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Hypergraph Packing and Graph Embedding

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We provide sufficient conditions for packing two hypergraphs. The emphasis is on the asymptotic case when one of the hypergraphs has a bounded degree and the other is dense. As an application, we give an alternative proof for the bipartite case of the recently developed Blow-up Lemma [12].

1. Introduction

A theorem by Sauer and Spencer [15] states that any two graphs G_1, G_2 on n vertices with $2\Delta(G_1)\Delta(G_2) < n$ can be packed. This is only a factor of 2 away from the best possible condition conjectured by Bollobás and Eldridge [6]. Sauer and Spencer used a simple algorithmic argument which repeatedly improves a current imperfect packing by a simple switching operation, and after finitely many steps it ends up with a perfect packing.

This switching argument applied to hypergraphs yields good results only when both hypergraphs are rather sparse. In particular it implies a packing of two Steiner triple systems proved by Teirlinck [17].

In this paper we show how the switching argument can be refined to give packing theorems applicable when one hypergraph is dense. The refinement is based on the probabilistic method.

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In the second part we apply our hypergraph packing result to a graph embedding problem, the Blow-up Lemma of Komlós, Sárkőzy and Szemerédi [12]. Our alternative proof will be given in the bipartite case only. It is likely, however, that the same 'packing' approach can be applied in the general case.

2. Hypergraph packing

Consider two hypergraphs: $\mathscr{B}$ on a vertex set V and $\mathscr{N}$ on a vertex set V', with $|V| = |V'| = n$. We say that a bijection $\varphi : V' \to V$ is a *packing* of $\mathscr{N}$ and $\mathscr{B}$ if the edge-sets of $\mathscr{B}$ and $\mathscr{S}_\varphi := \varphi(\mathscr{N})$ are disjoint, that is, $\forall e \in \mathscr{B}, e' \in \mathscr{N} : e \neq \varphi(e')$. (For packing of graphs see [4, 9].)

For a hypergraph $\mathscr{H}$, let $\Delta_j(\mathscr{H})$ be the maximum j-tuple degree in $\mathscr{H}$, that is,

$$\Delta_j(\mathscr{H}) = \max_{T \in [V]^j} \deg_{\mathscr{H}}(T),$$

where

$$\deg_{\mathscr{H}}(T) = |\{H \in \mathscr{H} : T \subset H\}|.$$

We set $\Delta(\mathscr{H}) = \Delta_1(\mathscr{H})$ to conform with the standard notation.

Proposition 2.1. *Let $\mathscr{B}$ and $\mathscr{N}$ be two k-uniform hypergraphs on n vertices. If*

$$\Delta(\mathscr{N})\Delta_{k-1}(\mathscr{B}) + \Delta(\mathscr{B})\Delta_{k-1}(\mathscr{N}) < n - k + 2, \tag{2.1}$$

then there is a packing of $\mathscr{B}$ and $\mathscr{N}$.

Proof. We say that any bijection $\varphi : V' \to V$ produces an *attempted packing* of $\mathscr{N}$ and $\mathscr{B}$. We set $\mathscr{S} = \mathscr{S}_\varphi = (V, \varphi(\mathscr{N}))$, where $\varphi(\mathscr{N})$ stands for the image-set of the edges of $\mathscr{N}$. Finally, we say that an attempted packing $\mathscr{S}'_\varphi$ is obtained from another attempted packing $\mathscr{S}_\varphi$ by a *switching* of x and y (where $x, y \in V$), if φ' is obtained from φ by only switching the preimages of x and y.

Any set $T \in \mathscr{B} \cap \mathscr{S}_\varphi$ is called a *conflict*. In this proof we will follow the strategy of Sauer and Spencer and eliminate all conflicts by a sequence of switchings beginning with an arbitrary initial attempted packing φ_0. Let T be a conflict of a current attempted packing φ. Let x be any vertex in T. Consider possible candidates for a switching partner y. Obviously, $y \notin T$. To avoid new conflicts, $y \notin T$ is not a good candidate if and only if either of the following two cases holds:

$$\exists T' : \ T' \cup \{x\} \in \mathscr{S}, \ T' \cup \{y\} \in \mathscr{B} \quad \text{or} \quad \exists T'' : \ T'' \cup \{x\} \in \mathscr{B}, \ T'' \cup \{y\} \in \mathscr{S}.$$

In the first case the vertex x can be contained in at most $\Delta(\mathscr{S}) = \Delta(\mathscr{N})$ many sets $T' \cup \{x\}$, and, in turn, the set T' is contained in at most $\Delta_{k-1}(\mathscr{B})$ sets $T' \cup \{y\} \in \mathscr{B}$. Excluding $y = x$, this forbids at most

$$\Delta(\mathscr{N})\Delta_{k-1}(\mathscr{B}) - 1$$

vertices. In the second case, conversely, there are at most $\Delta(\mathscr{B})$ sets $T'' \cup \{x\} \in \mathscr{B}$ and at

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most $\Delta_{k-1}(\mathscr{S}) = \Delta_{k-1}(\mathscr{N})$ extensions to $T'' \cup \{y\} \in \mathscr{S}$, a total of at most

$$\Delta(\mathscr{B})\Delta_{k-1}(\mathscr{N}) - 1$$

forbidden vertices. Summing up these two cases, we see, by inequality (2.1), that, for each conflict T and $x \in T$, there will always be among the remaining $n-k$ vertices at least one switching partner y. Switching x and y around avoids the conflict in question without introducing a new one. □

The parameter $\Delta(\mathscr{H})$ may be as large as $\binom{n}{k-1}$. Therefore, a serious drawback of the above proposition is that if $\Delta(\mathscr{B})$ or $\Delta(\mathscr{N})$ has order of magnitude larger than n, then it does not apply, because inequality (2.1) is not satisfied. (Note that this cannot happen in the graph case $k = 2$.) On the other hand, when both $\Delta(\mathscr{B}) = O(n)$ and $\Delta(\mathscr{N}) = O(n)$, then $|\mathscr{B}| = O(n^2)$ and $|\mathscr{N}| = O(n^2)$, and already for $k = 5$ a random bijection will be a perfect packing with probability at least $1 - O(1/n)$. In general, the probabilistic method yields much more: *if* $|\mathscr{B}||\mathscr{N}| < \binom{n}{k}$ *then there is a packing of* $\mathscr{B}$ *and* $\mathscr{N}$. Thus, for instance, as was observed by Alon [1] and proved by another method in [7], every pair of partial $(1, t, k, n)$-designs with $k \geqslant 2$ can be packed.

Yet, Proposition 2.1 is not completely useless. One immediate implication is a theorem of Teirlinck [17] from 1977 stating that all pairs of Steiner triple systems, that is, $(1, 2, 3, n)$-designs can be packed. In fact, the proof given here is Teirlinck's original argument in disguise.

Corollary 2.1. *Let* $\mathscr{B}$ *and* $\mathscr{N}$ *be two* 3*-uniform hypergraphs on* n *vertices. If* $\Delta_2(\mathscr{N}) = \Delta_2(\mathscr{B}) = 1$*, then there is a packing of* $\mathscr{B}$ *and* $\mathscr{N}$.

Proof. We have $\Delta(\mathscr{N}) \leqslant (n-1)/2$ and $\Delta(\mathscr{B}) \leqslant (n-1)/2$. Thus, if n is even, then inequality (2.1) is satisfied and $\mathscr{B}$ and $\mathscr{N}$ can be packed. Assume further that n is odd. Then we only have $\Delta(\mathscr{N}) + \Delta(\mathscr{B}) \leqslant n - 1$ and (2.1) may fail by just one.

Referring to the proof of Proposition 2.1, let T be a current conflict and $x \in T$. For every switching candidate $y \neq x$, each instance of

$$\{x, a, b\} \in \mathscr{B} \text{ and } \{y, a, b\} \in \varphi(\mathscr{N})$$

or

$$\{x, a, b\} \in \varphi(\mathscr{N}) \text{ and } \{y, a, b\} \in \mathscr{B}$$

constitutes one reason for rejecting y. If there is no switching partner for x then

$$\deg_{\varphi(\mathscr{N})}(x) = \deg_{\mathscr{B}}(x) = (n-1)/2.$$

Moreover, both vertices of $T \setminus \{x\}$ have no reason for rejecting (other than that they belong to T), and all remaining $n-3$ vertices each have exactly one reason for rejecting.

Suppose this is indeed the case. Pick any pair T' such that $T' \cup \{x\} \in \varphi(\mathscr{N})$. Let v be such that $T' \cup \{v\} \in \mathscr{B}$ and let u be such that $\{x, v, u\} \in \varphi(\mathscr{N})$. Finally, let B be the unique triple of $\mathscr{B}$ containing $\{v, u\}$ and let z be the third element of B. (The existence of v and z follows from the assumption that each $y \notin T$ has exactly one reason for rejecting, while the existence of u is guaranteed by the fact that $\deg_{\varphi(\mathscr{N})}(x) = (n-1)/2$.)

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Let φ' be the bijection obtained from φ by switching x and z. It avoids conflict T and introduces the new conflict B. Clearly, no more conflicts are introduced, since otherwise there would be a vertex $y \notin T$ with more than one reason for rejecting. But for φ', we have $\{z\} \cup T' \in \varphi'(\mathcal{N})$ and $\{v\} \cup T' \in \mathcal{B}$, which is a reason for rejecting $v \in B$. Thus we can find a good switching partner y for z and reduce the number of conflicts by 1. Hence, if there were originally C conflicts, after applying a sequence of at most $2C$ switchings, we will arrive at a perfect packing of $\mathcal{B}$ and $\mathcal{N}$. □

Packing $\mathcal{B}$ and $\mathcal{N}$ when $|\mathcal{B}||\mathcal{N}| \gg n^k$ becomes a much harder problem and is sometimes impossible. Alon [1] proved probabilistically the existence of pairs of partial $(1, t, k, n)$-designs for all $t \leqslant k \leqslant 2t-2$ which cannot be packed, leaving the case $k = 2t-1$ open. Note that in this case $|\mathcal{B}||\mathcal{N}| = O(n^{k+1})$. Below we propose a packing lemma, which can be applied to this case, provided one of the hypergraphs is very sparse and the other is reasonably dense.

So, we are now introducing some asymmetry to the problem by assuming that one hypergraph is *nice*, say $\mathcal{N}$, while the other is *bad*, at least as far as the packing goes. We will be more harsh on the hypergraph $\mathcal{B}$, therefore.

The idea is to tighten the bound (2.1) of Proposition 2.1 considerably, by counting, in the second case, only those sets T'' which are simultaneously contained in a set of $\mathcal{B}$ and in a set of $\varphi(\mathcal{N})$. Let $\mathcal{N}^{k-1}$ be the shadow of $\mathcal{N}$ on the $(k-1)$st level, that is, the family of all $(k-1)$-element sets that are contained in any hyperedge of $\mathcal{N}$. Define

$$\mathcal{B}_v = \{B \in [V]^{k-1} : B \cup \{v\} \in \mathcal{B}\}$$

and

$$D_\varphi(\mathcal{B}) = \max_{v \in V} |\mathcal{B}_v \cap \varphi(\mathcal{N}^{k-1})|.$$

If the term $\Delta(\mathcal{B})$ in (2.1) is replaced by $D_\varphi(\mathcal{B})$, we can certainly still make a switch without introducing new conflicts. However, this switch may cause $D_\varphi(\mathcal{B})$ to grow by as much as $(k-1)\Delta(\mathcal{N})$. More precisely, if φ' is obtained from φ by a single switching, then

$$D_{\varphi'}(\mathcal{B}) - D_\varphi(\mathcal{B}) \leqslant (k-1)\Delta(\mathcal{N}).$$

The reason is that a vertex of $\mathcal{N}$ may belong to as many as $(k-1)\Delta(\mathcal{N})$ sets of $\mathcal{N}^{k-1}$. To accommodate this growth, we need to add to the left-hand side of (2.1) an extra term

$$|\mathcal{B} \cap \varphi(\mathcal{N})|(k-1)\Delta(\mathcal{N})\Delta_{k-1}(\mathcal{N}).$$

It represents the number of conflicts of the initial attempted packing φ (which is $|\mathcal{B} \cap \varphi(\mathcal{N})|$), multiplied by a maximum value of the increase

$$D_{\varphi'}(\mathcal{B})\Delta_{k-1}(\mathcal{N}) - D_\varphi(\mathcal{B})\Delta_{k-1}(\mathcal{N})$$

due to a single switching. Hence, we arrive at the following general result.

Lemma 2.1. *Let $\mathcal{B}$ and $\mathcal{N}$ be two k-uniform hypergraphs on n vertices. If there exists an attempted packing φ such that*

$$\Delta(\mathcal{N})\Delta_{k-1}(\mathcal{B}) + D_\varphi(\mathcal{B})\Delta_{k-1}(\mathcal{N}) + |\mathcal{B} \cap \varphi(\mathcal{N})|(k-1)\Delta(\mathcal{N})\Delta_{k-1}(\mathcal{N}) < n - k + 2, \quad (2.2)$$

then there is a packing of $\mathcal{B}$ and $\mathcal{N}$. □

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The asymmetry between $\mathscr{N}$ and $\mathscr{B}$ appears naturally in applications where one of the hypergraphs has bounded degree while the other has size of the order of n^k. Our next results address such cases.

Theorem 2.1. *For all integers $k \geqslant 2$ and $\Delta \geqslant 1$, and for all $\gamma > 0$, there exists $\varepsilon > 0$ and n_0 such that the following holds. If $\mathscr{B}$ and $\mathscr{N}$ are two k-uniform hypergraphs on $n > n_0$ vertices satisfying*

(i) $\Delta(\mathscr{N}) \leqslant \Delta$,
(ii) $\Delta(\mathscr{B}) \leqslant \varepsilon n^{k-1}$,
(iii) $\Delta_{k-1}(\mathscr{B}) \leqslant \varepsilon n$,

then

(a) *there is a packing of $\mathscr{N}$ and $\mathscr{B}$,*
(b) *all but $\gamma n!$ initial attempted packings φ can be turned into a perfect packing by a sequence of at most $(2\Delta\varepsilon(k-2)!/\gamma)n$ switchings.*

Proof. For $k = 2$ the theorem follows quickly with $\varepsilon = 1/(3\Delta)$ and $n_0 = 2$ from Proposition 2.1. Indeed, then inequality (2.1) is satisfied and part (a) follows. Moreover, for all but $\gamma n!$ bijections φ, the number of initial conflicts $|\mathscr{B} \cap \varphi(\mathscr{N})|$ is at most

$$\frac{|\mathscr{N}||\mathscr{B}|}{\gamma\binom{n}{2}} \leqslant \Delta\varepsilon n,$$

and part (b) follows as well.

For $k \geqslant 3$, making no attempt to find the best possible constant, we prove the theorem with $\varepsilon = \frac{\gamma}{6k!\Delta^3}$, and n_0 such that the rightmost inequality in (2.7) and $\binom{n}{k} > \frac{n^k}{k^2(k-2)!}$ are satisfied for all $n > n_0$. We will apply Lemma 2.1. Consider a random bijection $\varphi : V' \to V$. Then the expected value of the random variable $X = |\mathscr{B} \cap \varphi(\mathscr{N})|$, counting the conflicts, equals

$$\mathrm{Ex}(X) = \mu = \frac{|\mathscr{N}||\mathscr{B}|}{\binom{n}{k}}.$$

By Markov's inequality,

$$\Pr(X > (2/\gamma)\mu) < \gamma/2.$$

Since we have $|\mathscr{N}| \leqslant \Delta n/k$ and $|\mathscr{B}| \leqslant \Delta(\mathscr{B})n/k$, this sets the promised upper bound on the number of iterations. Indeed, it follows that for at least $(1 - \gamma/2)n!$ bijections φ we have

$$|\mathscr{B} \cap \varphi(\mathscr{N})| \leqslant (2\Delta(k-2)!\varepsilon/\gamma)n. \tag{2.3}$$

To show that all these initial conflicts can actually be avoided by a sequence of switchings, we need to verify condition (2.2) of Lemma 2.1. We already have, by our assumptions (i) and (iii),

$$\Delta(\mathscr{N})\Delta_{k-1}(\mathscr{B}) < \varepsilon\Delta n \tag{2.4}$$

and, by (i) and (2.3), for all but $(\gamma/2)n!$ bijections φ,

$$|\mathscr{B} \cap \varphi(\mathscr{N})|(k-1)\Delta(\mathscr{N})\Delta_{k-1}(\mathscr{N}) < (2\Delta^3(k-1)!\varepsilon/\gamma)n. \tag{2.5}$$

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We will now show that the remaining term of the left-hand side of (2.2) is also small for almost all φ.

In order to bound $D_\varphi(\mathscr{B})$, consider, for every $v \in V$, the random variable

$$X_v = |\mathscr{B}_v \cap \varphi(\mathscr{N}^{k-1})|.$$

Set $b = |\mathscr{B}_v|$ and $r = |\mathscr{N}^{k-1}|$, and observe that

$$n/k \leqslant r = |\mathscr{N}^{k-1}| \leqslant \Delta n$$

and

$$b = |\mathscr{B}_v| \leqslant \Delta(\mathscr{B}) \leqslant \varepsilon n^{k-1}.$$

(For the lower bound on r we may assume that $\mathscr{N}$ has no isolated vertices.) The expectation of X_v is equal to

$$\mathrm{Ex}(X_v) = \frac{|\mathscr{B}_v||\mathscr{N}^{k-1}|}{\binom{n}{k-1}} = \frac{br}{\binom{n}{k-1}} < k!\Delta\varepsilon n.$$

If we show that

$$\text{with probability at least } 1-\gamma/(2n), \text{ we have } X_v < 2k!\Delta\varepsilon n, \tag{2.6}$$

then for all but $(1-\gamma)n!$ bijections φ, we have (2.5) and

$$D_\varphi(\mathscr{B}) < 2k!\Delta\varepsilon n.$$

These two inequalities, together with (2.4), bound the left-hand side of (2.2) from above by

$$\varepsilon\Delta n + 2k!\Delta^2\varepsilon n + (2\Delta^3 k!\varepsilon/\gamma)n$$

and thus make it smaller than $n-k+2$.

To prove (2.6), define the intersection graph R with vertex set $V(R) := \mathscr{N}^{k-1}$ and edge set

$$E(R) := \{\{S_1, S_2\} : S_1 \cap S_2 \neq \emptyset\}.$$

Notice that $\Delta(R) < (k-1)^2\Delta =: t$. Thus, by the Hajnal–Szemerédi theorem [8], there is a partition $V(R) = U_1 \cup \ldots \cup U_t$, such that the U_is are independent and have cardinality $\lfloor \frac{r}{t} \rfloor$ or $\lceil \frac{r}{t} \rceil$.

This corresponds to partitioning the family $\mathscr{N}^{k-1}$ into $\mathscr{E}_1 \cup \ldots \cup \mathscr{E}_t$, where each $\mathscr{E}_i$ consists of $\ell \in \{\lfloor \frac{r}{t} \rfloor, \lceil \frac{r}{t} \rceil\}$ pairwise disjoint edges.

If $X_v \geqslant 2k!\Delta\varepsilon n$, then there must be an index i such that $|\mathscr{B}_v \cap \varphi(\mathscr{E}_i)| \geqslant 2k!\Delta\varepsilon n/t$. Let $\mathscr{E}_i = \{E_1, \ldots, E_\ell\}$.

Suppose that we randomly generate an embedding φ in a sequential fashion, and that the images of $V(\mathscr{E}_i)$ are being chosen first, in the order $\varphi(E_1), \ldots, \varphi(E_\ell)$. Define I_j to be a random indicator variable that takes value 1 if $\varphi(E_j) \in \mathscr{B}_v$ and value 0 otherwise. We would like to give an upper bound for the probability that

$$Z_i := \sum I_j = |\mathscr{B}_v \cap \varphi(\mathscr{E}_i)| \geqslant 2k!\Delta\varepsilon n/t.$$

Choosing the image of E_j can be interpreted as drawing a ball out of $\binom{n-(j-1)(k-1)}{k-1}$

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balls in a bin, where the balls represent all $(k-1)$-subsets of the still available vertex set $V \setminus \{\varphi(E_1) \cup \ldots \cup \varphi(E_{j-1})\}$. After a ball is drawn, we remove all balls from the bin that represent those $(k-1)$-sets which intersect with the one we have just drawn. Suppose that at the beginning the balls representing the sets of $\mathscr{B}_v$ are *red*, the others are *white*, and no ball is *pink*. Then $I_j = 1$ if and only if the jth ball is red.

Since the I_js are not independent, we find a way out by defining another sequence J_j of indicators as follows. Imagine the same balls-and-bins experiment as before, but now, after each drawing, we count how many red or pink balls have been removed from the bin and recolour this many of the remaining white balls in the bin by pink. Now let $J_j = 1$ if the jth selected ball is red or pink, and $J_j = 0$ otherwise. Clearly, $I_j \leqslant J_j$, the total number of red and pink balls in the bin remains equal to b throughout the entire experiment, and, most importantly, the J_js are independent. Set $Y := \sum J_j$ and recall that $\ell \leqslant r/t + 1 \leqslant n/(k-1)^2 + 1$ and $b \leqslant \varepsilon n^{k-1}$. Then

$$\Pr(J_j = 1) \leqslant \Pr(J_\ell = 1) \leqslant \frac{b}{\binom{n-(\ell-1)(k-1)}{k-1}} < \varepsilon k!.$$

Hence, $\mathrm{Ex}(Y) \leqslant \varepsilon k!\ell$, and therefore

$$\Pr(Z_i \geqslant 2k!\Delta\varepsilon n/t) \leqslant \Pr(Z_i \geqslant 1.5\varepsilon k!\ell) \leqslant \Pr(Y \geqslant 1.5\varepsilon k!\ell) \leqslant \Pr(Y \geqslant \mathrm{Ex}(Y) + \varepsilon k!\ell/2),$$

which can be further bounded from above by the Chernoff bound

$$\exp\left(\frac{-(\frac{1}{2}\varepsilon k!\ell)^2}{2(\mathrm{Ex}(Y) + \frac{1}{2}\varepsilon k!\ell/3)}\right) \leqslant \exp\left(-\frac{3}{28}\varepsilon k!\ell\right).$$

Consequently,

$$\Pr(X_v \geqslant 2k!\Delta\varepsilon n) \leqslant \sum_{i=1}^{t} \Pr(Z_i \geqslant 2k!\Delta\varepsilon n/t) < t\exp\left(-\frac{3}{28}\varepsilon k!\ell\right).$$

Since $\ell \geqslant \frac{n}{k^3\Delta}$, we have, for sufficiently large n,

$$t\exp\left(-\frac{3}{28}\varepsilon k!\ell\right) \leqslant (k-1)^2\Delta\exp\left(-\frac{3\varepsilon k!}{28k^3\Delta}n\right) < \frac{\gamma}{2n} \tag{2.7}$$

and statement (2.6) is proved. This completes the proof of Theorem 2.1. □

If condition (iii) of Theorem 2.1 fails to be true, but just by a little, that is, all but at most εn^{k-1} sets $T \in [n]^{k-1}$ do have $\deg_{\mathscr{B}}(T) < \varepsilon n$ (call the exceptions *bad*), we may still be able to pack the two hypergraphs. What we need is that the property is hereditary in a sense that all but εn^{k-2} sets $T \in [n]^{k-2}$ have each at most εn extensions to bad sets of size $k-1$, and so on, until we reach the vertices. For vertices, however, to ensure a good start we would rather assume that all of them are good. Then, in fact, condition (ii) would follow. Let us now formulate this concept precisely.

For a k-uniform hypergraph $\mathscr{B}$ and a real number $\varepsilon > 0$, denote by $\mathscr{B}^{(\varepsilon)}_{k-1}$ the $(k-1)$-uniform hypergraph of all $(k-1)$-element sets T with $\deg_{\mathscr{B}}(T) > \varepsilon n$. Now, recursively, for every $2 \leqslant j \leqslant k$, define $\mathscr{B}^{(\varepsilon)}_{j-1}$ as the $(j-1)$-uniform hypergraph of all $(j-1)$-element sets T with $\deg_{\mathscr{B}^{(\varepsilon)}_j}(T) > \varepsilon n$. Note that $\mathscr{B}^{(\varepsilon)}_1$ is a set of singletons. If this set is empty then

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one can prove by induction on j that, for all $j = 2, \ldots, k$,

$$\Delta(\mathscr{B}_j^{(\varepsilon)}) < \varepsilon n^{j-1},$$

and thus

$$|\mathscr{B}_j^{(\varepsilon)}| < \varepsilon n^j / j.$$

To cover the (trivial) case $k = 2$ set $\mathscr{B}_k^{(\varepsilon)} = \mathscr{B}$.

Theorem 2.2. *For all integers $k \geqslant 2$ and $\Delta \geqslant 1$, and for all $\gamma > 0$ and $C > 0$, there exists $\varepsilon > 0$ and n_0 such that the following holds. If $\mathscr{B}$ and $\mathscr{N}$ are two k-uniform hypergraphs on $n > n_0$ vertices satisfying*

(i) $\Delta(\mathscr{N}) \leqslant \Delta$, *and*
(ii) $\Delta(\mathscr{B}_2^{(\varepsilon)}) < \varepsilon n$ (i.e., $\mathscr{B}_1^{(\varepsilon)} = \emptyset$),

then

(a) *there is a packing of $\mathscr{N}$ and $\mathscr{B}$,*
(b) *all but $\gamma n!$ initial attempted packings φ can be turned into a perfect packing by a sequence of at most Cn switchings.*

Proof. For $k = 2$ this theorem coincides with the case $k = 2$ of Theorem 2.1. For $k \geqslant 3$, we will now try to pack $\mathscr{B}_j^{(\varepsilon)}$ and $\mathscr{N}^j$ for each $j = 2, \ldots, k$. Arguing as in the proof of Theorem 2.1, separately for each j, and with γ replaced by $\gamma/(k-1)$, we obtain that for all but at most $\gamma n!$ bijections φ, for each $j = 2, \ldots, k$, there are at most

$$\frac{(2(k-1)/\gamma)|\mathscr{N}^j||\mathscr{B}_j^{(\varepsilon)}|}{\binom{n}{j}} = O(1)\varepsilon n$$

conflicts between $\mathscr{B}_j^{(\varepsilon)}$ and $\mathscr{S}_\varphi^j$, and that, at the same time,

$$D_\varphi(\mathscr{B}_j^{(\varepsilon)}) < \frac{2\Delta n \varepsilon n^{j-1}}{\binom{n}{j-1}} = O(1)\varepsilon n,$$

where the constants in $O(1)$ depend on k, Δ, γ and C but not on ε or n.

Our strategy is to eliminate all conflicts of order two first, then of order three, and so on. When handling conflicts of order j we may add some new conflicts of higher order but never of smaller order. A single switching may create only at most $\Delta(\mathscr{N}^l) = O(1)$ new conflicts of order $l > j$. So, after we are done with the conflicts of order j, we may be facing just a few more conflicts of order $j+1$ than existed originally. But with such room to spare we will handle them as well. Therefore, this process will terminate in a perfect packing of $\mathscr{N}$ and $\mathscr{B}$ after some Cn steps.

To achieve this goal, we need to have inequality (2.2) of Lemma 2.1 satisfied for all pairs of hypergraphs $\mathscr{N}^j$ and $\mathscr{B}_j^{(\varepsilon)}$, $j = 2, \ldots, k$, and with some extra slack to accommodate the possible growth of the number of conflicts at the higher levels. However, this inequality can be now relaxed a little by replacing the term $\Delta_{j-1}(\mathscr{B}_j^{(\varepsilon)})$ with

$$\max\{\deg_{\mathscr{B}_j^{(\varepsilon)}}(T) \ : \ T \in [V]^{j-1} \setminus \mathscr{B}_{j-1}^{(\varepsilon)}\}.$$

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We proceed as follows. Since there are no conflicts of size 1 (because of (ii)), it thus suffices to prove by induction on $j = 2, \ldots, k$ that, if there are no conflicts of size less than $j \geqslant 2$ and $T \in \mathscr{B}_j^{(\varepsilon)} \cap \mathscr{S}^j$ is a conflict of size j, then there are vertices $x, y \in V$ such that the new attempted packing obtained by switching x and y avoids T without producing any new conflicts of size at most j. Let $x \in T$. As in the proof of Proposition 2.1, if y is not a good candidate for a switching partner of x then, for some $i \leqslant j$, one of the two cases must hold:

$$\exists T' : \ T' \cup \{x\} \in \mathscr{S}^i, \ T' \cup \{y\} \in \mathscr{B}_i^{(\varepsilon)}, \quad \text{or} \quad \exists T'' : \ T'' \cup \{x\} \in \mathscr{B}_i^{(\varepsilon)}, \ T'' \cup \{y\} \in \mathscr{S}^i.$$

But, owing to our assumption of the absence of smaller conflicts, in the first case we have $T' \notin \mathscr{B}_{i-1}^{(\varepsilon)}$, and thus the number of extensions $T' \cup \{y\}$ can be bounded by $\deg_{\mathscr{B}_i^{(\varepsilon)}}(T') < \varepsilon n$. Hence, by choosing ε small enough, we will force inequality (2.2) to hold and thus ensure that there is always a choice of y to eliminate the conflict T. □

3. Application: an alternative proof of the Blow-up Lemma

Recently, Komlós, Sárkőzy and Szemerédi [12, 13] have developed a new powerful tool in extremal graph theory, the so-called Blow-up Lemma. Roughly speaking, it says that any graph H with bounded maximum degree can be embedded into any super-regular graph G, even if H and G are of the same order. Together with the Regularity Lemma [16], which guarantees that every graph has a super-regular partition, this has proved to be the key instrument for the solution of many difficult problems in graph theory such as a tree-packing conjecture of Bollobás on spanning trees [9], the Pósa–Seymour conjecture on powers of Hamiltonian cycles [10] or the Alon–Yuster conjecture on perfect F-matchings [11].

The idea of the original proof of the Blow-up Lemma is to embed the vertices of H into G sequentially via a randomized greedy algorithm until the choices for the remaining vertices become too limited, and then complete the embedding at once using the König–Hall theorem. Rödl and Ruciński [14] have proposed an alternative proof of the Blow-up Lemma: they embed H into G in a constant number of rounds. Here we give yet another proof based on hypergraph packing. We will deal with the bipartite case only, though it is likely that the same 'packing' approach can be applied in the general case. Roughly speaking, this approach aims right from the beginning at the König–Hall theorem and thus finds the embedding 'at once'.

We now define regularity and super-regularity and state the Blow-up Lemma in the bipartite case.

For a graph G and two disjoint subsets V_1 and V_2 of its vertex set $V(G)$, denote by $e_G(V_1, V_2)$ the number of edges of G with one endpoint in V_1 and the other in V_2, and set $d_G(V_1, V_2) = \frac{e_G(V_1, V_2)}{|V_1||V_2|}$ for the density of the pair (V_1, V_2) in G.

Given $\varepsilon > 0$, a bipartite graph $G = (V_1 \cup V_2, E)$ with $|V_1| = |V_2| = n$ is called *ε-regular* if, for every pair of sets U_1, U_2, where $U_i \subseteq V_i$ and $|U_i| > \varepsilon n$, $i = 1, 2$,

$$|d_G(U_1, U_2) - d_G(V_1, V_2)| < \varepsilon.$$

Let $\delta(G)$ stand for the minimum degree in graph G. Given $\varepsilon > 0$ and $0 < \delta < 1$, an

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ε-regular bipartite graph is called *super* (δ, ε)*-regular*, if

$$\delta(G) \geqslant (\delta - \varepsilon)n.$$

An *embedding* of a graph $H = (V', E')$ into a graph $G = (V, E)$ is an injection $\varphi : V' \to V$ such that the edges of H are mapped onto the edges of G: $\forall \{u, w\} \in E' : \{\varphi(u), \varphi(w)\} \in E$. Simply, an embedding of H into G points out a specific subgraph of G isomorphic to H. The theorem below is the bipartite case of the result in [12].

Theorem 3.1. *For every choice of* Δ *and* δ *there exists an* $\varepsilon > 0$ *such that every bipartite graph* $H = (X_1 \cup X_2, E_H)$*, with* $|X_1| = |X_2| = n$ *and maximum degree* $\Delta(H) = \Delta$*, can be embedded into any super* (δ, ε)*-regular graph* $G = (V_1 \cup V_2, E_G)$ *with* $|V_1| = |V_2| = n$.

We first state without proof an elementary lemma. It lists some simple properties of ε-regular graphs and super (δ, ε)-regular graphs.

Let $G = (V_1 \cup V_2, E)$ be an ε-regular bipartite graph with $|V_1| = |V_2| = n$. Set $N_G(v)$ for the neighbourhood of vertex v in graph G. For $S \subseteq V_1$ denote $N_G(S) := \bigcap_{v \in S} N_G(v)$ (please note the non-standard notation). For a real number δ, $0 < \delta < 1$, a set $S \subseteq V_1$ is called δ*-good*, or simply *good*, if

$$|N_G(S)| \geqslant (\delta - \varepsilon)^{|S|} n,$$

and *bad* otherwise.

Lemma 3.1. *For every* $k \geqslant 1$ *and* $0 < \delta < 1$ *there exists an* $\varepsilon > 0$ *such that the following statements are true for any* ε*-regular bipartite graph* $G = (V_1 \cup V_2, E)$ *with density at least* δ *and* $|V_1| = |V_2| = n$*, and for every* $1 \leqslant j \leqslant k$.

(a) *Every good* j*-set is contained in at most* εn *bad* $(j+1)$*-sets, and*
(b) *all but at most* $j\varepsilon\binom{n}{j}$ j*-sets are good.*
(c) *If* G *is super* (δ, ε)*-regular, then every* $v \in A$ *is contained in at most* εn^j *bad* $(j+1)$*-sets.*
(d) *If* G *is super* (δ, ε)*-regular with* $\delta \geqslant 2\varepsilon$ *then* G *fulfils the König–Hall condition and therefore has a perfect matching.* □

Proof of Theorem 3.1. It suffices to prove the theorem for all n greater than an arbitrary constant n_0. Indeed, if we choose $\varepsilon < \frac{1}{n_0}$ and $\varepsilon < \delta$, then for all $n \leqslant n_0$ the definition of super (δ, ε)-regularity of G already implies that $G = K_{n,n}$. Our choice of n_0 will be such that Theorem 2.2 can be applied and that inequality (3.1) will hold.

It is easy to show that there exists a graph H' on the vertex set $X_1 \cup X_2$ that contains H as a spanning subgraph, has $\Delta(H') \leqslant \Delta + 1$ and all vertices in X_2 have degree exactly Δ. Obviously, if H' can be embedded into G, then so can H, and hence we may assume without loss of generality that H already has the above properties of H'.

Before generating an embedding $\varphi^* : X_1 \cup X_2 \to V_1 \cup V_2$, we will first construct a mapping $\varphi : X_1 \to V_1$ that satisfies some particular properties which will allow for an easy extension to the final embedding φ^*.

Denote the vertices of V_2 by $v_1, v_2, \ldots, v_n$, the vertices of X_2 by $x_1, x_2, \ldots, x_n$ and their neighbourhoods by $N_i := N_H(x_i)$.

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For any $\varphi : X_1 \to V_1$ construct an auxiliary bipartite graph

$$A_\varphi := (X_2 \cup V_2, E_\varphi),$$

where $\{x_i, v_j\} \in E_\varphi$ if and only if $\varphi(N_i) \subseteq N_G(v_j)$. Notice that, if A_φ has a perfect matching, say $(\{x_i, v_i\})_{i\in[n]}$, then we can extend $\varphi : X_1 \to V_1$ to $\varphi^* : X_1 \cup X_2 \to V_1 \cup V_2$ by setting $\varphi^*(x_i) := v_i$ for all $x_i \in X_2$, and $\varphi^*(x) := \varphi(x)$ for all $x \in X_1$. This is obviously an embedding of H into G, since for any $x \in X_1$ and $x_i \in X_2$

$$\{x, x_i\} \in E_H \Rightarrow x \in N_i \Rightarrow \varphi^*(x) \in \varphi^*(N_i) \subseteq N_G(v_i) \Rightarrow \{\varphi^*(x), \varphi^*(x_i)\} \in E_G.$$

We will prove the existence of a perfect matching in A_φ using Lemma 3.1(d). To verify that A_φ is ε_1-regular for some ε_1, we will rely on the following criterion from [2].

Lemma 3.2. *If $G = (X, Y, E)$ is a bipartite graph, $|X| = |Y| = n$, with at least $(1-5\epsilon)n^2/2$ pairs of vertices $v, u \in X$ satisfying*

$$\deg_G(v), \deg_G(u) > (d - \epsilon)n,$$

and

$$|N_G(\{v, u\})| < (d + \epsilon)^2 n,$$

then G is $(16\epsilon)^{1/5}$-regular.

The assumptions of Lemma 3.2 (with ε_2) will be verified for graph A_φ in the next claim. The same claim will also imply that $\delta(A_\varphi) \geqslant (\delta^\Delta - \varepsilon_2)$.

Claim 3.1. *For every $d, \delta, \Delta, \varepsilon_2$, $\delta \leqslant d$, there exists an $\varepsilon < \varepsilon_2$ such that, if G and H are like in Theorem 3.1 and $d = d_G(V_1, V_2)$, then there exists a mapping $\varphi : X_1 \to V_1$ with the following properties.*

(1) *For all $x \in V_1$, $\deg_{A_\varphi}(x) \geqslant (\delta^\Delta - \varepsilon_2)n$.*

(2) *For all $v \in V_2$, $\deg_{A_\varphi}(v) \geqslant (\delta^\Delta - \varepsilon_2)n$.*

(3) *For all but at most εn vertices $v \in V_2$, $\deg_{A_\varphi}(v) \geqslant (d^\Delta - \varepsilon_2)n$.*

(4) *For all but at most $\varepsilon\binom{n}{2}$ pairs $v, w \in V_2$, $|N_{A_\varphi}(\{v, w\})| \leqslant (d^{2\Delta} + \varepsilon_2)n$.*

From the properties (3)–(4) guaranteed by Claim 3.1, it follows by Lemma 3.2 that, for some constant $\varepsilon_1 > \varepsilon_2$ depending on ε_2 only, A_φ is ε_1-regular. Moreover, it follows from properties (1)–(2) of Claim 3.1 that

$$\delta(A_\varphi) \geqslant (\delta^\Delta - \varepsilon_2)n \geqslant (\delta^\Delta - \varepsilon_1)n.$$

Hence, A_φ is super (δ, ε_1)-regular and, by Lemma 3.1(d), A_φ has a perfect matching.

It only remains to prove Claim 3.1. We will prove it via Theorem 2.2. Consider the hypergraph $\mathcal{N} = (X_1, \{N_1, \ldots, N_n\})$ and notice that it is Δ-uniform and has maximum degree at most $\Delta + 1$. Furthermore, consider the Δ-uniform hypergraph $\mathcal{B}$ on the vertex set V_1 whose hyperedges consist of the bad Δ-sets $B \subseteq V_1$, that is, of those sets B for which $|N_G(B)| < (\delta - \varepsilon)^\Delta n$. Our first task is to pack $\mathcal{N}$ and $\mathcal{B}$.

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Why are we interested in packing $\mathscr{N}$ and $\mathscr{B}$? Consider property (1) and recall that $\deg_{A_\varphi}(x_i)$ is the number of vertices $v \in V_2$, such that $\varphi(N_i)$ is contained in $N_G(v)$, that is,

$$\deg_{A_\varphi}(x_i) = |N_G(\varphi(N_i))|.$$

Therefore, if $\varphi(N_i) \notin \mathscr{B}$ then $\deg_{A_\varphi}(x_i) \geqslant (\delta - \varepsilon)^\Delta n \geqslant (\delta^\Delta - \varepsilon_2)n$. Thus, if φ is a packing of $\mathscr{N}$ and $\mathscr{B}$, then it has property (1).

By Lemma 3.1(a), for every $j = \Delta - 1, \ldots, 2$, the family $\mathscr{B}_j^{(\varepsilon)}$ contains only bad sets. Thus, by Lemma 3.1(c), $\mathscr{B}_1^{(\varepsilon)} = \emptyset$ and the assumptions of Theorem 2.2 with $k = \Delta$, $\Delta(\mathscr{N}) \leqslant \Delta + 1$ and with arbitrary γ and C, are fulfilled. Consequently, all but $\gamma n!$ initial attempted packings φ can be turned into a perfect packing by a sequence of at most Cn switchings. It remains to show that for a vast majority of bijections φ properties (2)–(4) also hold, but with, say, $\varepsilon_2/2$ in place of ε_2. This will be done again using the probabilistic method, that is, choosing a bijection $\varphi : X_1 \to V_1$ uniformly at random and bounding from above the probability Π that it does not have the required properties. If

$$\Pi + \gamma < 1, \tag{3.1}$$

then there will exist a bijection φ that satisfies both conclusion (b) of Theorem 2.2 and properties (2)–(4) of Claim 3.1 with $\varepsilon_2/2$. Since each switching can affect the left-hand sides of the inequalities appearing in properties (2)–(4) only by a constant amount, say s, the inequalities will hold with ε_2 after the complete switching procedure has been applied, provided $Cs < \varepsilon_2/2$.

For a random φ and a fixed subset $U \subseteq V_1$, denote by $Y(U)$ the number of hyperedges $\varphi(N_i)$ of $\mathscr{S}_\varphi$ that are contained in U. Notice that for $v, w \in V_2$ we have $\deg_{A_\varphi}(v) = Y(N_G(v))$ and $|N_{A_\varphi}(\{v, w\})| = Y(N_G(\{v, w\}))$. We are interested in the expected value of $Y(U)$ and its concentration, when $u = |U|$ is of the order of n.

Equivalently to asking how many $N_i \subset X_1$ are placed by a random φ inside a set $U \subset V_1$, one can ask how many N_i are contained in the preimage $\varphi^{-1}(U) \subset X_1$ for a randomly chosen φ^{-1}.

So, we abort the probability space of all bijections, and instead consider the space $[X_1]_u$ of all u-element subsets of X_1, equipped with the uniform distribution. A randomly chosen element of that space will also be designated by $[X_1]_u$. For every subset $W \subseteq X_1$, let $f(W)$ stand for the number of N_is contained in W.

Then, clearly, the random variable $f([X_1]_u)$ is identical with $Y(U)$. For convenience, we will switch the space once again and consider the binomial version of $[X_1]_u$, that is, a randomly chosen subset $[X_1]_p$ of X_1, generated by picking each possible vertex from X_1 independently with probability $p = \frac{u}{n}$.

Set $\mu = \mathrm{Ex}\left(f([X_1]_p)\right) = (u/n)^\Delta n$. Let us now define a family of sets

$$\mathscr{Q} := \{W \subseteq X_1, \ : \ |f(W) - \mu| < n^{0.6}\}.$$

By Pittel's inequality (see [5])

$$\Pr\left([X_1]_u \notin \mathscr{Q}\right) \leqslant 3\sqrt{u}\,\Pr\left([X_1]_{\frac{u}{n}} \notin \mathscr{Q}\right). \tag{3.2}$$

Observe that the random subset $[X_1]_p$ is generated by and thus can be identified with the random vector $(Z_1, \ldots, Z_n)$, where Z_i is the random indicator variable that assumes value

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1 if the ith vertex of X_1 is in $[X_1]_p$ and 0 otherwise. Observe also that, if two subsets W' and W'' of X_1 differ in only one element, then

$$|f(W') - f(W'')| \leqslant \Delta + 1,$$

because that one vertex could have been contained in at most $\Delta + 1$ hyperedges of $\mathcal{N}$. Thus, by Azuma's inequality (see [3]),

$$\Pr\left([X_1]_{\frac{u}{n}} \notin \mathcal{Q}\right) = \Pr\left(|f([X_1]_{\frac{u}{n}}) - \mu| \geqslant n^{0.6}\right) \leqslant 2\exp\left(\frac{-n^{0.2}}{2(\Delta+1)^2}\right).$$

By Pittel's inequality, switching back to the original space of all bijections, we thus have

$$\Pr\left(|Y(U) - \mu| \geqslant n^{0.6}\right) \leqslant 6\sqrt{u}\exp\left(\frac{-n^{0.2}}{2(\Delta+1)^2}\right). \tag{3.3}$$

Now take $v \in V_2$. Since G is super (δ, ε)-regular, $|N_G(v)| \geqslant (\delta - \varepsilon)n$ and we pick any subset $U \subseteq N_G(v)$ of size $u = |U| = (\delta - \varepsilon)n$. Then

$$\deg_{A_\varphi}(v) = Y(N_G(v)) \geqslant Y(U)$$

and $\mathrm{Ex}(Y(U)) = \mu = (\delta - \varepsilon)^\Delta n$. With some foresight, choose ε so that

$$(\delta - \varepsilon)^\Delta > \delta^\Delta - \varepsilon_2/2 + n^{-0.4},$$

$$(d - \varepsilon)^\Delta > d^\Delta - \varepsilon_2/2 + n^{-0.4},$$

and

$$(d + \varepsilon)^{2\Delta} < d^{2\Delta} + \varepsilon_2/2 + n^{-0.4}.$$

Then, for a random mapping φ, we have by (3.3)

$$\begin{aligned}
&\Pr\left(\exists v \in V_2 \ : \ \deg_{A_{\varphi(v)}} < (\delta^\Delta - \varepsilon_2/2)n\right) \\
&\leqslant \quad n\Pr(Y(U) < (\delta - \varepsilon)^\Delta n - n^{0.6}) \leqslant n\Pr(|Y(U) - \mu| \geqslant n^{0.6}) \\
&\leqslant \quad 6n\sqrt{(\delta - \varepsilon)n}\exp\left(\frac{-n^{0.2}}{2(\Delta+1)^2}\right) =: \Pi_2 = o(1).
\end{aligned}$$

(No Π_1, sorry.)

By Lemma 3.1(b) applied with $\delta = d$ we have that for all but at most εn vertices $v \in V_1$, $|N_G(v)| \geqslant (d - \varepsilon)n$. Thus, similarly, with probability at least $1 - \Pi_3 = 1 - o(1)$, the inequality of property (3) (with $\varepsilon_2/2$) holds for all these vertices.

To derive a bound on the probability of property (4) holding with $\varepsilon_2/2$, again apply (3.3). It can be proved in an analogous way that Lemma 3.1(a)–(b) holds if we redefine a set S to be bad whenever $|N(S)| > (\delta + \varepsilon)^{|S|}n$. In particular, this means that for at least $(1 - 2\varepsilon)\binom{n}{2}$ pairs $\{v, w\}$ we have $|N_G(\{v, w\})| \leqslant (d + \varepsilon)^2 n$. For any such pair, pick a superset $U \supseteq N_G(\{v, w\})$ of size $|U| = (d + \varepsilon)^2 n$. Now

$$|N_{A_\varphi}(\{v, w\})| = Y(N_G(\{v, w\})) \leqslant Y(U)$$

and $\mathrm{Ex}(Y(U)) = \mu = (d + \varepsilon)^{2\Delta}n$, and, as in the previous discussion, we conclude that a random mapping φ satisfies property (4) (with $\varepsilon_2/2$) with probability $1 - \Pi_4$ very close to

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1. Setting $\Pi = \Pi_2 + \Pi_3 + \Pi_4$ and choosing n_0 large enough, we see that inequality (3.1) is satisfied. This completes the proof of Theorem 3.1. □

In conclusion, let us describe how the constants ε_i are related to each other, and thereby summarize the arguments of the above proof. A mapping φ can be easily extended to form the required embedding of H into G if the auxiliary graph A_φ has a perfect matching. For a given δ, there exists an ε_1 so that such a perfect matching is guaranteed by Lemma 3.1(d) if A_φ is ε_1-regular and has $\delta(A_\varphi) \geqslant (\delta - \varepsilon_1)n$. For given $\delta, \Delta, \varepsilon_1$, the criterion in Lemma 3.2 shows that there exists an ε_2 such that these conditions are met if properties (1)–(4) of Claim 3.1 are fulfilled. Then, for given $\delta, \Delta, \varepsilon_2$, Theorem 2.2 and the above probabilistic argument imply that there exists an ε such that these properties are satisfied.

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