Ergod. Th. & Dynam. Sys. (2006), **26**, 783–786
doi:10.1017/S0143385706000113
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Printed in the United Kingdom

Adding machines as invariant sets for homeomorphisms of a disk into the plane

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(*Received* 3 *November* 2005 *and accepted in revised form* 7 *February* 2006)

Abstract. Let $h : D \to \mathbb{R}^2$ be a homeomorphism of a disk into the plane. Let $C \subset \operatorname{int} D$ be a Cantor set such that $(C, h|_C)$ is an adding machine. We give a condition under which there are points outside of C which have their entire orbits contained in D.

1. *Introduction*

Solenoids as invariant sets for continuous-time dynamical systems in $\mathbb{R}^3$ have been studied in the past. Under relatively weak assumptions, it is possible to prove that, in dimension three, a solenoid cannot be an isolated invariant set for the flow (see, for example, [**1**]).

The method used is that of sections. Since the section of a minimal flow on a solenoid is an adding machine, it is natural to study the existence of minimal sets around an adding machine in the plane. In this paper we analyze a very general situation: $h : D \to \mathbb{R}^2$ is a homeomorphism of a disk into the plane that, when restricted to some Cantor set $C \subset \operatorname{int} D$, becomes an adding machine.

The question we ask is: is there an $x \in D \backslash C$ such that every iteration of h in both time directions is defined on x?

Some results for continuous-time flows can be applied here to give a positive answer (for example, [**2**] when h allows a smooth suspension).

The answer to the question in its whole generality is not known. We give a condition under which one can answer the question positively (Theorem 2.4). The condition is topological in nature and is not related in any way to the smoothness of h.

2. *Main result*

Let $p_1, p_2, \ldots$ be natural numbers, $p_i > 1$. Consider the space $\prod_{i=1}^{\infty} \mathbb{Z}_{p_i}$ and equip it with a product topology generated by discrete topologies on every $\mathbb{Z}_{p_i}$. It is a standard result that $\prod_{i=1}^{\infty} \mathbb{Z}_{p_i}$ with this topology is a Cantor set.

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Let $a : \prod_{i=1}^{\infty} \mathbb{Z}_{p_i} \to \prod_{i=1}^{\infty} \mathbb{Z}_{p_i}$ be the 'add one' operation defined as follows:

$$a(p_1 - 1, p_2 - 1, p_3 - 1, \dots) = (0, 0, 0, \dots).$$

$$a(p_1 - 1, p_2 - 1, \dots, p_i - 1, b_1, b_2, \dots) = (0, 0, \dots (i \text{ times}) \dots, 0, b_1 + 1, b_2, b_3, \dots) \quad \text{when } b_1 \neq p_{i+1} - 1.$$

It is elementary to prove that a is a homeomorphism.

Definition. Let C be a Cantor set and let $h : C \to C$ be a homeomorphism. We say that (C, h) is an adding machine if there exist numbers $p_1, p_2, \dots$ such that $p_i > 1$ and a homeomorphism $f : \prod_{i=1}^{\infty} \mathbb{Z}_{p_i} \to C$ such that the following diagram commutes:

$$\begin{array}{ccc} \prod_{i=1}^{\infty} \mathbb{Z}_{p_i} & \xrightarrow{a} & \prod_{i=1}^{\infty} \mathbb{Z}_{p_i} \\ \downarrow f & & \downarrow f \\ C & \xrightarrow{h} & C \end{array}$$

In the rest of this paper we will assume that the following objects are given.

Let D be a disk in $\mathbb{R}^2$. Let $h : D \to \mathbb{R}^2$ be a homeomorphism of D and $h(D)$. Assume that there is a Cantor set $C \subset \operatorname{int} D$ such that $(C, h|_C)$ is an adding machine.

Definition. We introduce the following notation:

$$A^+ = \{x \in D : h^i(x) \text{ is defined for } i = 1, 2, 3, \dots\}$$

$$A^- = \{x \in D : h^i(x) \text{ is defined for } i = -1, -2, -3, \dots\}.$$

The following seven properties follow in an elementary way from the definition of the sets A^+ and A^-.

LEMMA 2.1.

(a) $C \subset A^+ \cap A^-$;

(b) $h(A^+) \subset A^+$;

(c) $h(A^-) \cap D \subset A^-$;

(d) $A^+ = D \cap h^{-1}(D) \cap h^{-2}(D) \cap \dots$;

(e) $A^- = D \cap h(D) \cap h^2(D) \cap \dots$;

(f) A^+ *is closed;*

(g) A^- *is closed.*

LEMMA 2.2. *If P and Q are compact subsets of a metric space X then the union of components of P intersecting Q is a compact set.*

This is a standard result. The reader is directed to the material in [**3**, Ch. VII].

LEMMA 2.3. *If A_0 is a component of A^+ such that there are $x, y \in A_0 \cap C$, $x \neq y$, then $A^+ \cap A^- \neq C$.*

Proof. The homeomorphism f from the definition of an adding machine allows us to associate with every point in C an infinite sequence of natural numbers. Let the sequences associated with x and y differ on the ith place.

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It is possible to divide C into disjoint sets $C_1, C_2, \ldots, C_r$ such that:

- for some $p \neq q$ we have $x \in C_p,\ y \in C_q$;
- for every j the set C_j is compact;
- $h(C_1) = C_2, h(C_2) = C_3, \ldots, h(C_r) = C_1$.

Indeed, given the fact that C is homeomorphic to $\prod_{n=1}^{\infty} \mathbb{Z}_{p_n}$, it is enough to divide this second set into parts of the form $\{p_1\} \times \cdots \times \{p_i\} \times \prod_{n=i+1}^{\infty} \mathbb{Z}_{p_n}$, and then have the homeomorphism induce a corresponding division on C.

Let Δ be the minimum possible distance between a pair of points belonging to different sets C_j, C_k. Since A_0 is connected it has to contain a point a_0 distanced precisely $\Delta/2$ from C.

Since x and y belonged to different sets C_j, then $h(x), h(y)$ also belong to different sets C_j. Note that $h(A_0)$ is a connected subset of A^+ containing both $h(x)$ and $h(y)$. As before we can select $a_1 \in h(A_0)$ that is distanced precisely $\Delta/2$ from C.

We continue the procedure to obtain a sequence of points $\{a_i\}_{i\in\mathbb{N}} \subset D$, all at a distance precisely $\Delta/2$ from C, such that $a_i \in h^i(A_0)$.

Choose an increasing sequence of numbers $\{b_i\}_{i\in\mathbb{N}}$ and $a \in D$ such that $a = \lim_{i\to\infty} a_{b_i}$. Since A^+ is closed we observe that $a \in A^+$.

Notice that for any $k \in \mathbb{N}$ the point $h^{-k}(a)$ is the limit of the sequence $\{h^{-k}(a_{b_i})\} \subset D$ starting with i such that $b_i > k$, so $h^{-k}(a) \in D$. Therefore $a \in A^-$. At the same time we know that the distance between a and C is $\Delta/2$, so $a \in (A^+ \cap A^-)\backslash C$. □

THEOREM 2.4. *Assume that no component of A^- intersects both C and δD (the boundary of D). Then there exists $x \in D\backslash C$ such that for all $i \in \mathbb{Z},\ h^i(x) \in D$.*

Proof. Define $G_i = \{x \in D$ such that $x, h^{-1}(x), h^{-2}(x), \ldots, h^{-i}(x) \in D\}$. The set G_i is the collection of all points in D that have their past defined at least i steps back in time.

Notice that every G_i is closed.

Note that $G_i \supset G_{i+1}$ and $\bigcap_{i=1}^{\infty} G_i = A^-$.

If for every i there is a component of G_i that intersects both C and δD, then denote by P_i the collection of all points in C that belong to components of G_i intersecting δD.

Every P_i is non-empty. Additionally $P_i \supset P_{i+1}$.

The sets P_i are closed by Lemma 2.2 applied first to G_i and δD and then to C and the compact set just obtained.

It follows that there exists $q \in \bigcap_{i=1}^{\infty} P_i$. The component of A^- containing q is an intersection of components of every G_i containing the point q.

Each such component is a continuum containing both q and a point from δD. If follows that the component of A^- containing q intersects δD contradicting the assumptions. Therefore, for some i every component of G_i containing points from C has to be contained in int D. Fix such i.

The set that will be of interest to us now will be the union of all those components of G_i that contain at least one point from C. Denote this set by K. K is compact by Lemma 2.2 applied to G_i and C.

Note that h maps points from G_i either into G_i or out of D.

Notice that every component of K contains points from C, and as such its image through h needs to intersect some component of K. But every component of K is disjoint from δD, so h maps every component of K into K. Therefore, $h(K) \subset K$. It follows that $K \subset A^+$.

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By continuity of h the points from C are contained in the interior of G_i. Since no point in a Cantor set is isolated we can find a component of K (and therefore also of A^+) that contains two points from C. Then Lemma 2.3 concludes the proof. □

Acknowledgements. The author is grateful to Professor Krystyna Kuperberg for her help during his work on this project. He also wishes to thank the referee for suggesting several changes that made the paper easier to read.

The author was supported in part by the Baskerville fellowship at Auburn University. The contents of this paper are part of the PhD dissertation written by the author at Auburn University.

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