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A novel curvature-based method for analyzing the second-order immobility of frictionless grasp

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SUMMARY
This paper presents a new method to analyze frictionless grasp immobility based on defined surface-to-surface signed distance function. Distance function's differential properties are analyzed and its second-order Taylor expansion with respect to differential motion is deduced. Based on the non-negative condition of the signed distance function, the first- and second-order free motions are defined and the corresponding conditions for immobility of frictionless grasp are derived. As one benefit of the proposed method, the second-order immobility check can be formulated as a nonlinear programming problem. Numerical examples are used to verify the proposed method.

KEYWORDS: Signed distance function; Grasp; Frictionless contact; Free motion.

1. Introduction

In multiple-finger grasping and fixture analysis, first-order theories,[1–3] such as Screw Theory,[4] are usually applied to investigate the mobility of grasped or constrained objects, in which the locations of the contact points and the contact normal are examined. However, first-order theories do not account for higher order geometrical effects, i.e., surface curvature of contacting bodies, and thus are often inadequate in practice. For example, in Fig. 1,[5–8] a smooth planar object is held by three point fingers. First-order theories would indicate that the object in both maximal and minimal grasp cases can instantaneously rotate about the vertical axis passing through the point where the contact normals intersect. However, if the fingers are rigidly immobile, the triangular object is in fact totally immobilized in the minimal grasp, and this immobility needs to be addressed by second-order theories.

Hanafusa and Asada[9] first introduced curvature or second-order effects into the analysis of grasp stability. Montana[10] obtained a condition, which determines whether or not a two-fingered grasp is contact stable. Howard and Kumar[11] provided simple criteria to determine the stability of a grasped object and pointed out that the stability depends on local curvature of contacting bodies, as well as the magnitude of contact forces and friction. Trinkle *et al.*[12] defined second-order stable grasps and formulated second-order stability cells. More recently, Myeong and Lee[13] leveraged the concept of translational and rotational acceleration convex polytopes to calculate the resistance forces and moments for quantifying the grasp stability. Yamada *et al.*[14,15] analyzed the static grasp stability for multiple objects in 2D and derived the corresponding grasp stiffness matrices. Later, they[16] analyzed the grasp stability in spatial case in which the effects of contact geometry were considered and the contact friction condition was discussed. Here, it needs to point out that stability study and immobility research in robotic grasping have different research focus. The former regards the grasp as a dynamic system and studies its properties under a perturbation while the latter is concerned with the mobility when a rigid body is in frictionless contact with rigid and stationary fingers. Van der Stappen *et al.*[17] and Czyzowicz *et al.*[18] proposed algorithms, respectively, for computing immobilizing grasps of polygonal parts. Rimon and Burdick[5–8] developed the concept of first- and second-order immobility of a grasp. They defined the first- and second-order mobility index of an equilibrium grasp and proved that when the first-order mobility index was greater than zero, a lower mobility index could be obtained by computing the second-order mobility index. When either mobility index is equal to zero, the grasp is immobile.

This paper concentrates on analyzing whether the grasp is immobile (or mobile) given the contact geometries of objects and fingers. Rimon and Burdick[5–8] defined a signed distance from a configuration point to the smooth boundary of C-obstacle associated fingers and then deduced its second-order Taylor expansion. Since the distance is defined in C-space, the second-order expansion has an acceleration term, which complicates the proof of second-order immobility. The failure of first-order theory for the case in Fig. 1 results from the fact of taking into account only the tangential aspects of infinitesimal motion. If other aspects are included as well, the infinitesimal motion approach can be used to address the evaluation of second-order immobility, which is the focus of this paper. Since the signed distance function here is with respect to differential motion, the acceleration term is dropped from its second-order expansion and this leads to a simpler method for analyzing immobility.

In this paper, the signed distance function between two surfaces is defined. Then, its differential properties are analyzed and the second-order Taylor expansion with respective to infinitesimal motion is derived. Based on non-negative signed distance function due to rigid body

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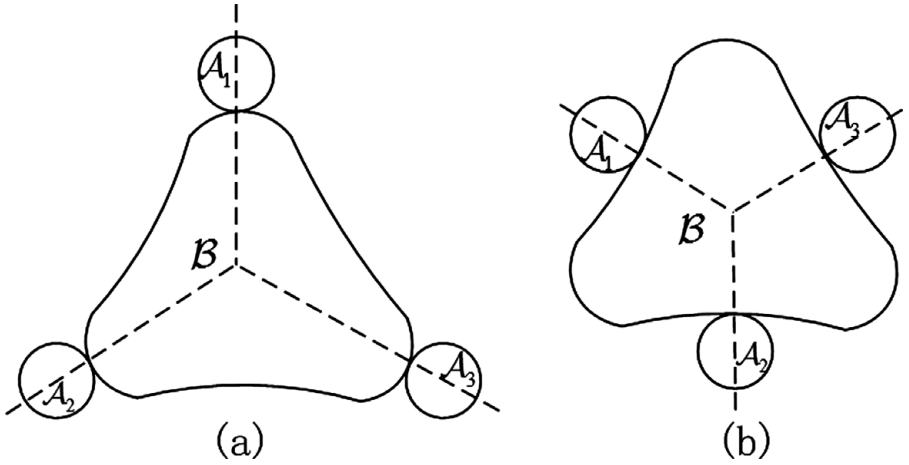


Fig. 1. (a) Maximal three-finger equilibrium grasp. (b) Minimal three-finger grasp.

impenetrability, the first- and second-order free motions are defined, and subsequently, the corresponding conditions for immobility of frictionless grasp are derived. This evaluation rule constitutes the first contribution of this paper, in which the developed method is inherently coordinate invariant and the distance function is defined in user familiar 3D Euclidean space compared with the existing Rimon's configuration space approach whose individual C-obstacle curvature form is, in general, not coordinate invariant. To provide a convenient and some cases more effective alternative to eigenvalue method, the second-order immobility check is also formulated as a nonlinear programming problem and this brings the second contribution of this paper where the second-order immobility depends on the sign of optimal objective value. Furthermore, we give a novel proof of the kinematic equation in the process of developing this method compared with the existing methods, which only directly apply the result, and this could be counted as the third contribution of this paper.

The method proposed in this paper has important theoretical and practical applications. One implication from this paper's immobility analysis is that contact geometry plays a vital role in reliable grasps, and this was also verified by Kragten *et al.*[19] through multibody dynamic simulations. Designing robotic hands in dealing with slippery objects could be one potential area to apply the proposed method. Another potential application comes in dealing with workpiece fixture design, the possibility of using fewer fingers provided by our method than predicted by conventional first-order theories can lead to more adaptable or efficient fixturing techniques and fixture planning algorithms.

The paper is organized as follows. In Section 2, we present a brief description of point to surface signed distance function and detailed surface-to-surface signed distance function and its differentiability. In Section 3, we analyze immobility of multiple-finger frictionless grasp and give two approaches to determine whether or not a grasp is immobile. Numerical examples are provided in Section 4 to show the validity of the proposed method. Conclusion and future research are presented in Section 5. Most of the proofs of the results in Section 2 can be found in the Appendices A–D.

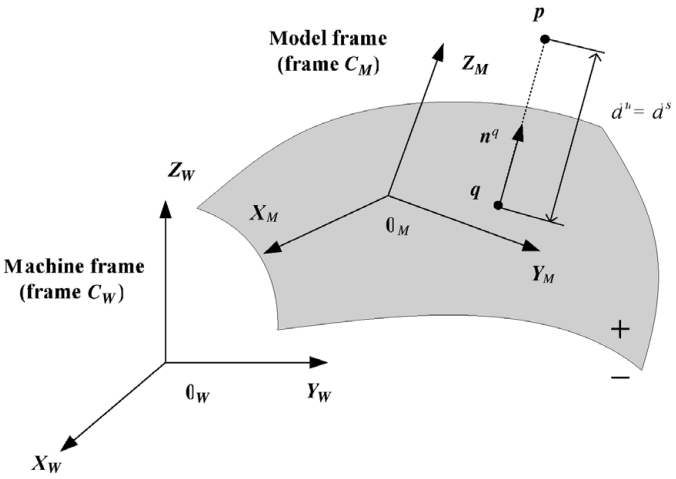


Fig. 2. Point-to-surface signed distance function.

2. Fundamentals of Distance Function and Its Differentiability

As surface-to-surface signed distance function is defined by drawing the benefits from point-to-surface signed distance function, a few basic facts about the distance function from point to surface are briefly outlined here (see Zhu[20–22] for details).

Given two coordinate frames, i.e., model frame C_M attached to rigid body and machine frame C_W as shown in Fig. 2, the location of frame C_M relative to frame C_W can be represented by the vector matrix pair $g = (\boldsymbol{t}, \boldsymbol{R}) \in SE(3)$, where $\boldsymbol{t} \in \Re^3$ is the position vector and $\boldsymbol{R} \in SO(3)$ is the rotation matrix. Here, $SO(3)$ denotes the rotation group of $\Re^3$. The set of all such locations form the special Euclidean group $SE(3)$. On the one hand, g serves as a coordinate transformation taking the coordinate of a point from frame C_M to frame C_W, i.e., if $\boldsymbol{p} \in \Re^3$ is a point expressed in C_M, then $g\boldsymbol{p} = \boldsymbol{R}\boldsymbol{p} + \boldsymbol{t}$ is its expression in C_W. On the other hand, g can be interpreted as a rigid body displacement taking the rigid body from the initial location, in which the frame C_M coincides with the frame C_W, to its present location. Therefore, the action of g on the point $\boldsymbol{p}$ has another interpretation, i.e., $\boldsymbol{p}$ is the initial position of a point, and $g\boldsymbol{p}$ is the position of that point after a rigid body motion is applied,

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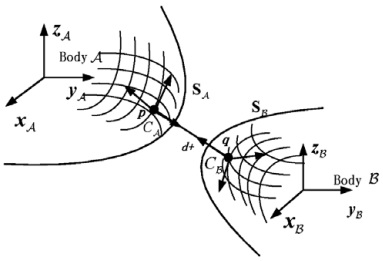


Fig. 3. Surface-to-surface signed distance function illustration.

both $\boldsymbol{p}$ and $g\boldsymbol{p}$ are specified in the frame C_W. The inverse of the element g^{-1}, $g^{-1} = (-\boldsymbol{R}^{\mathrm{T}}\boldsymbol{t}, \boldsymbol{R}^{\mathrm{T}})$, gives the location of frame C_W relative to frame C_M.[23]

Let Δx, Δy, and Δz be the differential translations of the nominal surface along X-, Y- and Z-axes of a reference frame, and δx, δy, and δz be the differential rotations of the geometry about those axes, then we can define the differential translation vector $\boldsymbol{v} = [\Delta x, \Delta y, \Delta z]^{\mathrm{T}}$ and the differential rotation vector $\boldsymbol{\omega} = [\delta x, \delta y, \delta z]^{\mathrm{T}}$. If such a differential motion is relative to the model frame C_M, then the location of the rigid body will become $\tilde{g}(\boldsymbol{v}, \boldsymbol{\omega}) \approx (\boldsymbol{R}v + \boldsymbol{t}, \boldsymbol{R}(\boldsymbol{I} + \hat{\boldsymbol{\omega}}))$, where $\hat{\boldsymbol{\omega}}$ is a 3×3 skew-symmetric matrix uniquely identified with the linear cross product operator $\boldsymbol{\omega}\times$, i.e., for $\forall \boldsymbol{x} \in \Re^3$, $\boldsymbol{\omega} \times \boldsymbol{x} = \hat{\boldsymbol{\omega}}\boldsymbol{x}$.[23]

Definition 1[20,21] **(Point-to-surface signed distance function):** Let $\boldsymbol{S} \in \Re^3$ be a smooth surface represented in its own model frame C_M, and $\boldsymbol{p} \in \Re^3$ be a point in the machine frame C_W. The location of C_M relative to C_W is denoted by g. If there exists one unique closest point $\boldsymbol{q} \in \boldsymbol{S}$, and $\boldsymbol{q}$ is not on the boundary of $\boldsymbol{S}$, the signed distance function is defined as $d^s_{p,s}(g, \boldsymbol{w}) = (g^{-1}\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{n}_{\boldsymbol{q}}$, where bold dot denotes scalar product and $\boldsymbol{q}$ is also called the foot point of $\boldsymbol{p}$ on surface $\boldsymbol{S}$.

As shown in Fig. 2, the signed distance between a point and the surface is positive if the point lies on the outer side of the surface, negative if the point lies on the inner side of the surface, and zero if the point happens to lie on the surface. It is worth noting that ambiguity exists where there are two or more foot points such as when $\boldsymbol{S}$ represents a spherical patch and $\boldsymbol{p}$ is located in the center of the sphere. Special cases like this are ignored in this paper because they hardly occur, especially when $\boldsymbol{p}$ is close to $\boldsymbol{S}$.

Similarly, the surface-to-surface signed distance function is defined and refers to the minimum value of all possible signed distances from points on one surface to the other surface.

As shown in Fig. 3, there exist two surfaces and their own model frames are $\mathcal{A}$, $\mathcal{B}$, in which $\mathcal{A}$ is assumed an inertial frame. $\boldsymbol{S}_{\mathcal{A}}$, $\boldsymbol{S}_{\mathcal{B}}$ are denoted as the smooth surface of rigid bodies.

Let $\boldsymbol{p}$ be an arbitrary point on surface $\boldsymbol{S}_{\mathcal{A}}$. The minimal signed distance from points on $\boldsymbol{S}_{\mathcal{A}}$ to $\boldsymbol{S}_{\mathcal{B}}$ is

$$d^s = \min_{\boldsymbol{p} \in \boldsymbol{S}_{\mathcal{A}}} \quad d^s_{\boldsymbol{p}, \boldsymbol{S}_{\mathcal{B}}} = \min_{\boldsymbol{p} \in \boldsymbol{S}_{\mathcal{A}}} \quad \left(g^{-1}_{\mathcal{AB}}\boldsymbol{p} - \boldsymbol{q}\right) \cdot \boldsymbol{n}_{\boldsymbol{q}}, \qquad (1)$$

where $d^s_{\boldsymbol{p}, \boldsymbol{S}_{\mathcal{B}}}$ is the signed distance function from point $\boldsymbol{p}$ to $\boldsymbol{S}_{\mathcal{B}}$, $\boldsymbol{q}$ is the foot point of $\boldsymbol{p}$ on surface $\boldsymbol{S}_{\mathcal{B}}$, $g_{\mathcal{AB}}$ is the location of $\mathcal{B}$ relative to $\mathcal{A}$. If there is only one point $\bar{\boldsymbol{p}}$ to satisfy the Eq. (1), the signed distance function from $\boldsymbol{S}_{\mathcal{A}}$ to $\boldsymbol{S}_{\mathcal{B}}$ is

$$d^s_{\boldsymbol{S}_{\mathcal{A}}\boldsymbol{S}_{\mathcal{B}}} = \left(g^{-1}_{\mathcal{AB}}\bar{\boldsymbol{p}} - \bar{\boldsymbol{q}}\right) \cdot \boldsymbol{n}_{\bar{\boldsymbol{q}}}. \qquad (2)$$

Similarly, we can define the signed distance function from $\boldsymbol{S}_{\mathcal{B}}$ to $\boldsymbol{S}_{\mathcal{A}}$. Let $\boldsymbol{q}$ be an arbitrary point on surface $\boldsymbol{S}_{\mathcal{B}}$. The minimal signed distance from points on $\boldsymbol{S}_{\mathcal{B}}$ to $\boldsymbol{S}_{\mathcal{A}}$ is

$$d^s = \min_{\boldsymbol{q} \in \boldsymbol{S}_{\mathcal{B}}} \quad d^s_{\boldsymbol{q}, \boldsymbol{S}_{\mathcal{A}}} = \min_{\boldsymbol{q} \in \boldsymbol{S}_{\mathcal{B}}} \quad (g_{\mathcal{AB}}\boldsymbol{q} - \boldsymbol{p}) \cdot \boldsymbol{n}_{\boldsymbol{p}}, \qquad (3)$$

where $d^s_{\boldsymbol{q}, \boldsymbol{S}_{\mathcal{A}}}$ is the signed distance function from $\boldsymbol{q}$ to $\boldsymbol{S}_{\mathcal{A}}$, $\boldsymbol{p}$ is the foot point of $\boldsymbol{q}$ on surface $\boldsymbol{S}_{\mathcal{A}}$. If there is only one point $\tilde{\boldsymbol{q}}$ to satisfy the Eq. (3), the signed distance function from $\boldsymbol{S}_{\mathcal{B}}$ to $\boldsymbol{S}_{\mathcal{A}}$ is

$$d^s_{\boldsymbol{S}_{\mathcal{B}}\boldsymbol{S}_{\mathcal{A}}} = (g_{\mathcal{AB}}\tilde{\boldsymbol{q}} - \tilde{\boldsymbol{p}}) \cdot \boldsymbol{n}_{\tilde{\boldsymbol{p}}}. \qquad (4)$$

Clearly, the quantities concerned in Eqs. (2) and (4) should satisfy $d_{\boldsymbol{S}_{\mathcal{B}}\boldsymbol{S}_{\mathcal{A}}} = d_{\boldsymbol{S}_{\mathcal{A}}\boldsymbol{S}_{\mathcal{B}}}$, $\bar{\boldsymbol{p}} = \tilde{\boldsymbol{p}}, \bar{\boldsymbol{q}} = \tilde{\boldsymbol{q}}$; namely, the distance between two surfaces being independent of the order of the surfaces. First, if $d_{\boldsymbol{S}_{\mathcal{A}}\boldsymbol{S}_{\mathcal{B}}} \neq d_{\boldsymbol{S}_{\mathcal{B}}\boldsymbol{S}_{\mathcal{A}}}$, then we assume that $d_{\boldsymbol{S}_{\mathcal{A}}\boldsymbol{S}_{\mathcal{B}}} > d_{\boldsymbol{S}_{\mathcal{B}}\boldsymbol{S}_{\mathcal{A}}}$ and it would yield the same result by assuming the opposite. Substituting the corresponding point pair $\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{q}}$ satisfying Eq. (4) for the point pair $\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}$ in Eq. (2), we will obtain a new $\tilde{d}_{\boldsymbol{S}_{\mathcal{A}}\boldsymbol{S}_{\mathcal{B}}}$. Then, based on the premise of Eq. (4), $\tilde{d}_{\boldsymbol{S}_{\mathcal{A}}\boldsymbol{S}_{\mathcal{B}}} < d_{\boldsymbol{S}_{\mathcal{A}}\boldsymbol{S}_{\mathcal{B}}}$ will be followed. This contradicts with the premise of Eq. (2); hence, $d_{\boldsymbol{S}_{\mathcal{B}}\boldsymbol{S}_{\mathcal{A}}} = d_{\boldsymbol{S}_{\mathcal{A}}\boldsymbol{S}_{\mathcal{B}}}$. Second, if $d_{\boldsymbol{S}_{\mathcal{B}}\boldsymbol{S}_{\mathcal{A}}} = d_{\boldsymbol{S}_{\mathcal{A}}\boldsymbol{S}_{\mathcal{B}}}$, but $\bar{\boldsymbol{p}} \neq \tilde{\boldsymbol{p}}, \bar{\boldsymbol{q}} \neq \tilde{\boldsymbol{q}}$, this contradicts the unique point pairs in Eqs. (2) and (4), and thus, $\bar{\boldsymbol{p}} = \tilde{\boldsymbol{p}}$, $\bar{\boldsymbol{q}} = \tilde{\boldsymbol{q}}$. Subsequently, we will no long distinguish between $\bar{\boldsymbol{p}}$ and $\tilde{\boldsymbol{p}}$, $\bar{\boldsymbol{q}}$ and $\tilde{\boldsymbol{q}}$, $d_{\boldsymbol{S}_{\mathcal{A}}\boldsymbol{S}_{\mathcal{B}}}$ and $d_{\boldsymbol{S}_{\mathcal{B}}\boldsymbol{S}_{\mathcal{A}}}$, and sum up surface-to-surface signed distance function as follows.

Definition 2 (Surface-to-surface signed distance function): Let $\boldsymbol{S}_{\mathcal{A}}$, $\boldsymbol{S}_{\mathcal{B}}$ be two smooth surfaces. If there exists a unique point $\boldsymbol{p}$ to satisfy Eq. (1), the signed distance function between $\boldsymbol{S}_{\mathcal{A}}$ and $\boldsymbol{S}_{\mathcal{B}}$ is

$$d^s = (g^{-1}_{\mathcal{AB}}\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{n}^{\boldsymbol{q}}, \qquad (5)$$

where $\boldsymbol{p}$, $\boldsymbol{q}$ are the foot point of each other. Then, it easily follows that $g^{-1}_{\mathcal{AB}}\boldsymbol{p} - \boldsymbol{q} = \lambda\boldsymbol{n}^{\boldsymbol{q}}$, $g_{\mathcal{AB}}\boldsymbol{q} - \boldsymbol{p} = \lambda\boldsymbol{n}^{\boldsymbol{p}}$ and the line between $\boldsymbol{p}$ and $\boldsymbol{q}$ is perpendicular to the two surfaces.

The signed distance between both surfaces is positive if the two surfaces are separated from each other; negative if the two surfaces are penetrated into each other; and zero if the two surfaces are in contact with each other.

Proposition 1: *Assume surface $\boldsymbol{S}_{\mathcal{B}}$ undergoes a differential rigid body motion $[\boldsymbol{v}, \boldsymbol{\omega}]$, then one has*

$$\Delta d^s(g) \approx -\boldsymbol{n}^{\boldsymbol{q}} \cdot \boldsymbol{v} - (\boldsymbol{q} \times \boldsymbol{n}^{\boldsymbol{q}}) \cdot \boldsymbol{\omega}, \qquad (6)$$

where $\boldsymbol{q}$ is the foot point of $\boldsymbol{p}$ in Eq. (5).

The proof of Proposition 1 is provided in Appendix C and this is similar to the one of differentiability of point-to-surface signed distance function (see Zhu *et al.*[20–22] for details). In fact, $[\,(\boldsymbol{n}^{\boldsymbol{q}})^{\mathrm{T}}\ (\boldsymbol{q} \times \boldsymbol{n}^{\boldsymbol{q}})^{\mathrm{T}}\,]$ is the Plücker coordinate at the normal vector at point $\boldsymbol{q}$. Note that the parameters concerned in Eq. (6) are measured in the coordinate frame $\mathcal{B}$. If the parameters concerned are measured in the coordinate

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frame $\mathcal{A}$, Eq. (6) has the same expression, which is called the coordinate invariance and was proved in Lou *et al.*[24]

Coordinate invariance is an important and more desirable property that our developed method inherently owns. Practically speaking, coordinate invariance ensures that the results will be the same regardless of the choice of the reference frame.

Proposition 2: *Assume that surfaces $\boldsymbol{S}_{\mathcal{A}}$, $\boldsymbol{S}_{\mathcal{B}}$ have locally parametric descriptions $\boldsymbol{\psi}(u, v)$ and $\boldsymbol{\varphi}(s, t)$ whose representations are orthogonal, and the surface $\boldsymbol{S}_{\mathcal{B}}$ undergoes a differential rigid body motion $[\boldsymbol{v}, \boldsymbol{\omega}]$. If the matrices of $\tilde{\boldsymbol{K}}_{\mathcal{A}}$, $\boldsymbol{K}_{\mathcal{B}}$, and $\tilde{\boldsymbol{K}}_{\mathcal{A}} + \boldsymbol{K}_{\mathcal{B}} + d_0\tilde{\boldsymbol{K}}_{\mathcal{A}}\boldsymbol{K}_{\mathcal{B}}$ are invertible, where $\boldsymbol{K}_{\mathcal{A}}$, $\boldsymbol{K}_{\mathcal{B}}$ are the curvature tensors,[23] and $\tilde{\boldsymbol{K}}_{\mathcal{A}} = \boldsymbol{R}_{\phi}\boldsymbol{K}_{\mathcal{A}}\boldsymbol{R}_{\phi}$, and $\boldsymbol{R}_{\phi}$ is the orientation of the x- and y-axes of the instantaneous frame $C_{\mathcal{B}}$ relative to the x- and y-axes of $C_{\mathcal{A}}$ in Fig. 3, the second-order differential matrix of signed distance function is*

$$\nabla^2 d^s(g) = \begin{bmatrix} \boldsymbol{\Phi} \\ \hat{\boldsymbol{q}}\boldsymbol{\Phi} + \hat{\boldsymbol{n}}^{\boldsymbol{q}}\boldsymbol{\Phi}\left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d1\right) \end{bmatrix} \left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + \boldsymbol{K}_{\mathcal{B}}^{-1} + d_0 1\right)^{-1} \left[\boldsymbol{\Phi}^{\mathrm{T}}, \quad [\hat{\boldsymbol{q}}\boldsymbol{\Phi}]^{\mathrm{T}} + \left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d_0 1\right)[\hat{\boldsymbol{n}}^{\boldsymbol{q}}\boldsymbol{\Phi}]^{\mathrm{T}}\right] + \frac{1}{2}\begin{bmatrix} 0 & \boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\hat{n}^{q} \\ -\hat{\boldsymbol{n}}^{\boldsymbol{q}}\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} & \hat{\boldsymbol{n}}^{\boldsymbol{q}}\boldsymbol{\Phi}\left\{\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{q}} + \left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d_0 1\right)\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{n}}^{\boldsymbol{q}}\right\} + \left\{\boldsymbol{q}\hat{\Phi} + \hat{\boldsymbol{n}}^{\boldsymbol{q}}\boldsymbol{\Phi}\left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d_0 1\right)\right\}\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{n}}^{\boldsymbol{q}} \end{bmatrix}, \tag{7}$$

where $\boldsymbol{\Phi} = (\frac{\boldsymbol{\varphi}_s}{\|\boldsymbol{\varphi}_s\|}, \frac{\boldsymbol{\varphi}_t}{\|\boldsymbol{\varphi}_t\|})$ and 1 is a 2×2 unit matrix, d_0 is the initial distance, and the expression of curvature tensors K is given in Appendix C. If $d_0 = 0$, Eq. (7) can be further simplified as

$$\nabla^2 d^s(g) = \begin{bmatrix} \boldsymbol{\Phi} \\ \hat{\boldsymbol{q}}\boldsymbol{\Phi} + \hat{\boldsymbol{n}}^{\boldsymbol{q}}\boldsymbol{\Phi}\tilde{K}_{\mathcal{A}}^{-1} \end{bmatrix} \left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + \boldsymbol{K}_{\mathcal{B}}^{-1}\right)^{-1} \left[\boldsymbol{\Phi}^{\mathrm{T}}, \quad [\hat{\boldsymbol{q}}\boldsymbol{\Phi}]^{\mathrm{T}} + \tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1}[\hat{\boldsymbol{n}}^{\boldsymbol{q}}\boldsymbol{\Phi}]^{\mathrm{T}}\right] + \frac{1}{2}\begin{bmatrix} 0 & \boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\hat{n}^{q} \\ -\hat{\boldsymbol{n}}^{\boldsymbol{q}}\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} & \hat{\boldsymbol{n}}^{\boldsymbol{q}}\boldsymbol{\Phi}\left\{\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{q}} + \tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1}\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{n}}^{\boldsymbol{q}}\right\} + \left\{\hat{\boldsymbol{q}}\boldsymbol{\Phi} + \hat{\boldsymbol{n}}^{\boldsymbol{q}}\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1}\boldsymbol{\Phi}\right\}\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{n}}^{\boldsymbol{q}} \end{bmatrix}. \tag{8}$$

Appendix A gives the detailed proof of Proposition 2. Two points need to be followed. First, the invertibility of $\tilde{\boldsymbol{K}}_{\mathcal{A}}$, $\boldsymbol{K}_{\mathcal{B}}$ is not prerequisite in Proposition 2, but the introduction of this condition helps to simplify the expression and give a unified form. Appendix A also provides the second-order Hessian matrix when $\tilde{\boldsymbol{K}}_{\mathcal{A}}$, $\boldsymbol{K}_{\mathcal{B}}$ is singular. Second, a new proof of surface kinematic equation is provided in Appendix A, which is based on the above defined surface-to-surface signed distance function and is different from Montana' method.

From Propositions 1 and 2, the second-order Taylor series expansion of the signed distance function is

$$d(g) = d_0 - [(\boldsymbol{n}^{\boldsymbol{q}})^{\mathrm{T}} \quad (\boldsymbol{q} \times \boldsymbol{n}^{\boldsymbol{q}})^{\mathrm{T}}] \cdot \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} + \frac{1}{2}[\boldsymbol{v}^{\mathrm{T}} \quad \boldsymbol{\omega}^{\mathrm{T}}]\nabla^2 d^s(g)\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} + O^3\left(\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}\right), \tag{9}$$

where $\nabla^2 d^s(g)$ is the expression in Eq. (7). If $\mathcal{B}$ and $\mathcal{A}$ contact each other, i.e., $d_0 = 0$, $\nabla^2 d^s(g)$ corresponds to the expression of Eq. (8) and the constant term is dropped.

This second-order differential increment of the signed distance function has the same desired property of coordinate invariance as the first-order one, which is proved in Appendix B. Note that Eq. (9) is the second-order Taylor expansion on differential motion and it does not have any acceleration term, compared with Rimon's second-order Taylor expansion about the configuration. Therefore, the first and second terms in Eq. (9) are different from the ones of Rimon's second-order Taylor expansion.

3. Immobility Analysis

If one rigid body $\mathcal{B}$ is in frictionless contact with k fingers or locators $\mathcal{A}_1, \ldots \mathcal{A}_k$, the initial distance is equal to zero. After the rigid body $\mathcal{B}$ undergoes a differential motion, the distances from $\mathcal{B}$ to every finger change. If the distance from $\mathcal{B}$ to the ith finger is negative, i.e., $\mathcal{B}$ and the ith finger penetrate into each other, the differential motion is unreasonable, since both $\mathcal{B}$ and fingers are assumed to be rigid. Hence, the viable motion should assure the distance between $\mathcal{B}$ and every finger is non-negative. Then, we can define the first- and second-order free motions based on the first- and second-order terms of Taylor expansion of the signed distance function.

3.1. First- and second-order free motions

3.1.1. First-order free motions. Let the rigid body $\mathcal{B}$ contact with k frictionless fingers/locators $\mathcal{A}_1, \ldots \mathcal{A}_k$. First, we consider the case of contacting with one finger and then expand to a multifinger case. The first-order Taylor expansion of the signed distance function between $\mathcal{B}$ and $\mathcal{A}_i$

$$d(g(\boldsymbol{v}, \boldsymbol{\omega})) = d_0 - \begin{bmatrix} (\boldsymbol{n}^{\boldsymbol{q}})^{\mathrm{T}} & (\boldsymbol{q} \times \boldsymbol{n}^{\boldsymbol{q}})^{\mathrm{T}} \end{bmatrix} \boldsymbol{V} + O^2(\boldsymbol{V}), \tag{10}$$

where $\boldsymbol{V} = [\boldsymbol{v}^{\mathrm{T}} \quad \boldsymbol{\omega}^{\mathrm{T}}]^{\mathrm{T}}$. Because $\mathcal{B}$ is in contact with $\mathcal{A}_i$, thus $d_0 = 0$. If higher order terms are ignored, the above equation can be simplified as

$$d(g(\boldsymbol{v}, \boldsymbol{\omega})) \approx -[\,(\boldsymbol{n}^{\boldsymbol{q}})^{\mathrm{T}} \; (\boldsymbol{q} \times \boldsymbol{n}^{\boldsymbol{q}})^{\mathrm{T}}\,]\boldsymbol{V}. \tag{11}$$

Undergoing a differential motion $\boldsymbol{V}$, $\mathcal{B}$ is apart from $\mathcal{A}_i$ when the distance function is positive; $\mathcal{B}$ maintains contact with $\mathcal{A}_i$ and makes a roll and/or slide motion on $\mathcal{A}_i$ when the distance function is zero; $\mathcal{B}$ and $\mathcal{A}_i$ penetrate into each other when

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the distance function is negative. But the third case scenario should not occur as both $\mathcal{B}$ and $\mathcal{A}_i$ are rigid bodies. Thus, we can define the first two cases as the first-order free motions and sum it up as follows.

Definition 3: The first-order free motions of $\mathcal{B}$ relative to $\mathcal{A}_i$ can be defined as

$$\mathcal{I}^1_{\mathcal{B}}(\mathcal{A}_i) \underline{\underline{\Delta}} \left\{ \boldsymbol{V} : \left[(\boldsymbol{n^q})_i^{\mathrm{T}} \ (\boldsymbol{q} \times \boldsymbol{n^q})_i^{\mathrm{T}} \right] \boldsymbol{V} \leq 0 \right\}. \tag{12}$$

Equation (12) defines a half space. The boundary of this half space, i.e., $\{\boldsymbol{V} : [\,(\boldsymbol{n^q})_i^{\mathrm{T}} \ (\boldsymbol{q} \times \boldsymbol{n^q})_i^{\mathrm{T}}\,]\boldsymbol{V} = 0\}$, is called first-order roll–slide free motions, and the interior of this half space, i.e., $\{\boldsymbol{V} : [\,(\boldsymbol{n^q})_i^{\mathrm{T}} \ (\boldsymbol{q} \times \boldsymbol{n^q})_i^{\mathrm{T}}\,]\boldsymbol{V} < 0\}$, is called first-order escape free motions. Note that the first-order free motions given here is similar to the one in Rimon and Burdick.[5] If there are k-finger $\mathcal{A}_1, \ldots \mathcal{A}_k$ keeps frictionless contact with $\mathcal{B}$, the first-order free motions can be expressed as

$$\mathcal{I}^1_{\mathcal{B}}(\mathcal{A}_1, \ldots, \mathcal{A}_k) \underline{\underline{\Delta}} \bigcap_{i=1}^{n} \mathcal{I}^1_{\mathcal{B}}(\mathcal{A}_i). \tag{13}$$

From Section 1, the first-order free motion is inherently coordinate invariant since the first-order term of signed distance function has this property.

3.1.2. Second-order free motions. From Eq. (9), the sign of the signed distance function d depends on the sign of first-order term when the first-order term is not zero and the constant term is zero. Furthermore, when both the first-order term and the constant are equal to zero, the sign of the signed distance function depends on the sign of second-order term, and so on. Thus, when a first-order free motion is the first-order roll–slide motion, i.e., the first-order term is equal to zero, we can define second-order free motion. It is worth noting that the initial distance is equal to zero due to the fact that the rigid body and fingers are in contact originally.

Definition 4: The second-order free motions of $\mathcal{B}$ relative to $\mathcal{A}_i$ is

$$\begin{aligned} \mathcal{I}^2_{\mathcal{B}}(\mathcal{A}_i) \underline{\underline{\Delta}} \{ \boldsymbol{V} : \left[(\boldsymbol{n^q})_i^{\mathrm{T}} \quad (\boldsymbol{q} \times \boldsymbol{n^q})_i^{\mathrm{T}} \right] \boldsymbol{V} \\ = 0 \ \& \ \boldsymbol{V}^{\mathrm{T}} \nabla^2 d_i^S(g) \boldsymbol{V} \geq 0 \}, \end{aligned} \tag{14}$$

where $\nabla^2 d_i^S(g)$ can be computed according to Eq. (8). The condition of $[\,(\boldsymbol{n^q})_i^{\mathrm{T}} \ (\boldsymbol{q} \times \boldsymbol{n^q})_i^{\mathrm{T}}\,]\boldsymbol{V} = 0$ ensures the first-order roll–slide motion, and the one of $\boldsymbol{V}^{\mathrm{T}} \nabla^2 d_i^S(g) \boldsymbol{V} \geq 0$ ensures the second term is non-negative. Similar to the first-order free motions, the second-order free motions can be subdivided into the following: (1) $\{\boldsymbol{V} : [(\boldsymbol{n^q})_i^{\mathrm{T}} \quad (\boldsymbol{q} \times \boldsymbol{n^q})_i^{\mathrm{T}}]\boldsymbol{V} = 0 \ \& \ \boldsymbol{V}^{\mathrm{T}} \nabla^2 d_i^S(g) \boldsymbol{V} = 0\}$, called the second-order roll–slide motions; (2) $\{\boldsymbol{V} : [(\boldsymbol{n^q})_i^{\mathrm{T}} \quad (\boldsymbol{q} \times \boldsymbol{n^q})_i^{\mathrm{T}}]\boldsymbol{V} = 0 \ \& \ \boldsymbol{V}^{\mathrm{T}} \nabla^2 d_i^S(g) \boldsymbol{V} > 0\}$, called the second-order escape motions. Note that the second-order free motions given here are different from the one in Rimon and Burdick.[5] If rigid body $\mathcal{B}$ is grasped by k fingers $\mathcal{A}_1, \ldots \mathcal{A}_k$, the second-order free motions can be expressed as

$$\mathcal{I}^2_{\mathcal{B}}(\mathcal{A}_1, \ldots, \mathcal{A}_k) \underline{\underline{\Delta}} \bigcap_{i=1}^{n} \mathcal{I}^2_{\mathcal{B}}(\mathcal{A}_i). \tag{15}$$

From Section 1, the second-order free motion is inherently coordinate invariant because the second-order term of signed distance function has this property.

3.2. Immobility analysis

3.2.1. First immobility analysis. Let rigid body $\mathcal{B}$ be grasped by k-finger or constrained by k-locator $\mathcal{A}_1, \ldots \mathcal{A}_k$. From Section 3.1, the first- and second-order immobilities are subdivided depending on the first- and second-order free motions. From the definition of first-order free motion space, its complementary spaces is

$$\begin{aligned} \overline{\mathcal{I}^1_{\mathcal{B}}(\mathcal{A}_1, \ldots, \mathcal{A}_k)} \underline{\underline{\Delta}} \bigcup_{i=1}^{k} \overline{\mathcal{I}^1_{\mathcal{B}}(\mathcal{A}_i)} \\ = \bigcup_{i=1}^{k} \left\{ \boldsymbol{V} : \left[(\boldsymbol{n^q})_i^{\mathrm{T}} \ (\boldsymbol{q} \times \boldsymbol{n^q})_i^{\mathrm{T}} \right] \boldsymbol{V} > 0 \right\}. \end{aligned} \tag{16}$$

If first free motion space is null or $\overline{\mathcal{I}^1_{\mathcal{B}}(\mathcal{A}_1, \ldots, \mathcal{A}_k)} = \Re^6$, the corresponding grasp is first-order immobile grasp. So, we have the following result: the grasp is first-order immobile when there exists at least one finger $\mathcal{A}_i$ satisfying $[\,(\boldsymbol{n^q})_i^{\mathrm{T}} \ (\boldsymbol{q} \times \boldsymbol{n^q})_i^{\mathrm{T}}\,]\boldsymbol{V} > 0$ for all differential motions $\boldsymbol{V} \neq 0$; the grasp is mobile if there exists a $\mathbf{V} \neq 0$ to make every finger $\mathcal{A}_i$ satisfying $[\,(\boldsymbol{n^q})_i^{\mathrm{T}} \ (\boldsymbol{q} \times \boldsymbol{n^q})_i^{\mathrm{T}}\,]\boldsymbol{V} \leq 0$ and there exists at least one finger $\mathcal{A}_j$ such that $[\,(\boldsymbol{n^q})_j^{\mathrm{T}} \ (\boldsymbol{q} \times \boldsymbol{n^q})_j^{\mathrm{T}}\,]\boldsymbol{V} < 0$; the second-order immobility analysis is needed if every finger $\mathcal{A}_i$ satisfying $[\,(\boldsymbol{n^q})_i^{\mathrm{T}} \ (\boldsymbol{q} \times \boldsymbol{n^q})_i^{\mathrm{T}}\,]\boldsymbol{V} = 0$ for some differential motion $\boldsymbol{V} \neq 0$.

3.2.2. Second-order immobility analysis of multifinger grasp. If the rigid body can make first-order roll–slide motion on fingers $\mathcal{A}_1, \ldots \mathcal{A}_k$, i.e., there exists a $\boldsymbol{V}$ to make every finger $\mathcal{A}_i$ satisfying $[\,(\boldsymbol{n^q})_i^{\mathrm{T}} \ (\boldsymbol{q} \times \boldsymbol{n^q})_i^{\mathrm{T}}\,]\boldsymbol{V} = 0$, we need analyze the second-order immobility. All nonzero differential motion $\boldsymbol{V}$s satisfying the above condition form a linear subspace $\mathcal{V}^{sr} \triangleq \{\boldsymbol{V} \in \mathcal{V}^{sr} : \boldsymbol{V} = \sum_{i=1}^{m} \lambda_i \boldsymbol{V}_i, \sum_{i=1}^{m} |\lambda_i| \neq 0\}$, $\boldsymbol{V}_i$ are m linearly independent vectors of those nonzero differential motions and there exists at least a λ_i that is not equal to zero. Based on previous second-order free motion space, the complementary second-order free motion space in subspace $\mathcal{V}^{sr}$

$$\begin{aligned} \overline{\mathcal{I}^2_{\mathcal{B}}(\mathcal{A}_1, \ldots, \mathcal{A}_k)} \underline{\underline{\Delta}} \bigcup_{i=1}^{k} \overline{\mathcal{I}^2_{\mathcal{B}}(\mathcal{A}_i)} \\ = \bigcup_{i=1}^{k} \left\{ \boldsymbol{V} : \boldsymbol{V}^{\mathrm{T}} \nabla^2 d_i^S(g) \boldsymbol{V} < 0, \boldsymbol{V} \in \mathcal{V}^{sr} \right\}. \end{aligned} \tag{17}$$

Under first-order roll–slide motion, if the second-order free space is null or $\overline{\mathcal{I}^2_{\mathcal{B}}(\mathcal{A}_1, \ldots, \mathcal{A}_k)} = \mathcal{V}^{sr}$, the corresponding

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grasp is immobile, which is called second-order immobile grasp. Then, Proposition 3 is easily followed.

Proposition 3: *The grasp is second-order immobile when there is at least one finger $\mathcal{A}_i$ satisfying $\boldsymbol{V}^{\mathrm{T}}\nabla^2 d_i^S(g)\boldsymbol{V} < 0$ for all $\boldsymbol{V} \in \mathcal{V}^{sr}$; the grasp is not immobile when there exists a $\boldsymbol{V}^* \in \mathcal{V}^{sr}$ to make every $\mathcal{A}_i$ satisfying $(\boldsymbol{V}^*)^{\mathrm{T}}\nabla^2 d_i^S(g)\boldsymbol{V}^* \geq 0$, $i = 1, \ldots, k$ and at least one finger $\mathcal{A}_j$ satisfying $(\boldsymbol{V}^*)^{\mathrm{T}}\nabla^2 d_j^S(g)\boldsymbol{V}^* > 0$. If there exists a $\boldsymbol{V} \in \mathcal{V}^{sr}$ such that $\mathbf{V}^{\mathrm{T}}\nabla^2 d_i^S(g)\mathbf{V} = 0$ for every $\mathcal{A}_i$, we need consider the third-order term of distance function.*

Proposition 3 gives the evaluation rule. However, it is not computable and not easy to be applied. One solution to this is to transform it into analyzing eigenvalues. Since all first-order roll–slide motions can be denoted by $\boldsymbol{V} = [\,\boldsymbol{V}_1 \cdots \boldsymbol{V}_m\,][\,\lambda_1 \cdots \lambda_m\,]^{\mathrm{T}}$, which is the general solution of a system of linear homogeneous equations by setting Eq. (11) equal to zero, where $\boldsymbol{V}_1 \cdots \boldsymbol{V}_m$ are m linearly independent solutions and $\lambda_1 \cdots \lambda_m$ are not all zero, we have

$$\boldsymbol{V}^{\mathrm{T}} H_i \boldsymbol{V} = [\lambda_1 \quad \cdots \quad \lambda_m] \begin{bmatrix} \boldsymbol{V}_1^{\mathrm{T}} \\ \vdots \\ \boldsymbol{V}_m^{\mathrm{T}} \end{bmatrix} \nabla^2 d_i^S(g) [\boldsymbol{V}_1 \quad \cdots \quad \boldsymbol{V}_m] \times \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix} = [\lambda_1 \quad \cdots \quad \lambda_m] \tilde{\boldsymbol{H}}_i \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix}, \tag{18}$$

and the second-order immobility can be evaluated by analyzing the eigenvalues of

$$\tilde{\boldsymbol{H}}_i = \begin{bmatrix} \boldsymbol{V}_1^{\mathrm{T}} \\ \vdots \\ \boldsymbol{V}_m^{\mathrm{T}} \end{bmatrix} \nabla^2 d_i^S(g) [\boldsymbol{V}_1 \quad \cdots \quad \boldsymbol{V}_m].$$

Since $\tilde{H}_i$ is a real symmetric matrix, there exists a real orthogonal matrix $\boldsymbol{X}_i$ such that $\Lambda_i = \boldsymbol{X}_i^{-\mathrm{T}} \tilde{\boldsymbol{H}}_i \boldsymbol{X}_i^{-1}$ is a diagonal matrix. Then, we have the following Proposition 4.

Proposition 4: *Let $h_1^{\mathcal{A}_i} \cdots h_m^{\mathcal{A}_i}$ be the eigenvalues of $\tilde{\boldsymbol{H}}_i$, $\bar{\boldsymbol{\lambda}}^{\mathcal{A}_i} = \boldsymbol{X}_i \boldsymbol{\lambda}$ where $\boldsymbol{X}_i$ is a full matrix whose columns are formed by the eigenvectors of $\tilde{\boldsymbol{H}}_i$. (i)* Second-order immobility. *If for all nonzero vector $\boldsymbol{\lambda} = [\lambda_1 \quad \cdots \quad \lambda_m]^{\mathrm{T}}$, there is at least one finger $\mathcal{A}_i$ satisfying $\sum_{l=1}^m h_l^{\mathcal{A}_i} (\bar{\lambda}_l^{\mathcal{A}_i})^2 < 0$, then the grasp is second-order immobile. (ii)* Second-order mobility. *If for some nonzero vector $\boldsymbol{\lambda}$, every finger $\mathcal{A}_i$ satisfies the condition of $\sum_{l=1}^m h_l^{\mathcal{A}_i} (\bar{\lambda}_l^{\mathcal{A}_i})^2 \geq 0$ and there exists at least one finger $\mathcal{A}_j$ satisfying the condition of $\sum_{l=1}^m h_l^{\mathcal{A}_j} (\bar{\lambda}_l^{\mathcal{A}_j})^2 > 0$, then the grasp is mobile. (iii)* Third- or higher order immobility(or mobility) analysis. *If for some nonzero vector $\boldsymbol{\lambda}$, every finger $\mathcal{A}_i$ satisfies the condition of $\sum_{l=1}^m h_l^{\mathcal{A}_i} (\bar{\lambda}_l^{\mathcal{A}_i})^2 = 0$, then the third-order term of distance function should be analyzed.*

Remark: *If every $\tilde{\boldsymbol{H}}_i$ is seminegative, all $\tilde{\boldsymbol{H}}_i$s have at least one null eigenvalue, and then, we need to analyze whether there exists a certain set of constants $\lambda_1 \quad \cdots \quad \lambda_m$ such that $\sum_{l=1}^m h_l^{\mathcal{A}_i} (\bar{\lambda}_l^{\mathcal{A}_i})^2 = 0$ for each finger $\mathcal{A}_i$. First, assume every $\tilde{\boldsymbol{H}}_i$ has ONLY one null eigenvalue and every $\boldsymbol{X}_i$ is chosen such that*

$$\tilde{\boldsymbol{H}}_i = \boldsymbol{X}_i \begin{bmatrix} h_1^{\mathcal{A}_i} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \boldsymbol{X}_i^{-1},$$

namely, the null eigenvalue is placed in the last column and last row of the diagonal. We have the following conclusion: If $\boldsymbol{e}_m = [0, \cdots, 0, 1]^{\mathrm{T}}$ is not the eigenvector of $\boldsymbol{X}_j \boldsymbol{X}_i^{-1}$, $i \neq j$, there does not exist a $\boldsymbol{\lambda}$ such that both $\sum_{l=1}^m h_l^{\mathcal{A}_i} (\bar{\lambda}_l^{\mathcal{A}_i})^2 = 0$ and $\sum_{l=1}^m h_l^{\mathcal{A}_j} (\bar{\lambda}_l^{\mathcal{A}_j})^2 = 0$, and furthermore, there does not exist a $\boldsymbol{\lambda}$ enabling $\sum_{l=1}^m h_l^{\mathcal{A}_i} (\bar{\lambda}_l^{\mathcal{A}_i})^2 = 0$ for every finger $\mathcal{A}_i$. Hence, the grasp is immobile (see Appendix D for detailed proof on this). Second, if $\tilde{H}_i$ has more than one null eigenvalue, similarly $\boldsymbol{X}_i \boldsymbol{\lambda}$ must be in the linear space defined by

$$\boldsymbol{e}_{m-m_i} = \left[0, \cdots, 0, 1, \underbrace{0, \cdots 0}_{m_i - 1}\right]^{\mathrm{T}}, \ldots, \boldsymbol{e}_m = [0, \cdots, 0, 1]^{\mathrm{T}}$$

except for the original point, where m_i is the number of null eigenvalues. If the linear spaces

$$\boldsymbol{X}_j \boldsymbol{X}_i^{-1} \begin{pmatrix} \mathbf{0}_{(m-m_i)\times m_i} \\ \boldsymbol{I}_{m_i \times m_i} \end{pmatrix} \boldsymbol{u} \quad \text{and} \quad \begin{pmatrix} \mathbf{0}_{(m-m_j)\times m_j} \\ \boldsymbol{I}_{m_j \times m_j} \end{pmatrix} \tilde{\boldsymbol{u}}$$

have no other common spaces except for the original point, there does not exist a $\boldsymbol{\lambda}$ enabling both $\sum_{l=1}^m h_l^{\mathcal{A}_i} (\bar{\lambda}_l^{\mathcal{A}_i})^2 = 0$ and $\sum_{l=1}^m h_l^{\mathcal{A}_j} (\bar{\lambda}_l^{\mathcal{A}_j})^2 = 0$. Hence, the grasp is immobile.

Additionally, if the linear space $\mathcal{V}^{sr}$ is only one-dimensional, which corresponds to the essential equilibrium grasp[1],[5] then $\tilde{\boldsymbol{H}}_i$ will degenerate a scale and its sign is used to check immobility (mobility). If this value is positive, then the grasp is immobile. If negative, then the grasp is mobile, and if it is equal to zero, we need to analyze the third order immobility.

Also, it is worth pointing out that the condition given by Proposition 4 is sufficient and necessary for second-order immobility (or mobility) check. Moreover, a sufficient condition for second-order immobility check is given as follows.

Proposition 5 (Sufficient condition for second-order immobility): *When rigid body $\mathcal{B}$ undergoes first-order roll–slide free motions on multifinger $\mathcal{A}_1, \cdots \mathcal{A}_k$ and there exists one finger $\mathcal{A}_i$ satisfying that all eigenvalues of $\tilde{\boldsymbol{H}}_i$ are strictly negative, this grasp is second-order immobile.*

For all nonzero vector $\boldsymbol{\lambda}$, $\bar{\boldsymbol{\lambda}} = \boldsymbol{X}_i \boldsymbol{\lambda}$ must not be a null vector due to the fact that $\boldsymbol{X}_i$ is a full matrix. Since there exists one finger $\mathcal{A}_i$ such that all the eigenvalues of $\tilde{\boldsymbol{H}}_i$ are negative, i.e., $h_l^{\mathcal{A}_i} < 0$, $l = 1, \cdots, m$, thus $\sum_{l=1}^m h_l^{\mathcal{A}_l} (\bar{\lambda}_l^{\mathcal{A}_i})^2 < 0$. This finger can resist any first-order roll–slide free motions.

[1] If an equilibrium k finger grasp is essential, then the Plücker coordinates of any $k-1$ normal vectors are linearly independent, i.e., the first-order free motions are one-dimensional.

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Table I. The relationship of free motions, signed distance function, and immobility.

First-order motion analysis	Second-order motion analysis	Signed distance function	Optimum of NLP	Is it immobile
First-order escape motion		$d > 0$	$\lambda_0^* > 0$	No
First-order roll–slide motion	Second-order escape motion			
	Second-order roll–slide motion	$d = 0$	$\lambda_0^* = 0$	Need consider the third-order free motion
	Unreasonable motion	$d < 0$	$\lambda_0^* < 0$	Yes
Unreasonable motion				

3.2.3. Nonlinear programming problem on second-order immobility analysis. In the above second-order immobility analysis, the sign of Eq. (18) turns out to be important. When there exists at least one finger $\mathcal{A}_i$ leading to the negative value of Eq. (18), the grasp is second-order immobile. Then, in the concerned k fingers, we only need to consider the minimum value of Eq. (18) for all nonzero vector $\boldsymbol{\lambda}$. If all minimum ones are less than zero, the frictionless grasp is second-order immobile. In other words, if the maximum of all the minimum ones is less than zero, the frictionless grasp is second-order immobile. It can be described as the problem

$$\text{P1} \quad \max_{(\lambda_1,\cdots,\lambda_m)} \min_{1\le i\le k} [\lambda_1 \quad \cdots \quad \lambda_m]\tilde{\boldsymbol{H}}_i[\lambda_1 \quad \cdots \quad \lambda_m]^{\mathrm{T}} \quad s.t. \quad \sum_{j=1}^{m} \lambda_j^2 = 1,$$

where the condition $\sum_{j=1}^{m} \lambda_j^2 = 1$ ensures that $\lambda_1 \quad \cdots \quad \lambda_m$ are not all zero. Clearly, if a new variable λ_0 is defined by $\lambda_0 = \min_{1\le i\le k} [\lambda_1 \quad \cdots \quad \lambda_m]\tilde{\boldsymbol{H}}_i[\lambda_1 \quad \cdots \quad \lambda_m]^{\mathrm{T}}$, then the problem P1 of maximizing the minimum can be transformed into a nonlinear programming problem as follows:

$$\text{NLP} \quad \max_{(\lambda_0,\lambda_1,\cdots,\lambda_m)} \lambda_0 \quad s.t. \ [\lambda_1 \ \cdots \ \lambda_m]\tilde{\boldsymbol{H}}_i[\lambda_1 \ \cdots \ \lambda_m]^{\mathrm{T}} \ge \lambda_0, i = 1, \ldots, k \quad \sum_{j=1}^{m} \lambda_j^2 = 1.$$

The second-order immobility check is now being reshaped as analyzing the solution of the nonlinear programming problem.

Proposition 6: Assume λ_0^* be the optimum of NLP above. When $\lambda_0^* < 0$, the multiple-finger frictionless grasp is second-order immobile; when $\lambda_0^* > 0$, it is mobile; when $\lambda_0^* = 0$, it needs to dig deep into third-order free motions.

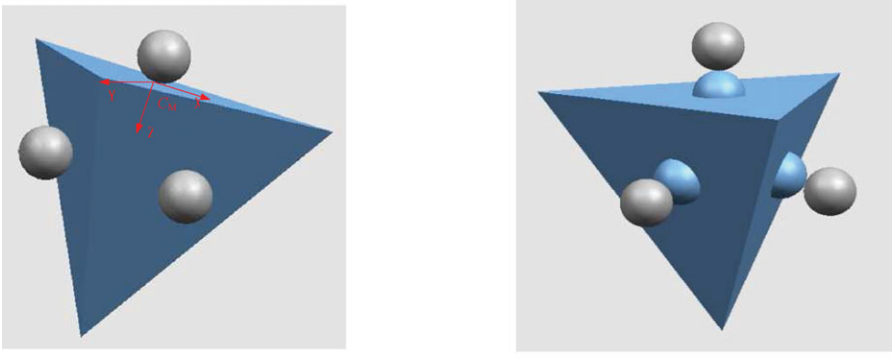
Proposition 6 gives a sufficient and necessary condition for the second-order immobility. If $\tilde{\boldsymbol{H}}_i$ is negative definite, the optimum must be less than zeros, and the corresponding grasp is second-order immobile, which is in line with **Proposition 4**.

Owing to the fact that the objection and constraint functions are differentiable, we can get the necessary and sufficient condition of the locally optimal value λ_0 by Karush–Kuhn–Tucher (KKT) multipliers.[27] It is worth noting that if the local optimal value λ_0 satisfies the rule for mobility, we do not need to find out the globally optimal one for checking the grasp.

Lastly, the relationship of free motions, signed distance function, and immobility is summarized in Table I.

4. Examples

The numerical examples given here are similar to maximal four-finger grasp and minimal four-finger grasp in 3D space. As shown in Fig. 4, four spherical fingers grasp a rigid object: (a) Tetrahedron: every finger is located in contact



(a) (b)

Fig. 4. (Colour online) A tetrahedron grasped by four fingers.

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with the central point of every triangular face; (b) Deformed tetrahedron: there is a semisphere at the center part of every triangular face and four figures are located in contact with the semisphere. According to traditional first-order theory, i.e., Screw Theory, it is concluded that the two objects could rotate about the central point of the two objects. However, for the first case, i.e., Case (a), the rigid body should be totally restricted by four fingers from geometric analysis. This contradiction can be explained by the second-order immobility analysis.

Let the edge length of the tetrahedron be 6 mm and the radius of every finger be 1 mm. Assume the machine frame C_W accords with the model frame C_M, where Plane XY is in line with the bottom plane of tetrahedron and axis Z passes through a vertex. Hence, the original point of the frames locates at the central point of the bottom surface. First, we compute the first-order roll–slide motions. From Section 3, the first-order roll–slide motions must satisfy $[(\boldsymbol{n}^{\boldsymbol{q}})_i^{\mathrm{T}} \quad (\boldsymbol{q} \times \boldsymbol{n}^{\boldsymbol{q}})_i^{\mathrm{T}}]\boldsymbol{V} = 0, i = 1, \ldots, 4$. By solving the system of linear equations

$$\begin{bmatrix} (\boldsymbol{n}^{\boldsymbol{q}})_1^{\mathrm{T}} & (\boldsymbol{q} \times \boldsymbol{n}^{\boldsymbol{q}})_1^{\mathrm{T}} \\ & \vdots \\ (\boldsymbol{n}^{\boldsymbol{q}})_4^{\mathrm{T}} & (\boldsymbol{q} \times \boldsymbol{n}^{\boldsymbol{q}})_4^{\mathrm{T}} \end{bmatrix} V = 0,$$

the linearly independent solutions form a solution space, i.e.,

$$\boldsymbol{V} = \begin{bmatrix} -0.3196 & -0.6939 & 0.1278 \\ -0.6443 & 0.3442 & 0.2577 \\ 0 & 0 & 0 \\ -0.5260 & 0.2811 & 0.2104 \\ 0.2610 & 0.5666 & -0.1044 \\ 0.3714 & 0 & 0.9285 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \bar{\boldsymbol{V}} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix},$$

where λ_1, λ_2, and λ_3 are a set of coefficient. Obviously, V is the null space of the coefficient matrix

$$\begin{bmatrix} (\boldsymbol{n}^{\boldsymbol{q}})_1^{\mathrm{T}} & (\boldsymbol{q} \times \boldsymbol{n}^{\boldsymbol{q}})_1^{\mathrm{T}} \\ & \vdots \\ (\boldsymbol{n}^{\boldsymbol{q}})_k^{\mathrm{T}} & (\boldsymbol{q} \times \boldsymbol{n}^{\boldsymbol{q}})_k^{\mathrm{T}} \end{bmatrix}.$$

There are many ways to compute $\bar{\boldsymbol{V}}$ in the matrix theory. Every $\nabla^2 d_i^S(g)$ can be computed according to Eq. (8). Subsequently, we get $\tilde{\boldsymbol{H}}_i = \bar{\boldsymbol{V}}^{\mathrm{T}} \nabla^2 d_i^S(g) \bar{\boldsymbol{V}}$.

We use the symbol

$$\boldsymbol{V}(1:3) = \begin{bmatrix} -0.3196 & -0.6939 & 0.1278 \\ -0.6443 & 0.3442 & 0.2577 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

to represent the vector composed of the first three components of $\boldsymbol{V}$, and

$$\boldsymbol{V}(4:6) = \begin{bmatrix} -0.5260 & 0.2811 & 0.2104 \\ 0.2610 & 0.5666 & -0.1044 \\ 0.3714 & 0 & 0.9285 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

the vector composed of the remaining three components of $\boldsymbol{V}$. Obviously, for any nonzero vector

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}, \quad \|\boldsymbol{V}(4:6)\| \neq 0$$

and $\boldsymbol{V}(1:3)^{\mathrm{T}}\boldsymbol{V}(4:6) = 0$ hold. Hence, first-order roll–slide motions $\boldsymbol{V}$ are a rotation motion according to Chasles's theorem.

Eigenvalue method

Case a: Calculate every finger's matrix $\tilde{\boldsymbol{H}}_i$ and its corresponding eigenvalues. The eigenvalues of $\tilde{\boldsymbol{H}}_1$ are -2.0764, -0.8899, 0; -2.0764, -0.8899, 0 for $\tilde{\boldsymbol{H}}_2$; -2.0764, -0.8899 ,0 for $\tilde{\boldsymbol{H}}_3$; and -0.8899, -0.8899, 0 for $\tilde{\boldsymbol{H}}_4$, and $[0, 0, 1]^{\mathrm{T}}$ is not the eigenvector of $\boldsymbol{X}_4\boldsymbol{X}_1^{-1}$, where $\boldsymbol{X}_1, \boldsymbol{X}_4$ are the eigenvector matrix of $\tilde{\boldsymbol{H}}_1$ and $\tilde{\boldsymbol{H}}_4$, respectively. Evidently, for every set of $\lambda_1, \lambda_2, \lambda_3$, there exists at least one finger $\mathcal{A}_i$ to make $\sum_{l=1}^{m} h_l^{\mathcal{A}_i}(\bar{\lambda}_l^{\mathcal{A}_i})^2 < 0$, thus this grasp is immobile.

Case b: Calculate every finger's matrix $\tilde{\boldsymbol{H}}_i$ and its corresponding eigenvalues. The eigenvalues of $\tilde{\boldsymbol{H}}_1$ are 0.2333, 0.1000, 0; 0.2333, 0.1000, 0 for $\tilde{\boldsymbol{H}}_2$; 0.2333, 0.1000, 0 for $\tilde{\boldsymbol{H}}_3$; and 0.1000, 0.1000, 0 for $\tilde{\boldsymbol{H}}_4$, and $[0, 0, 1]^{\mathrm{T}}$ is not the eigenvector of $\boldsymbol{X}_4\boldsymbol{X}_1^{-1}$, where $\boldsymbol{X}_1, \boldsymbol{X}_4$ are the eigenvector matrix of $\tilde{\boldsymbol{H}}_1$ and $\tilde{\boldsymbol{H}}_4$, respectively. Evidently, for every finger $\mathcal{A}_i$, $\sum_{l=1}^{m} h_l^{\mathcal{A}_i}(\bar{\lambda}_l^{\mathcal{A}_i})^2 \geq 0$ holds, and for an arbitrary set of $\lambda_1, \lambda_2, \lambda_3$, there exists at least one finger $\mathcal{A}_i$ to make $\sum_{l=1}^{m} h_l^{\mathcal{A}_i}(\bar{\lambda}_l^{\mathcal{A}_i})^2 > 0$, thus this grasp is mobile.

Nonlinear programming method

Case a: Using function "fmincon" in Matlab to get the optimal value -0.8475 for λ_0, this value leads to the same conclusion: the grasp in Case a is immobile.

Case b: Using function "fmincon" in Matlab to get the optimal value $0.0833 > 0$ for λ_0, and this value brings the same conclusion: the grasp in Case b is mobile.

This example highlights again the need to check beyond first-order theory. The proposed method, which accounts for geometrical properties, yields different conclusion from first-order theory. Compared with Eigenvalue method, Nonlinear programming method in the above example avoided eigenvalues computation and the process of analyzing eigenvalues to reach the conclusion.

5. Conclusions and Future Research

This paper extended point-to-surface signed distance function to surface-to-surface signed distance function, analyzed its differential properties, and deduced its second-order Taylor expansion. Based on this theory, the first- and second-order free motions of rigid body $\mathcal{B}$ relative to multifinger $\mathcal{A}_1, \ldots \mathcal{A}_k$ were defined, and a new method was put forward to evaluate the immobility of multifinger frictionless grasp. In particular, for second-order immobility check, two methods were presented in this paper. Compared with existing methods in this area, the second-order term of

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signed distance function is inherently coordinate invariant, and the second-order immobility check is also being formulated as a nonlinear programming problem, which avoids the process of detailed eigenvalue analysis to reach the conclusion. Furthermore, we give a new proof of kinematic equation through the defined distance function while existing methods only directly apply the kinematic equation.

The developed method is based on frictionless contact assumption; however, it is also useful when friction is present as friction always acts to enhance the immobility of the grasped object or workpiece. The immobilization based on contact geometry consideration is always guaranteed to work no matter what the particular friction is at the contacts, and the focus of our future research is to apply the developed tools to this area, i.e., workpiece fixture design and analysis.[28,29]

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Appendix A

Proof. In Fig. 5, on each rigid body, let $C_{\mathcal{A}}$, $C_{\mathcal{B}}$ be the instantaneous frame of contact on each body, and $L_{\mathcal{A}}(t)$, $L_{\mathcal{B}}(t)$, respectively, be the family of frame associated with each body such that at time t we have $L_A(t) = C_A$, $L_{\mathcal{B}}(t) = C_{\mathcal{B}}$. Let $\boldsymbol{\psi}(u, v)$, $\boldsymbol{\varphi}(s, t)$ represent local coordinates $\boldsymbol{S}_{\mathcal{A}}$, $\boldsymbol{S}_{\mathcal{B}}$ around contact point with respect to $\mathcal{A}$, $\mathcal{B}$, where $\mathcal{A}$, $\mathcal{B}$ are frame associated with each body, respectively. Thus, $\bar{\boldsymbol{\psi}}(u, v) = g_{L_{\mathcal{A}}\mathcal{A}}\boldsymbol{\varphi}(u, v)$, $\bar{\boldsymbol{\varphi}}(s, t) = g_{L_{\mathcal{B}}\mathcal{B}}\boldsymbol{\phi}(s, t)$ are local coordinates with respect to $L_{\mathcal{A}}(t)$, $L_{\mathcal{B}}(t)$, respectively. Note that the parameters u, v of $\boldsymbol{\psi}(u, v)$ are the same parameters as the ones of $\bar{\boldsymbol{\psi}}(u, v)$, and this rule is also

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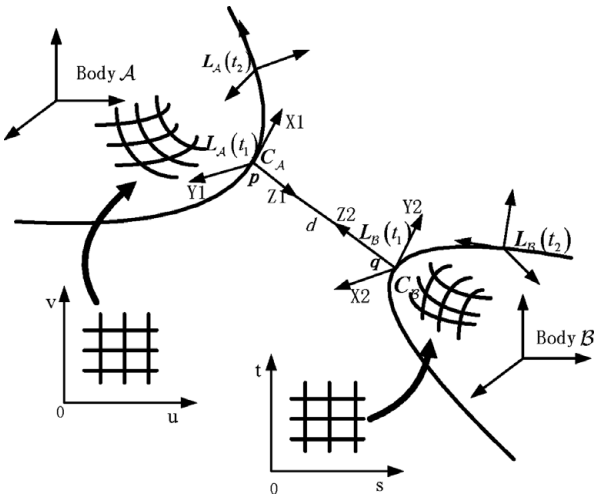


Fig. 5. Surface-to-Surface signed distance function.

applied to the parameters s and t. The configuration of the $\mathcal{B}$ frame relative to the $\mathcal{A}$ frame is

$$g_{\mathcal{AB}} = g_{\mathcal{A}L_{\mathcal{A}}} g_{L_{\mathcal{A}}L_{\mathcal{B}}} g_{L_{\mathcal{B}}\mathcal{B}},\ g_{L_{\mathcal{A}}L_{\mathcal{B}}} = \begin{bmatrix} \boldsymbol{R}_{\phi} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & d \end{bmatrix},$$

where

$$\boldsymbol{R}_{\phi} = \begin{bmatrix} \cos\phi & -\sin\phi \\ -\sin\phi & -\cos\phi \end{bmatrix},$$

and ϕ represents the angle between frame $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$. Note that configuration of the $L_{\mathcal{A}}(t)$ frame relative to $\mathcal{A}$ and the $L_{\mathcal{B}}(t)$ frame relative to $\mathcal{B}$ is constant. If $\boldsymbol{\psi}(u, v)$ and $\boldsymbol{\varphi}(s, t)$ are the orthogonal representation of surfaces, $\bar{\boldsymbol{\psi}}(u, v)$ and $\bar{\boldsymbol{\varphi}}(s, t)$ are also the orthogonal representation of surfaces. Let $\bar{\boldsymbol{\psi}}_u$ and $\bar{\boldsymbol{\psi}}_v$ accord with X- and Y-axes of $L_{\mathcal{A}}$, and $\bar{\boldsymbol{\varphi}}_s$ and $\bar{\boldsymbol{\varphi}}_t$ accord with the X- and Y-axes of $L_{\mathcal{B}}(t)$. We have

$$\Delta \boldsymbol{n}_{\boldsymbol{q}} = \left(\frac{\partial \boldsymbol{n}}{\partial s}, \frac{\partial \boldsymbol{n}}{\partial t}\right) \cdot \begin{pmatrix} \Delta s \\ \Delta t \end{pmatrix} = (\boldsymbol{\varphi}_s, \boldsymbol{\varphi}_t)\boldsymbol{C}_{\varphi} \begin{pmatrix} \Delta s \\ \Delta t \end{pmatrix}, \quad \text{(A 1)}$$

$$\begin{aligned} \Delta(\boldsymbol{q} \times \boldsymbol{n}_{\boldsymbol{q}}) &= \Delta \boldsymbol{q} \times \boldsymbol{n}_{\boldsymbol{q}} + \boldsymbol{q} \times \Delta \boldsymbol{n}_{\boldsymbol{q}} = (\boldsymbol{\varphi}_s, \boldsymbol{\varphi}_t) \begin{pmatrix} \Delta s \\ \Delta t \end{pmatrix} \\ &\quad \times \boldsymbol{n}_{\boldsymbol{q}} + \boldsymbol{q} \times \left[(\boldsymbol{\varphi}_s, \boldsymbol{\varphi}_t)\boldsymbol{C}_{\varphi} \begin{pmatrix} \Delta s \\ \Delta t \end{pmatrix} \right], \\ &= [\hat{\boldsymbol{q}}(\boldsymbol{\varphi}_s, \boldsymbol{\varphi}_t)\boldsymbol{C}_{\varphi} - \hat{\boldsymbol{n}}_{\boldsymbol{q}}(\boldsymbol{\varphi}_s, \boldsymbol{\varphi}_t)] \begin{pmatrix} \Delta s \\ \Delta t \end{pmatrix}. \quad \text{(A 2)} \end{aligned}$$

Combine (A 1) and (A 2)

$$\begin{bmatrix} \Delta \boldsymbol{n}_{\boldsymbol{q}} \\ \Delta(\boldsymbol{q} \times \boldsymbol{n}_{\boldsymbol{q}}) \end{bmatrix} = \begin{bmatrix} (\boldsymbol{\varphi}_s, \boldsymbol{\varphi}_t)\boldsymbol{C}_{\varphi} \\ \hat{\boldsymbol{q}}(\boldsymbol{\varphi}_s, \boldsymbol{\varphi}_t)\boldsymbol{C}_{\varphi} - \hat{\boldsymbol{n}}_{\boldsymbol{q}}(\boldsymbol{\varphi}_s, \boldsymbol{\varphi}_t) \end{bmatrix} \begin{pmatrix} \Delta s \\ \Delta t \end{pmatrix}. \quad \text{(A 3)}$$

To build the relationship between second-order differential matrix of signed distance function and differential motion, we need to build the relationship between the parameter coordinate of contact points and differential motion.

From Eqs. (6) and (7),

$$\begin{cases} \boldsymbol{R}^{\mathrm{T}} \bar{\boldsymbol{\psi}}(u^*, v^*) - \boldsymbol{R}^{\mathrm{T}} \boldsymbol{t} - \bar{\boldsymbol{\varphi}}(s^*, t^*) = d\boldsymbol{n}(s^*, t^*), \\ \boldsymbol{R} \bar{\boldsymbol{\varphi}}(s^*, t^*) + \boldsymbol{t} - \bar{\boldsymbol{\psi}}(u^*, v^*) = d\boldsymbol{n}(u^*, v^*), \end{cases} \quad \text{(A 4)}$$

where $(\boldsymbol{t}, \boldsymbol{R})$ is the configuration of $L_{\mathcal{A}}(t)$ relative to $L_{\mathcal{B}}(t)$, i.e., $g_{L_{\mathcal{A}}L_{\mathcal{B}}} = (\boldsymbol{t}, \boldsymbol{R})$. Because

$$\boldsymbol{n}(s, t) = \frac{\bar{\boldsymbol{\varphi}}_s \times \bar{\boldsymbol{\varphi}}_t}{\|\bar{\boldsymbol{\varphi}}_s \times \bar{\boldsymbol{\varphi}}_t\|}, \boldsymbol{n}(u, v) = \frac{\bar{\boldsymbol{\psi}}_u \times \bar{\boldsymbol{\psi}}_v}{\|\bar{\boldsymbol{\psi}}_u \times \bar{\boldsymbol{\psi}}_v\|},$$

thus

$$\begin{cases} \dfrac{\partial \bar{\boldsymbol{\varphi}}(s^*, t^*)}{\partial s} \cdot \boldsymbol{n}(s^*, t^*) = 0, \\ \dfrac{\partial \bar{\boldsymbol{\varphi}}(s^*, t^*)}{\partial t} \cdot \boldsymbol{n}(s^*, t^*) = 0, \end{cases} \quad \text{(A 5)}$$

$$\begin{cases} \dfrac{\partial \bar{\boldsymbol{\psi}}(u^*, v^*)}{\partial u} \cdot \boldsymbol{n}(u^*, v^*) = 0, \\ \dfrac{\partial \bar{\boldsymbol{\psi}}(u^*, v^*)}{\partial v} \cdot \boldsymbol{n}(u^*, v^*) = 0. \end{cases} \quad \text{(A 6)}$$

Substitute Eq. (A 4) into Eqs. (A 5) and (A 6), we have

$$\begin{cases} [\boldsymbol{R}^{\mathrm{T}} \bar{\boldsymbol{\psi}}(u^*, v^*) - \boldsymbol{R}^{\mathrm{T}} \boldsymbol{t} - \bar{\boldsymbol{\varphi}}(s^*, t^*)] \cdot \bar{\boldsymbol{\varphi}}_s = 0, \\ [\boldsymbol{R}^{\mathrm{T}} \bar{\boldsymbol{\psi}}(u^*, v^*) - \boldsymbol{R}^{\mathrm{T}} \boldsymbol{t} - \bar{\boldsymbol{\varphi}}(s^*, t^*)] \cdot \bar{\boldsymbol{\varphi}}_t = 0, \end{cases} \quad \text{(A 7)}$$

$$\begin{cases} [\boldsymbol{R} \bar{\boldsymbol{\varphi}}(s^*, t^*) + \boldsymbol{t} - \bar{\boldsymbol{\psi}}(u^*, v^*)] \cdot \bar{\boldsymbol{\psi}}_u = 0, \\ [\boldsymbol{R} \bar{\boldsymbol{\varphi}}(s^*, t^*) + \boldsymbol{t} - \bar{\boldsymbol{\psi}}(u^*, v^*)] \cdot \bar{\boldsymbol{\psi}}_v = 0. \end{cases} \quad \text{(A 8)}$$

Differentiate Eq. (A 7), we get

$$\begin{cases} \left[\Delta \boldsymbol{R}^{\mathrm{T}} \bar{\boldsymbol{\psi}}(u^*, v^*) + \boldsymbol{R}^{\mathrm{T}}(\bar{\boldsymbol{\psi}}_u \Delta u + \bar{\boldsymbol{\psi}}_v \Delta v) - \Delta \boldsymbol{R}^{\mathrm{T}} \boldsymbol{t} - \boldsymbol{R}^{\mathrm{T}} \Delta \boldsymbol{t} - \bar{\boldsymbol{\varphi}}_s \Delta s - \bar{\boldsymbol{\varphi}}_t \Delta t\right] \cdot \bar{\boldsymbol{\varphi}}_s + d\boldsymbol{n}(s^*, t^*)(\bar{\boldsymbol{\varphi}}_{ss} \Delta s + \bar{\boldsymbol{\varphi}}_{st} \Delta t) = 0, \\ \left[\Delta \boldsymbol{R}^{\mathrm{T}} \bar{\boldsymbol{\psi}}(u^*, v^*) + \boldsymbol{R}^{\mathrm{T}}(\bar{\boldsymbol{\psi}}_u \Delta u + \bar{\boldsymbol{\psi}}_v \Delta v) - \Delta \boldsymbol{R}^{\mathrm{T}} \boldsymbol{t} - \boldsymbol{R}^{\mathrm{T}} \Delta \boldsymbol{t} - \bar{\boldsymbol{\varphi}}_s \Delta s - \bar{\boldsymbol{\varphi}}_t \Delta t\right] \cdot \bar{\boldsymbol{\varphi}}_t + d\boldsymbol{n}(s^*, t^*)(\bar{\boldsymbol{\varphi}}_{st} \Delta s + \bar{\boldsymbol{\varphi}}_{tt} \Delta t) = 0, \end{cases}$$

$$\begin{bmatrix} (\boldsymbol{R}^{\mathrm{T}} \bar{\boldsymbol{\psi}}_u) \cdot \bar{\boldsymbol{\varphi}}_s & (\boldsymbol{R}^{\mathrm{T}} \bar{\boldsymbol{\psi}}_v) \cdot \bar{\boldsymbol{\varphi}}_s \\ (\boldsymbol{R}^{\mathrm{T}} \bar{\boldsymbol{\psi}}_u) \cdot \bar{\boldsymbol{\varphi}}_t & (\boldsymbol{R}^{\mathrm{T}} \bar{\boldsymbol{\psi}}_v) \cdot \bar{\boldsymbol{\varphi}}_t \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} - \left\{ \boldsymbol{R}^{\mathrm{T}} [\bar{\boldsymbol{\psi}}(u^*, v^*) - \boldsymbol{t}] \times \begin{bmatrix} \bar{\boldsymbol{\varphi}}_s \\ \bar{\boldsymbol{\varphi}}_t \end{bmatrix} \right\} \cdot \bar{\boldsymbol{\omega}} - \begin{bmatrix} \bar{\boldsymbol{\varphi}}_s^{\mathrm{T}} \\ \bar{\boldsymbol{\varphi}}_t^{\mathrm{T}} \end{bmatrix} \cdot \bar{\boldsymbol{v}} - (\boldsymbol{I}_{\mathcal{B}} + d\boldsymbol{\Pi}_{\mathcal{B}}) \begin{bmatrix} \Delta s \\ \Delta t \end{bmatrix} = 0,$$

$$\boldsymbol{R}_{\phi} \boldsymbol{M}_{\mathcal{A}} \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} + d \begin{bmatrix} -\bar{\omega}_y \\ \bar{\omega}_x \end{bmatrix} - \begin{bmatrix} \bar{v}_x \\ \bar{v}_y \end{bmatrix} - (\boldsymbol{1} + d\boldsymbol{K}_{\mathcal{B}}) \boldsymbol{M}_{\mathcal{B}} \begin{bmatrix} \Delta s \\ \Delta t \end{bmatrix} = 0, \quad \text{(A 9)}$$

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where $\bar{\boldsymbol{v}}$ in the above deduction is $\boldsymbol{v}^b_{L_\mathcal{A}L_\mathcal{B}}$, $\bar{\boldsymbol{\omega}}$ is $\boldsymbol{\omega}^b_{L_\mathcal{A}L_\mathcal{B}}$, it is also the same case for following deduction. $\boldsymbol{I}$, $\boldsymbol{\Pi}$ are the first and second fundamental form matrices,[21] respectively. M is metric tensor given by the square root of the first fundamental form and K is curvature tensor defined by $\boldsymbol{K} = \boldsymbol{M}^{-\mathrm{T}}\boldsymbol{\Pi}\boldsymbol{M}^{-1}$. If the coordinate system is in line with the principal curvature curves, then curvature tensor is simplified as a diagonal matrix $\boldsymbol{K} = \mathrm{diag}(\kappa_1, \kappa_2)$, where κ_1, κ_2 is the maximum and minimum principal curvature, respectively. Two factors on the left-hand side of Eq. (A 8) left-multiplied by $\boldsymbol{R}^{\mathrm{T}}$, then the dot product is the same, and differentiate it, we obtain

$$
\begin{cases}
\left[\bar{\boldsymbol{\varphi}}_s\Delta s + \bar{\boldsymbol{\varphi}}_t\Delta t + \Delta\boldsymbol{R}^{\mathrm{T}}\boldsymbol{t} + \boldsymbol{R}^{\mathrm{T}}\Delta\boldsymbol{t} - \Delta\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}(u^*, v^*) - \boldsymbol{R}^{\mathrm{T}}(\bar{\boldsymbol{\psi}}_u\Delta u + \bar{\boldsymbol{\psi}}_v\Delta v)\right]\cdot(\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_u) + d\boldsymbol{R}^{\mathrm{T}}\boldsymbol{n}(u^*, v^*)\cdot\left[\Delta\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_u + \boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_{uu} + \boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_{uv}\right] = 0,\\
\left[\bar{\boldsymbol{\varphi}}_s\Delta s + \bar{\boldsymbol{\varphi}}_t\Delta t + \Delta\boldsymbol{R}^{\mathrm{T}}\boldsymbol{t} + \boldsymbol{R}^{\mathrm{T}}\Delta\boldsymbol{t} - \Delta\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}(u^*, v^*) - \boldsymbol{R}^{\mathrm{T}}\left(\bar{\boldsymbol{\psi}}_u\Delta u + \bar{\boldsymbol{\psi}}_v\Delta v\right)\right]\cdot\left(\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_v\right) + d\boldsymbol{R}^{\mathrm{T}}\boldsymbol{n}(u^*, v^*)\cdot\left[\Delta\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_v + \boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_{vu} + \boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_{vv}\right] = 0,
\end{cases}
$$

$$
\begin{bmatrix}
\bar{\boldsymbol{\varphi}}_s\cdot(\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_u) & \bar{\boldsymbol{\phi}}_t\cdot(\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_u)\\
\bar{\boldsymbol{\varphi}}_s\cdot(\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_v) & \bar{\boldsymbol{\phi}}_t\cdot(\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_v)
\end{bmatrix}
\begin{bmatrix}\Delta s\\ \Delta t\end{bmatrix}
- d\left\{\hat{\bar{\boldsymbol{\omega}}}[\boldsymbol{R}^{\mathrm{T}}\boldsymbol{n}(u^*, v^*)]\right\}\cdot
\begin{bmatrix}(\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_u)\\ (\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_v)\end{bmatrix}
+
\begin{bmatrix}(\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_u)\\ (\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_v)\end{bmatrix}\cdot\bar{\boldsymbol{v}}
$$

$$
-\begin{bmatrix}
(\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_u)\cdot(\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_u) & (\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_v)\cdot(\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_u)\\
(\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_u)\cdot(\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_v) & \left(\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_u\right)\cdot(\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_v)
\end{bmatrix}
\begin{bmatrix}\Delta u\\ \Delta v\end{bmatrix}
- d\boldsymbol{R}^{\mathrm{T}}\boldsymbol{n}(u^*, v^*)\cdot
\begin{bmatrix}\bar{\boldsymbol{\omega}}\hat{\boldsymbol{R}^{\mathrm{T}}}\bar{\boldsymbol{\psi}}_u\\ \hat{\bar{\boldsymbol{\omega}}}\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{\psi}}_v\end{bmatrix}
- d\boldsymbol{\Pi}_o
\begin{bmatrix}\Delta u\\ \Delta v\end{bmatrix} = 0,
$$

$$
\boldsymbol{R}_\phi\boldsymbol{M}_\mathcal{B}\begin{bmatrix}\Delta s\\ \Delta t\end{bmatrix} + \boldsymbol{R}_\phi\begin{bmatrix}\bar{v}_x\\ \bar{v}_y\end{bmatrix} - (\mathbf{1} + d\boldsymbol{K}_\mathcal{A})\boldsymbol{M}_\mathcal{A}\begin{bmatrix}\Delta u\\ \Delta v\end{bmatrix} = 0. \tag{A 10}
$$

Combine Eqs. (A 9) and (A 10)

$$
\begin{cases}
\boldsymbol{R}_\phi\boldsymbol{M}_\mathcal{A}\begin{bmatrix}\Delta u\\ \Delta v\end{bmatrix} + d\begin{bmatrix}-\bar{\omega}_y\\ \bar{\omega}_x\end{bmatrix} - \begin{bmatrix}\bar{v}_x\\ \bar{v}_y\end{bmatrix} - (\mathbf{1} + d\boldsymbol{K}_\mathcal{B})\boldsymbol{M}_\mathcal{B}\begin{bmatrix}\Delta s\\ \Delta t\end{bmatrix} = 0,\\
\boldsymbol{R}_\phi\boldsymbol{M}_\mathcal{B}\begin{bmatrix}\Delta s\\ \Delta t\end{bmatrix} + \boldsymbol{R}_\phi\begin{bmatrix}\bar{v}_x\\ \bar{v}_y\end{bmatrix} - (\mathbf{1} + d\boldsymbol{K}_\mathcal{A})\boldsymbol{M}_\mathcal{A}\begin{bmatrix}\Delta u\\ \Delta v\end{bmatrix} = 0.
\end{cases} \tag{A 11}
$$

Solve a system of linear equations (A 11)

$$
\begin{cases}
\begin{bmatrix}\Delta s\\ \Delta t\end{bmatrix} = \boldsymbol{M}_\mathcal{B}^{-1}(\tilde{\boldsymbol{K}}_\mathcal{A} + \boldsymbol{K}_\mathcal{B} + d\tilde{\boldsymbol{K}}_\mathcal{A}\boldsymbol{K}_\mathcal{B})^{-1}\\
\quad\left\{(\mathbf{1} + d\tilde{\boldsymbol{K}}_\mathcal{A})\begin{bmatrix}-\bar{\omega}_y\\ \bar{\omega}_x\end{bmatrix} - \tilde{\boldsymbol{K}}_\mathcal{A}\begin{bmatrix}\bar{v}_x\\ \bar{v}_y\end{bmatrix}\right\},\\
\begin{bmatrix}\Delta u\\ \Delta v\end{bmatrix} = \boldsymbol{M}_\mathcal{A}^{-1}\boldsymbol{R}_\phi(\boldsymbol{K}_\mathcal{B} + \tilde{\boldsymbol{K}}_\mathcal{A} + d\boldsymbol{K}_\mathcal{B}\tilde{\boldsymbol{K}}_\mathcal{A})^{-1}\\
\quad\left\{\boldsymbol{K}_\mathcal{B}\begin{bmatrix}\bar{v}_x\\ \bar{v}_y\end{bmatrix} + \begin{bmatrix}-\bar{\omega}_y\\ \bar{\omega}_x\end{bmatrix}\right\},
\end{cases} \tag{A 12}
$$

where $\tilde{\boldsymbol{K}}_\mathcal{A} = \boldsymbol{R}_\phi\boldsymbol{K}_\mathcal{A}\boldsymbol{R}_\phi$.

Note that Eq. (A 12) is same as the equation presented by Anitescu *et al.*[25] who derived it by expanding Montana[26] equation. Here, we provide a novel prove for the kinematic equations of contact. When distance set to be zero, Eq. (A 12) reduces to Montana Equation.

Substitute the first expression in (A 12) into Eq. (A 3)

$$
\begin{aligned}
\begin{bmatrix}\Delta\boldsymbol{n}_q\\ \Delta(\boldsymbol{q}\times\boldsymbol{n}_q)\end{bmatrix} &= \begin{bmatrix}(\boldsymbol{\varphi}_s, \boldsymbol{\varphi}_t)C_\varphi\\ \hat{\boldsymbol{q}}(\boldsymbol{\varphi}_s, \boldsymbol{\varphi}_t)C_\varphi - \hat{\boldsymbol{n}}_{\boldsymbol{q}}(\boldsymbol{\varphi}_s, \boldsymbol{\varphi}_t)\end{bmatrix}\\
&\times \boldsymbol{M}_\mathcal{B}^{-1}(\tilde{\boldsymbol{K}}_\mathcal{A} + \boldsymbol{K}_\mathcal{B} + d\tilde{\boldsymbol{K}}_\mathcal{A}\boldsymbol{K}_\mathcal{B})^{-1}\\
&\cdot\left[-\tilde{\boldsymbol{K}}_\mathcal{A}\ \ 0\ \ (1 + d\tilde{\boldsymbol{K}}_\mathcal{A})\Delta\ \ 0\right]\cdot\boldsymbol{A}d^{-1}_{g_{\mathcal{B}L_\mathcal{B}}}\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\omega}\end{bmatrix},
\end{aligned} \tag{A 13}
$$

where

$$
\Delta = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\cdot\boldsymbol{A}d^{-1}_{g_{\mathcal{B}L_\mathcal{B}}} = \begin{bmatrix}\boldsymbol{R}^{\mathrm{T}}_{\mathcal{B}L_\mathcal{B}} & -\boldsymbol{R}^{\mathrm{T}}_{\mathcal{B}L_\mathcal{B}}\hat{\boldsymbol{t}}_{\mathcal{B}L_\mathcal{B}}\\ \mathbf{0} & \boldsymbol{R}^{\mathrm{T}}_{\mathcal{B}L_\mathcal{B}}\end{bmatrix}
$$

in Eq. (A 13) can further be expanded as

$$
\begin{aligned}
\boldsymbol{A}d^{-1}_{g_{\mathcal{B}L_\mathcal{B}}} &= \begin{bmatrix}\boldsymbol{R}^{\mathrm{T}}_{\mathcal{B}L_\mathcal{B}} & -\boldsymbol{R}^{\mathrm{T}}_{\mathcal{B}L_\mathcal{B}}\hat{\boldsymbol{q}}\\ \mathbf{0} & \boldsymbol{R}^{\mathrm{T}}_{\mathcal{B}L_\mathcal{B}}\end{bmatrix}\\
&= \begin{bmatrix}\left[\dfrac{\boldsymbol{\varphi}_s}{\|\boldsymbol{\varphi}_s\|}, \dfrac{\boldsymbol{\varphi}_t}{\|\boldsymbol{\varphi}_s\|}, \boldsymbol{n}_\varphi\right]^{\mathrm{T}} & \left(\left[\dfrac{\boldsymbol{q}\times\boldsymbol{\varphi}_s}{\|\boldsymbol{\varphi}_s\|}, \dfrac{\boldsymbol{q}\times\boldsymbol{\varphi}_t}{\|\boldsymbol{\varphi}_s\|}, \boldsymbol{q}\times\boldsymbol{n}_\varphi\right]\right)^{\mathrm{T}}\\ \mathbf{0} & \left[\dfrac{\boldsymbol{\varphi}_s}{\|\boldsymbol{\varphi}_s\|}, \dfrac{\boldsymbol{\varphi}_t}{\|\boldsymbol{\varphi}_s\|}, \boldsymbol{n}_\varphi\right]^{\mathrm{T}}\end{bmatrix}.
\end{aligned} \tag{A 14}
$$

Substituting Eq. (A 14) into Eq. (A 13)

$$
\begin{aligned}
&\begin{bmatrix}\Delta\boldsymbol{n}_q\\ \Delta(\boldsymbol{q}\times\boldsymbol{n}_q)\end{bmatrix}\\
&= \begin{bmatrix}\left(\dfrac{\boldsymbol{\varphi}_s}{\|\boldsymbol{\varphi}_s\|}, \dfrac{\boldsymbol{\varphi}_t}{\|\boldsymbol{\varphi}_s\|}\right)\\ \hat{\boldsymbol{q}}\left(\dfrac{\boldsymbol{\varphi}_s}{\|\boldsymbol{\varphi}_s\|}, \dfrac{\boldsymbol{\varphi}_t}{\|\boldsymbol{\varphi}_s\|}\right) - \hat{\boldsymbol{n}}_{\boldsymbol{q}}\left(\dfrac{\boldsymbol{\varphi}_s}{\|\boldsymbol{\varphi}_s\|}, \dfrac{\boldsymbol{\varphi}_t}{\|\boldsymbol{\varphi}_s\|}\right)K_\mathcal{B}^{-1}\end{bmatrix}\\
&\times\left(\tilde{\boldsymbol{K}}_\mathcal{A}^{-1} + \boldsymbol{K}_\mathcal{B}^{-1} + d\mathbf{1}\right)^{-1}\\
&\times\left[-\left(\dfrac{\boldsymbol{\varphi}_s}{\|\boldsymbol{\varphi}_s\|}, \dfrac{\boldsymbol{\varphi}_t}{\|\boldsymbol{\varphi}_t\|}\right)^T, -\left[\left(\hat{\boldsymbol{q}}\dfrac{\boldsymbol{\varphi}_s}{\|\boldsymbol{\varphi}_s\|}, \dfrac{\boldsymbol{\varphi}_t}{\|\boldsymbol{\varphi}_s\|}\right)\right]^{\mathrm{T}}\right.\\
&\left.-\left(\tilde{\boldsymbol{K}}_\mathcal{A}^{-1} + d\mathbf{1}\right)\left[\hat{\boldsymbol{n}}_{\boldsymbol{q}}\left(\dfrac{\boldsymbol{\varphi}_s}{\|\boldsymbol{\varphi}_s\|}, \dfrac{\boldsymbol{\varphi}_t}{\|\boldsymbol{\varphi}_t\|}\right)\right]^{\mathrm{T}}\right]\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\omega}\end{bmatrix}.
\end{aligned}
$$

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Thus, the second-order Hessen matrix of surface-to-surface signed distance function is

$$\nabla^2 d^S(g) = \begin{bmatrix} \boldsymbol{\Phi} \\ \hat{\boldsymbol{q}}\boldsymbol{\Phi} + \hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}\left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d\mathbf{1}\right) \end{bmatrix} \left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + \boldsymbol{K}_{\mathcal{B}}^{-1} + d\mathbf{1}\right)^{-1} \left[\boldsymbol{\Phi}^{\mathrm{T}}, \quad [\hat{\boldsymbol{q}}\boldsymbol{\Phi}]^{\mathrm{T}} + \left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d\mathbf{1}\right)[\hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}]^{\mathrm{T}}\right] + \frac{1}{2}\begin{bmatrix} 0 & \boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{n}}_{\boldsymbol{q}} \\ -\hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} & \hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}\left\{\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{q}} + \left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d\mathbf{1}\right)\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{n}}_{\boldsymbol{q}}\right\} + \left\{\hat{\boldsymbol{q}}\boldsymbol{\Phi} + \hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}\left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d\mathbf{1}\right)\right\}\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{n}}_{\boldsymbol{q}} \end{bmatrix},$$

where

$$\boldsymbol{\Phi} = \left(\frac{\boldsymbol{\varphi}_s}{\|\boldsymbol{\varphi}_s\|}, \frac{\boldsymbol{\varphi}_t}{\|\boldsymbol{\varphi}_t\|}\right).$$

When $\tilde{\boldsymbol{K}}_{\mathcal{A}} = 0$, i.e., $\mathcal{A}$ is locally planar, and $\boldsymbol{K}_{\mathcal{B}}$ is invertible, then $\tilde{\boldsymbol{K}}_{\mathcal{A}} + \boldsymbol{K}_{\mathcal{B}} + d\tilde{\boldsymbol{K}}_{\mathcal{A}}\boldsymbol{K}_{\mathcal{B}}$ is invertible, then the second-order Hessen matrix is

$$\nabla^2 d^S(g) = -\frac{1}{2}\begin{bmatrix} 0 & \boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{n}}^{\boldsymbol{q}} \\ -\hat{\boldsymbol{n}}^{\boldsymbol{q}}\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} & \hat{\boldsymbol{q}}\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{n}}^{\boldsymbol{q}} + \hat{\boldsymbol{n}}^{\boldsymbol{q}}\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{q}} - 2\hat{\boldsymbol{n}}^{\boldsymbol{q}}\boldsymbol{\Phi}K_{\mathcal{B}}^{-1}\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{n}}^{\boldsymbol{q}} \end{bmatrix},$$

when $\boldsymbol{K}_{\mathcal{B}} = 0$ and $\tilde{\boldsymbol{K}}_{\mathcal{A}}$ is invertible, then $\tilde{\boldsymbol{K}}_{\mathcal{A}} + \boldsymbol{K}_{\mathcal{B}} + d\tilde{\boldsymbol{K}}_{\mathcal{A}}\boldsymbol{K}_{\mathcal{B}}$ is invertible, then the second-order Hessen matrix is

$$\nabla^2 d^S(g) = \frac{1}{2}\begin{bmatrix} 0 & \boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{n}}^{\boldsymbol{q}} \\ -\hat{\boldsymbol{n}}^{\boldsymbol{q}}\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} & \hat{\boldsymbol{q}}\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{n}}^{\boldsymbol{q}} + \hat{\boldsymbol{n}}^{\boldsymbol{q}}\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{q}} + 2\hat{\boldsymbol{n}}^{\boldsymbol{q}}\boldsymbol{\Phi}\left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d\mathbf{1}\right)\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{n}}^{\boldsymbol{q}} \end{bmatrix}.$$

□

Appendix B

Proof. If two arbitrary frames $\mathcal{C}$ and $\mathcal{D}$ such that $\mathcal{C}$ is associated with object $\mathcal{A}$ and $\mathcal{D}$ is associated with object $\mathcal{B}$, then

$$\boldsymbol{V}_{\mathcal{CD}}^{b} = \boldsymbol{Ad}_{g_{\mathcal{BD}}}^{-1}\boldsymbol{V}_{\mathcal{AB}}^{b}, \tag{B 1}$$

$$[\boldsymbol{n}_{\boldsymbol{q}}\ \boldsymbol{q} \times \boldsymbol{n}_{\boldsymbol{q}}]_{\mathcal{D}} = [\boldsymbol{n}_{\boldsymbol{q}}\ \boldsymbol{q} \times \boldsymbol{n}_{\boldsymbol{q}}]_{\mathcal{B}}\boldsymbol{Ad}_{g_{\mathcal{BD}}}. \tag{B 2}$$

The second-order differential increment of surface-to-surface signed distance function is

$$\frac{1}{2}\begin{bmatrix} \boldsymbol{v}_{\mathcal{AB}} \\ \boldsymbol{\omega}_{\mathcal{AB}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{\Phi} \\ \hat{\boldsymbol{q}}\boldsymbol{\Phi} + \hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}\left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d\mathbf{1}\right) \end{bmatrix}_{\mathcal{B}} \left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + \boldsymbol{K}_{\mathcal{B}}^{-1} + d\mathbf{1}\right)^{-1} \left[\boldsymbol{\Phi}^{\mathrm{T}}, [\hat{\boldsymbol{q}}\boldsymbol{\Phi}]^{\mathrm{T}} + \left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d\mathbf{1}\right)[\hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}]^{\mathrm{T}}\right]_{\mathcal{B}} \begin{bmatrix} \boldsymbol{v}_{\mathcal{AB}} \\ \boldsymbol{\omega}_{\mathcal{AB}} \end{bmatrix} + \frac{1}{4}\begin{bmatrix} \boldsymbol{v}_{\mathcal{AB}} \\ \boldsymbol{\omega}_{\mathcal{AB}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & \boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\hat{n}_q \\ -\hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} & \hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}\left\{\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{q}} + \left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d\mathbf{1}\right)\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{n}}_{\boldsymbol{q}}\right\} + \left\{\hat{\boldsymbol{q}}\boldsymbol{\Phi} + \hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}\left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d\mathbf{1}\right)\right\}\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{n}}_{\boldsymbol{q}} \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} \boldsymbol{v}_{\mathcal{AB}} \\ \boldsymbol{\omega}_{\mathcal{AB}} \end{bmatrix}. \tag{B 3}$$

Substituting Eqs. (B 1) and (B 2) into Eq. (B 3) yields

$$\begin{aligned} &\frac{1}{2}\begin{bmatrix} \boldsymbol{v}_{\mathcal{CD}} \\ \boldsymbol{\omega}_{\mathcal{CD}} \end{bmatrix}^{\mathrm{T}} \boldsymbol{Ad}_{g_{\mathcal{BD}}}^{\mathrm{T}}\boldsymbol{Ad}_{g_{\mathcal{BD}}}^{-\mathrm{T}} \begin{bmatrix} \boldsymbol{\Phi} \\ \hat{\boldsymbol{q}}\boldsymbol{\Phi} + \hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}\left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d\mathbf{1}\right) \end{bmatrix}_{\mathcal{D}} \left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + \boldsymbol{K}_{\mathcal{B}}^{-1} + d\mathbf{1}\right)^{-1} \\ &\times \left[\boldsymbol{\Phi}^{\mathrm{T}}, [\hat{\boldsymbol{q}}\boldsymbol{\Phi}]^{\mathrm{T}} + \left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d\mathbf{1}\right)[\hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}]^{\mathrm{T}}\right]_{\mathcal{D}} \boldsymbol{Ad}_{g_{\mathcal{BD}}} \\ &\times \boldsymbol{Ad}_{g_{\mathcal{BD}}}^{-1}\begin{bmatrix} \boldsymbol{v}_{\mathcal{CD}} \\ \boldsymbol{\omega}_{\mathcal{CD}} \end{bmatrix} + \frac{1}{4}\begin{bmatrix} \boldsymbol{v}_{\mathcal{CD}} \\ \boldsymbol{\omega}_{\mathcal{CD}} \end{bmatrix}^{\mathrm{T}} \boldsymbol{Ad}_{g_{\mathcal{BD}}}^{\mathrm{T}}\boldsymbol{Ad}_{g_{\mathcal{BD}}}^{-\mathrm{T}} \\ &\times \begin{bmatrix} 0 & \boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\hat{n}_q \\ -\hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} & \hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}\left\{\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{q}} + \left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d\mathbf{1}\right)\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{n}}_{\boldsymbol{q}}\right\} + \left\{\hat{\boldsymbol{q}}\boldsymbol{\Phi} + \hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}\left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d\mathbf{1}\right)\right\}\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{n}}_{\boldsymbol{q}} \end{bmatrix}_{\mathcal{D}} \\ &\times \boldsymbol{Ad}_{g_{\mathcal{BD}}}\boldsymbol{Ad}_{g_{\mathcal{BD}}}^{-1}\begin{bmatrix} \boldsymbol{v}_{\mathcal{CD}} \\ \boldsymbol{\omega}_{\mathcal{CD}} \end{bmatrix}. \end{aligned} \tag{B 4}$$

Equation (B 4) is further simplified as

$$\begin{aligned} &\frac{1}{2}\begin{bmatrix} \boldsymbol{v}_{\mathcal{CD}} \\ \boldsymbol{\omega}_{\mathcal{CD}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{\Phi} \\ \hat{\boldsymbol{q}}\boldsymbol{\Phi} + \hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}\left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d\mathbf{1}\right) \end{bmatrix}_{\mathcal{D}} \left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + \boldsymbol{K}_{\mathcal{B}}^{-1} + d\mathbf{1}\right)^{-1} \\ &\times \left[\boldsymbol{\Phi}^{\mathrm{T}}, [\hat{\boldsymbol{q}}\boldsymbol{\Phi}]^{\mathrm{T}} + \left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d\mathbf{1}\right)[\hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}]^{\mathrm{T}}\right]_{\mathcal{D}} \begin{bmatrix} \boldsymbol{v}_{\mathcal{CD}} \\ \boldsymbol{\omega}_{\mathcal{CD}} \end{bmatrix} + \frac{1}{4}\begin{bmatrix} \boldsymbol{v}_{\mathcal{CD}} \\ \boldsymbol{\omega}_{\mathcal{CD}} \end{bmatrix}^{\mathrm{T}} \\ &\times \begin{bmatrix} 0 & \boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\hat{n}_q \\ -\hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} & \hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}\left\{\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{q}} + \left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d\mathbf{1}\right)\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{n}}_{\boldsymbol{q}}\right\} + \left\{\hat{\boldsymbol{q}}\boldsymbol{\Phi} + \hat{\boldsymbol{n}}_{\boldsymbol{q}}\boldsymbol{\Phi}\left(\tilde{\boldsymbol{K}}_{\mathcal{A}}^{-1} + d\mathbf{1}\right)\right\}\boldsymbol{\Phi}^{\mathrm{T}}\hat{\boldsymbol{n}}_{\boldsymbol{q}} \end{bmatrix}_{\mathcal{D}} \\ &\times \begin{bmatrix} \boldsymbol{v}_{\mathcal{CD}} \\ \boldsymbol{\omega}_{\mathcal{CD}} \end{bmatrix}. \end{aligned} \tag{B 5}$$

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Comparing Eq. (B 3) with Eq. (B 5), two expressions are quite similar. Therefore, the second-order differential increment of signed distance function is coordinate invariant. □

Appendix C. The first-order differentiability of surface-to-surface signed distance function

Proof. Let $\psi(u, v)$, $\boldsymbol{\varphi}(s, t)$ represent local coordinates $\boldsymbol{S}_{\mathcal{A}}$, $\boldsymbol{S}_{\mathcal{B}}$ around contact points $\boldsymbol{p}$, $\boldsymbol{q}$ with respect to $\mathcal{A}$, $\mathcal{B}$. Then, $\boldsymbol{p} = \boldsymbol{\psi}(u^*, v^*), \boldsymbol{q} = \boldsymbol{\varphi}(s^*, t^*)$ and substitute into Eq. (5), then we have

$$\begin{aligned} d^s &= (g_{\mathcal{AB}}^{-1}\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{n}^{\boldsymbol{q}} \\ &= \left[\boldsymbol{R}_{\mathcal{AB}}^{\mathrm{T}}\psi(u^*, v^*) - \boldsymbol{R}_{\mathcal{AB}}^{\mathrm{T}}\boldsymbol{t}_{\mathcal{AB}} - \boldsymbol{\varphi}(s^*, t^*)\right] \cdot \boldsymbol{n}(s^*, t^*). \end{aligned} \tag{C 1}$$

After a differential motion, $\tilde{g}(\boldsymbol{v}, \boldsymbol{\omega}) \approx (\boldsymbol{R}_{\mathcal{AB}}\boldsymbol{v} + \boldsymbol{t}, \boldsymbol{R}_{\mathcal{AB}}(\boldsymbol{I} + \hat{\boldsymbol{\omega}}))$ and the corresponding contact points change into $\boldsymbol{\psi}\,(u^* + \Delta u, v^* + \Delta v)$, $\boldsymbol{\varphi}\,(s^* + \Delta s, t^* + \Delta t)$. Then, we have

$$\begin{aligned} \Delta d^S = \Big[&\Delta \boldsymbol{R}_{\mathcal{AB}}^{\mathrm{T}}\boldsymbol{\psi}(u^*, v^*) - \Delta \boldsymbol{R}_{\mathcal{AB}}^{\mathrm{T}}\boldsymbol{t}_{\mathcal{AB}} - \boldsymbol{R}_{\mathcal{AB}}^{\mathrm{T}}\Delta \boldsymbol{t}_{\mathcal{AB}} \\ &+ \boldsymbol{R}_{\mathcal{AB}}^{\mathrm{T}}\frac{\partial \boldsymbol{\psi}(u^*, v^*)}{\partial u}\Delta u + \boldsymbol{R}_{\mathcal{AB}}^{\mathrm{T}}\frac{\partial \boldsymbol{\psi}(u^*, v^*)}{\partial v}\Delta v \\ &- \frac{\partial \boldsymbol{\varphi}(s^*, t^*)}{\partial s}\Delta s - \frac{\partial \boldsymbol{\varphi}(s^*, t^*)}{\partial t}\Delta t\Big] \cdot \boldsymbol{n}(s^*, t^*) \\ &+ \left(\boldsymbol{R}_{\mathcal{AB}}^{\mathrm{T}}\boldsymbol{\psi}(u^*, v^*) - \boldsymbol{R}_{\mathcal{AB}}^{\mathrm{T}}\boldsymbol{t}_{\mathcal{AB}} - \boldsymbol{\varphi}(s^*, t^*)\right) \\ &\times \left[\frac{\partial \boldsymbol{n}(s^*, t^*)}{\partial s}\Delta s + \frac{\partial \boldsymbol{n}(s^*, t^*)}{\partial t}\Delta t\right]. \end{aligned} \tag{C 2}$$

Since $\boldsymbol{n}(s^*, t^*)$ is a unit vector,

$$\boldsymbol{n}(s^*, t^*) \cdot \frac{\partial \boldsymbol{n}(s^*, t^*)}{\partial s} = \boldsymbol{n}(s^*, t^*) \cdot \frac{\partial \boldsymbol{n}(s^*, t^*)}{\partial t} = 0,$$

and

$$\begin{cases} \dfrac{\partial \boldsymbol{\varphi}(s^*, t^*)}{\partial s} \cdot \boldsymbol{n}(s^*, t^*) = 0 \\ \dfrac{\partial \boldsymbol{\varphi}(s^*, t^*)}{\partial t} \cdot \boldsymbol{n}(s^*, t^*) = 0 \end{cases},$$

$$\begin{cases} \left[\boldsymbol{R}_{\mathcal{AB}}^{\mathrm{T}}\dfrac{\partial \boldsymbol{\psi}(u^*, v^*)}{\partial u}\right] \cdot \boldsymbol{n}(s^*, t^*) = 0, \\ \left[\boldsymbol{R}_{\mathcal{AB}}^{\mathrm{T}}\dfrac{\partial \boldsymbol{\psi}(u^*, v^*)}{\partial v}\right] \cdot \boldsymbol{n}(s^*, t^*) = 0. \end{cases}$$

Equation (C 2) can be simplified as

$$\begin{aligned} \Delta d^S &= \left[\Delta \boldsymbol{R}_{\mathcal{AB}}^{\mathrm{T}}\boldsymbol{\psi}(u^*, v^*) - \Delta \boldsymbol{R}_{\mathcal{AB}}^{\mathrm{T}}\boldsymbol{t}_{\mathcal{AB}} - \boldsymbol{R}_{\mathcal{AB}}^{\mathrm{T}}\Delta \boldsymbol{t}_{\mathcal{AB}}\right] \cdot \boldsymbol{n}(s^*, t^*) \\ &= \left[\hat{\boldsymbol{\omega}}^{\mathrm{T}}\boldsymbol{R}_{\mathcal{AB}}^{\mathrm{T}}(\boldsymbol{\psi}(u^*, v^*) - \boldsymbol{t}_{\mathcal{AB}}) - \boldsymbol{v}\right] \cdot \boldsymbol{n}(s^*, t^*) \\ &= -\boldsymbol{n}^{\boldsymbol{q}} \cdot \boldsymbol{v} - (\boldsymbol{q} \times \boldsymbol{n}^{\boldsymbol{q}}) \cdot \boldsymbol{\omega} \end{aligned}$$

□

Appendix D

Proof. Assume there is a $\boldsymbol{\lambda}$ satisfying $\sum_{l=1}^{m} h_l^{\mathcal{A}_i}(\bar{\lambda}_l^{\mathcal{A}_i})^2 = 0$ for every finger $\mathcal{A}_i$. Then, for any two fingers $\mathcal{A}_i$ and $\mathcal{A}_j$ $(i \neq j)$, we have $\sum_{l=1}^{m} h_l^{\mathcal{A}_i}(\bar{\lambda}_l^{\mathcal{A}_i})^2 = 0$ and $\sum_{l=1}^{m} h_l^{\mathcal{A}_j}(\bar{\lambda}_l^{\mathcal{A}_j})^2 = 0$. Every $\tilde{H}_i$ has ONLY one null eigenvalue and every $\boldsymbol{X}_i$ is chosen such that

$$\tilde{\boldsymbol{H}}_i = \boldsymbol{X}_i \begin{bmatrix} h_1^{\mathcal{A}_i} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \boldsymbol{X}_i^{-1}.$$

Then, $\boldsymbol{X}_i\boldsymbol{\lambda}$ and $\boldsymbol{X}_j\boldsymbol{\lambda}$ must be in the linear space defined by $\boldsymbol{e}_m = [0, \cdots, 0, 1]^{\mathrm{T}}$ except for the original point. Hence, $\boldsymbol{X}_i\boldsymbol{\lambda} = y\boldsymbol{e}_m$ $(y \neq 0)$ and $\boldsymbol{X}_j\boldsymbol{\lambda} = y'\boldsymbol{e}_m$ $\left(y' \neq 0\right)$. Then, $\boldsymbol{\lambda} = y\boldsymbol{X}_i^{-1}\boldsymbol{e}_m = y'\boldsymbol{X}_j^{-1}\boldsymbol{e}_m$. We have $\boldsymbol{X}_j\boldsymbol{X}_i^{-1}\boldsymbol{e}_m = \frac{y'}{y}\boldsymbol{e}_m$. Thus, $\boldsymbol{e}_m$ is the eigenvector of $\boldsymbol{X}_j\boldsymbol{X}_i^{-1}$ $(i \neq j)$, which contradicts the prerequisite that $\boldsymbol{e}_m$ is not the eigenvector of $\boldsymbol{X}_j\boldsymbol{X}_i^{-1}$ $(i \neq j)$. Thus, the conclusion holds. □

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