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Exact wave solutions for a cylindrically bounded magnetized plasma

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Mathematically exact solutions describing propagating waves are constructed for a cold magnetized plasma bounded by a rigid cylindrical dielectric. Possible particle sources and sinks due to chemical reactions are allowed. It is found that the volume waves can be either amplified or damped because of their density dependence, while the surface waves are unaffected by the sources and sinks.

1. Introduction

Wave propagation in bounded low-temperature plasmas is of interest (Boardman 1982; Moisan *et al.* 1982; Gradov and Stenflo 1983; Hasegawa 1990; Stenflo and Gradov 1994) because of their many practical applications (Moisan *et al.* 1982; Henderson 1986; Hasegawa 1990; Selwyn *et al.* 1990; Chen 1995) such as laboratory plasma production and diagnostics, sources of coherent radiation and particle beams, devices for plasma-assisted material processing, as well as signal control and transmission in optical devices and waveguides. Bounded plasmas differ from unbounded ones in that surface waves, which are localized near the interface between the plasma and the wall, can appear. In the linear limit, the surface waves are independent modes that propagate with their own characteristics. However, at large amplitudes, often the surface and volume waves can interact and form global eigenmodes.

Non-perturbative volume and surface plasma-wave solutions of arbitrary amplitude in a cold-plasma cylinder bounded by a dielectric have been investigated recently (Stenflo 1990; Stenflo and Yu 1992, 1993, 1995; Vladimirov *et al.* 1994; Murtaza and Yu 1996). In these studies, the derivation of the evolution equations for the space–time behaviour of the wave dynamics involves no approximation on the basic equations, namely the cold-electron-fluid and the Maxwell equations. These solutions are of interest because, besides characterizing the nonlinear properties of the problem accurately, they can also be used in verifying new approximation or numerical methods. In this paper, we extend our earlier study (Murtaza and Yu 1996) of nonlinear waves in plasmas with particle sources and sinks to include external magnetic field and current. The electron source is assumed to be due to ionization, while the ion source is chosen to satisfy the physical constraints, such as the boundary conditions and particle conservation. The latter assumption is necessary because for (non-perturbative) exact solutions there is no corresponding steady or

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equilibrium state. The evolution equations are derived from the governing equations and boundary conditions by a process similar to the separation-of-variables method for solving linear partial differential equations. The resulting system of nonlinear ordinary differential equations is solved numerically. It is found that the surface and volume modes are strongly coupled, but the growth or damping due to the sources and sinks does not affect the surface waves.

2. Governing equations

We consider a cold-electron plasma in a positively charged background. The plasma is bounded cylindrically at $r = R$ by a rigid linear dielectric of constant permittivity ε_d. For non-perturbative solutions there does not exist a steady equilibrium, or lowest-order, state. That is, a wave solution also describes a dynamic equilibrium of the system. Furthermore, the system does not have to be neutral.

We assume that the electron density n_e and fluid velocity $\mathbf{v}$ are governed by the cold-fluid conservation equations

$$\partial_t n_e + \nabla \cdot (n_e \mathbf{v}) = S_e, \tag{1}$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}_0), \tag{2}$$

where S_e, $q = -e$ and m are the electron source, charge and mass, $\mathbf{E} \equiv -\nabla\varphi$ is the wave electric field, and $\mathbf{B}_0$ is the constant external magnetic field. Pressure effects are neglected. We shall assume that the electron source is from ionization, so that $S_e \propto n$ if the details of the ionization process are neglected.

The immobile (in the scale of interest) ion background is also assumed to have a source, so that the ion density $n_i(\mathbf{r}, t)$ is governed by

$$\partial_t n_i = S_i, \tag{3}$$

where the ion source S_i is to be determined later. In the absence of a steady equilibrium state, S_i is to be chosen such that the governing equations and our *Ansätze* in the next section are consistent.

We also need the Poisson equation

$$\varepsilon_0 \nabla^2 \varphi = -q(n_e - n_i), \tag{4}$$

Since the plasma is sharply bounded, the nonlinear boundary condition is (Stenflo 1990)

$$[qnv_r - \varepsilon_0 \partial_t \partial_r \varphi]_{r=R_-} = [-\varepsilon_0 \varepsilon_d \partial_t \partial_r \varphi]_{r=R_+}, \tag{5}$$

representing the continuity of the total radial current density across the interface at $r = R$ between the plasma and the dielectric. In cold-plasma theories, the thickness of the surface layer at the interface where the particle densities rapidly vanish is usually smaller than any other characteristic dimension, such as the attenuation length of the surface fields. Consistent with the cold-fluid theory, the infinitesimally thin boundary layer acts as source and sink for particles as well as momentum.

3. The evolution equations

To derive the evolution equations, we follow the method of Stenflo (1990), who studied large-amplitude electron waves in a plasma cylinder. The method is similar

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to that used by Lorenz (1963) in investigating nonlinear atmospheric waves and deterministic chaos. However, here one does not make any truncation of the higher spatial harmonics, nor make any other approximation. Instead, one chooses by trial and error the spatial wave structure such that the time and space variables can be completely separated, in a manner similar to the method of separation of variables for linear partial differential equations (Arfken 1985). Thus, for the spatial structure in the plasma, we make the *Ansätze*

$$n_e(\mathbf{r}, t) = n(t), \tag{6}$$

$$v_r(\mathbf{r}, t) = \nu_0 + (v_2 + v_1 \cos 2\theta + v_4 \sin 2\theta)\ \frac{r}{R}, \tag{7}$$

$$v_\theta(\mathbf{r}, t) = -(v_3 - v_4 \cos 2\theta + v_1 \sin 2\theta)\ \frac{r}{R}, \tag{8}$$

$$\left[\frac{q\varphi(\mathbf{r}, t)}{m}\right]_{r<R} = (\phi_c + \phi_m \cos 2\theta + \phi_n \sin 2\theta)\ \frac{r^2}{R^2} - \phi_c\ - \nu_0\,(v_2 + v_1 \cos 2\theta + v_4 \sin 2\theta)\ \frac{r}{R}, \tag{9}$$

where ν_0 is a constant radial velocity, and v_1, v_2, v_3, v_4, ϕ_c, ϕ_m and ϕ_n are functions of time only.

For the electron and ion sources, we assume

$$S_e(\mathbf{r}, t) = \nu_0 n_e / r, \tag{10}$$

$$S_i(\mathbf{r}, t) = \frac{n_0 \nu_0}{\omega_p R}\left[2\left(\frac{\omega_p n_e}{n_0} + \frac{d_\tau v_2}{R}\right) + \frac{1}{r}\,(3\, d_\tau v_1\ \cos 2\theta - d_\tau v_2)\right], \tag{11}$$

where $\tau = \omega_p t$, and we have defined the plasma frequency $\omega_p = (n_0 e^2/\varepsilon_0 m)^{1/2}$, with n_0 an arbitrary reference density. It is clear from (10) that ν_0 is related to the ionization coefficient if we assert that the electron source is due to ionization. Here we use the simplest ionization model. Using a realistic model would involve a much more complicated representation. The ions do not participate in the waves dynamics, and their source has been chosen such that the governing equations and boundary conditions can be satisfied under our *Ansätze*.

Finally, for the potential inside the dielectric, we set

$$\left[\frac{q\varphi(\mathbf{r}, t)}{m}\right]_{r>R} = (\phi_m\ \cos 2\theta + \phi_n \sin 2\theta)\ \frac{R^2}{r^2} - \nu_0\left[v_2 + (v_1\ \cos 2\theta\ + v_4 \sin 2\theta)\,\frac{R^2}{r^2}\right], \tag{12}$$

which satisfies the Laplace equation.

The *Ansätze* (6)–(12) have been chosen such that the spatial and time variations of the physical quantities are separable. In fact, substituting them into (1)–(5) and equating the coefficients of terms with similar spatial dependences, one obtains

$$d_\tau N = -2NV_2, \tag{13}$$

$$d_\tau V_1 = -2V_1V_2 - 2\Phi_m + bV_4, \tag{14}$$

$$d_\tau V_2 = -V_1^2 - V_2^2 + V_3^2 - V_4^2 - 2\Phi_c - bV_3, \tag{15}$$

$$V_3 = \tfrac{1}{2}b, \tag{16}$$

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$$d_\tau V_4 = -2V_2V_4 - 2\Phi_n - bV_1, \tag{17}$$

$$2d_\tau \Phi_c = NV_2 + \nu\,(N + d_\tau V_2)\,, \tag{18}$$

$$d_\tau \Phi_{m,n} = \frac{NV_{1,4}}{2(1+\varepsilon_d)} + \frac{1}{2}\frac{1+2\varepsilon_d}{1+\varepsilon_d}\nu d_\tau V_{1,4}, \tag{19}$$

where

$$\tau = \omega_p(t), \quad N = \frac{n}{n_0}, \quad V_j = \frac{v_j}{\omega_p R} \quad (j = 1, \ldots, 4), \quad \nu = \frac{\nu_0}{\omega_p R}, \quad \Phi_j = \frac{q\phi_j}{m\omega_p^2 R^2},$$

and $b = qB_0/m\omega_p$ is the ratio of cyclotron to plasma frequencies.

It should be mentioned that the equation for $d_t v_3$ has the original form

$$\left(d_t v_3 + \frac{2v_2v_3}{R} - \frac{qB_0}{m}v_2\right)\frac{r}{R} = -\nu_0\left(\frac{2v_3}{R} - \frac{qB_0}{m}\right), \tag{20}$$

which leads to two equations after matching the r-dependent terms. Both of the resulting equations are satisfied by (16), which would correspond to a special solution of the sourceless case (Stenflo and Yu 1995, (13)). However, it turned out that in the numerical solutions (Stenflo and Yu 1995) for the sourceless case, all the well-behaved solutions correspond to this special case.

The temporal evolution of finite-amplitude surface and volume waves in the plasma cylinder is thus governed by the coupled nonlinear first-order ordinary differential equations (13)–(18). The system is exact and has no steady state. That is, the solutions represent a dynamic equilibrium. However, for the sourceless case ($\nu = 0$) the linear limit of this system describes two upper-hybrid volume modes with frequencies $\omega^2 = 1 + b^2$, and two upper-hybrid surface modes with frequencies given by the dispersion relation $[\omega^2 - 1/(1+\varepsilon_d)]^2 - b^2\omega^2 = 0$.

4. Numerical solutions

The set of ordinary differential equations (13)–(18) can be solved numerically. A typical solution is shown in Fig. 1, where we have set $\varepsilon_d = 2$, $\nu = -0.003$ and $b = 0.01$. Figure 1(a) gives the time evolution of $N - 1$ (an eigenfunction of the a volume wave, shown as a solid line) and V_1 (an eigenfunction of the a surface wave, shown as a dashed line). Figure 1(b) shows the phase diagram ($N - 1$, V_2) of the upper-hybrid mode, and (c) shows the phase diagram (Φ_m, V_1) of the surface mode. In Fig. 1(d), the corresponding power spectra (arbitrary units) of N and V_1 (minus their mean values) are given. Note that the volume mode ($N - 1$, V_2) is modulated by the surface mode, but the reverse does not occur. For completeness, we mention that the initial conditions are $V_1 = 0.01$, $V_2 = 0.01$, $V_4 = 0.01$, $\Phi_m = 0.01$, $\Phi_n = 0.01$ and $\Phi_c = 0.01$ at $\tau = 0$.

As the ions do not participate in the high-frequency motion considered, the damping of the upper-hybrid (density) wave is due to the removal of electrons by the electron sink. For the same reason, growing density-wave solutions can appear if we reverse the sign of ν, so that there is an electron source in the plasma. The surface wave, which does not involve density fluctuations, is not affected by sources or sinks. This result is analogous to the fact that external waves can affect the surface waves but not the density waves (Stenflo and Yu 1993).

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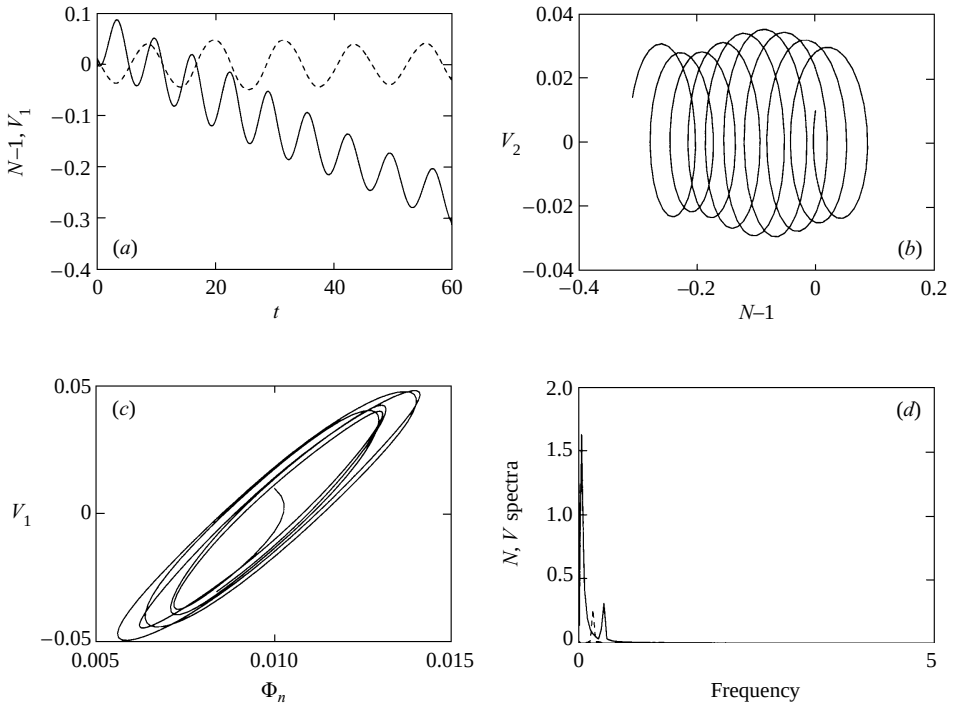


Figure 1. Periodic wave solutions. In (*a*) and (*d*), the solid and dashed lines correspond to N and V_1 and their spectra respectively. Note that the damping of the volume wave is due to the negative ν (= −0.003).

5. Conclusions

We have shown that exact three-dimensional nonlinear wave solutions exist for a magnetized cold-electron plasma bounded by a cylindrical dielectric. Linearly, four distinct modes, two volume and two surface, exist in such a system if there are no sources and sinks. Nonlinearly, the volume waves are strongly, but not resonantly, coupled to the surface waves, but not vice versa. Thus addition or removal of electrons to the system can affect only the volume mode.

The present results are useful in surface-wave generation and control of plasmas for material processing (Boardman 1982; Moisan *et al.* 1982; Henderson 1986; Hasegawa 1990; Selwyn *et al.* 1990; Chen 1995), as well as for investigating nonlinear wave instabilities and their evolution (Davidson 1974), as well as deterministic chaos (Lorenz 1963). Being mathematically exact, they are also useful for verifying new perturbation and numerical methods. However, since the type of solutions here depends crucially on the separability of the temporal and spatial dependences of the physical quantities, and the ion source term is severely restricted by this condition, the results presented here is expected to be applicable to only a very limited class of physical problems.

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