is well to point out the significance of $\psi_+(\theta) = \{(\bar{\xi} - \bar{\xi}_+)\cos\theta\} + \{(\bar{\eta} - \bar{\eta}_+)\sin\theta\}$ vanishing. This can come about in two ways. First, $\psi_+(\theta)$ will certainly vanish when $\bar{\xi} = \bar{\xi}_+$ and $\bar{\eta} = \bar{\eta}_+$, i.e. $(\bar{\xi}, \bar{\eta})$ on W_+. But $\psi_+(\theta)$ will also vanish for all values of $(\bar{\xi}, \bar{\eta})$ on the line l:

$$\{(\bar{\xi} - \bar{\xi}_+)\cos\theta\} + \{(\bar{\eta} - \bar{\eta}_+) \times \sin\theta\} = 0,$$

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θ being fixed. This line passes through the point $(\bar{\xi}_+, \bar{\eta}_+)$ on W_+ and has the slope $-\cot\theta$. Now the tangent vector to W_+ at an ordinary point has the slope

$$\frac{d\bar{\eta}}{d\bar{\xi}} = \frac{\bar{\eta}'_\pm}{\bar{\xi}'_\pm} = \frac{(v_\pm + v''_\pm)\cos\theta}{-(v_\pm + v''_\pm)\sin\theta} = -\cot\theta.$$

Hence the line l is tangent to $W_\pm$ at $(\bar{\xi}_\pm, \bar{\eta}_\pm)$ and *does not cross* $W_\pm$ *at an ordinary point.* For this reason the formulae to be given below in this section necessarily assume that the left-hand side approach of $(\bar{\xi}, \bar{\eta})$ to W is not along l.

From equation (3·16) the perturbed poles of $f^{-1}(p|\bar{\xi}, \bar{\eta})$ are given by

LEFT of $W_\pm$ $$p = p_0 \pm i\frac{\sqrt{2}}{\sin^2\theta}|\chi_\pm|^{\frac{1}{2}}$$

and the upper half p-plane pole is

$$p = p_0 + i\frac{\sqrt{2}}{\sin^2\theta}|\chi_\pm|^{\frac{1}{2}}. \tag{4·2}$$

RIGHT of $W_\pm$ $$p = p_0 \pm \frac{\sqrt{2}}{\sin^2\theta}\sqrt{[\chi_\pm(\theta)]},$$

hence there are no (perturbed) upper half p-plane poles.

From equation (1·11) a residue calculation now gives

$$u_1|_{\text{LEFT}} = -\frac{1}{\pi\tau}\Im\frac{Q_{22}(p_0, 1, \bar{\xi}_\pm p_0 + \bar{\eta}_\pm)}{\pm 2v_\pm\sqrt{(B^2-4A)}\,(v_\pm + v''_\pm)\,i\dfrac{\sqrt{2}}{\sin^2\theta}}|\chi_\pm|^{-\frac{1}{2}},$$

$$= \pm\frac{Q_{22}(\cos\theta, \sin\theta, v_\pm)}{2\sqrt{2\pi\tau v_\pm}\sqrt{(B^2-4A)}\,(v_\pm + v''_\pm)}|\chi_\pm|^{-\frac{1}{2}}. \tag{4·3}$$

Set $$h_\pm(\theta) = \pm[2\sqrt{2\pi\tau}\sqrt{(B^2-4A)}\,(v_\pm + v''_\pm)]^{-1}, \tag{4·4}$$

then the displacement component jumps near an ordinary point on $W_\pm$ are given by

$$[\overline{u_1}]_\pm = h_\pm(\theta)\,Q_{22}(\cos\theta, \sin\theta, v_\pm)\,|\chi_\pm(\theta)|^{-\frac{1}{2}}, \tag{4·5}$$

$$[\overline{u_2}]_\pm \equiv [\overline{v_1}]_\pm = -h_\pm(\theta)\,Q_{12}(\cos\theta, \sin\theta)\,|\chi_\pm(\theta)|^{-\frac{1}{2}}, \tag{4·6}$$

and $$[\overline{v_2}]_\pm = h_\pm(\theta)\,Q_{11}(\cos\theta, \sin\theta, v_\pm)\,|\chi_\pm(\theta)|^{-\frac{1}{2}}. \tag{4·7}$$

Equations (4·5)–(4·7) are to be interpreted in the sense that if $Q_{ij}(\theta) \neq 0$, then the displacement jumps at an ordinary point of $W_\pm$ are singular as $d^{-\frac{1}{2}}$, d measuring distance from $(\bar{\xi}_\pm, \bar{\eta}_\pm)$ on the left-hand side of W.

When $Q_{12}(\theta) = 0$ then u_2 ($\equiv v_1$), being an odd function of ξ and η (3), vanishes identically. On the other hand a necessary condition (for the sake of brevity the necessary and sufficient conditions will not be written down here) for the vanishing of Q_{11} or Q_{22} is that $[\gamma - (1+\alpha\beta)] \times \cos^2\theta\sin^2\theta = 0$. In all the relevant cases where this condition is satisfied, direct calculations (omitted here) reveal that the displacement components u_1 and v_2 are continuous across W.

It is perhaps worthwhile to note that when $\gamma = (\alpha\beta + 1)$, $\alpha > 1$ and $0 < \beta < 1$ then $Q_{11}(\cos\theta, \sin\theta, v_-) = 0$ for all values of θ and $v_2(\xi, \eta, \tau) \equiv 0$ for $(\bar{\xi}, \bar{\eta})$ to the right of

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W_+ which in this case is the ellipse $\bar{\xi}^2+\bar{\eta}^2/\beta = 1$. The exterior domain to W_+ in the $(\bar{\xi},\bar{\eta})$ plane is thus a lacuna for v_2 so that W_- is a phantom front. Of course what happens in this case is that $Q_{11}(p, 1, \bar{\xi}p+\bar{\eta})$ is a factor of $Q(p, 1, \bar{\xi}p+\bar{\eta})$. Similar remarks apply to the situation when $\gamma = (\alpha\beta+1)$, $0 < \alpha < 1, \beta > 1$ so that $Q_{22}(\cos\theta, \sin\theta, v_-) = 0$. Here $u_1(\xi,\eta,\tau) \equiv 0$ for $(\bar{\xi},\bar{\eta})$ to the right of W_+: $\bar{\xi}^2/\alpha+\bar{\eta}^2 = 1$.

Thus at an ordinary point of W where $Q_{ij}(\theta) = 0$, the Q_{ij} term dominates the singular term $|\chi_\pm|^{-\frac{1}{2}}$ so that the corresponding displacement jump is zero, at least in so far as W is crossed along its normal at the point in question.

5. *Displacement singularity at a cuspoidal point of* W. For certain combinations of (α,β,γ) the front W_+ will contain cuspoidal points (there are no such points on W_-) and the corresponding point of N_+ will be a (simple) inflection point (1, 2). At such a point the line L (see section 2) will intersect $N_\pm$ in such a way that $f(p|\bar{\xi}_+,\bar{\eta}_+)$ will have a real triple p root at $p = p_0$. Proceeding as in sections 2 and 3

$$f(p|\bar{\xi},\bar{\eta}) \approx H + f'''(p_0|\bar{\xi}_+,\bar{\eta}_+)(p-p_0)^3/3!, \tag{5·1}$$

where
$$f'''(p_0|\bar{\xi}_+,\bar{\eta}_+) = \frac{24\cos\theta\sin^3\theta}{(B^2-4A)^2}[\gamma-(\alpha\beta+1)][\gamma-(\alpha+\beta)] \times [(1-\beta)^2\sin^4\theta-(1-\alpha)^2\cos^4\theta]. \tag{5·2}$$

or, alternatively
$$f'''(p_0|\bar{\xi}_+,\bar{\eta}_+) = -2\sin^2\theta\sqrt{(B^2-4A)}\,v_+(v'_+ + v'''_+). \tag{5·3}$$

If one proceeds formally here, as in section 2, the cusp tip singularity for the displacement functions will be proportional to $|H|^{-\frac{2}{3}}$, which agrees with that found by Bazer and Yen (5) for a similar situation. However, this is not the singularity that one encounters upon crossing W_+ (i.e. on passing from the exterior to the interior of the associated lacuna) along a line through the cusp. The reason for this is as follows. The only line which crosses W_+ at a cuspoidal point is the (limiting) tangent line l. (although $v_+ + v''_+ = 0$ here, $v'_+ + v'''_+ \neq 0$ so that the limiting slope, when θ is approached from above, is still $-\cot\theta$.) But for $(\bar{\xi},\bar{\eta})$ on l, $H = 0$ so that a more accurate approximation to $f(p|\bar{\xi},\bar{\eta})$, than that given above is required. By analysing the perturbed roots of f, it can be shown that when $(\bar{\xi},\bar{\eta})$ is near a (simple) cusp of W_+ and on the line l
$$f(p|\bar{\xi},\bar{\eta}) \approx a_0 + a_1(p-p_0) + \tfrac{1}{2}a_2(p-p_0)^2 + \tfrac{1}{6}a_3(p-p_0)^3, \tag{5·4}$$
where

$$a_0 = \tfrac{1}{2}[f_{\bar{\xi}\bar{\xi}}(p_0|\bar{\xi}_+,\bar{\eta}_+)(\bar{\xi}-\bar{\xi}_+)^2 + 2f_{\bar{\xi}\bar{\eta}}(p_0|\bar{\xi}_+,\bar{\eta}_+)(\bar{\xi}-\bar{\xi}_+)(\bar{\eta}-\bar{\eta}_+) + f_{\bar{\eta}\bar{\eta}}(p_0|\bar{\xi}_+,\bar{\eta}_+)(\bar{\eta}-\bar{\eta}_+)^2],$$

$$a_1 = [f'_{\bar{\xi}}(\bar{\xi}-\bar{\xi}_+) + f'_{\bar{\eta}}(\bar{\eta}-\bar{\eta}_+)] = -\frac{2v_+}{\sin^3\theta}\sqrt{(B^2-4A)}\,(\bar{\xi}-\bar{\xi}_+), \tag{5·5}$$

$$a_2 = [f''_{\bar{\xi}}(\bar{\xi}-\bar{\xi}_+)f''_{\bar{\eta}} + (\bar{\eta}-\bar{\eta}_+)], \tag{5·6}$$

and
$$a_3 = f'''. \tag{5·7}$$

In the subsequent analysis a_0, being $0[(\bar{\xi}-\bar{\xi}_+)^2]$ can be neglected. It is helpful here to introduce the terms
$$\phi(\theta) = \frac{6a_1}{a_3} = \frac{6(\bar{\xi}-\bar{\xi}_+)}{\sin^5\theta(v'_+ + v'''_+)} \tag{5·8}$$

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and $$\lambda(\theta) = \frac{a_2}{2a_1 \sin\theta} - \frac{f^{\mathrm{iv}}}{2f''' \sin\theta} = \frac{2[(\alpha+1)\cos\theta - 2v_+\bar{\xi}_+]}{\sqrt{(B^2-4A)}} + \frac{\bar{\xi}_+}{v_+} + \frac{6(\alpha-\bar{\xi}_+^2)(1-\bar{\xi}_+^2)}{\sin^3\theta\sqrt{(B^2-4A)}\,v_+(v'_+ + v'''_+)} \quad (5{\cdot}9)$$

it being understood that $\bar{\xi}$ and $\bar{\eta}$ lie on l.

The same notation for the displacement that was introduced in equation (4·1) can again be used at a cuspoidal point of W_+, with the understanding that $\bar{\xi}$ and $\bar{\eta}$ immediately to the left of W_+ should be interpreted as $\bar{\xi}$ and $\bar{\eta}$ on l and immediately to the left of (i.e. exterior to) the lacuna. From the observation of typical W_+ and N_+ curves, paying particular attention to the sign of the curvature slope at an inflexion point of N_+, the conclusion may be drawn that $\phi(\theta) > 0$ when $\bar{\xi}$ is on l and immediately to the left of a cusp and $\phi(\theta) < 0$ to the right.

From equation (5·4) the perturbed upper half p-plane pole of $f^{-1}(p|\bar{\xi},\bar{\eta})$ is given by

LEFT of W_+. $$p = p_0 + i\sqrt{[\phi(\theta)]}. \quad (5{\cdot}10)$$

RIGHT of W_+. All poles are real, hence there are no upper half p-plane poles.

From equation (1·11) a residue calculation now gives

$$u_1\Big|_{\text{LEFT}} = -\frac{1}{\pi\tau}\Im\frac{Q_{22}(p,1,\bar{\xi}p+\bar{\eta})}{f'(p|\bar{\xi},\bar{\eta})}\Big|_{p=p_0+i\sqrt{\phi(\theta)}} \quad (5{\cdot}11)$$

$$= \frac{\lambda(\theta)\sin\theta Q_{22} + Q'_{22}}{\frac{1}{3}\pi\tau f'''}|\phi|^{-\frac{1}{2}}. \quad (5{\cdot}12)$$

Set $$g(\theta) = -\tfrac{1}{3}[2\pi\tau v_+(v'_+ + v'''_+)\sin^3\theta\sqrt{(B^2-4A)}]. \quad (5{\cdot}13)$$

Then the jumps for the various displacement components near a cuspoidal point of W_+, when $\bar{\xi}$ is on the line l: $(\bar{\xi}-\bar{\xi}+)\cos\theta + (\bar{\eta}-\bar{\eta}+)\sin\theta = 0$ and just outside the lacuna, are given by

$$\overline{[u_1]}_+ = \frac{\lambda(\theta)\,Q_{22}(\cos\theta,\sin\theta,v_+) + 2(\alpha\cos\theta - \bar{\xi}_+ v_+)}{g(\theta)}|\phi(\theta)|^{-\frac{1}{2}}, \quad (5{\cdot}14)$$

$$\overline{[u_2]}_+ \equiv \overline{[v_1]}_+ = -\frac{\lambda(\theta)\,Q_{12}(\cos\theta,\sin\theta) + \kappa\sin\theta}{g(\theta)}|\phi(\theta)|^{-\frac{1}{2}}, \quad (5{\cdot}15)$$

and $$\overline{[v_2]}_+ = \frac{\lambda(\theta)\,Q_{11}(\cos\theta,\sin\theta,v_+) + 2(\cos\theta - \bar{\xi}_+ v_+)}{g(\theta)}|\phi(\theta)|^{-\frac{1}{2}}. \quad (5{\cdot}16)$$

Equations (5·14)–(5·16) thus predict that when a (simple) cuspoidal point of W_+ is approached along l from the exterior side of the lacuna, the displacement jumps are singular like $d^{-\frac{1}{2}}$.

6. *Displacement singularity at a point lacuna on W.* The only situation previously excluded from the analysis of displacement singularities at the wave front is that corresponding to the shrinkage of an otherwise triangular-shaped lacuna down to a single point. This is a specialized situation in which only certain combinations of (α,β,γ) are allowed, and in particular where γ has a specific dependence on α and β

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(1, 2). At such a point the line L will intersect N_+ in such a way that $f(p|\bar{\xi}_+ \bar{\eta}_+)$ will have a real quadruple p root at $p = p_0$. This condition is made possible by the coalescing of two inflexion points on N_+ at the point where L touches N_+. Then

$$f(p|\bar{\xi}, \bar{\eta}) \approx H + f^{\mathrm{iv}}(p_0|\bar{\xi}_+, \bar{\eta}_+)\,(p-p_0)^4/4! \tag{6·1}$$

where
$$f^{\mathrm{iv}}(p_0|\bar{\xi}_+, \bar{\eta}_+) = 24(\alpha - \bar{\xi}_+^2)\,(1-\bar{\xi}_+^2). \tag{6·2}$$

There are several different explicit expressions for f^{iv} depending upon the mode by which f''', as given in equation (5·2), vanishes.

(i) $\cos\theta = 0$, then
$$(4!\sin^4\theta)^{-1} f^{\mathrm{iv}} = \alpha; \tag{6·3}$$

(ii) $\sin\theta = 0$, then
$$\lim_{\sin\theta \to 0} (4!\sin^4\theta)^{-1} f^{\mathrm{iv}} = \beta, \tag{6·4}$$

and

(iii) $[(1-\beta)^2\sin^4\theta - (1-\alpha)^2\cos^4\theta] = 0$, then

$$(4!\sin^4\theta)^{-1} f^{\mathrm{iv}} = \frac{(2-\alpha-\beta)^2}{4}\left[-1 + \frac{\{\gamma_0(\alpha,\beta)+\alpha+\beta\}^2}{4\{\gamma_0^2(\alpha,\beta) - 4\alpha\beta\}}\right]. \tag{6·5}$$

In cases (i) and (ii) the point lacunas on W_+ will, respectively, lie on the $\bar{\eta}$ and $\bar{\xi}$-axes, while for case (iii) the point lacunas will lie off the axes. In all these cases it should be noted that $f^{\mathrm{iv}} > 0$. An alternative expression for f^{iv} in terms of $v(\theta)$ is given by

$$f^{\mathrm{iv}}(p_0|\bar{\xi}_+, \bar{\eta}_+) = 2\sqrt{(B^2 - 4A)}\, v_+ \sin^4\theta (v''_+ + v^{\mathrm{iv}}_+), \tag{6·6}$$

so that, from equations (6·1) and (3·12)

$$f(p|\bar{\xi}, \bar{\eta}) \approx \frac{f^{\mathrm{iv}}}{4!}\left[\frac{4!\psi_+(\theta)}{f^{\mathrm{iv}}}\left\{-\frac{2v_+\sqrt{(B^2-4A)}}{\sin^4\theta}\right\} + (p-p_0)^4\right], \tag{6·7}$$

when $\bar{\xi}$, $\bar{\eta}$ and p are near $\bar{\xi}_+, \bar{\eta}_+$ and p_0 respectively

There is one important difference between the character of the displacement singularity near a point lacuna and that of an ordinary point or cuspoidal point on W_+. In the latter two situations, the singular part of the displacement was found to the left of W_+, whereas for a point lacuna the displacement is singular or both sides of W_+. This stems from the fact that now all four roots of $f(p|\bar{\xi}, \bar{\eta}) = 0$ are perturbed when $(\bar{\xi}, \bar{\eta})$ moves away from the point lacuna on W_+. Here also, as in section 4, the line l does not cross W_+ locally and the assumption is made that $(\bar{\xi}, \bar{\eta})$ does not lie on l.

From equation (6·7) the perturbed poles of $f^{-1}(p|\bar{\xi}, \bar{\eta})$ which lie in the upper half p-plane are given by

LEFT of W_+
$$p = p_0 + e^{\frac{1}{4}i\pi} K(\theta)|\psi_{+L}(\theta)|^{\frac{1}{4}}$$

and
$$p = p_0 + e^{\frac{1}{4}i3\pi} K(\theta)|\psi_{+L}(\theta)|^{\frac{1}{4}}. \tag{6·8}$$

RIGHT of W_+
$$p = p_0 + iK(\theta)|\psi_{+R}(\theta)|^{\frac{1}{4}}, \tag{6·9}$$

where
$$K(\theta) = \left[\frac{48 v_+\sqrt{(B^2-4A)}}{\sin^4\theta f^{\mathrm{iv}}}\right]^{\frac{1}{4}}. \tag{6·10}$$

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Residue calculations now yield,

$$u_{1L} \sim \frac{Q_{22}(\cos\theta, \sin\theta, v_+)}{2\sqrt{2\pi\tau j(\theta)}} |\psi_{+L}(\theta)|^{-\frac{3}{4}}, \tag{6·11}$$

$$u_{2L} \equiv v_{1L} \sim -\frac{Q_{12}(\cos\theta, \sin\theta)}{2\sqrt{2\pi\tau j(\theta)}} |\psi_{+L}(\theta)|^{-\frac{3}{4}}, \tag{6·12}$$

and

$$v_{2L} \sim \frac{Q_{11}(\cos\theta, \sin\theta, v_+)}{2\sqrt{2\,\pi\tau j(\theta)}} |\psi_{+L}(\theta)|^{-\frac{3}{4}}, \tag{6·13}$$

when the point $(\bar{\xi}, \bar{\eta})$ approaches the point lacuna from the left. Here

$$j(\theta) = [2v_+\sqrt{(B^2 - 4A)}]^{\frac{3}{4}} \left[\frac{f^{\mathrm{iv}}}{4! \sin^4\theta}\right]^{\frac{1}{4}}. \tag{6·14}$$

On the other hand when the point lacuna is approached from the right-hand side of W_+

$$u_{1R} \sim -\frac{Q_{22}(\cos\theta, \sin\theta, v_+)}{4\pi\tau j(\theta)} |\psi_{+R}(\theta)|^{-\frac{3}{4}}, \tag{6·15}$$

$$u_{2R} \equiv v_{1R} \sim \frac{Q_{12}(\cos\theta, \sin\theta)}{4\pi\tau j(\theta)} |\psi_{+R}(\theta)|^{-\frac{3}{4}}, \tag{6·16}$$

and

$$v_{2R} \sim -\frac{Q_{11}(\cos\theta, \sin\theta, v_+)}{4\pi\tau j(\theta)} |\psi_{+R}(\theta)|^{-\frac{3}{4}}. \tag{6·17}$$

In the above formulae, equation (6·10) is undefined when $\sin\theta = 0$, however, it can be shown that the final results remain valid in this case also, provided equation (6·4) is used in evaluating $j(\theta)$.

Finally, the possibility of $Q_{ij} = 0$ must be investigated. For a point lacuna, this possibility can only be realized on the $\bar{\xi}$ and $\bar{\eta}$-axes. If $Q_{ij} = 0$ at the particular θ-value for which the singularity is evaluated, then the formulae of equations (6·11)–(6·17) become indeterminate as the singularity is approached. Now $Q_{12}(\cos\theta, \sin\theta) = 0$ on $\theta = 0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi$ (i.e. on the $\bar{\xi}$ and $\bar{\eta}$-axes) but here $u_2 \equiv 0$ and $v_1 \equiv 0$ so that the zero of Q_{12} dominates. However, when $Q_{11} = 0$ or $Q_{22} = 0$, this is not the case.

$Q_{22} = 0$

$$u_{1L}(\xi, 0, \tau) \sim \{2\sqrt{2\pi\tau}[\beta^3(1-\alpha)]^{\frac{1}{4}}\}^{-1}(\alpha - \bar{\xi}^2)^{-\frac{1}{4}}, \tag{6·18}$$

and

$$u_{1R}(\xi, 0, \tau) \sim \{4\pi\tau[\beta^3(1-\alpha)]^{\frac{1}{4}}\}^{-1}(\bar{\xi}^2 - \alpha)^{-\frac{1}{4}}, \tag{6·19}$$

where $0 < \alpha < 1$ and $\gamma = \alpha(\beta + 1)$.

$$u_{1L}(0, \eta, \tau) \sim \frac{1}{2\sqrt{2\pi\tau}} \left[\frac{\alpha}{\beta - 1}\right]^{\frac{1}{4}} (1 - \bar{\eta}^2)^{-\frac{1}{4}}, \tag{6·20}$$

and

$$u_{1R}(0, \eta, \tau) \sim \frac{1}{4\pi\tau} \left[\frac{\alpha}{\beta - 1}\right]^{\frac{1}{4}} (\bar{\eta}^2 - 1)^{-\frac{1}{4}}, \tag{6·21}$$

where $\beta > 1$ and $\gamma = (\alpha + 1)$.

$Q_{11} = 0$

$$v_{2L}(\xi, 0, \tau) \sim \frac{1}{2\sqrt{2\pi\tau}} \left[\frac{\beta}{\alpha - 1}\right]^{\frac{1}{4}} (1 - \bar{\xi}^2)^{-\frac{1}{4}}, \tag{6·22}$$

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and $$v_{2R}(\xi, 0, \tau) \sim \frac{1}{4\pi\tau}\left[\frac{\beta}{\alpha-1}\right]^{\frac{1}{4}}(\bar{\xi}^2-1)^{-\frac{1}{4}}, \tag{6·23}$$

where $$\alpha > 1 \quad \text{and} \quad \gamma = (\beta+1).$$

$$v_{2L}(0, \eta, \tau) \sim \{2\sqrt{2}\pi\tau[\alpha^3(1-\beta)]^{\frac{1}{4}}\}^{-1}(\beta-\bar{\eta}^2)^{-\frac{1}{4}}, \tag{6·24}$$

and $$v_{2R}(0, \eta, \tau) \sim \{4\pi\tau[\alpha^3(1-\beta)]^{\frac{1}{4}}\}^{-1}(\bar{\eta}^2-\beta)^{-\frac{1}{4}}, \tag{6·25}$$

where $$0 < \beta < 1 \quad \text{and} \quad \gamma = \beta(\alpha+1).$$

The remarks of this section permit the conclusion that the displacement functions become singular like $d^{-\frac{3}{4}}$ near a point lacuna on W_+. However, if the point lacuna lies on the coordinate axes then the singularity may be weakened to $d^{-\frac{1}{4}}$.

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