Bounds on sums of cubes

A. F. BEARDON

1. *Introduction*

Here we are interested in the familiar identity

$$(1 + 2 + \ldots + n)^2 = 1^3 + 2^3 + \ldots + n^3$$

but only in as far as it provides the solution $(1, 2, \ldots, n)$ of the Diophantine equation

$$(a_1 + \ldots + a_n)^2 = a_1^3 + \ldots + a_n^3, \tag{1}$$

where n and the a_j are positive integers. In fact, not only is $(1, 2, \ldots, n)$ a solution of (1), it has the remarkable property that, up to a permutation of its entries, it is the only solution of (1) in which the a_j are distinct (see Section 4). Thus any other solution will necessarily contain repetitions among the a_j and, for this reason, a solution must be defined to be a *sequence* (which allows repetitions) rather than a set. However, we shall not distinguish between two solutions whose entries are permutations of each other.

To help the reader feel more at ease with (1), we begin with a few elementary solutions of it. First, $(n, n, \ldots, n)$ is a solution of (1). Next, if $n = 2k$ then $(k, \ldots, k, 2k, \ldots, 2k)$, where k and $2k$ each occur k times, is a solution. Also, if n is twice a triangular number, say $n = 2(1 + 2 + \ldots + k) = k(k + 1)$, then $(k, k, \ldots, k, n)$ is a solution of (1). Later we shall be interested in more general solutions of the form $a, \ldots, a, b)$, where a < b and a occurs n − 1 times, for this reduces the problem to solving the much simpler Diophantine equation

$$(ma + b)^2 = ma^3 + b^3, \qquad m = n - 1, \tag{2}$$

where m, a and b are positive integers.

One example of a solution of the form $(a, \ldots, a, b)$ is (3, 3, 3, 3, 6), where $n = 5$, and this example is of interest because it shows that there are some values of n with solutions $(a_1, \ldots, a_n)$ that satisfy $\max\{a_1, \ldots, a_n\} > n$. By contrast, it is shown in [1] that if $\alpha = \max\{a_1, \ldots, a_n\}$ then

$$\alpha^3 < \sum a_j^3 = \left(\sum a_j\right)^2 \leqslant (n\alpha)^2,$$

so that, for all values of n, and all solutions $(a_1, \ldots, a_n)$, we have $\max\{a_1, \ldots, a_n\} < n^2$. Having proved this, the authors of [1] then go on to say that 'numerical evidence indicates that a much better bound (than n^2) can be established', but no bound is suggested. They also suggest that good bounds on the *number* of positive solutions of (1) for each n 'seem plausible', but, at present, it seems that no (non-trivial) bounds of this type are known.

https://doi.org/10.1017/mag.2025.7 Published online by Cambridge University Press

Here we shall establish the two inequalities

$$\max\{a_1, \ldots, a_n\} < n^{4/3}, \qquad a_1 + \ldots + a_n \leqslant n^2,$$

each of which is stronger than their inequality $\max\{a_1, \ldots, a_n\} < n^2$. We shall also show that our 'much better upper bound' of $n^{4/3}$ is best possible. The proof that

$$a_1 + \ldots + a_n \leqslant n^2 \tag{3}$$

is easy. The function x^3 is convex on the interval $(0, +\infty)$, so if $(a_1, \ldots, a_n)$ is a solution of (1), then (by Jensen's inequality for convex functions)

$$\left(\frac{a_1 + \ldots + a_n}{n}\right)^3 \leqslant \frac{a_1^3 + \ldots + a_n^3}{n} = \frac{(a_1 + \ldots + a_n)^2}{n},$$

and this gives (3). It follows that, for a given n, (1) has only finitely many solutions and so, for relatively small values of n, all solutions can easily be obtained from a computer. One of the unresolved problems here is to obtain a better understanding of the general structure of solutions (or even of subclasses of solutions) of (1), particularly for large n.

We shall now present universal bounds on the solutions of (1) as n varies. For brevity, we shall write **a** for $(a_1, \ldots, a_n)$, and our bounds are as follows.

Theorem 1: Let **a** be a solution of (1), and s a positive number. Then

$$a_1^s + \ldots + a_n^s \leqslant \begin{cases} n^{s+1} & \text{if } 1 \leqslant s \leqslant 3, \\ n^{4s/3} & \text{if } s \geqslant 3. \end{cases} \tag{4}$$

If we put $s = 3$ in (4) we obtain $a_1^3 + \ldots + a_n^3 \leqslant n^4$, and this establishes the inequality (but not the last assertion) in the next result.

Corollary 1: Let **a** be a solution of (1). Then

$$\max\{a_1, \ldots, a_n\} < n^{4/3},$$

and this bound is best possible.

The origin of the discussion of (1) appears to be a result proved by the French mathematician Liouville in 1857 [1, 2, 3]. For each positive integer m, $\tau(m)$ is the number of positive divisors of m. If p and q are coprime positive integers then $\tau(pq) = \tau(p)\tau(q)$, so if m has the prime factorisation $m = p_1^{m_1} \ldots p_r^{m_r}$, then

$$\tau(m) = (m_1 + 1) \ldots (m_r + 1).$$

As an example, $\tau(12) = 6$, and the six divisors of 12 are 1, 2, 3, 4, 6, 12. Now the sequence of τ-values of these divisors is 1, 2, 2, 3, 4, 6, and as

$$(1 + 2 + 2 + 3 + 4 + 6)^2 = 1^3 + 2^3 + 2^3 + 3^3 + 4^3 + 6^3,$$

we see that (1, 2, 2, 3, 4, 6) is a solution of (1) when $n = 6$. Liouville observed that this is a general phenomenon.

https://doi.org/10.1017/mag.2025.7 Published online by Cambridge University Press

Theorem 2: Let $d_1, \ldots, d_r$ be the distinct, positive, divisors of some positive integer N. Then

$$[\tau(d_1) + \ldots + \tau(d_r)]^2 = \tau(d_1)^3 + \ldots + \tau(d_r)^3; \tag{5}$$

equivalently, $(\tau(d_1), \ldots, \tau(d_r))$ is a solution of (1) with $n = r$.

Liouville's theorem is not as mysterious as it might seem at first sight. Indeed, if $n = p^r$, where p is a prime, then the divisors of n are $1, p, \ldots, p^r$, and the sequence of τ-values of these divisors is $1, 2, \ldots, r + 1$. Thus (5) holds when n is a power of a prime precisely because of the identity at the start of this Article. The fact that (5) holds for any positive integer n depends (in an elementary way) on the fact that $\tau(pq) = \tau(p)\tau(q)$ whenever p and q are coprime; see [4] for the details. It is perhaps worth noting, though, that if $d_1, \ldots, d_r$ are the divisors of N, then $\max\{\tau(d_1), \ldots, \tau(d_r)\} = \tau(N)$. Now it is known [5, p. 260] that, for each positive δ, $\tau(N)/N^{\delta} \to 0$ as $N \to \infty$, and this suggests that Liouville's result is most unlikely to play a significant role in most discussions of the solutions of (1).

Finally, we note that the idea behind Liouville's theorem was generalised in [6] where it was noted that if $(a_1, \ldots, a_p)$ and $(b_1, \ldots, b_q)$ satisfy (1) with $n = p$ and $n = q$, respectively, and if $(c_1, c_2, \ldots, c_{pq})$ is any listing of the terms $a_i b_j$, $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$, then

$$\begin{aligned}(c_1 + \ldots + c_{pq}) &= (a_1 + \ldots + a_p)^2 (b_1 + \ldots + b_q)^2 \\ &= (a_1^3 + \ldots + a_p^3)(b_1^3 + \ldots + b_q^3)^2 \\ &= c_1^3 + \ldots + c_{pq}^3.\end{aligned}$$

In conclusion, $(c_1, c_2, \ldots, c_{pq})$ satisfies (1) with n replaced by pq. Thus, in this way, we can generate more and more solutions of (1) (albeit with increasing n).

2. *The proof of Theorem* 1

Although we may profess to be only interested in integer solutions of the Diophantine equation (1), there is much to be gained by allowing the a_j to be positive numbers and considering the function $s \to a_1^s + \ldots + a_n^s$ of the real variable s (essentially because convex functions satisfy many interesting inequalities). Here we shall assume that the a_j are positive, and that $\mathbf{a}$ is a solution of (1); nevertheless, some of the following inequalities hold *without* the assumption that $\mathbf{a}$ is a solution of (1).

For what follows, we refer to reader to [7, Chapter II], or to the modern account of this in [8, Chapter 1]. For a given sequence $a_1, \ldots, a_n$ of positive numbers, and a non-zero number s, we define the s-mean of $(a_1, \ldots, a_n)$ as

$$M_s(\mathbf{a}) = \left(\frac{a_1^s + \ldots + a_n^s}{n}\right)^{1/s}.$$

https://doi.org/10.1017/mag.2025.7 Published online by Cambridge University Press

Now it is known that

$$M_{-\infty}(\mathbf{a}) = \lim_{s \to -\infty} M_s(\mathbf{a}) = \min\{a_1, \ldots, a_n\},$$

$$M_0(\mathbf{a}) = \lim_{s \to 0} M_s(\mathbf{a}) = (a_1 \ldots a_n)^{1/n},$$

$$M_{+\infty}(\mathbf{a}) = \lim_{s \to +\infty} M_s(\mathbf{a}) = \max\{a_1, \ldots, a_n\},$$

and that, with these definitions, for each $\mathbf{a}$, $s \to M_s(\mathbf{a})$ is a continuous and increasing function of s on the extended real line $[-\infty, +\infty]$. A more detailed study of the function $M_s(\mathbf{a})$ leads to the well-known Hölder's inequality for sequences $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ of positive numbers, and for positive numbers p and q that satisfy $1/p + 1/q = 1$ (or, equivalently, $(p-1)(q-1) = 1$), namely

$$\sum_{j=1}^{n} a_j b_j \leqslant \left(\sum_{j=1}^{n} a_j^p\right)^{1/p} \left(\sum_{j=1}^{n} b_j^q\right)^{1/q}.$$

If we now put $b_1 = \ldots = b_n = 1$, $p = 3$ and $q = 3/2$ in this we find that

$$\left(\sum_{j=1}^{n} a_j\right)^3 \leqslant n^2 \sum_{j=1}^{n} a_j^3.$$

If we now combine this with the fact that $\mathbf{a}$ is a solution of (1), namely

$$\left(\sum_{j=1}^{n} a_j\right)^2 = \sum_{j=1}^{n} a_j^3,$$

we immediately obtain the inequalities that correspond to $s = 1$ and $s = 3$ in (4).

The inequalities for $1 < s < 3$ now follow from the properties of convex functions. Indeed, it is known that if $0 < r < s < t$ then

$$[M_s(\mathbf{a})]^s \leqslant [M_r(\mathbf{a})]^{ru} [M_t(\mathbf{a})]^{tv}, \qquad u = \frac{t-s}{t-r}, v = \frac{s-r}{t-r},$$

and if we put $r = 1$ and $t = 3$ then, for $1 < s < 3$, we have, since $\mathbf{a}$ satisfies (1),

$$\begin{aligned} a_1^s + \ldots + a_n^s &\leqslant (a_1 + \ldots + a_n)^{(3-s)/2} \left(a_1^3 + \ldots + a_n^3\right)^{(s-1)/2} \\ &= (a_1 + \ldots + a_n)^{(3-s)/2} (a_1 + \ldots + a_n)^{s-1} \\ &= (a_1 + \ldots + a_n)^{(1+s)/2} \\ &\leqslant n^{1+s}, \end{aligned}$$

which is the first inequality in (4).

https://doi.org/10.1017/mag.2025.7 Published online by Cambridge University Press

Finally, we suppose that $s > 3$. Now it is shown in [7, p. 28] that the function

$$G_s(\mathbf{a}) : s \to \left(a_1^s + \ldots + a_n^s\right) = n^{1/s} M_s(\mathbf{a})$$

is a decreasing function of s on $(0, +\infty)$. It follows that $G_s(\mathbf{a}) \leqslant G_3(\mathbf{a})$: thus

$$\left(a_1^s + \ldots + a_n^s\right)^{1/s} \leqslant \left(a_1^3 + \ldots + a_n^3\right)^{1/3} \leqslant n^{4/3},$$

and this completes our proof of Theorem 1.

The ideas above also lead to the following two corollaries – but we leave the proofs to the reader.

Corollary 2: If $\mathbf{a}$ is a solution of (1) with $a_j \geqslant n$ for all j, then $a_j = n$ for all j. In particular, except for the solution $(n, \ldots, n)$, we have $\min\{a_1, \ldots, a_n\} < n$.

The inequality $a_1^s + \ldots + a_n^s \leqslant n^{s+1}$ sometimes fails, but we do have the next result.

Corollary 3: In the notation above, $a_1^s + \ldots + a_n^s \leqslant n^{s+1}$ for all s in $(1, +\infty)$ if, and only if, $\max\{a_1, \ldots, a_n\} \leqslant n$.

3. *The upper bound $n^{4/3}$ is best possible*

We shall now show that the upper bound of $n^{4/3}$ is best possible in the following sense. Suppose that $\mathbf{a}$ is a solution of (1), and let $\alpha = \max\{a_1, \ldots a_n\}$. Then we can define t by $\alpha = n^t$, and we claim that $t \leqslant 4/3$ and that this bound is best possible. Since $t = \log \alpha / \log n$ (where log is the natural logarithm), this means that we are claiming that

$$\sup\left(\frac{\log(\max\{a_1 \ldots a_n\}}{\log n}\right) = \frac{4}{3}, \tag{6}$$

where the supremum is taken over all n, and all solutions $(a_1, \ldots, a_n)$, of (1). Now we already know that, for each n, $(n, n, \ldots, n)$ is a solution of (1), and also that, for all n, and for all solutions $\mathbf{a}$ of (1), we have $\max\{a_1, \ldots, a_n\} \leqslant n^{4/3}$. Thus, if we denote the supremum in (6) by γ, then, at present, we know that $1 \leqslant \gamma \leqslant 4/3$.

In order to verify (6) it is sufficient to find an infinite set, say N, of positive integers such that, for each n in N, there is a solution $\mathbf{a}$ of (1) such that

$$\frac{\log(\max\{a_1, \ldots, a_n\})}{\log n} \to \frac{4}{3}$$

as $n \to \infty$ through the set N. We shall now show that this is possible if we restrict our attention to solutions of the form $(a, a, \ldots, a, b)$.

https://doi.org/10.1017/mag.2025.7 Published online by Cambridge University Press

Initially, the author searched for such solutions on a computer, and found the solution

$$(a, b, n) = (5476, 88985, 8282) \tag{7}$$

(in which 5476 occurs 8281 times), and this shows that $\gamma > 1.26318\ldots$. Moreover, the numerical evidence strongly indicated that we would need ever larger values of n to approach the desired value of $1.333\ldots$ but, fortunately, the referee of the original version of this paper showed how to avoid further computations. The referee noted that we can parametrise the solutions of the 'reduced' equation $(ma + b)^2 = ma^3 + b^3$ in the form

$$a = \tfrac{1}{2}(x^3 - x), \qquad b = \tfrac{1}{2}(x^4 - x^2), \qquad m = x^3 - 2x, \qquad n = m + 1. \tag{8}$$

Of course, one can check directly (and laboriously) that this parametrisation is valid, but it is simpler to write $x = b/a$ for then the equation becomes $a^2(m + x)^2 = a^3(m + x^3)$, and this will certainly hold if we require x to satisfy the simultaneous equations (i) $m + x = 2a$ and (ii) $m + x^3 = 4a$. These give $a = (x^3 - x)/2$, and (8) now follows. It is also clear that with this parametrisation of the solutions $(a, \ldots, a, b)$, we have

$$n = x^3 - 2x + 1, \qquad \max\{a_1, \ldots, a_n\} = b = \tfrac{1}{2}(x^4 - x^2).$$

If we now let x run through the integers $2, 3, \ldots$, then n will run through the infinite set $\{x^3 - 2x + 1 : x = 2, 3, \ldots\}$, and, for these n, we see that $b^3/n^4 \to 1/2^3$ as $x \to \infty$. It follows that $b/n^{4/3} \to 1/2$, and this implies that $\log b/\log n \to 4/3$ as $x \to \infty$ so the bound of $4/3$ cannot be improved. Finally, these ideas are consistent with the results found in searching for the solution (7), and they support the view that if we use this method to find a solution for which b is approximately $n^{4/3}$, then we need to take n very large indeed.

4. *The uniqueness of strictly increasing solutions*

We have said that $(1, 2, \ldots, n)$ is, up to a permutation of its entries, the only integer solution of (1) in which the a_j are distinct. Here, for the reader's convenience, we present a version of the proof of this given in [1]. However, before we do this it is worth observing that this result does depend (in some way) on the fact that the a_j are integers (rather than real numbers). Indeed, if $a = 3/2$ and $b = (5 + \sqrt{13})/4$ then $1 < a < b$ and $(a + b)^2 = a^3 + b^3$. Note that if we regard this last equation as a cubic equation in b (for a given a) then one solution is $b = -a$, and the remaining solutions can be found by solving a quadratic equation in b.

Theorem 2: Suppose that the integers $a_1, \ldots, a_n$ satisfy $1 \leqslant a_1 < a_2 < \ldots < a_n$. Then

$$[a_1 + \ldots + a_n]^2 \leqslant a_1^3 + \ldots + a_n^3, \tag{9}$$

with, for $n \geqslant 2$, equality if, and only if, $(a_1, a_2, \ldots, a_n) = (1, 2, \ldots, n)$.

https://doi.org/10.1017/mag.2025.7 Published online by Cambridge University Press

Proof: The proof is by induction. The inequality (9) is trivially true when $n = 1$, so we suppose that (9) is true when $n = m$, and consider integers $a_1, \ldots, a_{m+1}$ with $1 \leqslant a_1 < a_2 < \ldots < a_{m+1}$. The crucial fact is that

$$a_{r+s} \geqslant a_r + s, \tag{10}$$

and the key step is the observation that, by comparison with

$$1 + 2 + \ldots + n = \tfrac{1}{2}(n+1)\big([n+1] - 1\big),$$

we have

$$a_1 + a_2 + \ldots + a_n \leqslant \tfrac{1}{2}a_{n+1}\big(a_{n+1} - 1\big),$$

or, equivalently,

$$a_{m+1}^2 \geqslant a_{m+1} + 2\big(a_1 + \ldots + a_m\big). \tag{11}$$

In order to prove (11), we start with the self-evident inequality

$$\big(a_{m+1} - m\big)\big(a_{m+1} - (m+1)\big) \geqslant 0.$$

This, with (10), yields

$$\begin{aligned} a_{m+1}^2 &\geqslant a_{m+1} + 2ma_{m+1} - m(m+1) \\ &= a_{m+1} + 2ma_{m+1} - 2(1 + \ldots + m) \\ &= a_{m+1} + 2\big((a_{m+1} - 1) + \ldots + (a_{m+1} - m)\big) \\ &\geqslant a_{m+1} + 2\big(a_m + \ldots + a_1\big). \end{aligned} \tag{12}$$

It follows from this and the induction hypothesis that

$$\begin{aligned} a_1^3 + \ldots + a_{m+1}^3 &= \big(a_1^3 + \ldots + a_m^3\big) + a_{m+1}^3 \\ &\geqslant (a_1 + \ldots + a_m)^2 + a_{m+1}^2 + 2a_{m+1}\big(a_1 + \ldots + a_m\big) \\ &= (a_1 + \ldots + a_{m+1})^2 \end{aligned}$$

which is (9).

Finally, we show that equality in (9) (with $n = m+1$) implies that $(a_1, \ldots, a_{m+1})$ is $(1, 2, \ldots, m+1)$. Now an analysis of the argument above shows that if equality holds in (9) (with $n = m+1$), then equality must hold in (12), and hence in (10) for each k. This completes the proof.

5. *Other Diophantine equations*

Finally, we write $\sigma_k(n)$ for $1^k + 2^k + \ldots + n^k$, and comment on a quite different aspect of the basic identity $\sigma_3(n) = \sigma_1(n)^2$ in the title. This relation shows that all of the points $\big(\sigma_1(n), \sigma_3(n)\big)$, $n = 1, 2, 3, \ldots$, lie on the parabola $y = x^2$ in the Euclidean plane $\mathbb{R}^2$. In fact (see [9]), there is a corresponding result for *any* two functions $\sigma_p(n)$ and $\sigma_q(n)$, where p and q are positive integers, and it follows from this that, in principle, the problem discussed above generalises to any pair of positive integers p and q.

https://doi.org/10.1017/mag.2025.7 Published online by Cambridge University Press

We give an explicit example. For $n = 1, 2, \ldots$ the functions

$$\sigma_3(n) = 1^3 + 2^3 + \ldots + n^3, \qquad \sigma_5(n) = 1^5 + 2^5 + \ldots + n^5$$

satisfy the algebraic relation

$$16x^3 - x^2 - 6xy - 9y^2 = 0, \qquad x = \sigma_3,\ y = \sigma_5,$$

(see [9]), so, for $n = 1, 2, \ldots,$ the points $\big(\sigma_3(n), \sigma_5(n)\big)$ lie on the algebraic curve given by $16x^3 = (x + 3y)^2$. Thus, for example, $\big(1, 2, \ldots, n\big)$ is a solution of the (formidable) Diophantine equation

$$16\left(\sum_{j=1}^{n} a_j^3\right)^3 = \left(\sum_{j=1}^{n} a_j^3 + 3\sum_{j=1}^{n} a_j^5\right)^2.$$

Note that $(n, n, \ldots, n)$ is not a solution of this equation unless $n = 1$.

References

1. E. Barbeau and S. Seraj, Sum of cubes is square of sum, *Notes on Number Theory and Discrete Mathematics*, **19**(1) pp. 1-13, available at https://nndtm.net/papers/nntdm-19/NNTDM–19–1–01–13.pdf
2. J. Liouville, *Jour. de Mathématiques* **2**(2), Comptes Rendus, (1857) pp. 425-432.
3. L. Dickson, *History of the theory of numbers*, Vol. 1, Chelsea [1910] (republished 1971).
4. D. Pagni, An interesting number fact, *Math. Gaz.* **82** (July 1998) pp. 271-273.
5. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers* (5th edn) Clarendon Press (1979).
6. J. H. Mason, Generalising 'Sums of cubes equal to squares of sums', *Math. Gaz.* **85** (March 2001) pp. 50-58.
7. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge (1978).
8. B. Bollobas, *Linear Analysis*, Cambridge (1990).
9. A. F. Beardon, Sums of powers of integers, *Amer. Math. Monthly* **103** (1996) pp. 201-213.

10.1017/mag.2025.7 © The Authors, 2025
Published by Cambridge University Press on behalf of The Mathematical Association

ALAN BEARDON
DPMMS,
University of Cambridge,
CB3 0WB
e-mail: *afb@dpmms.cam.ac.uk*

https://doi.org/10.1017/mag.2025.7 Published online by Cambridge University Press