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Chebyshev-Legendre Spectral Domain Decomposition Method for Two-Dimensional Vorticity Equations

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Abstract. We extend the Chebyshev-Legendre spectral method to multi-domain case for solving the two-dimensional vorticity equations. The schemes are formulated in Legendre-Galerkin method while the nonlinear term is collocated at Chebyshev-Gauss collocation points. We introduce proper basis functions in order that the matrix of algebraic system is sparse. The algorithm can be implemented efficiently and in parallel way. The numerical analysis results in the case of one-dimensional multi-domain are generalized to two-dimensional case. The stability and convergence of the method are proved. Numerical results are given.

AMS subject classifications: 65M12, 65M70

Key words: Multi-domain spectral method(spectral element method), Chebyshev-Legendre spectral method, vorticity equations.

1 Introduction

Spectral method has so called "infinity order" convergence rate but there are also some limits in applications. One main obstacle of spectral method is of treating discontinuous problems. The other is of treating complex domain problems. Therefore, how to overcome these obstacles has become the focus and hotspot in the research of the spectral method in recent years. Some progress has been achieved so far. For example, to the discontinuous problems, appropriate filter and reconstruction method is already used to restore the spectral approximation accuracy [1]. For the complex domain problems, if the whole domain is mapped into a regular area, it may increase the complexity of the problem and bring more errors. Then the domain decomposition skills were introduced. The basic idea of the domain decomposition method is to divide the original domain into several sub areas. If necessary, the sub-domain is mapped into regular area. And then using

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the spectral method in every sub-domain. Domain decomposition methods have been extensively used for numerical simulations involving spectral methods, as they can offer more flexibility in treating complicated domains as well as computational savings. Even for simple domain there are some good reasons to employ the domain decomposition spectral methods. Firstly, the spectral collocation points can be distributed a little more freedom so that the resolution on each subinterval can be chosen suitably. Secondly, the corresponding algebraic systems are better-conditioned because of using polynomials of smaller degrees. In addition, the algorithms allow for the best use of parallel computers which makes domain decomposition (multi-domain methods more and more popular.

The spectral domain decomposition method was first proposed in [2] in 1980, which it is also one of pioneer work of the spectral element method. Ref. [2] considered the Poisson equation with L-shaped domain. The L-shaped domain was decomposed into three non-overlapping sub-domains. In every sub-domain, the unknown functions were solved by expanding them into Chebyshev series and giving some connection conditions of interface. Such kind of spectral domain decomposition method is also referred to as the spectral element method or multi-domain spectral method. In [2] and [3] two kinds of spectral domain decomposition methods are studied. In the first kind, the domain is divided into several un-overlapped sub-domains and solution is connected through neighbouring interface conditions. This kind includes two cases. One is called spectral element method which is based on Galerkin projection methods. The other is the matching method which is based on collocation points and also called pseudo-spectral element method. In the second kind, they divided the domain into several overlapped sub-domains and solve the equations in stagger in every sub-domain. This kind of domain decomposition method is also known as the Schwarz method.

So far, the study of multi-domain spectral methods is very active and some progress has been achieved in algorithms and error estimates [4, 5] and Quarteroni [6]. Now the multi-domain spectral has been successfully applied into many practical fields such as fluid mechanics, oceanography and aerodynamics [7–9]. The study of spectral domain decomposition method and extending applications have great significance. This might eventually lead to overcoming of the biggest obstacle in the application of spectral method into complex domain problem.

In this paper, we consider the Chebyshev-Legendre spectral domain decomposition method for solving the following two-dimensional vorticity equations

$$\begin{cases} \partial_t \xi + J(\xi,\psi) - \nu\nabla^2\xi = f_1, & x\in\Omega,\ t\in(0,T], \\ -\nabla^2\psi = \xi + f_2, & x\in\Omega,\ t\in(0,T], \\ \xi(x,0) = \xi_0(x), & x\in\Omega\bigcup\partial\Omega, \end{cases} \tag{1.1}$$

where $I=(-1,1)$,$\Omega=I\times I$, and $x=(x_1,x_2)\in\Omega$, and $\xi=\xi(x,t)$ and $\psi=\psi(x,t)$ are the vorticity function and stream function respectively, $f_i=f_i(x,t)$ $(i=1,2)$ and $\xi_0(x)$ are given functions, $\nu>0$ is the kinematic viscosity, and $J(\xi,\psi)=\partial_2\psi\partial_1\xi-\partial_1\psi\partial_2\xi$ with $\partial_i:=\partial_{x_i}$ $(i=1,2)$. Suppose that the boundary is non-slip and $\psi|_{\partial\Omega}=0$. In addition, similar to the treatment

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in [10], we assume $\xi|_{\partial\Omega}=0$ for simplicity in analysis, although the meaningful boundary conditions for the equations are on the stream function and its normal derivative.The problem is mathematically well posed [11, 12].Vorticity equation is one of the typical equations of incompressible flow problems. The vorticity equation is one of important model equations in weather forecast. Nowadays, using dynamic feature of atmosphere motion to forecast weather has become the main direction of numerical weather forecast. There are many developments on solving vorticity equations with numerical methods. The early work was based on difference methods. With regard to the spectral method, Guo and his collaborators did a lot of work [13–19]. In [20], the authors presented an efficient, direct, and highly parallel spectral-element algorithm for elliptic equations in separable domains. The direct solver is based on a matrix decomposition approach which reduces multi-dimensional separable problems to a sequence of one dimensional problems. In [21], a spectral-element direction splitting scheme for the incompressible Navier-Stokes equations is developed. And it can be presented an efficient parallel implementation. The unconditional stability of the full discrete scheme is rigorously proved for the two-dimensional case. The Chebyshev-Legendre(C-L) spectral method is a mixed method combined with Chebyshev method and Legendre method. That is, the whole scheme is in the Legendre Galerkin form while the nonlinear term is interpolated by Chebyshev method. The C-L spectral method can combine the advantages of Chebyshev and Legendre spectral method on computing and numerical analysis [10, 22–27]. In [28], the authors applied the multi-domain C-L method into solving the convection-diffusion equation. In [10], the authors used the C-L method to two-dimensional vorticity equations and constructed the frame of numerical analysis of two-dimensional C-L method and made the convergence and stability analysis for the schemes.

In this paper, we consider the multi-domain C-L method for solving two-dimensional vorticity equations. This equation is the two-dimensional Navier-Stokes equation in vorticity-stream formation. The vorticity equation is also the important model equation for describing the dynamics features of atmosphere motion. The rest of the paper is organized as follows. In Section 2, we give some notations and the discrete schemes of the C-L method for the two-dimensional vorticity equations. In Section 3, we describe our algorithm implementing in parallel way in detail. Then in Section 4, we introduce some preliminaries which will be used in the Section 5 for analyzing the stability and convergence of the schemes. In Section 6, we give some numerical tests of our method. Finally, we will summarize our results and make some conclusions.

2 Notations and schemes

Let $J_i=I=(-1,1),(i=1,2)$ be partitioned into K non-overlap subintervals

$$I=\bigcup_{k=0}^{K} I_k,\quad I_k=(a_{k-1},a_k],\quad h_k=a_k-a_{k-1},$$

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where $-1=a_0<a_1<\cdots<a_K=1$. Set $\mathcal{N}=(N_1,\cdots,N_k)$, with N_i $(i=0,\cdots,K)$ being positive integers. For simplicity, we do the same partition for J_1 and J_2. Let $v^k\equiv v|_{I_k}$, $1\le k\le K$, and $\mathbb{P}_{N_k}(I)$ be the space of polynomials of degree at most N_k on interval I. Let

$$\mathbb{P}_{\mathcal{N}}(I)=\{v:v|_{I_k}\in\mathbb{P}_{N_k}(I_k),\ 1\le k\le K\}$$

and define

$$\mathbb{P}_{\mathcal{N}}(\Omega)=\mathbb{P}_{\mathcal{N}}(I)\otimes\mathbb{P}_{\mathcal{N}}(I).$$

We then set the following relation

$$v(x)=\overline{v}(\overline{x}),\quad x=\frac{1}{2}(h_k\overline{x}+a_{k-1}+a_k),\quad x\in I_k,\ \ \overline{x}\in I. \tag{2.1}$$

Denote $\mathcal{N}=(N_1,N_2,\cdots,N_k,\cdots,N_K)$. Define $\Pi^C_{\mathcal{N},i}:C(\overline{I_i})\to\mathbb{P}_{\mathcal{N}}(J_i)$ be the Chebyshev-Gauss interpolation operator generated by $\Pi^C_{N_k}:C(\overline{I_i})\to\mathbb{P}_{N_k}(J_i)$, $1\le k\le K$, $1\le i\le d$ and satisfied with

$$\overline{(\Pi^C_{\mathcal{N},i}u)^k}=\Pi^C_{N_k}\overline{u}^k.$$

Set $\Pi^C_{\mathcal{N}}=\Pi^C_{\mathcal{N},1}\circ\Pi^C_{\mathcal{N},2}\circ\cdots\circ\Pi^C_{\mathcal{N},d}$.

Define the approximation space

$$V_{\mathcal{N}}=\mathbb{P}_{\mathcal{N}}(\Omega)\cap H^1_0(\Omega),$$

where $H^1_0(\Omega)=\{v\in H^1(\Omega):v|_{\partial\Omega}=0\}$. Denote two-dimensional multi-domain interpolation operator based on Chebyshev-Gauss points by $\Pi^C_{\mathcal{N}}$. For discretization of time, we adopt the 2-order Crank-Nicolson/leapfrog (CNLF) scheme.

Let τ be the time step and $S_t=\{m\tau:\ m=1,2,\cdots,n_t,t=n_t\tau\}$. Denote

$$v_{\hat{t}}(t)=\frac{v(t+\tau)-v(t-\tau)}{2\tau},\quad \hat{v}(t)=\frac{1}{2}[v(t+\tau)+v(t-\tau)].$$

For simplicity, we do the same partition in different space direction. Set $K=K_{x_1}=K_{x_2}$, where K_{x_1} and K_{x_2} are the numbers of the subinterval in x_1 and x_2 directions. Then the semi-discrete scheme of (1.1) is to find $\eta,\phi\in V_{\mathcal{N}}$ such that for any $t\in(0,T]$

$$\begin{cases}(\partial_t\eta,v)+(J_C(\eta,\phi),v)+\nu(\nabla\eta,\nabla v)=(\Pi^C_{\mathcal{N}}f_1,v), & \forall v\in V_{\mathcal{N}},\ \ t\in(0,T],\\ -(\nabla^2\phi,\omega)=(\eta+\Pi^C_{\mathcal{N}}f_2,\omega), & \omega\in V_{\mathcal{N}},\ \ t\in(0,T],\end{cases} \tag{2.2}$$

where $J_C(u,v)=\partial_1\Pi^C_{\mathcal{N}}(\partial_2 vu)-\partial_2\Pi^C_{\mathcal{N}}(\partial_1 vu)$. The fully discrete scheme is

$$\begin{cases}(\eta_{\hat{t}},v)+\nu(\nabla\hat{\eta},\nabla v)=(\Pi^C_{\mathcal{N}}f_1,v)-(J_C(\eta,\phi),v), & v\in V_{\mathcal{N}},\ \ t\in S_T,\\ -(\nabla^2\phi,\omega)=(\eta+\Pi^C_{\mathcal{N}}f_2,\omega), & \omega\in V_{\mathcal{N}},\ \ t\in S_T,\\ \eta(\tau)=\Pi^C_{\mathcal{N}}(\xi_0+\tau\partial_t\xi(0)),\\ \eta(0)=\Pi^C_{\mathcal{N}}\xi_0.\end{cases} \tag{2.3}$$

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In order to make the matrix of the algebraic equations be sparse, the proper base functions are adopted which are similar to those in [22, 29, 30]. Let L_l be the l-th order Legendre polynomial. For $i=1,2,\cdots,K$ and $l=0,1,\cdots,N_i-2$, define

$$\phi_l^{(i)}(x)=\begin{cases} L_l(\hat{x})-L_{l+2}(\hat{x}), & x^{(i)}=:x=\frac{1}{2}(h_i\hat{x}+a_{i-1}+a_i)\in[a_{i-1},a_i],\\ 0, & \text{otherwise.}\end{cases}$$

Then $\{\phi_l^{(i)}|_{I_i}\}_{l=0}^{N_i-2}$ is the base function on the subinterval $S_{0,N_i}^{(i)}:=\mathbb{P}_{N_i}(I_i)\cap H_0^1(I_i)$. For $i=1,2,\cdots,K-1$, let

$$\phi^{(i)}(x)=\begin{cases} \frac{1}{2}(L_0(\hat{x})+L_1(\hat{x})), & x=\frac{1}{2}(h_i\hat{x}+a_{i-1}+a_i)\in[a_{i-1},a_i],\\ \frac{1}{2}(L_0(\hat{x})-L_1(\hat{x})), & x=\frac{1}{2}(h_{i+1}\hat{x}+a_i+a_{i+1})\in(a_i,a_{i+1}],\\ 0, & \text{otherwise,}\end{cases}$$

and $\phi^{(0)}(x)=\phi^{(K)}(x)\equiv 0$. These functions $\phi^{(i)}(x)$ are called support function and $\phi_l^{(i)}(x)$ are the inner base functions. Next we describe how to solve the equations (2.3). We first arrange the base function according to the subinterval as the following

$$\begin{aligned}&\phi^{(0)}(x),\phi_0^{(1)}(x),\phi_1^{(1)}(x),\phi_2^{(1)}(x),\cdots,\phi_{N_1-2}^{(1)}(x),\phi^{(1)}(x),\\ &\phi_0^{(2)}(x),\phi_1^{(2)}(x),\cdots,\phi_{N_2-1}^{(2)}(x),\phi^{(3)}(x),\cdots,\phi_{N_{K-1}-2}^{(K-1)}(x),\phi^{(K)}(x).\end{aligned}$$

It is easy to see that there are totally $\sum_{i=1}^{k}N_i-1$ base functions. We number them from 0,1 up to KN and denote these renumbered base functions as the following

$$\phi_l(x)=(\phi_0(x),\phi_1(x),\cdots,\phi_{KN}(x)).$$

We introduce matrixes $A=(a_{j,k})_{KN\times KN}$, $B=(b_{j,k})_{KN\times KN}$, where $a_{j,k}=(\phi_j'(x),\phi_k'(x))$, $b_{j,k}=(\phi_j(x),\phi_j(x))$. The matrix $B=(b_{j,k})_{0\le j,k\le KN}$ is pentadiagonal symmetric matrix. We also set

$$\begin{aligned}&\phi_{j,k}(x):=\phi_j(x_1)\phi_k(x_2),\quad 0\le j,k\le KN,\\ &U(x,t)=\sum_{0\le j,k\le KN}\hat{u}_{j,k}(t)\phi_{j,k}(x),\\ &V(x,t)=\sum_{0\le j,k\le KN}\hat{v}_{j,k}(t)\phi_{j,k}(x).\end{aligned}$$

Then noting that

$$\begin{aligned}(\eta(x,t),\phi_{p,q}(x))&=\left(\sum_{j,k=0}^{KN}\hat{u}_{j,k}(t)\phi_{j,k}(x),\phi_{p,q}(x)\right)\\ &=\sum_{j,k=0}^{KN}\hat{u}_{j,k}(t)(\phi_{j,k}(x),\phi_{p,q}(x))\end{aligned}$$

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$$
\begin{aligned}
&= \sum_{j,k=0}^{KN} \hat{u}_{j,k}(t)(\phi_j(x_1),\phi_p(x_1))(\phi_k(x_2),\phi_q(x_2)) \\
&= [(\phi_p,\phi_0),\cdots,(\phi_p,\phi_{KN})]\hat{U}[(\phi_q,\phi_0),\cdots,(\phi_q,\phi_{KN})]^T,
\end{aligned}
$$

and making use of matrix B, we can get

$$
\begin{aligned}
(\eta(x,t),\tilde{\phi}) &= \left(\sum_{j,k=0}^{KN} \hat{u}_{j,k}(t)\phi_{j,k}(x),\tilde{\phi}\right) \\
&= B\hat{U}B,
\end{aligned}
$$

where

$$
B = \begin{bmatrix}
(\phi_0,\phi_0) & (\phi_0,\phi_1) & \cdots & (\phi_0,\phi_{KN}) \\
(\phi_1,\phi_0) & (\phi_1,\phi_1) & \cdots & (\phi_1,\phi_{KN}) \\
\vdots & \vdots & \ddots & \vdots \\
(\phi_{KN},\phi_0) & (\phi_{KN},\phi_1) & \cdots & (\phi_{KN},\phi_{KN})
\end{bmatrix}
$$

and $\hat{U}$ is the coefficient matrix of the unknown function $U(x,t)$ at time t.

Meanwhile,

$$
\begin{aligned}
(\partial_x U,\partial_x\phi_{p,q}(x)) &= \left(\sum_{j,k=0}^{KN} \hat{u}_{j,k}(t)\partial_x\phi_{j,k}(x),\partial_x\phi_{p,q}(x)\right) \\
&= \sum_{j,k=0}^{KN} \hat{u}_{j,k}(t)(\partial_x\phi_{j,k}(x),\partial_x\phi_{p,q}(x)) \\
&= \sum_{j,k=0}^{KN} \hat{u}_{j,k}(t)(\partial_x\phi_j(x_1),\partial_x\phi_p(x_1))(\phi_k(x_2),\phi_q(x_2)) \\
&= [(\partial_x\phi_p,\partial_x\phi_0),\cdots,(\partial_x\phi_p,\partial_x\phi_{KN})]\hat{U}[(\phi_q,\phi_0),\cdots,(\phi_q,\phi_{KN})]^T.
\end{aligned}
$$

Let p,q take all values. We arrive at

$$
\begin{aligned}
(\partial_x U(x,t),\partial_x\phi) &= \left(\sum_{j,k=0}^{KN} \hat{u}_{j,k}(t)\partial_x\phi_{j,k}(x),\partial_x\phi\right) \\
&= A\hat{U}B,
\end{aligned}
$$

where

$$
A = \begin{bmatrix}
(\partial_x\phi_0,\partial_x\phi_0) & (\partial_x\phi_0,\partial_x\phi_1) & \cdots & (\partial_x\phi_0,\partial_x\phi_{KN}) \\
(\partial_x\phi_1,\partial_x\phi_0) & (\partial_x\phi_1,\partial_x\phi_1) & \cdots & (\partial_x\phi_1,\partial_x\phi_{KN}) \\
\vdots & \vdots & \ddots & \vdots \\
(\partial_x\phi_{KN},\partial_x\phi_0) & (\partial_x\phi_{KN},\partial_x\phi_1) & \cdots & (\partial_x\phi_{KN},\partial_x\phi_{KN})
\end{bmatrix}.
$$

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It turns out that for the every time level the algebraic equations of (2.3) we need to solve the following equations

$$B\hat{U}(t+\tau)B+\tau\nu A\hat{U}(t+\tau)B+\tau\nu B\hat{U}(t+\tau)A=G(t), \tag{2.4}$$

$$A\hat{V}(t+\tau)B+B\hat{V}(t+\tau)A=H(t), \tag{2.5}$$

where $\hat{U}=(\hat{u}_{j,k})$, $\hat{V}=(\hat{v}_{j,k})$, $G=(g_{j,k})$, $H=(h_{j,k})$. The matrixes A and B are the stiff matrix and mass matrix respectively.

Set

$$g_{j,k}:=(U(t-\tau),\phi_{j,k})-\tau\nu(\nabla U(t-\tau),\nabla\phi_{j,k})-2\tau(J_C(U(t),V(t)),\phi_{j,k})+2\tau(\Pi_N^C f_1(t),\phi_{j,k}),$$
$$h_{j,k}:=(U(t+\tau)+\Pi_N^C f_2(t+\tau),\phi_{j,k}).$$

Then (2.4) and (2.5) can be written as the following form

$$\gamma BXB+\alpha AXB+\beta BXA=F. \tag{2.6}$$

Eq. (2.6) can be written as the tensor product form

$$(\gamma B\otimes B+\alpha A\otimes B+\beta B\otimes A)\bar{u}=\bar{f}, \tag{2.7}$$

where $\bar{u}$ and $\bar{f}$ are the column vector form of X and F. Denote the matrix tensor product by $\otimes$. Then $A\otimes B=(Ab_{ij})_{i,j=1}^{KN+1}$.

We use the matrix diagonalization technique to solve Eq. (2.6). We adopt the method similar as in Section 6.2 of [31]. Consider the generalized eigenvalue problem:

$$B\bar{x}=\lambda A\bar{x}. \tag{2.8}$$

According to the definition of matrix A and B, the two matrices are both symmetric positive definite matrices. So all the eigenvalues are real, positive and distinct. Let Λ be the diagonal matrix. The diagonal entries λ_p of Λ are the eigenvalues of (2.8). Let E be the matrix whose columns are the eigenvectors of (2.8). Then it can lead to

$$BE=AE\Lambda. \tag{2.9}$$

Let $X=EY$. Then due to (2.9), Eq. (2.6) becomes

$$\gamma AE\Lambda YB+\alpha AEYB+\beta AE\Lambda YA=F. \tag{2.10}$$

Multiplying $E^{-1}A^{-1}$ to (2.10), we arrive at

$$\gamma\Lambda YB+\alpha YB+\beta\Lambda YA=E^{-1}A^{-1}F\equiv G. \tag{2.11}$$

Let $y_p=(y_{p0},y_{p1},\cdots,y_{p,KN})$, $g_p=(g_{p0},g_{p1},\cdots,g_{p,KN})$, $p=0,1,\cdots,KN$. Then the p-th column of (2.11) can be written as

$$\{(\gamma\lambda_p+\alpha)B+\beta\lambda_p A\}y_p=g_p,\quad p=0,1,2,\cdots,KN. \tag{2.12}$$

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3 Implementing in parallel way

In this section, we describe how to implement the scheme in parallel way. Firstly, for the first equation of (2.3), the equation for each time level can be written as

$$A_N(\eta(t+\tau),v)=g_N(v),\quad v\in V_{\mathcal{N}},$$

where

$$\begin{aligned}A_N&=(\eta(t+\tau),v)+\tau\nu(\nabla\eta(t+\tau),\nabla v),\\ g_N&=(\eta(t-\tau),v)-\tau\nu(\nabla\eta(t-\tau),\nabla v)\\&\quad-2\tau(J_C(\eta(t),\phi(t)),v)+2\tau(\Pi^C_{\mathcal{N}}f_1(t),v).\end{aligned}$$

The unknown function $\omega(x):=\eta(x,t+\tau)$ can be expressed as

$$\begin{aligned}\omega(x)=&\sum_{i,j=1}^{K}\sum_{k=0}^{N_i-2}\sum_{l=0}^{N_j-2}c_{kl}^{(ij)}\phi_k^{(i)}(x_1)\phi_l^{(j)}(x_2)+\sum_{i,j=0}^{K}\eta_{ij}\phi^{(i)}(x_1)\phi^{(j)}(x_2)\\&+\sum_{i,j=1}^{K}\sum_{l=0}^{N_j-2}p_{ij}\phi^{(i)}(x_1)\phi_l^{(j)}(x_2)+\sum_{i,j=1}^{K}\sum_{k=0}^{N_i-2}q_{ij}\phi^{(j)}(x_2)\phi_k^{(i)}(x_1),\end{aligned}\tag{3.1}$$

where $\eta_{ij}=\omega(a_i,a_j)=\eta(a_i,a_j,t+\tau),c_{kl}^{(ij)},p_{ij},q_{ij}$ are unknowns.

Let $v^{(ij)}(x):=\sum_{k=0}^{N_i}\sum_{l=0}^{N_j}c_{kl}^{(ij)}\phi_k^{(i)}(x_1)\phi_l^{(j)}(x_2)$. Then the following equations hold

$$\begin{aligned}A_N(v^{(ij)},\phi_k^{(i)}\phi_l^{(j)})=&g_N(\phi_k^{(i)}\phi_l^{(j)})-\eta_{i-1,j-1}A_N(\phi^{(i-1)}\phi^{(j-1)},\phi_k^{(i)}\phi_l^{(j)})\\&-\eta_{i,j-1}A_N(\phi^{(i)}\phi^{(j-1)},\phi_k^{(i)}\phi_l^{(j)})-\eta_{i-1,j}A_N(\phi^{(i-1)}\phi^{(j)},\phi_k^{(i)}\phi_l^{(j)})\\&-\eta_{i,j}A_N(\phi^{(i)}\phi^{(j)},\phi_k^{(i)}\phi_l^{(j)})-\eta_{i-1,j}A_N(\phi^{(i-1)}\phi_l^{(j)},\phi_k^{(i)}\phi_l^{(j)})\\&-\eta_{i,j}A_N(\phi^{(i)}\phi_l^{(j)},\phi_k^{(i)}\phi_l^{(j)})-\eta_{i,j-1}A_N(\phi^{(j-1)}\phi_k^{(i)},\phi_k^{(i)}\phi_l^{(j)})\\&-\eta_{i,j}A_N(\phi^{(j)}\phi_k^{(i)},\phi_k^{(i)}\phi_l^{(j)}).\end{aligned}\tag{3.2}$$

For $v_m^{(ij)}(x):=\sum_{k=0}^{N_i}\sum_{l=0}^{N_j}c_{mkl}^{(ij)}\phi_k^{(i)}(x_1)\phi_l^{(j)}(x_2),m=0,1,2,3,4,5,6,7,8$, Eq. (3.2) can be decomposed into the following equations

$$A_N(v_0^{(ij)},\phi_k^{(i)}\phi_l^{(j)})=g_N(\phi_k^{(i)}\phi_l^{(j)}),\tag{3.3a}$$
$$A_N(v_1^{(ij)},\phi_k^{(i)}\phi_l^{(j)})=-A_N(\phi^{(i-1)}\phi^{(j-1)},\phi_k^{(i)}\phi_l^{(j)}),\tag{3.3b}$$
$$A_N(v_2^{(ij)},\phi_k^{(i)}\phi_l^{(j)})=-A_N(\phi^{(i)}\phi^{(j-1)},\phi_k^{(i)}\phi_l^{(j)}),\tag{3.3c}$$
$$A_N(v_3^{(ij)},\phi_k^{(i)}\phi_l^{(j)})=-A_N(\phi^{(i-1)}\phi^{(j)},\phi_k^{(i)}\phi_l^{(j)}),\tag{3.3d}$$

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$$A_N(v_4^{(ij)},\phi_k^{(i)}\phi_l^{(j)}) = -A_N(\phi^{(i)}\phi^{(j)},\phi_k^{(i)}\phi_l^{(j)}), \tag{3.3e}$$
$$A_N(v_5^{(ij)},\phi_k^{(i)}\phi_l^{(j)}) = -A_N(\phi^{(i-1)}\phi_l^{(j)},\phi_k^{(i)}\phi_l^{(j)}), \tag{3.3f}$$
$$A_N(v_6^{(ij)},\phi_k^{(i)}\phi_l^{(j)}) = -A_N(\phi^{(i)}\phi_l^{(j)},\phi_k^{(i)}\phi_l^{(j)}), \tag{3.3g}$$
$$A_N(v_7^{(ij)},\phi_k^{(i)}\phi_l^{(j)}) = -A_N(\phi^{(j-1)}\phi_k^{(i)},\phi_k^{(i)}\phi_l^{(j)}), \tag{3.3h}$$
$$A_N(v_8^{(ij)},\phi_k^{(i)}\phi_l^{(j)}) = -A_N(\phi^{(j)}\phi_k^{(i)},\phi_k^{(i)}\phi_l^{(j)}). \tag{3.3i}$$

Therefore the solutions of (3.2) can be expressed as

$$v^{(ij)} = v_0^{(ij)} + \eta_{i-1,j-1}v_1^{(ij)} + \eta_{i,j-1}(v_2^{(ij)} + v_7^{(ij)}) + \eta_{i-1,j}(v_3^{(ij)} + v_5^{(ij)}) + \eta_{i,j}(v_4^{(ij)} + v_6^{(ij)} + v_8^{(ij)}).$$

It is easy to know that the equations (3.3a) can be solved on the sub-domain $I_i \times I_j$ respectively. Therefore it can be implemented in parallel way. The coefficient matrix of the equation set can be decoupled into two tri-diagonal submatrices (that is two tri-diagonal subsystems for the odd and even components). Especially, for the four equations (3.3b), (3.3c), (3.3d) and (3.3g), since they are time independent we only need to solve them once. With the above discussion, (3.1) can be expressed as

$$\begin{aligned}\omega(x) &= \sum_{i,j=0}^{K}(v_0^{(ij)}(x) + \eta_{i-1,j-1}v_1^{(ij)}(x) + \eta_{i,j-1}(v_2^{(ij)}(x) + v_7^{(ij)}(x)) + \eta_{i-1,j}(v_3^{(ij)}(x) + v_7^{(ij)}(x))\\ &\quad + \eta_{i,j}(v_4^{(ij)}(x) + v_6^{(ij)}(x) + v_8^{(ij)}(x)) + \sum_{i,j=1}^{K-1}\eta_{ij}\phi^{(i)}(x_1)\phi^{(j)}(x_2)\\ &= \sum_{i,j=0}^{K} v_0^{(ij)}(x) + \sum_{i,j=1}^{K-1}\Big(v_1^{(i+1,j+1)}(x) + v_2^{(i,j-1)}(x) + v_3^{(i+1,j)}(x) + v_4^{(ij)}(x) + v_5^{(i+1,j)}(x)\\ &\qquad + v_6^{(ij)}(x) + v_7^{(i,j+1)}(x) + v_8^{(ij)}(x) + \phi^{(i)}(x_1)\phi^{(j)}(x_2)\Big)\eta_{ij}.\end{aligned}$$

At last, the unknowns η_{ij} can be solved through the following equation

$$\begin{aligned}&\sum_{i,j=1}^{K-1} A_N(v_1^{(i+1,j+1)} + v_2^{(i,j-1)} + v_3^{(i+1,j)} + v_4^{(ij)} + v_5^{(i+1,j)} + v_6^{(ij)} + v_7^{(i,j+1)} + v_8^{(ij)}, \phi^{(i)}\phi^{(j)})\eta_{ij}\\ &= g_N(\phi^{(i)}\phi^{(j)}) - \sum_{i,j=0}^{K} A_N(v_0^{(ij)}, \phi^{(i)}\phi^{(j)}).\end{aligned} \tag{3.4}$$

Thus we can solve the fist equation of (2.3). For the second equation of (2.3), we have similar parallel schemes. Let

$$\begin{aligned}B_N &= (\nabla\phi, \nabla\omega),\\ h_N &= (\eta(t+\tau) + \Pi_N^C f_2(t+\tau), \omega).\end{aligned}$$

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Then we can determine ϕ_{ij} through the following equation

$$\sum_{i,j=1}^{K-1} B_N(v_1^{(i+1,j+1)}+v_2^{(i,j-1)}+v_3^{(i+1,j)}+v_4^{(ij)}+v_5^{(i+1,j)}+v_6^{(ij)}+v_7^{(i,j+1)}+v_8^{(ij)},\phi^{(i)}\phi^{(j)})\phi_{ij}$$
$$=h_N(\phi^{(i)}\phi^{(j)})-\sum_{i,j=0}^{K} B_N(v_0^{(ij)},\phi^{(i)}\phi^{(j)}). \tag{3.5}$$

where the coefficient matrix is tri-diagonal since for $i,j=1,2,\cdots,K$, supp $v_m^{ij}\subset(a_{i-1},a_i)$, $m=0,1,2,3,4,5,6,7,8$ and supp $\phi^{(i)}\subset(a_{i-1},a_{i+1})$.

4 The stability and convergence of the full discrete scheme

In the following we consider the stability and convergence of the full discrete schemes (2.3).

4.1 Preliminary

Set

$$\hbar=\max_{1\le k\le K.} h_k N_k^{-1}. \tag{4.1}$$

Hereafter by $\|\cdot\|_\sigma$ and $|\cdot|_\sigma$ we respectively denote the usual Sobolev space $H^\sigma(\Omega)$ norm and semi-norm. Particularly $\|\cdot\|$ stands for $\|\cdot\|_{\sigma=0}$, i.e. the usual L^2-norm. We have the following lemmas whose proofs are similar to that of Lemma 4.5, Lemma 4.6 in [10].

Lemma 4.1. *If $u\in H^\sigma(\Omega)$, $\sigma>1$, then*

$$||u-\Pi_{\mathcal{N}}^C u||\le C\hbar^\sigma||u||_\sigma. \tag{4.2}$$

Set

$$h^*=\max_{1\le k\le K}\left\{\frac{M_k}{N_k}\right\}. \tag{4.3}$$

Then it arrives at the following lemma.

Lemma 4.2. *If $u\in\mathbb{P}_{\mathcal{M}}(\Omega)$, then*

$$||\Pi_{\mathcal{N}}^C u||\le C(1+h^*)^2||u||. \tag{4.4}$$

Set the two-dimensional Legendre polynomials as

$$L_{km}(x)=L_k(x_1)L_m(x_2).$$

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Expand $u\in\mathbb{P}_{\mathcal{N}}(\Omega)$ into Legendre series

$$u(x)=\sum_{k=0}^{N_i}\sum_{m=0}^{N_j}\hat{u}_{km}^{(ij)}L_{km}(\hat{x}),\quad x=x^{(ij)}(\hat{x}):=(x^{(i)}(\hat{x}_1),x^{(j)}(\hat{x}_2))\in I_i\times I_j,\quad 1\le i,j\le K,$$

where $x^{(i)}(t):=\frac{1}{2}(h_it+a_{i-1}+a_i)$. It is known that $|L_k(x_i)|\le 1$, $x_i\in\bar{I}$, $k\ge 0$, where K is the number of the subintervals partitioned on every direction.

Denote $\bar{N}=\max_{1\le k\le K}\frac{N_k}{\sqrt{h_k}}$. Then it exists positive constant C_0^* such that

$$\begin{aligned}||u||_{L^\infty(\Omega)}^2&=\max_{1\le i,j\le k}\Big\|\sum_{k=0}^{N_i}\sum_{m=0}^{N_j}\hat{u}_{km}^{(ij)}L_{km}\Big\|_{L^\infty(I_i\times I_j)}^2\le\max_{1\le i,j\le k}\Big(\sum_{k=0}^{N_i}\sum_{m=0}^{N_j}|\hat{u}_{km}^{(ij)}|\Big)^2\\&\le\max_{1\le i,j\le k}\Big(\sum_{k=0}^{N_i}\sum_{m=0}^{N_j}|\hat{u}_{km}^{(ij)}|^2\Big(k+\frac{1}{2}\Big)^{-1}\Big(m+\frac{1}{2}\Big)^{-1}\frac{h_ih_j}{4}\sum_{k=0}^{N_i}\sum_{m=0}^{N_j}\frac{4(k+\frac{1}{2})(m+\frac{1}{2})}{h_ih_j}\Big)\\&\le\max_{1\le i,j\le k}\frac{(N_i+1)^2(N_j+1)^2}{h_ih_j}\|u\|_{L^2(I_i\times I_j)}\le(C_0^*\bar{N}^2)^2||u||^2.\end{aligned}$$

Therefore we have the following lemma.

Lemma 4.3. *If $u\in\mathbb{P}_{\mathcal{N}}(\Omega)$, then*

$$||u||_{L^\infty(\Omega)}\le C_0^*\bar{N}^2||u||_{L^2(\Omega)}.\tag{4.5}$$

Let $\hat{P}_{N_k}:L^2(I)\to\mathbb{P}_{N_k}(I)$ be the Legendre project operator which is satisfied $(\hat{P}_{N_k}u-u,v)=0$. For $1\le s\le 2$, define the projection operators $\hat{P}_{s,N_k}:H^s(I)\mapsto\mathbb{P}_{N_k}(I)$ such that

$$\hat{P}_{s,N_k}v(x)=v(-1)+\int_{-1}^{x}\hat{P}_{s-1,N_k-1}\partial_y\bar{v}(y)dy,$$

where $\hat{P}_{0,N_k}=\hat{P}_{N_k}$. Therefore we have $\hat{P}_{s,N_k}u-u\in H_0^s(I_k)$ and

$$(\partial_x^s\hat{P}_{s,N_k}u,\partial_x^sv)=(\hat{P}_{N_k-s}\partial_x^su,\partial_x^sv)=(\partial_x^su,\partial_x^sv),\quad v\in\mathbb{P}_{N_k}(I).$$

Now define the multi-domain projection $\hat{P}_{s,\mathcal{N}}:H^s(I)\to\mathbb{P}_{\mathcal{N}}(I)$ such that

$$\overline{(\hat{P}_{s,\mathcal{N}}u)^k}(\bar{x})=\hat{P}_{s,N_k}\bar{u}^k(\bar{x}),\quad s=1,2.$$

It is readily checked that $\hat{P}_{s,\mathcal{N}}u\in H^s(I)$ and

$$\begin{aligned}&(\partial_x^s\hat{P}_{s,\mathcal{N}}u,\partial_x^sv)=(\partial_x^su,\partial_x^sv),\quad v\in\mathbb{P}_{\mathcal{N}}(I)\cap H^s(I),\\&\sum_{k=1}^{K}(\partial_x^s\hat{P}_{s,\mathcal{N}}u,\partial_x^sv)_{I_k}=(\partial_x^su,\partial_x^sv),\quad v\in\mathbb{P}_{\mathcal{N}}(I).\end{aligned}$$

For such $\hat{P}_{s,N_k}$ we have the following lemma.

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Lemma 4.4. (§9.7.2 in [32]) *For $u\in H^\sigma(I)$, $\sigma\geq s$, we have*

$$\|u-\hat{P}_{s,N}u\|_l\leq CN^{l-\sigma}\|u\|_\sigma,\quad 0\leq l\leq s\leq\sigma, \tag{4.6}$$

$$\|u-\hat{P}_{s,\mathcal{N}}u\|_l\leq C\hbar^{\sigma-l}\|u\|_\sigma,\quad 0\leq l\leq s\leq\sigma. \tag{4.7}$$

Define the multi-domain projection $P_{1,\mathcal{N}}:H_0^1(\Omega)\to V_{\mathcal{N}}(\Omega)$ such that

$$(\nabla P_{1,\mathcal{N}}u-\nabla u,\nabla v)=0,\quad v\in V_{\mathcal{N}}.$$

Then according to Lemma 4.4 and by using the same way as that of proof in Lemma 4.5 or 4.6 in [10], we can get the following results.

Lemma 4.5. *For $u\in H^\sigma(\Omega)\cap H_0^1(\Omega)$, $\sigma\geq 1$,*

$$||u-P_{1,\mathcal{N}}u||_l\leq C\hbar^{\sigma-l}||u||_\sigma,\quad l=0,1. \tag{4.8}$$

Define the multi-domain projection $P_{2,\mathcal{N}}:H_0^1(\Omega)\to V_{\mathcal{N}}(\Omega)$ such that

$$(\nabla^2(u-P_{2,\mathcal{N}}u),\nabla^2 v)+(u-P_{2,\mathcal{N}}u,v)=0,v\in H^2(\Omega)\cap\mathbb{P}_{\mathcal{N}}(\Omega).$$

Lemma 4.6. *For any $u\in H^\sigma(\Omega)$, $\sigma\geq 2$*

$$||u-P_{2,\mathcal{N}}u||_l\leq C\hbar^{\sigma-l}||u||_\sigma,\quad l=0,1,2. \tag{4.9}$$

Next we give the definition of the two-dimensional negative norm

$$||u||_{-1}:=\sup_{v\in H_0^1(\Omega),v\neq 0}\frac{|(u,v)|}{||v||_1}. \tag{4.10}$$

Set $\hat{P}_{\mathcal{N}}$ be the one dimensional multi-domain Legendre project operator produced by $\hat{P_{N_k}}$ that is $\overline{(\hat{P}_{\mathcal{N}}u(x))}|_{I_k}=\hat{P_{N_k}}\bar{u}^k(\bar{x})$, $1\leq k\leq K$. Let $P_{\mathcal{N}}=\hat{P}_{\mathcal{N}}^{(1)}\circ\hat{P}_{\mathcal{N}}^{(2)}$ be the two-dimensional multi-domain project which is produced by doing the one-dimensional project both in x_1 and x_2 directions. It can be verified easily that $P_{\mathcal{N}}u\in\mathbb{P}_{\mathcal{N}}(\Omega)$ and $(u-P_{\mathcal{N}}u,v)=0,\forall v\in\mathbb{P}_{\mathcal{N}}(\Omega)$. Then we have the following approximation results.

Lemma 4.7. *If $u\in H^\sigma(\Omega)$,$\sigma\geq 0$, then*

$$||u-P_{\mathcal{N}}u||\leq C\hbar^\sigma|u|_\sigma. \tag{4.11}$$

Lemma 4.8. *Set $u\in V_{\mathcal{N}}$,$g\in L^2(\Omega)$ be satisfied with*

$$(\nabla u,\nabla v)=(g,v),\quad\forall v\in V_{\mathcal{N}}. \tag{4.12}$$

Then

$$|u|_{1,\infty}\leq C(C_0^*\bar{N}^2\hbar+1)\|g\|. \tag{4.13}$$

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Proof. Assume U be satisfied with the Poisson equation $\Delta U = g, U_{\partial\Omega} = 0$. Then $\|U\|_2 \leq C\|g\|$, and $u = P_{1,\mathcal{N}}U$. According to Lemmas 4.3, 4.5 and 4.6, we can get the following results.

$$\begin{aligned}|u|_{1,\infty} &= \|P_{1,\mathcal{N}}U\|_{1,\infty} \leq \|P_{1,\mathcal{N}}U - P_{2,\mathcal{N}}U\|_{1,\infty} + \|P_{2,\mathcal{N}}U\|_{1,\infty}\\ &\leq C_0^*\bar{N}^2\|P_{1,\mathcal{N}}U - P_{2,\mathcal{N}}U\|_1 + C\|P_{2,\mathcal{N}}U\|_2\\ &\leq C_0^*\bar{N}^2(\|P_{1,\mathcal{N}}U - U\|_1 + \|P_{2,\mathcal{N}}U - U\|_1) + C\|U\|_2\\ &\leq CC_0^*\bar{N}^2(\hbar\|U\|_2 + \hbar\|U\|_2) + C\|U\|_2 = C(C_0^*\bar{N}^2\hbar + 1)\|g\|.\end{aligned}$$

This gives the proof. □

Lemma 4.9. *Let $u \in \mathbb{P}_{\mathcal{N}}(\Omega)$, $v \in \mathbb{P}_{\mathcal{N}}(\Omega)$, $\omega \in H_0^1(\Omega)$, then we have*

$$|(J_C(u,v),\omega)| \leq C||u||\ |v|_{1,\infty}|\omega|_1, \tag{4.14a}$$

$$|(J_C(u,v),\omega)| \leq C||u||_{0,\infty}|v|_1|\omega|_1. \tag{4.14b}$$

Proof. It can be proved by using the integration by parts method and Lemma 4.2. □

Lemma 4.10 ([33]). *If the following is satisfied,*

(i) Set $E(t)$ is the non-negative function defined on S_T,

(ii) ρ, M_1 and M_2 are nonnegative constant,

(iii) For all $t \in S_T$,

$$E(t) \leq \rho + M_1\tau \sum_{s\in S_{t-\tau}} \{E(s) + M_2E^2(s)\},$$

(iv) $E(0) \leq \rho$ and

$$\rho e^{2M_1t} \leq \frac{1}{M_2},$$

then for any $t \in S_T$, we have

$$E(t) \leq \rho e^{2M_1t}.$$

4.2 The stability

Instead of the usual strong stability, what we consider in this paper is the generalized stability in a weak sense (see [34,35], also see [36, pp. 96-99]). We assume that all functions below are valued at time s unless specified. Denote the errors of the right hand side of the first two equations of (2.3) by $\tilde{f}_1$ and $\tilde{f}_2$ respectively. The solutions of (2.3) corresponding to $\Pi_{\mathcal{N}}^C f_1 + \tilde{f}_1$ and $\Pi_{\mathcal{N}}^C f_2 + \tilde{f}_2$ are η^* and ϕ^*. That means η^* and ϕ^* satisfy the following equations

$$\begin{cases}(\eta_{\hat{t}}^*, v) + \nu(\nabla\hat{\eta}^*, \nabla v) = (\Pi_{\mathcal{N}}^C f_1 + \tilde{f}_1, v) - (J_C(\eta^*, \phi^*), v), & v \in V_{\mathcal{N}},\ t \in S_T,\\ -(\nabla^2\phi^*, \omega) = (\eta^* + \Pi_{\mathcal{N}}^C f_2 + \tilde{f}_2, \omega), & \omega \in V_{\mathcal{N}},\ t \in S_T,\end{cases} \tag{4.15}$$

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The errors of η and ϕ induced by $\tilde{f}_1$ and $\tilde{f}_2$ are denoted by $\tilde{\eta}$ and $\tilde{\phi}$ respectively, that is $\tilde{\eta}=\eta^*-\eta, \tilde{\phi}=\phi^*-\phi$. Then the error equations of (2.3) are

$$\begin{cases} (\tilde{\eta}_{\hat{t}},v)+\nu(\nabla\hat{\tilde{\eta}},\nabla v)=(\tilde{f}_1,v)-(J_C(\tilde{\eta},\tilde{\phi})+J_C(\tilde{\eta},\phi)+J_C(\eta,\tilde{\phi}),v), & \forall v\in V_{\mathcal{N}}, \\ (\nabla\tilde{\phi},\nabla\omega)=(\tilde{\eta}+\tilde{f}_2,\omega), & \forall\omega\in V_{\mathcal{N}}. \end{cases} \tag{4.16}$$

In (4.16), taking $v=\hat{\tilde{\eta}}$ we have

$$\frac{1}{2}||\tilde{\eta}||^2_{\hat{t}}+\nu|\hat{\tilde{\eta}}|^2_1=(\tilde{f}_1,\hat{\tilde{\eta}})-\sum_{i=1}^{3}F_i, \tag{4.17}$$

where

$$F_1=(J_C(\tilde{\eta},\tilde{\phi}),\hat{\tilde{\eta}}),\quad F_2=(J_C(\tilde{\eta},\phi),\hat{\tilde{\eta}}),\quad F_3=(J_C(\eta,\tilde{\phi}),\hat{\tilde{\eta}}). \tag{4.18}$$

Therefore

$$|(\tilde{f}_1,\hat{\tilde{\eta}})|\le\frac{1}{8}\nu|\hat{\tilde{\eta}}|^2_1+\frac{C}{\nu}||\tilde{f}_1||^2_{-1}.$$

Next we estimate F_i $(i=1,2,3)$. Taking $\omega=\tilde{\phi}$ in (4.16) yields

$$(\nabla\tilde{\phi},\nabla\tilde{\phi})=(\tilde{\eta}+\tilde{f}_2,\tilde{\phi}).$$

According to the Poincare inequality we get

$$|\tilde{\phi}|^2_1\le C||\tilde{\eta}+\tilde{f}_2||\ ||\tilde{\phi}||\le C||\tilde{\eta}+\tilde{f}_2||\ |\tilde{\phi}|_1.$$

Consequently, we have

$$|\tilde{\phi}|_1\le C(||\tilde{\eta}||+||\tilde{f}_2||). \tag{4.19}$$

From the above discussion, the first equation of Lemma 4.9 and Lemma 4.8 we can obtain

$$\begin{aligned} |F_1|&\le\frac{1}{8}\nu|\hat{\tilde{\eta}}|^2_1+\frac{C}{\nu}||\tilde{\eta}||^2|\tilde{\phi}|^2_{1,\infty} \\ &\le\frac{1}{8}\nu|\hat{\tilde{\eta}}|^2_1+\frac{C(C_0^*\bar{N}^2\hbar+1)^2}{\nu}||\tilde{\eta}||^2|\tilde{\phi}|_1 \\ &\le\frac{1}{8}\nu|\hat{\tilde{\eta}}|^2_1+\frac{C(C_0^*\bar{N}^2\hbar+1)^2}{\nu}||\tilde{\eta}||^2(||\tilde{\eta}||^2+||\tilde{f}_2||^2) \\ &\le\frac{1}{8}\nu|\hat{\tilde{\eta}}|^2_1+\frac{C(C_0^*\bar{N}^2\hbar+1)^2}{\nu}||\tilde{\eta}||^4+\frac{C(C_0^*\bar{N}^2\hbar+1)^2}{\nu}||\tilde{\eta}||^2||\tilde{f}_2||^2. \end{aligned}$$

Similarly, from Lemma 4.9 we can get

$$\begin{aligned} |F_2|&\le\frac{1}{8}\nu|\hat{\tilde{\eta}}|^2_1+\frac{C}{\nu}||\tilde{\eta}||^2|\phi|^2_{1,\infty}, \\ |F_3|&\le\frac{1}{8}\nu|\hat{\tilde{\eta}}|^2_1+\frac{C}{\nu}||\eta||^2_{0,\infty}|\tilde{\phi}|^2_1 \\ &\le\frac{1}{8}\nu|\hat{\tilde{\eta}}|^2_1+\frac{C}{\nu}||\eta||^2_{0,\infty}(||\tilde{\eta}||^2+||\tilde{f}_2||^2) \\ &\le\frac{1}{8}\nu|\hat{\tilde{\eta}}|^2_1+\frac{C}{\nu}||\eta||^2_{0,\infty}||\tilde{\eta}||^2+\frac{C}{\nu}||\eta||^2_{0,\infty}||\tilde{f}_2||^2. \end{aligned}$$

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All the above estimations for F_i $(i=1,2,3)$ together with (4.17), yield

$$\frac{1}{2}||\tilde{\eta}||_{\hat{t}}^2+\frac{1}{2}\nu|\hat{\tilde{\eta}}|_1^2\leq\frac{C}{\nu}(||\tilde{f}_1||_{-1}^2+||\eta||_{0,\infty}^2||\tilde{f}_2||^2)$$
$$+\frac{C}{\nu}((C_0^*\bar{N}^2\hbar+1)^2||\tilde{f}_2||^2+|\phi|_{1,\infty}^2+||\eta||_{0,\infty}^2)||\tilde{\eta}||^2+\frac{C(C_0^*\bar{N}^2\hbar+1)^2}{\nu}||\tilde{\eta}||^4. \tag{4.20}$$

Summing up the above equation with respect to $s\in S_{t-\tau}$ yields

$$E(\tilde{\eta},t)\leq\rho(\tilde{\eta},\tilde{f}_1,\tilde{f}_2,t)+\left\{C_1\tau\sum_{s\in S_{t-\tau}}E(\tilde{\eta},s)+\frac{C_2}{C_1}E(\tilde{\eta},s)^2\right\}, \tag{4.21}$$

where

$$E(\tilde{\eta},t)=||\tilde{\eta}(t)||^2+||\tilde{\eta}(t-\tau)||^2+\nu\tau\sum_{s\in S_{t-\tau}}|\hat{\tilde{\eta}}|_1^2,$$
$$\rho(\tilde{\eta},\tilde{f}_1,\tilde{f}_2,t)=||\tilde{\eta}(0)||^2+||\tilde{\eta}(\tau)||^2+\tau\sum_{s\in S_{t-\tau}}\left\{\frac{C}{\nu}(||\tilde{f}_1(s))||_{-1}^2+||\eta(s)||_{0,\infty}^2||\tilde{f}_2(s)||^2\right\},$$

where C_1,C_2 are constant that satisfied with

$$C_1\geq\max_{s\in S_{t-\tau}}\frac{C}{\nu}((C_0^*\bar{N}^2\hbar+1)^2||\tilde{f}_2(s)||^2+|\phi(s)|_{1,\infty}^2+||\eta(s)||_{0,\infty}^2),\ C_2\geq\frac{C(C_0^*\bar{N}^2\hbar+1)^2}{\nu}. \tag{4.22}$$

Based on the above discussion and in light of Lemma 4.10 we can come to the following stability theorem.

Theorem 4.1. *Assume the* (4.22) *is satisfied and for any T the following*

$$\rho(\tilde{\eta},\tilde{f}_1,\tilde{f}_2,T)e^{2C_1T}\leq\frac{C_1}{C_2}$$

holds, then there is

$$E(\tilde{\eta},t)\leq\rho(\tilde{\eta},\tilde{f}_1,\tilde{f}_2,t)e^{2C_1t},\quad\forall t\in S_T. \tag{4.23}$$

4.3 The Convergence

Next we investigate the convergence. Set $\xi^*=P_{1,\mathcal{N}}\xi$, $\psi^*=P_{1,\mathcal{N}}\psi$, $\tilde{\xi}=\eta-\xi^*$, $\tilde{\psi}=\phi-\psi^*$. From (1.1) and (2.3), we can obtain the following equations.

$$\begin{cases}(\tilde{\xi}_{\hat{t}},v)+\nu(\nabla\hat{\tilde{\xi}},\nabla v)=\sum_{j=1}^4A_j(v)-(J_C(\tilde{\xi},\tilde{\psi})+J_C(\tilde{\xi},\psi^*)+J_C(\xi^*,\tilde{\psi}),v), & \forall v\in V_{\mathcal{N}},\\ (\nabla\tilde{\psi},\nabla\omega)=(\tilde{\xi},\omega)+A_5(\omega), & \forall\omega\in V_{\mathcal{N}},\\ \tilde{\xi}(\tau)=\Pi_{\mathcal{N}}^C(\xi_0+\tau\partial_t\xi(0))-P_{1,\mathcal{N}}^1\xi(\tau),\\ \tilde{\xi}(0)=\Pi_{\mathcal{N}}^C\xi_0-P_{1,\mathcal{N}}^1\xi(0),\end{cases} \tag{4.24}$$

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where $A_j(v)=A_j(v,t)$ and

$$\begin{aligned}
&A_1(v)=(\partial_t\xi,v)-(\xi^*_{\hat{t}},v),\quad A_2(v)=(J(\xi,\psi),v)-(J_C(\xi^*,\psi^*),v),\\
&A_3(v)=-\nu(\nabla^2\xi,v)+\nu(\nabla^2\hat{\xi}^*,v),\quad A_4(v)=-(f_1,v)+(\Pi^C_{\mathcal{N}}f_1,v),\\
&A_5(\omega)=(\xi^*-\xi+\Pi^C_{\mathcal{N}}f_2-f_2+\nabla^2\psi^*-\nabla^2\psi,\omega)=(\tilde{f}_3,\omega).
\end{aligned}$$

Similar to the discussion of the stability theorem, we have the following theorem for convergence.

Theorem 4.2. *Suppose that ξ,ψ and η,ϕ are the solutions of* (1.1) *and* (2.3) *respectively. If*

$$\begin{aligned}
&\sigma\geq 2,\ \xi\in H^1(0,T;H^{\sigma+1}(\Omega)\cap H^1_0(\Omega))\cap H^2(0,T;H^1(\Omega))\cap H^3(0,T;L^2(\Omega)),\\
&\psi\in C(0,T;H^{\sigma+1}(\Omega)\cap H^1_0(\Omega)),\ \partial_t\xi_0\in H^\sigma(\Omega),
\end{aligned}$$

there exists a positive constant C which depends on ν^{-1} and the regularity of ξ and ψ such that if τ and $\hbar$ are small enough, then

$$||\xi-\eta||\leq C(\tau^2+\hbar^\sigma). \tag{4.25}$$

Proof. Compare (2.3) with (4.24). Similar to the analysis of stability, take $v=\hat{\tilde{\xi}}$ in the first equation and $\omega=\tilde{\psi}$ in the second equation of (4.24). Then by summing up the first equation of (4.24) with respect to $s\in S_{t-\tau}$ we can obtain the following equation

$$\|\tilde{\xi}(t)\|+\|\tilde{\xi}(t-\tau)\|^2+\nu\tau\sum_{s\in S_{t-\tau}}|\hat{\tilde{\xi}}|_1^2=\|\tilde{\xi}(0)\|^2+\|\tilde{\xi}(\tau)\|^2+\tau\sum_{s\in S_{t-\tau}}\sum_{j=1}^{4}A_j(\hat{\tilde{\xi}})-\tau\sum_{s\in S_{t-\tau}}\sum_{j=1}^{3}F_j(\hat{\tilde{\xi}}).$$

According to Taylor expansion, Cauchy-Schwartz inequality and Lemma 4.5, we have the following estimation.

$$\begin{aligned}
\tau\sum_{s\in S_{t-\tau}}|A_1(\hat{\tilde{\xi}})|&=\tau\sum_{s\in S_{t-\tau}}|(\partial_t\xi-\xi_{\hat{t}},\hat{\tilde{\xi}})+(\xi_{\hat{t}}-\xi^*_{\hat{t}},\hat{\tilde{\xi}})|\\
&\leq C\tau^4||\xi||^2_{H^3(0,T;L^2(\Omega))}+C\tau\hbar^{2\sigma}\sum_{s\in S_{t-\tau}}||\xi_{\hat{t}}||^2_\sigma+\frac{1}{32}\nu\tau\sum_{s\in S_{t-\tau}}|\hat{\tilde{\xi}}|_1^2\\
&\leq C\tau^4||\xi||^2_{H^3(0,T;L^2(\Omega))}+C\tau\hbar^{2\sigma}||\xi||^2_{H^1(0,T;H^\sigma(\Omega))}+\frac{1}{32}\nu\tau\sum_{s\in S_{t-\tau}}|\hat{\tilde{\xi}}|_1^2.
\end{aligned}$$

Rewrite $A_2(v)$ as the following form

$$A_2(v)=(J(\xi,\psi)-J(\xi^*,\psi^*),v)+((J-J_C)(\xi^*,\psi^*),v)\equiv\sum_{i=1}^{2}B_i,$$

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and we make estimation for B_1 and B_2. For B_1, by using partial integral, we can obtain the following results.

$$\begin{aligned}
B_1 &= (\partial_2\psi\partial_1\xi,v)-(\partial_1\psi\partial_2\xi,v)-(\partial_2\psi^*\partial_1\xi^*,v)+(\partial_1\psi^*\partial_2\xi^*,v)| \\
&= (\partial_2(\psi-\psi^*)\partial_1\xi),v)+(\partial_2\xi^*(\partial_1\xi-\partial_1\xi^*),v) \\
&\quad +(\partial_1\psi^*\partial_2(\psi^*-\psi),v)+(\partial_1(\psi^*-\psi)\partial_2\xi,v), \\
|B_1| &\le \|\psi-\psi^*\|^2\|\partial_1\xi\|^2_{0,\infty}+\|\xi^*\|^2_{0,\infty}\|\partial_1(\xi-\xi^*)\|^2 \\
&\quad +\|\psi^*\|^2_{0,\infty}\|\partial_2(\psi^*-\psi)\|^2+\|\psi^*-\psi\|^2\|\partial_2\xi\|^2_{0,\infty}+\frac{1}{64}\nu|v|^2_1 \\
&\le C\hbar^{2\sigma}\|\psi\|^2_\sigma|\xi|^2_{1,\infty}+C\hbar^{2\sigma}\|\xi\|^2_\sigma\|\xi\|^2+C\hbar^{2\sigma}\|\psi\|^2_{\sigma+1}\|\psi\|^2+\frac{1}{64}\nu|v|^2_1,
\end{aligned}$$

where the constant C depends on ν^{-1}. Next we make the estimation of B_2. From Lemma 4.1 and using partial integral, and similar treatment as B_1, we can get following estimation

$$\begin{aligned}
|B_2| &\le C(\|\Pi^C_{\mathcal{N}}(\psi^*\partial_1\xi^*)-\psi^*\partial_1\xi^*\|^2+\|\Pi^C_{\mathcal{N}}(\psi^*\partial_2\xi^*)-\psi^*\partial_2\xi^*\|^2)+\frac{1}{64}\nu|v|^2_1 \\
&\le C\hbar^{2\sigma}\|\psi\|^2_\sigma|\xi|^2_{1,\infty}+\hbar^{2\sigma}\|\psi\|^2_\sigma|\xi|^2{}_2 \\
&\quad +C\hbar^{2\sigma}\|\psi\|^2_{0,\infty}\|\xi\|^2_\sigma+\hbar^{2\sigma}(\|\psi\partial_1\xi\|^2+\|\psi\partial_2\xi\|^2)+\frac{1}{64}\nu|v|^2_1.
\end{aligned}$$

Next, from Lemma 4.5 and using Taylor expansion, we find

$$\begin{aligned}
\tau\sum_{s\in S_{t-\tau}}|A_3(\hat{\tilde{\xi}})| &\le \tau\sum_{s\in S_{t-\tau}}|\nu(\nabla^2(\hat{\xi}^*-\xi^*),\hat{\tilde{\xi}})+\nu(\nabla^2(\xi^*-\xi),\hat{\tilde{\xi}})| \\
&\le C\tau^5||\partial_t^2(\xi^*-\xi)||^2_{L^2(0,T;H^1(\Omega))}+C\tau^5||\partial_t^2\xi||^2_{L^2(0,T;H^1(\Omega))} \\
&\quad +C\tau\sum_{s\in S_{t-\tau}}\|\nabla\xi^*-\nabla\xi\|^2+\frac{1}{32}\nu\tau\sum_{s\in S_{t-\tau}}|\hat{\tilde{\xi}}|^2_1 \\
&\le C\tau^5||\partial_t^2\xi||^2_{L^2(0,T;H^1(\Omega))}+C\tau\hbar^{2\sigma}\sum_{s\in S_{t-\tau}}\|\xi\|^2_{\sigma+1}+\frac{1}{32}\nu\tau\sum_{s\in S_{t-\tau}}|\hat{\tilde{\xi}}|^2_1, \\
\tau\sum_{s\in S_{t-\tau}}|A_4(\hat{\tilde{\xi}})| &\le \tau\sum_{s\in S_{t-\tau}}\frac{C}{\nu}||\Pi^C_{\mathcal{N}}f_1(s)-f_1(s)||^2+\frac{1}{32}\nu\tau\sum_{s\in S_{t-\tau}}|\hat{\tilde{\xi}}|^2_1 \\
&\le C\tau\sum_{s\in S_{t-\tau}}\hbar^{2\sigma}||f_1(s)||^2_\sigma+\frac{1}{32}\nu\tau\sum_{s\in S_{t-\tau}}|\hat{\tilde{\xi}}|^2_1.
\end{aligned}$$

In the light of Lemma 4.5 and Lemma 4.1, it yields

$$||\tilde{f}_3||\le C\hbar^\sigma(||\xi||_\sigma+\|\psi\|_\sigma+||f_2||_\sigma).$$

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For the initial error, according to Lemma 4.1 and Taylor formula we have

$$\begin{aligned}
&||\tilde{\xi}(0)|| = ||\eta(0)-\xi^*(0)|| = ||\Pi_N^C\xi_0-\xi_0||+||\xi_0-\xi^*(0)|| \le C\hbar^\sigma||\xi_0||_\sigma,\\
&||\tilde{\xi}(\tau)|| = ||\eta(\tau)-\xi^*(\tau)|| = ||\Pi_N^C(\xi_0+\tau\partial_t\xi_0)-\xi^*(\tau)||\\
&\qquad \le ||\Pi_N^C\xi_0-\xi_0||+||\Pi_N^C\tau\partial_t\xi_0-\tau\partial_t\xi_0||+||\xi_0+\tau\partial_t\xi(0)-\xi(\tau)||+||\xi(\tau)-\xi^*(\tau)||\\
&\qquad \le C\hbar^\sigma||\xi||_\sigma+C\tau\hbar^\sigma||\partial_t\xi(0)||_\sigma+C\tau^2||\partial_t^2\xi||_{C(0,\tau;L^2(\Omega))}.
\end{aligned}$$

Then similar to the discussion as Theorem 4.1, making use of Lemma 4.10, Lemma 4.5 and triangle inequality, we can read the conclusion of the theorem. □

5 The numerical results

In this section, we give the numerical results of multi-domain Chebyshev-Legendre method of the scheme (2.3). We also make a comparison with the results of the case of single domain Chebyshev-Legendre method [10]. Let time step be small sufficiently so that the errors in space discrete are the main errors. It shows that at the same N, compared with the case of the single domain method on the case of the achieving the same precision, the multi-domain methods converge more rapidly.

We take the following testing functions as

$$\begin{aligned}
&\psi(x_1,x_2,t) = A(x_1^2-1)(x_1^2-5)\sin(\pi x_2)e^{-Bt}, \quad f_2(x_1,x_2,t)=0,\\
&\xi(x_1,x_2,t) = -\nabla^2\psi(x_1,x_2,t).
\end{aligned}$$

Let $U(x)$ be the analytic solution of (1.1). And $u(x)$ is the numerical solution. Define the maximum error on the points ($\xi_i = -1+\frac{2*i}{N}$, $1\le i\le N$) distributed uniform on $[-1,1]$ as

$$\epsilon = \max_{1\le i\le N}|U(\xi_i)-u(\xi_i)|.$$

In computing, we take $A=0.5$, $B=0.4$. For convenient of notation, we do the same partition on x_1 and x_2 direction respectively. We separate the interval $[-1,1]$ into two subintervals as $[-1,0]$ and $[0,1]$. The results of single domain see from [10].

In the following five cases, we make the comparison of precision with single-domain method and multi-domain method.

Table 1: The maximum error when $t=2$: $\tau=0.01$, $\nu=0.0001$.

single domain	ξ	ψ	multi-domain	ξ	ψ
(8,8)	1.3575e-3	2.0111e-3	(4,4,4,4)	1.2765e-3	3.3157e-5
(16,16)	1.7787e-5	3.6557e-6	(8,8,8,8)	1.0657e-5	2.1892e-6
(32,32)	1.7797e-5	3.6557e-6	(16,16,16,16)	1.0658e-5	2.1892e-6

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Table 2: The maximum error when $t=2$: $\tau=0.005$, $\nu=0.0001$.

single domain	ξ	ψ	multi-domain	ξ	ψ
(8,8)	1.3594e-3	1.8790e-5	(4,4,4,4)	1.3399e-3	3.3824e-5
(16,16)	4.4493e-6	9.139e-7	(8,8,8,8)	1.0658e-7	2.1892e-8
(32,32)	4.4492e-6	9.1392e-7	(16,16,16,16)	1.0658e-7	2.1892e-8

Table 3: The maximum error when$t=2$: $\tau=0.0005$, $\nu=0.0001$.

single domain	ξ	ψ	multi-domain	ξ	ψ
(8,8)	1.3584e-3	1.8302e-5	(4,4,4,4)	1.3437e-3	3.3893e-5
(16,16)	4.4494e-8	9.1396e-9	(8,8,8,8)	2.6645e-8	5.4730e-9
(32,32)	4.4500e-8	9.1412e-9	(16,16,16,16)	2.6644e-8	5.4729e-9

Table 4: The maximum error when $t=2$: $\tau=0.00005$, $\nu=0.0001$.

single domain	ξ	ψ	multi-domain	ξ	ψ
(8,8)	1.3581e-3	1.8288e-5	(4,4,4,4)	1.3472e-3	3.3959e-5
(16,16)	4.4214e-9	9.0500e-11	(8,8,8,8)	2.7993e-10	5.5041e-11
(32,32)	5.5196e-10	1.1852e-11	(16,16,16,16)	2.7993e-10	5.5041e-11

Table 5: The maximum error when $t=0.5$: $\tau=0.0005$, $\nu=0.001$.

single domain	ξ	ψ	multi-domain	ξ	ψ
(8,8)	9.2462e-4	1.1972e-5	(4,4,4,4)	1.5564e-3	1.6362e-5
(16,16)	1.4177e-8	2.9073e-9	(8,8,8,8)	6.3736e-9	6.5923e-10
(32,32)	1.4175e-8	2.9077e-9	(16,16,16,16)	6.5527e-9	6.5913e-10
(64,64)	1.4179e-8	2.9084e-9	(32,32,32,32)	6.5526e-9	6.5964e-10

6 The conclusion

In this paper we have extended the Chebyshev-Legendre spectral method to multi-domain case and solved the two-dimensional vorticity equations (1.1). The schemes were formulated in Legendre-Galerkin method while the nonlinear term was collocated at Chebyshev-Gauss collocation points. This scheme can be implemented in parallel way. We also proved the convergence and stability of the scheme.The numerical examples of the method are given. By comparing with single-domain method, our multi-domain method converge more rapidly. Our approach can be extended to the case of discontinuous Galerkin spectral multi-domain method, and will be considered elsewhere.

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