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Generalized fractions, Buchsbaum modules and generalized Cohen–Macaulay modules

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Introduction. Let A be a (commutative Noetherian) local ring (with identity) having maximal ideal $\mathfrak{m}$ and positive dimension. This note is concerned with, among other things, a complex $C(\mathscr{U}(A), A)$ of A-modules which was studied in [8] and which involves modules of generalized fractions derived from A and subsets of systems of parameters for A; in ([8], 3·5), the complex $C(\mathscr{U}(A), A)$ was shown to have connections with local cohomology. The complex is described as follows.

For each $i \in \mathbb{N}$ (throughout, we use $\mathbb{N}$ to denote the set of positive integers), set

$$U(A)_i = \{(s_1, \ldots, s_i) \in A^i: \text{ there exists } j \text{ with } 0 \leqslant j \leqslant i \text{ such that } s_1, \ldots, s_j \text{ form a s.s.o.p. for } A \text{ and } s_{j+1} = \ldots = s_i = 1\}.$$

('s.s.o.p.' stands for 'subset of a system of parameters'.) Then, for each $i \in \mathbb{N}$, $U(A)_i$ is a triangular subset ([7], 2·1) of A^i. We use $\mathscr{U}(A)$ to denote the family $(U(A)_i)_{i \in \mathbb{N}}$. We can ([8], 3·1 and 3·2) form the complex

$$0 \longrightarrow A \xrightarrow{d^0} U(A)_1^{-1}A \longrightarrow \ldots \longrightarrow U(A)_i^{-i}A \xrightarrow{d^i} U(A)_{i+1}^{-i-1}A \longrightarrow \ldots$$

of A-modules and A-homomorphisms, which we denote by $C(\mathscr{U}(A), A)$. Here, $U(A)_i^{-i}A$ is the module of generalized fractions ([7], §2) of A with respect to the triangular subset $U(A)_i$ of A^i, and the homomorphisms d^i (for $i \geqslant 0$) are given by the following: $d^0(a) = a/(1)$ for all $a \in A$, and, for each $i \in \mathbb{N}$,

$$d^i(a/(s_1, \ldots, s_i)) = a/(s_1, \ldots, s_i, 1)$$

for all $a \in A$ and $(s_1, \ldots, s_i) \in U(A)_i$.

It is a consequence of the exactness theorem ([9], 3·3) that $C(\mathscr{U}(A), A)$ is exact if and only if A is a Cohen–Macaulay ring. (Incidentally, O'Carroll has provided, in ([3], §3), a simpler proof of the exactness theorem which does not require the underlying ring to be Noetherian.) In recent years, there have appeared in the literature many papers which concern local rings, such as Buchsbaum local rings and generalized Cohen–Macaulay local rings, which are, in a sense, only a little worse than Cohen–Macaulay local rings. It is natural to ask whether generalized fractions have any role to play in the theory of such 'nearly' Cohen–Macaulay local rings. The purpose of this note is to present some results which connect the above-mentioned concepts.

The idea of an f-ring ([5], (2·3)) seems to be important here: A is an f-ring if each

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system of parameters $x_1, \ldots, x_h$ for A (where $h = \dim A$) is a filter-A-regular sequence, that is

$$\operatorname{Supp}(((Ax_1 + \ldots + Ax_{i-1}) : x_i)/(Ax_1 + \ldots + Ax_{i-1})) \subseteq \{\mathfrak{m}\}$$

for all $i = 1, \ldots, h$. (Here, $Ax_1 + \ldots + Ax_{i-1}$ is to be interpreted as 0 when $i = 1$, and $\operatorname{Supp} M$ (or $\operatorname{Supp}_A M$) denotes the support of the A-module M. The definition given here, although not identical to that of ([5], (2·1)), is easily seen to be equivalent to it.) Of course, A is Cohen–Macaulay if and only if, for each system of parameters $x_1, \ldots, x_h$ for A,

$$((Ax_1 + \ldots + Ax_{i-1}) : x_i)/(Ax_1 + \ldots + Ax_{i-1}) = 0$$

for all $i = 1, \ldots, h$. Note that the class of f-rings includes all Buchsbaum local rings and all generalized Cohen–Macaulay local rings: A is a Buchsbaum ring if, for each system of parameters $x_1, \ldots, x_h$ for A, and for all $i = 1, \ldots, h$,

$$\mathfrak{m}[((Ax_1 + \ldots + Ax_{i-1}) : x_i)/(Ax_1 + \ldots + Ax_{i-1})] = 0,$$

and A is a generalized Cohen–Macaulay ring ([5], §3) if there exists $r \in \mathbb{N}$ such that, for each system of parameters $x_1, \ldots, x_h$ for A, and for all $i = 1, \ldots, h$,

$$\mathfrak{m}^r[((Ax_1 + \ldots + Ax_{i-1}) : x_i)/(Ax_1 + \ldots + Ax_{i-1})] = 0.$$

Special cases of our results relate the above concepts and the complex $C(\mathscr{U}(A), A)$, and we mention two in detail here. First, it turns out that generalized fractions behave sufficiently well with respect to ordinary fraction formation for us to be able to deduce from the exactness theorem that A is an f-ring if and only if

$$\operatorname{Supp}(\ker d^i/\operatorname{im} d^{i-1}) \subseteq \{\mathfrak{m}\} \quad \text{for all} \quad i = 0, \ldots, h-2.$$

Secondly, we shall show that, when this is the case,

$$\ker d^i/\operatorname{im} d^{i-1} \cong H^i_{\mathfrak{m}}(A) \quad \text{for all} \quad i = 0, \ldots, h-1.$$

(Here, $H^i_{\mathfrak{m}}(A)$ is the ith local cohomology module of A with respect to $\mathfrak{m}$; it should be noted that, by ([8], 3·5), $H^h_{\mathfrak{m}}(A) \cong U(A)^{-h-1}_{h+1} A$.) These results lead to a characterization of generalized Cohen–Macaulay local rings in terms of modules of generalized fractions, and also to a description of the invariant $I(A)$ of a Buchsbaum local ring A ([4], Satz 2) in terms of the complex $C(\mathscr{U}(A), A)$.

1. *Notation and preparatory results.* Actually, we shall work in a situation more general than that discussed in the Introduction. Throughout, we shall use M to denote a non-zero finitely generated A-module of dimension $n \geqslant 1$. Recall that n elements $x_1, \ldots, x_n$ of $\mathfrak{m}$ form a system of parameters (s.o.p.) for M if $M/(Ax_1 + \ldots + Ax_n)M$ has finite length. We shall use the following elementary fact.

(1·1) LEMMA. *Set $\bar{A} = A/(0:M)$ and let the natural image in $\bar{A}$ of an element a of A be denoted by $\bar{a}$. Let $x_1, \ldots, x_n \in A$. Then $x_1, \ldots, x_n$ form a s.o.p. for M if and only if $\bar{x}_1, \ldots, \bar{x}_n$ form a s.o.p. for the local ring $\bar{A}$.*

The next lemma can be established by an argument similar to that needed for ([7], 3·7).

(1·2) LEMMA. *For each $i \in \mathbb{N}$, set*

$$U(M)_i = \{(s_1, \ldots, s_i) \in A^i : \text{there exists } j \text{ with } 0 \leqslant j \leqslant i \text{ such that } s_1, \ldots, s_j \text{ form a s.s.o.p. for } M \text{ and } s_{j+1} = \ldots = s_i = 1\}.$$

We have that $U(M)_i$ is a triangular subset of A^i.

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Next we shall need an easy change of rings result concerning generalized fractions. When discussing modules of generalized fractions, we shall use the notation of [7] and [8], except that we shall use slightly different notation concerning matrices, in that round brackets will now be used instead of square ones, we shall agree to use n-tuples $(u_1, \ldots, u_n)$ of elements of A and $1 \times n$ row matrices $(u_1 \ldots u_n)$ over A interchangeably, and we shall cease to use boldface letters to denote matrices: it is hoped that these changes will simplify the notation. We still use ^{T} to denote matrix transpose and, for $i \in \mathbb{N}$, $D_i(R)$ to denote the set of all $i \times i$ lower triangular matrices over a commutative ring R.

(1·3) LEMMA. *Let R be a commutative ring (with identity), and let X be an R-module. Let $\mathfrak{b}$ be a proper ideal of R contained in the annihilator of X, so that X may be regarded as an $R/\mathfrak{b}$-module in a natural way. Let $\pi: R \to R/\mathfrak{b}$ denote the natural homomorphism. Let $i \in \mathbb{N}$ and let U be a triangular subset of R^i.*

Then $\pi(U) = \{(\pi(u_1), \ldots, \pi(u_i)) : (u_1, \ldots, u_i) \in U\}$ is a triangular subset of $(R/\mathfrak{b})^i$ and there is an R-isomorphism $\psi: U^{-i}X \to (\pi(U))^{-i}X$ such that

$$\psi(x/(u_1, \ldots, u_i)) = x/(\pi(u_1), \ldots, \pi(u_i))$$

for all $x \in X$ and $(u_1, \ldots, u_i) \in U$.

Proof. That $\pi(U)$ is a triangular subset of $(R/\mathfrak{b})^i$ is clear, and it is routine to check that there is an R-epimorphism $\psi: U^{-i}X \to (\pi(U))^{-i}X$ given by the formula in the statement of the lemma. We show that ψ is injective. For $u = (u_1, \ldots, u_i) \in U$, it will be convenient to denote $(\pi(u_1), \ldots, \pi(u_i))$ by $\pi(u)$.

Let $x \in X$ and $u \in U$ be such that $\psi(x/u) = 0$. Then there exist $v = (v_1, \ldots, v_i) \in U$ and $H^* \in D_i(R/\mathfrak{b})$ such that $H^*\pi(u)^T = \pi(v)^T$ and $|H^*|\, x \in (\Sigma_{j=1}^{i-1} (R/\mathfrak{b})\, \pi(v_j))\, X$.

Since U is a triangular subset of R^i, there exist $w = (w_1, \ldots, w_i) \in U$ and $K, L \in D_i(R)$ such that $Ku^T = w^T = Lv^T$. Write D for the diagonal matrix $\operatorname{diag}(w_1, \ldots, w_i)$, and note that

$$DKu^T = (w_1^2, \ldots, w_i^2)^T.$$

For $G = (g_{rs}) \in D_i(R)$, we use $\pi(G)$ to denote the matrix $(\pi(g_{rs}))$ of $D_i(R/\mathfrak{b})$. Thus we have

$$\pi(K)\, \pi(u)^T = \pi(w)^T = \pi(L)\, \pi(v)^T = \pi(L)\, H^*\pi(u)^T,$$

and, by ([7], 2·2),

$$|\pi(L)|\, |H^*|\, x \in (\Sigma_{j=1}^{i-1} (R/\mathfrak{b})\, \pi(w_j))\, X.$$

Next, by ([7], (2·3)),

$$|\pi(D)|\, |\pi(K)| - |\pi(D)|\, |\pi(L)|\, |H^*| \in \sum_{j=1}^{i-1} (R/\mathfrak{b})\, (\pi(w_j))^2.$$

It therefore follows that $|\pi(D)|\, |\pi(K)|\, x \in (\Sigma_{j=1}^{i-1} (R/\mathfrak{b})\, (\pi(w_j))^2)\, X$, that is

$$|DK|\, x \in \left(\sum_{j=1}^{i-1} Rw_j^2\right) X.$$

As $DKu^T = (w_1^2, \ldots, w_i^2)^T$, it follows that $x/u = 0$ in $U^{-i}X$, as required.

(1·4) *Notation.* We use $\mathscr{U}(M)$ to denote the family $(U(M)_i)_{i \in \mathbb{N}}$, where $U(M)_i$ is the triangular subset of A^i of (1·2). Note that $\mathscr{U}(M)$ is a chain of triangular subsets on A in the sense of ([3], p. 420).

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It will be convenient for us to introduce and work with a more general chain $\mathscr{V} = (V_i)_{i\in\mathbb{N}}$ of triangular subsets on A which are such that

(i) $V_i \subseteq U(M)_i$ for all $i \in \mathbb{N}$, and

(ii) there exists an s.o.p. $y_1, \ldots, y_n$ for M such that $(y_1, \ldots, y_n) \in V_n$.

For example, if $x_1, \ldots, x_n$ is a fixed s.o.p. for M, then $\mathscr{V}$ could be the family

$$\mathscr{U}(x) = (U(x)_i)_{i\in\mathbb{N}}$$

where (x denotes $(x_1, \ldots, x_n)$ and), for all $i \in \mathbb{N}$, $U(x)_i$ is the expansion ([7], 3·2) of the triangular subset $\{(x_1^{\alpha_1}, \ldots, x_i^{\alpha_i})\colon \alpha_j \in \mathbb{N} \text{ for all } j = 1, \ldots, i\}$ of A^i, where x_r is interpreted as 1 whenever $r > n$.

By ([8], 3·2) and Lemmas (1·1) and (1·3) above (or, alternatively, by ([3], p. 420)), $\mathscr{V}$ gives rise to a complex

$$0 \xrightarrow{e^{-1}} M \xrightarrow{e^0} V_1^{-1}M \longrightarrow \ldots \longrightarrow V_i^{-i}M \xrightarrow{e^i} V_{i+1}^{-i-1}M \longrightarrow \ldots$$

of A-modules and A-homomorphisms, which we denote by $C(\mathscr{V}, M)$, in which $e^0(m) = m/(1)$ for all $m \in M$ and, for each $i \in \mathbb{N}$,

$$e^i(m/(u_1, \ldots, u_i)) = m/(u_1, \ldots, u_i, 1)$$

for all $m \in M$ and $(u_1, \ldots, u_i) \in V_i$.

We shall write the complex $C(\mathscr{U}(M), M)$ as

$$0 \xrightarrow{d^{-1}} M \xrightarrow{d^0} U(M)_1^{-1}M \longrightarrow \ldots \longrightarrow U(M)_i^{-i}M \xrightarrow{d^i} U(M)_{i+1}^{-i-1}M \longrightarrow \ldots.$$

We can obtain the next lemma quickly from ([8], §3).

(1·5) LEMMA. *For all $i = 1, \ldots, n$, $H^j_{\mathfrak{m}}(V_i^{-i}M) = 0$ for all $j \in \mathbb{N} \cup \{0\}$.*

Proof. It is immediate from (1·1) and ([8], 3·4(ii)) that, for an integer i with $1 \leqslant i \leqslant n$, $H^j_{\mathfrak{m}}(\overline{V}_i^{-i}M) = 0$ for all $j \geqslant 0$, where

$$\overline{V}_i = \{(\overline{u}_1, \ldots, \overline{u}_i)\colon (u_1, \ldots, u_i) \in V_i\}$$

and $\overline{\mathfrak{m}}$ is the maximal ideal of $\bar{A}$. Hence, by ([6], 4·3), $H^j_{\mathfrak{m}}(\overline{V}_i^{-i}M) = 0$ for all $j \geqslant 0$. The result therefore follows from (1·3).

Generalized fractions behave well with respect to ordinary fraction formation. This is shown in the next result, which is proved in [1].

(1·6) PROPOSITION ([1], 2·1). *Let R be a commutative ring (with identity), let $n \in \mathbb{N}$, let W be a triangular subset of R^n, let S be a multiplicatively closed subset of R and let X be an R-module. Denote by $\phi\colon R \to S^{-1}R$ the natural mapping.*

Then $\phi(W) = \{(\phi(w_1), \ldots, \phi(w_n))\colon (w_1, \ldots, w_n) \in W\}$ is a triangular subset of $(S^{-1}R)^n$, and there is an $S^{-1}R$-isomorphism

$$\psi\colon S^{-1}(W^{-n}X) \to (\phi(W))^{-n}(S^{-1}X)$$

which is such that $\psi((x/(w_1, \ldots, w_n))/s) = (x/s)/(\phi(w_1), \ldots, \phi(w_n))$ for all $x \in X$, $(w_1, \ldots, w_n) \in W$ and $s \in S$.

(1·7) *Remarks.* In the situation of (1·6), when S is the complement $R\backslash\mathfrak{p}$ of a prime ideal $\mathfrak{p}$ of R, we shall write $W_{\mathfrak{p}}$ instead of the $\phi(W)$ of (1·6).

Now suppose that $\mathscr{W} = (W_i)_{i\in\mathbb{N}}$ is a chain of triangular subsets on R and that

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$\mathfrak{p} \in \operatorname{Spec}(R)$. Let $\mathscr{W}_{\mathfrak{p}}$ denote the induced chain $(W_{i\mathfrak{p}})_{i \in \mathbb{N}}$ of triangular subsets on $R_{\mathfrak{p}}$. Consider, for an R-module X, the associated complexes $C(\mathscr{W}, X)$ and $C(\mathscr{W}_{\mathfrak{p}}, X_{\mathfrak{p}})$ ([3], p. 420). Now (1·6) provides us with natural isomorphisms

$$\psi^i\colon (W_i^{-i} X)_{\mathfrak{p}} \to (W_{i\mathfrak{p}})^{-i} X_{\mathfrak{p}}$$

for all $i \in \mathbb{N}$, and it is easy to see that, if we define $\psi^{-1}\colon X_{\mathfrak{p}} \to X_{\mathfrak{p}}$ to be the identity map, then $\Psi = (\psi^i)_{i \in \mathbb{N}}$ provides an isomorphism of complexes of $R_{\mathfrak{p}}$-modules and $R_{\mathfrak{p}}$-homomorphisms

$$\Psi\colon C(\mathscr{W}, X)_{\mathfrak{p}} \to C(\mathscr{W}_{\mathfrak{p}}, X_{\mathfrak{p}}).$$

Our final task in this preparatory section is to recall the definitions of f-module, Buchsbaum module and generalized Cohen–Macaulay module over A from [5] and to give the definition of quasi-Buchsbaum A-module. For an A-module Y, we have that each element of Y is annihilated by some power of $\mathfrak{m}$ if and only if $\operatorname{Supp} Y \subseteq \{\mathfrak{m}\}$. The definitions of the above concepts may therefore be given in the following way.

(1·8) *Reminders and Definitions.* (i) ([5], (2·1)) A sequence $x_1, \ldots, x_r$ of elements of $\mathfrak{m}$ is said to be a *filter-M-regular sequence* if

$$\operatorname{Supp}\left(((Ax_1 + \ldots + Ax_{i-1})M : x_i)/(Ax_1 + \ldots + Ax_{i-1})M\right) \subseteq \{\mathfrak{m}\}$$

for all $i = 1, \ldots, r$.

(ii) ([5], (2·3)) We say that M is an *f-module* if every system of parameters for M is a filter-M-regular sequence.

(iii) ([5], (3·2) and (3·3)) We say that M is a *generalized Cohen–Macaulay A-module* if there exists $r \in \mathbb{N}$ such that, for each system of parameters $x_1, \ldots, x_n$ for M, and for all $i = 1, \ldots, n$,

$$\mathfrak{m}^r[((Ax_1 + \ldots + Ax_{i-1})M : x_i)/(Ax_1 + \ldots + Ax_{i-1})M] = 0.$$

(iv) ([5], (4·1)) We say that M is a *Buchsbaum A-module* if, for each system of parameters $x_1, \ldots, x_n$ for M and for all $i = 1, \ldots, n$,

$$\mathfrak{m}[((Ax_1 + \ldots + Ax_{i-1})M : x_i)/(Ax_1 + \ldots + Ax_{i-1})M] = 0.$$

(v) Note that, in view of ([5], (3·2) and (3·3)), M is a generalized Cohen–Macaulay A-module if and only if there exists $s \in \mathbb{N}$ such that $\mathfrak{m}^s H^i_{\mathfrak{m}}(M) = 0$ for all $i = 0, \ldots, n-1$. It is true that, if M is a Buchsbaum A-module, then $\mathfrak{m} H^i_{\mathfrak{m}}(M) = 0$ for all $i = 0, \ldots, n-1$: see ([4], Hilfssatz 3) and modify the arguments used there. However, the converse statement is not true, as is shown by example 2 of [10]: accordingly, we shall say that M is a *quasi-Buchsbaum A-module* if $\mathfrak{m} H^i_{\mathfrak{m}}(M) = 0$ for all $i = 0, \ldots, n-1$.

2. *The results.* In our first proposition below, R denotes a general commutative ring (with identity).

(2·1) Proposition. *Let $\mathscr{W} = (W_i)_{i \in \mathbb{N}}$ be a chain of triangular subsets on R and let X be an R-module. Write the associated complex $C(\mathscr{W}, X)$ as*

$$0 \xrightarrow{f^{-1}} X \xrightarrow{f^0} W_1^{-1} X \longrightarrow \ldots \longrightarrow W_i^{-i} X \xrightarrow{f^i} W_{i+1}^{-i-1} X \longrightarrow \ldots.$$

Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then $(\ker f^i / \operatorname{im} f^{i-1})_{\mathfrak{p}} = 0$ for all $i \in \mathbb{N} \cup \{0\}$ if and only if, for all $i \in \mathbb{N}$ and for all $(w_1, \ldots, w_i) \in W_i$, we have

$$(((Aw_1 + \ldots + Aw_{j-1})X : w_j)/(Aw_1 + \ldots + Aw_{j-1})X)_{\mathfrak{p}} = 0$$

for all $j = 1, \ldots, i$.

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Proof. By ([2], chapter 6, theorem 1), $(\ker f^i/\operatorname{im} f^{i-1})_{\mathfrak{p}} = 0$ for all $i \in \mathbb{N} \cup \{0\}$ if and only if $C(\mathscr{W}, X)_{\mathfrak{p}}$ is exact; by (1·7), this is the case if and only if $C(\mathscr{W}_{\mathfrak{p}}, X_{\mathfrak{p}})$ is exact; by the general exactness theorem ([3], 3·1), $C(\mathscr{W}_{\mathfrak{p}}, X_{\mathfrak{p}})$ is exact if and only if, for all $i \in \mathbb{N}$, every member of $W_{i\mathfrak{p}}$ is a poor $X_{\mathfrak{p}}$-sequence. The result is an easy consequence of this.

We now revert to our local situation and the complexes $C(\mathscr{V}, M)$ and $C(\mathscr{U}(M), M)$. We need a lemma before we draw conclusions from (2·1).

(2·2) LEMMA. (i) $\operatorname{Supp}(\ker e^{n-1}/\operatorname{im} e^{n-2}) \subseteq \{\mathfrak{m}\}$.

(ii) $\ker e^i/\operatorname{im} e^{i-1} = 0$ *for all* $i \geqslant n$.

Proof. (i) Once again, we work with the sets

$$\overline{V}_i = \{\overline{u}_1, \ldots, \overline{u}_i) : (u_1, \ldots, u_i) \in V_i\}$$

for $i \in \mathbb{N}$. If we set $\overline{\mathscr{V}} = (\overline{V}_i)_{i \in \mathbb{N}}$, a chain of triangular subsets on $\bar{A}$, then we can obtain from (1·1) and ([8], 3·3) the result for $C(\overline{\mathscr{V}}, M)$ analogous to the one desired for $C(\mathscr{V}, M)$. It is then easy to use (1·3) to complete the proof.

(ii) A similar argument shows that $\ker e^n/\operatorname{im} e^{n-1} = 0$. Next, e^n is surjective, since every member of V_{n+1} has its final co-ordinate equal to 1. Finally, for $i > n+1$, $V_{\bar{i}}^{-i} M = 0$ by ([7], (3·3)(ii)).

Thus we see that, for the complex $C(\mathscr{V}, M)$, it is automatic that

$$\operatorname{Supp}(\ker e^i/\operatorname{im} e^{i-1}) \subseteq \{\mathfrak{m}\} \quad \text{for all} \quad i \geqslant n-1.$$

With the aid of this comment, we can now make the following deduction from (2.1).

(2·3) COROLLARY. $\operatorname{Supp}(\ker e^i/\operatorname{im} e^{i-1}) \subseteq \{\mathfrak{m}\}$ *for all* $i = 0, \ldots, n-2$ *if and only if, for each* $i = 1, \ldots, n$, *every s.s.o.p.* $x_1, \ldots, x_i$ *for* M *for which* $(x_1, \ldots, x_i) \in V_i$ *is a filter-*M*-regular sequence.*

In particular, M *is an* f*-module if and only if*

$$\operatorname{Supp}(\ker d^i/\operatorname{im} d^{i-1}) \subseteq \{\mathfrak{m}\} \quad \textit{for all} \quad i = 0, \ldots, n-2.$$

Of course, in the special case in which $n = 1$, the last condition is vacuously satisfied and M is automatically an f-module.

Our next aim is to show that, when the conditions of the first part of (2·3) are satisfied, each local cohomology module $H^i_{\mathfrak{m}}(M)$ (for $0 \leqslant i \leqslant n-1$) actually 'appears' in the complex $C(\mathscr{V}, M)$ in the sense that it is isomorphic to $\ker e^i/\operatorname{im} e^{i-1}$, the ith cohomology module of the complex $C(\mathscr{V}, M)$. (That $H^n_{\mathfrak{m}}(M) \cong V_{n+1}^{-n-1} M$ follows from (1·1), (1·3) and ([8], 3·5).)

(2·4) THEOREM. *Suppose that, for each* $i = 0, \ldots, n-2$,

$$\operatorname{Supp}(\ker e^i/\operatorname{im} e^{i-1}) \subseteq \{\mathfrak{m}\}.$$

Then $\ker e^i/\operatorname{im} e^{i-1} \cong H^i_{\mathfrak{m}}(M)$ *for all* $i = 0, \ldots, n-1$.

Proof. Note that $\operatorname{Supp}(\ker e^{n-1}/\operatorname{im} e^{n-2}) \subseteq \{\mathfrak{m}\}$ also, by (2·2)(i).

Choose an integer i with $1 \leqslant i \leqslant n$. Consider the exact sequences

$$0 \to \ker e^i \to V_{\bar{i}}^{-i} M \to \operatorname{im} e^i \to 0$$

and

$$0 \to \operatorname{im} e^{i-1} \to V_{\bar{i}}^{-i} M \to \operatorname{coker} e^{i-1} \to 0$$

and use (1·5) to see that

$$H^0_{\mathfrak{m}}(\ker e^i) = 0 = H^0_{\mathfrak{m}}(\operatorname{im} e^{i-1}) \tag{1}$$

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and

$$H^j_{\mathfrak{m}}(\operatorname{im} e^i) \cong H^{j+1}_{\mathfrak{m}}(\ker e^i) \quad \text{for all} \quad j \in \mathbb{N} \cup \{0\}. \tag{2}$$

Hence

$$H^1_{\mathfrak{m}}(\ker e^i) = 0 \quad \text{for all} \quad i = 1, \ldots, n-1. \tag{3}$$

Next, it follows from the hypothesis and ([**6**], 6·1) that, for each $k = 0, \ldots, n-1$,

$$H^0_{\mathfrak{m}}(\ker e^k / \operatorname{im} e^{k-1}) \cong \ker e^k / \operatorname{im} e^{k-1} \tag{4}$$

and

$$H^j_{\mathfrak{m}}(\ker e^k / \operatorname{im} e^{k-1}) = 0 \quad \text{for all} \quad j \in \mathbb{N}. \tag{5}$$

We can now deduce from (1), (3), (4), (5) and the exact sequence

$$0 \to \operatorname{im} e^{i-1} \to \ker e^i \to \ker e^i / \operatorname{im} e^{i-1} \to 0$$

that, for each $i = 1, \ldots, n-1$,

$$H^1_{\mathfrak{m}}(\operatorname{im} e^{i-1}) \cong \ker e^i / \operatorname{im} e^{i-1} \tag{6}$$

and

$$H^j_{\mathfrak{m}}(\operatorname{im} e^{i-1}) \cong H^j_{\mathfrak{m}}(\ker e^i) \quad \text{for all} \quad j \geqslant 2. \tag{7}$$

It now follows from (2), (6) and (7) that

$$H^i_{\mathfrak{m}}(\operatorname{im} e^0) \cong \ker e^i / \operatorname{im} e^{i-1} \quad \text{for all} \quad i = 1, \ldots, n-1. \tag{8}$$

Next, the exact sequence $0 \to \ker e^0 \to M \to \operatorname{im} e^0 \to 0$ induces the long exact sequence

$$\begin{aligned} 0 \to H^0_{\mathfrak{m}}(\ker e^0) \to H^0_{\mathfrak{m}}(M) \to H^0_{\mathfrak{m}}(\operatorname{im} e^0) \\ \to H^1_{\mathfrak{m}}(\ker e^0) \to H^1_{\mathfrak{m}}(M) \to H^1_{\mathfrak{m}}(\operatorname{im} e^0) \to \ldots, \end{aligned}$$

in which $H^0_{\mathfrak{m}}(\operatorname{im} e^0) = 0$ (by (1)) and $H^j_{\mathfrak{m}}(\ker e^0) = 0$ for all $j \in \mathbb{N}$ (by (5)). Therefore $\ker e^0 \cong H^0_{\mathfrak{m}}(\ker e^0) \cong H^0_{\mathfrak{m}}(M)$ and $H^j_{\mathfrak{m}}(\operatorname{im} e^0) \cong H^j_{\mathfrak{m}}(M)$ for all $j \in \mathbb{N}$. We now use (8) to complete the proof.

We can now combine (2·3) and (2·4) to obtain characterizations, in terms of modules of generalized fractions, of generalized Cohen–Macaulay modules and quasi-Buchsbaum modules. (See (1·8) for definitions of these concepts.)

(2·5) THEOREM. (*M is a non-zero finitely generated A-module of dimension $n \geqslant 1$.*)

(i) *We have that M is a generalized Cohen–Macaulay A-module if and only if there exists $r \in \mathbb{N}$ such that*

$$\mathfrak{m}^r(\ker e^i / \operatorname{im} e^{i-1}) = 0 \quad \textit{for all} \quad i = 0, \ldots, n-1,$$

that is such that all the cohomology modules of the complex $C(\mathscr{V}, M)$ are annihilated by $\mathfrak{m}^r$.

(ii) *Likewise, M is a quasi-Buchsbaum A-module if and only if all the cohomology modules of the complex $C(\mathscr{V}, M)$ are annihilated by $\mathfrak{m}$.*

(2·6) *Remarks.* (i) Note that, in (2·5), $\mathscr{V}$ can be any chain of triangular subsets on A that satisfies the conditions (1·4)(i), (ii). In particular, $\mathscr{V}$ could be $\mathscr{U}(M)$, or $\mathscr{V}$ could be $\mathscr{U}(x)$ where $x = (x_1, \ldots, x_n)$ for a fixed s.o.p. $x_1, \ldots, x_n$ for M. Indeed, we can deduce from (2·5)(i) and (2·4) that, for $r \in \mathbb{N}$, $\mathfrak{m}^r$ annihilates all the cohomology modules of the complex $C(\mathscr{V}, M)$ if and only if $\mathfrak{m}^r$ annihilates all the cohomology modules of the complex $C(\mathscr{U}(M), M)$.

(ii) Entirely similar comments apply in the case of quasi-Buchsbaum modules and (2·5)(ii).

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(iii) The above two comments immediately raise a question about (2·3). That result shows that we can characterize f-modules in terms of the complex $C(\mathscr{U}(M), M)$, for M is an f-module if and only if all the cohomology modules of the complex $C(\mathscr{U}(M), M)$ have support contained in $\{\mathfrak{m}\}$. An obvious question is whether this always remains true if we replace $\mathscr{U}(M)$ by a general $\mathscr{V}$ as described in (1·4).

To see that this is not the case, consider the example $B = C_{\mathfrak{n}}/\mathfrak{a}C_{\mathfrak{n}}$, where C is the polynomial ring $K[X_1, X_2, X_3]$ in indeterminates X_1, X_2, X_3 with coefficients in a field K, where $\mathfrak{n} = CX_1 + CX_2 + CX_3$ and where $\mathfrak{a} = CX_1 \cap (CX_1^2 + CX_2)$. Use x_i to denote the natural image of X_i in B (for $i = 1, 2, 3$). Then it is not difficult to see that the s.o.p. x_2, x_3 for B is not a filter-B-regular sequence, so that B is not an f-ring, but that, for all $n_2, n_3 \in \mathbb{N}$, $x_3^{n_3}$, $x_2^{n_2}$ is a filter-B-regular sequence, so that, by (2·3), the complex $C(\mathscr{U}((x_3, x_2)), B)$ does have the property that all its cohomology modules have support contained in $\{\mathfrak{m}\}$.

(iv) Suppose that M is a generalized Cohen–Macaulay module. Then the constant $C(M)$ of ([1], Satz (3·7)) can be described in terms of lengths of cohomology modules of the complex $C(\mathscr{V}, M)$. This follows from (2·4) and the definition of $C(M)$.

In particular, if A itself is a Buchsbaum local ring, then its invariant $I(A)$ ([4], Satz 2) may be described in terms of vector space dimensions (over $A/\mathfrak{m}$) of cohomology modules of the complex $C(\mathscr{U}(A), A)$, or of cohomology modules of the complex $C(\mathscr{U}(x), A)$, where $x = (x_1, \ldots, x_h)$ for a fixed s.o.p. $x_1, \ldots, x_h$ for A.

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