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Partitioning Two-Coloured Complete Graphs into Two Monochromatic Cycles

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We prove that there exists n_0, such that, for every $n \geqslant n_0$ and every 2-colouring of the edges of the complete graph K_n, one can find two vertex-disjoint monochromatic cycles of different colours which cover all vertices of K_n.

1. Introduction

In this paper, we are concerned with a conjecture of J. Lehel, who asked whether the vertex set of the complete graph K_n, whose edges are coloured with red and blue, can be partitioned into two cycles of different colours, that is, whether there exist a red cycle C^r and a blue cycle C^b such that $V(C^r) \cap V(C^b) = \emptyset$ and $V(K_n) = V(C^r) \cup V(C^b)$. (In order to state our results in a compact form, here and below we treat edges, vertices, and the empty set as cycles of length two, one, and zero, respectively. Thus, in particular, a vertex and the empty set can be viewed as both red and blue cycles at the same time.) Although

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Lehel's conjecture has inspired many results concerning the case when more than two colours are used (see [2], [4] and [5] for some of them), it has been settled only for some special classes of colourings (see [1]). The most important observation concerning this problem has been made by Gyárfás, who proved in [3] that one can cover each such 2-coloured K_n by two cycles of different colours which have at most one vertex in common (see Lemma 4.1 below). The main result of this paper states that Lehel's conjecture is true, provided the order of the graph is large enough.

Theorem 1.1. *There exists n_0 such that, for every $n \geqslant n_0$, and every 2-colouring of the edges of K_n, there exists a partition of the vertices of K_n into two monochromatic cycles of different colours.*

The structure of this paper is as follows. In Section 2 we recall the notion of a regular pair and the statement of the Regularity Lemma, which is crucial for our argument. In Section 3 we give some results on the Hamiltonicity of graphs, the second main ingredient of our proof. In Section 4 we study the case when there exists a large complete bipartite graph in one of the colours: it then turns out that Theorem 1.1 follows from somewhat lengthy, but elementary, considerations (Lemma 4.2). Finally, the last section of the paper contains the proof of Theorem 1.1.

2. Regularity Lemma

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For every $A, B \subseteq V(G)$, we denote by $e(A,B) = e_G(A,B)$ the number of edges $\{v,w\}$ with $v \in A$ and $w \in B$. We say that a pair (A,B), where $A, B \subseteq V(G)$ and $A \cap B = \emptyset$, is (ϵ, G)-*regular* for some positive constant ϵ, if, for every $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geqslant \epsilon|A|$ and $|B'| \geqslant \epsilon|B|$, we have

$$\left| \frac{e(A',B')}{|A'||B'|} - \frac{e(A,B)}{|A||B|} \right| < \epsilon.$$

A partition $\Pi = (V_i)_{i=0}^k$ of the vertex set $V(G)$ of G is (ϵ,k)-*equitable* if $|V_0| \leqslant \epsilon|V(G)|$ and $|V_1| = \cdots = |V_k|$. An (ϵ,k)-equitable partition $\Pi = (V_i)_0^k$ is (ϵ,k,G)-*regular*, or simply (ϵ,G)-*regular*, if at most $\epsilon\binom{k}{2}$ pairs (V_i,V_j) with $1 \leqslant i < j \leqslant k$ are not (ϵ,G)-regular. In this paper, we use the Regularity Lemma in its standard form as given in [8] (for an extensive survey on different variants of the Regularity Lemma and its applications, see [7]).

Lemma 2.1. *For any given $\epsilon > 0$ and a natural number k_0 there exists $K_0 = K_0(\epsilon,k_0) \geqslant k_0$ such that every graph G with at least k_0 vertices admits an (ϵ,k,G)-regular partition $\Pi = (V_0, V_1, \ldots, V_k)$ of its vertex set for some $k_0 \leqslant k \leqslant K_0$.*

Let us remark that an (ϵ,G)-regular pair is at the same time (ϵ,G^c)-regular, where G^c is the complement of G. Thus, any (ϵ,k,G)-regular partition is obviously (ϵ,k,G^c)-regular as well.

Finally, we observe that every regular pair contains a large bipartite subgraph in which

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all degrees are not much smaller than the average; more precisely, the following fact is a straightforward consequence of the definition of a regular pair (see also [7]).

Fact 2.2. *Let $0 < \epsilon < 0.1$, and let (A, B), where $A \cap B = \emptyset$ and $|A| = |B|$, be an (ϵ, G)-regular pair of subsets of a graph G. Then there exist $A' \subseteq A$ and $B' \subseteq B$ such that:*

(i) $|A'|, |B'| \geqslant (1 - \epsilon)|A|$,

(ii) (A', B') *is* $(2\epsilon, G)$*-regular,*

(iii) *the bipartite graph spanned in G by the pair (A', B') has minimum degree larger than $(1 - 2\epsilon)e_G(A, B)/|A|$.*

3. Hamilton cycles and paths

Let us recall that the classical theorem of Dirac states that every graph on n vertices, with minimum degree at least $n/2$, is Hamiltonian. However, in our considerations, we shall deal mainly with bipartite graphs, which typically do not meet this sufficient condition. It turns out, however, that a bipartite graph G with a bipartition (V_1, V_2), $|V_1| = |V_2| = m$, is Hamiltonian provided that its minimum degree is large enough and edges between V_1 and V_2 are distributed in a 'uniform' way. In particular, a very special case of the Blow-up Lemma [6] (see also [7]) states that if such a bipartite graph G has large minimum degree and, furthermore, the pair (V_1, V_2) is (ϵ, G)-regular for some sufficiently small $\epsilon > 0$, then G is Hamiltonian. Here, however, we shall use Haxell's result from [5], which states that G is Hamiltonian under much weaker 'uniformity' assumptions.

Lemma 3.1. *Let $0 < \epsilon < 1/7$ be a constant. Furthermore, let G be a bipartite graph with bipartition (V_1, V_2) such that $|V_1| = |V_2| = m$, the minimum degree of G is at least $7\epsilon m$, and every pair of subsets $A \subseteq V_1$ and $B \subseteq V_2$, such that $|A|, |B| \geqslant \epsilon m$, is joined by an edge. Then G is Hamiltonian.* □

For our argument, we shall need a consequence of Lemma 3.1 which states that, given a bipartite graph G with uniformly distributed edges and a family of vertex-disjoint paths $\hat{\mathscr{P}}$, one can find in G a Hamilton path with ends at prescribed vertices v_1 and v_2 which, moreover, contains all edges of paths from $\hat{\mathscr{P}}$, provided that G is uniform enough, the family $\hat{\mathscr{P}}$ is small, and a natural necessary 'parity' condition holds. Unfortunately, although the proof of Lemma 3.2 is not too complicated, its precise statement is quite technical.

Lemma 3.2. *Let G be a graph that contains:*

(i) *a pair (V', V'') of disjoint subsets of vertices of G such that each pair (W', W''), where $W' \subseteq V'$, $W'' \subseteq V''$ and $|W'| \geqslant 10^{-6}|V'|$, $|W''| \geqslant 10^{-6}|V''|$, is joined by at least one edge of G, each vertex $v' \in V'$ has at least $0.15|V''|$ neighbours in V'', and each $v'' \in V''$ is joined to at least $0.15|V'|$ vertices of V',*

(ii) *a family $\mathscr{P}'$ of r' vertex-disjoint paths, such that each path $P' \in \mathscr{P}'$ starts and ends in V' and no internal vertex of P' belongs to $V' \cup V''$,*

(iii) *a family $\mathscr{P}''$ of r'' vertex-disjoint paths, such that each path $P'' \in \mathscr{P}''$ has both ends in V'' and no internal vertex of P'' belongs to $V' \cup V'' \cup \bigcup_{P' \in \mathscr{P}'} V(P')$,*

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(iv) *vertices* x' *and* y'' *such that* x' *belongs to* $V' \setminus \bigcup_{P' \in \mathscr{P}'} V(P')$, *while* y'' *is a vertex from* $V'' \setminus \bigcup_{P'' \in \mathscr{P}''} V(P'')$.

Furthermore, let us assume that

$$r' + r'' \leqslant 0.01m = 0.01 \min\{|V'|, |V''|\} \tag{3.1}$$

and

$$r'' - r' = |V''| - |V'|. \tag{3.2}$$

Then G *contains a path that starts at* x', *ends in* y'', *and goes through all edges of paths from* $\mathscr{P}'$ *and* $\mathscr{P}''$ *and all vertices of* V' *and* V''.

Proof. Clearly, we may and shall assume that G contains no edges besides those joining V', V'' and those belonging to paths from $\mathscr{P}'$ and $\mathscr{P}''$. Note that because of (i) each pair of vertices taken from either V' or V'' is joined by at least $0.14m$ internally vertex-disjoint paths of length three and any two vertices of V' (or V'') are joined by at least $0.14m$ internally vertex-disjoint paths of length four. Thus, one can greedily construct a path $P_{x'z''}$ such that $P_{x'z''}$ starts at x', ends in some $z'' \in V''$, contains all edges of paths from $\mathscr{P}'$ and $\mathscr{P}''$, and, finally, uses not too many vertices from V' and V'', that is,

$$|V(P_{x'z''}) \cap V'| \leqslant 0.04m \quad \text{and} \quad |V(P_{x'z''}) \cap V''| \leqslant 0.04m.$$

Moreover, (3.2) implies that

$$|V' \setminus V(P_{x'z''})| = |V'' \setminus V(P_{x'z''})|.$$

Now let $P_{y''z'}$ be another path on $\lfloor 0.04m \rfloor$ vertices that starts at y'', ends in some $z' \in V'$, $P_{y''z'}$ is vertex-disjoint with $P_{x'z''}$, and z' has at least $0.001|V''|$ neighbours in $V(P_{y''z'})$. One can easily construct such a path; it is enough to build greedily a path $P_{y''z'_0}$ of length $\lfloor 0.04m \rfloor - 2$ which starts at y'', and observe that at least $0.002|V''|$ among the vertices from $V'' \setminus V(P_{x'z''})$ have more than $0.001|V''|$ neighbours in $V(P_{y''z'_0}) \cup V''$, so one can use property (i) to add to $P_{y''z'_0}$ two more edges. Denote by S'' the set of neighbours of z' in $V(P_{y''z'}) \cup V''$ and let S' be the set of all vertices from $V(P_{y''z'}) \cup V'$ which succeed in $P_{y''z'}$ the vertices from S'. Then $|S'| = |S''| \geqslant 0.001m$. More importantly, for each $s' \in S'$, the graph G contains a path $P_{y''s'}$ that starts at y'', ends at s', and has precisely the same vertex set as $P_{y''z'}$ (in order to obtain $P_{y''z'}$ it is enough to add to $P_{y''z'}$ the edge that joins z' with the predecessor s'' of s' and remove from it the edge $\{s', s''\}$).

Now it is not hard to complete the proof. Note first that the bipartite graph spanned in G by $V' \setminus (V(P_{x'z''}) \cup V(P_{y''z'})$ and $V'' \setminus (V(P_{x'z''}) \cup V(P_{y''z'})$ fulfils the assumptions of Lemma 3.1, and thus contains a Hamilton cycle C. The vertex z'' has in C at least $0.004m$ neighbours; thus there exists a set $T'' \in V''$ such that $|T''| \geqslant 0.004m$, and for every $t'' \in T''$ there exists a path $P_{z''t''}$ from z' to t'' with vertex set $V(P_{z''t''}) = \{z''\} \cup V(C)$. Furthermore, due to (i), there exists at least one edge, say $\{s', t''\}$, between sets S' and T''. Thus, merging the paths $P_{x'z''}$, $P_{z''t''}$, the edge $\{s', t''\}$ and the path $P_{y''s'}$, we get the path $P_{x'y''}$ for which we are looking. □

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4. Colourings with large monochromatic complete bipartite graphs

In this part of the paper we prove that Theorem 1.1 is true for a special case of 2-colourings of K_n, when there exists a large complete bipartite graph in one of the colours. In our argument, we employ a simple consequence of this result, stated below as Fact 4.3, which says that it is enough to show Theorem 1.1 for colourings in which each pair of large sets is connected by a short path of a prescribed colour.

In the proof we shall use Gyárfás's result on monochromatic cycles, which we have already mentioned in the Introduction.

Lemma 4.1. *For every colouring of the edges of K_n with red and blue, there exist two cycles C^r and C^b such that all edges of C^r are red, all edges of C^b are coloured blue, $V(C^r) \cup V(C^b) = V(K_n)$ and $|V(C^r) \cap V(C^b)| \leqslant 1$.* □

The result we shall need goes as follows.

Lemma 4.2. *Suppose that the edges of K_n are coloured with red and blue in such a way that the vertices of K_n can be partitioned into three sets V_1, V_2, and V_3 for which the following holds:*

(i) *all edges between V_1 and V_2 are blue,*
(ii) $\min\{|V_1|, |V_2|\} \geqslant 5 + 2|V_3|$.

Then, the vertex set of K_n can be covered by two vertex-disjoint cycles C^r and C^b such that all edges of C^r are red and all edges of C^b are coloured blue.

Proof. Let us assume that V_1, V_2, and V_3 fulfil (i) and (ii). Since (i) and (ii) provide no information on the colour of the edges between $V_1 \cup V_2$ and V_3, we shall construct a new partition $V_1' \subseteq V_1$, $V_2' \subseteq V_2$, $V_3' \subseteq V_3$ and V_4', such that all edges between $V_1' \cup V_2'$ and V_3' are red and the additional set V_4' contains a blue path coming through all its vertices which can be easily dealt with (it will be treated as a 'long' edge between V_1' and V_2').

Thus, define recursively sets $V_1(k)$, $V_2(k)$, $V_3(k)$ and $V_4(k)$ for $k \geqslant 0$ in the following way. Set $V_i(0) = V_i$ for $i = 1, 2, 3$, and $V_4(0) = \emptyset$. Having constructed $V_i(k)$ for some $k \geqslant 0$, check whether all edges between $V_3(k)$ and $V_1(k) \cup V_2(k)$ are red. If this is the case, stop the procedure; if not, define sets $V_i(k+1)$, $i = 1, 2, 3, 4$, according to the following rules.

Case 1: *there is a vertex $w_3 \in V_3(k)$ that has at least two blue neighbours, say w_1, w_1', in $V_1(k)$.*
Then choose a vertex $w_2 \in V_2(k)$ and put $V_1(k+1) = V_1(k) \setminus \{w_1, w_1'\}$, $V_2(k+1) = V_2(k) \setminus \{w_2\}$, $V_3(k+1) = V_3(k) \setminus \{w_3\}$ and $V_4(k+1) = V_4(k) \cup \{w_1, w_1', w_2, w_3\}$.

Case 2: *there is a vertex $w_3 \in V_3(k)$ that has at least two blue neighbours, say w_2 and w_2', in $V_2(k)$.*
Then, as in the previous case, pick $w_1 \in V_1(k)$ and move vertices w_1, w_2, w_2' and w_3 to $V_4(k+1)$.

Case 3: *there exists $w_3 \in V_3(k)$ with one blue neighbour in $V_1(k)$ and/or one blue neighbour in $V_2(k)$.*

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Let w_1, w_2 denote the blue neighbours of w_3 in $V_1(k)$ and $V_2(k)$, respectively (if w_3 has no blue neighbours in $V_i(k)$ for some $i = 1, 2$, take as w_i any vertex from $V_i(k)$). Then, put $V_1(k+1) = V_1(k) \setminus \{w_1\}$, $V_2(k+1) = V_2(k) \setminus \{w_2\}$, $V_3(k+1) = V_3(k)$ and $V_4(k+1) = V_4(k) \cup \{w_1, w_2, \}$.

Note that, in each step of the above procedure, the number of vertices in $V_3(k)$ that have blue neighbours in $V_1(k) \cup V_2(k)$ drops down by at least one. Thus the process must terminate after $\ell \leqslant |V_3|$ steps. One can easily check that the resulting sets $V_i' = V_i(\ell)$, $i = 1, 2, 3, 4$, where, possibly, sets V_3 and V_4 are empty, form a partition of vertices of K_n such that:

(i′) all edges between V_1' and V_2' are blue,
(ii′) $\min\{|V_1'|, |V_2'|\} \geqslant 5 + |V_3'|$,
(iii′) all edges between V_3' and $V_1' \cup V_2'$ are red,
(iv′) there exists a blue path $v_1 v_2 \dots v_m$ that consists of all vertices of V_4' such that all edges between v_1 and V_1' and all edges between v_m and V_2' are coloured blue.

Thus, suppose that V_1', V_2', V_3', V_4' fulfil (i′)–(iv′), and let $n_i = |V_i'|$ for $i = 1, 2, 3$, where $n_1 \geqslant n_2$. We split our argument into several cases and subcases.

Case 1: $n_1 - n_2 \leqslant n_3$.
If n_3 is of the same parity as $n_1 - n_2$, choose subsets $W_1' \subseteq V_1'$ and $W_2' \subseteq V_2'$ such that $|W_1'| = (n_1 - n_2 + n_3)/2$ and $|W_2'| = (n_3 - n_1 + n_2)/2$. Note that $|W_1' \cup W_2'| = |V_3'|$, so the subgraph spanned by $W_1' \cup W_2' \cup V_3'$ contains a spanning complete balanced bipartite graph with all edges coloured red, and thus one can find in it a red Hamilton cycle. Similarly, the graph spanned by sets $V_1' \setminus W_1'$, $V_2' \setminus W_2'$ and V_4' contains a Hamilton cycle with all edges coloured blue.

If $n_3 - n_1 + n_2$ is odd, then one of two possibilities holds. When there exists a blue edge $\{v_1, v_1'\} \subseteq V_1'$, one can choose $v_2 \in V_2'$, replace V_1' by $V_1' \setminus \{v_1, v_1'\}$, V_2' by $V_2' \setminus \{v_2\}$, and V_4' by $V_4' \cup \{v_1, v_1', v_2\}$ and argue as in the previous case. If all edges contained in V_1' are red, pick $W_1' \subseteq V_1'$ and $W_2' \subseteq V_2'$ such that $|W_1'| = (n_1 - n_2 + n_3 + 1)/2$ and $|W_2'| = (n_3 - n_1 + n_2 + 1)/2$ and find a blue cycle coming through all vertices from $(V_1' \setminus W_1') \cup (V_2' \setminus W_2') \cup V_4'$ (note that $|V_1' \setminus W_1'| = |V_2' \setminus W_2'|$) and a red Hamilton cycle in the graph spanned by $W_1' \cup W_2' \cup V_3'$.

Remark. Note that the above argument works whenever sets $V_1' \setminus W_1'$ and $V_2' \setminus W_2'$ have at least one element each. Thus Case 1 also remains valid when 5 in (ii′) is replaced by 3.

Case 2: $n_1 - n_2 > n_3$.
We consider a few subcases according to the structure of the graph R_1' induced by the red edges contained in V_1'.

Subcase 2a: *R_1' contains a red cycle C_1' of length $n_1 - n_2$.*
Then there exists a red cycle with vertex set $V(C_1') \cup V_3'$ and a blue Hamilton cycle in the graph spanned by $(V_1' \setminus V(C_1')) \cup V_2' \cup V_4'$.

Subcase 2b: *R_1' contains no red cycles of length larger than $n_1 - n_2$.*
From Lemma 4.1 applied to the subgraph spanned by V_1', it follows that V_1' can be partitioned into two sets $\bar{W}^r$ and $\bar{W}^b$, such that there exists a red cycle coming through

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all vertices of $\bar{W}^r$ and a blue path containing all vertices of $\bar{W}^b$. Thus one can find a partition W^r, W^b of V_1' such that:

- $|W^r| \geqslant n_3$ and either $|W^r| = n_3$, or the red subgraph spanned by W^r is Hamiltonian,
- $|W^b| \geqslant n_2$ and the blue subgraph spanned by W^b contains a Hamilton path.

Now it is enough to note that there is a blue cycle coming through all vertices of $W^b \cup V_2' \cup V_4'$ and a red one whose vertex set is $W^r \cup V_3'$.

Subcase 2c: *R_1' contains red cycles longer than $n_1 - n_2$, but none of them has length $n_1 - n_2$.* Note that, if the graph spanned by V_1' contains two vertex-disjoint blue paths P_1^b and P_2^b such that

$$|V(P_1^b)| + |V(P_2^b)| = n_1 - n_2 + 2, \tag{4.1}$$

then we can pick two vertices $v_2, v_2' \in V_2'$ and replace V_1' by $V_1' \setminus (V(P_1^b) \cup V(P_2^b))$, V_2' by $V_2' \setminus \{v_2, v_2'\}$ and V_4' by $V_4' \cup V(P_1^b) \cup V(P_2^b) \cup \{v_2, v_2'\}$. For such modified sets, we have $|V_1'| = |V_2'|$, so one can argue as in Case 1. (Note that the value of $\min\{n_1, n_2\}$ decreases by two but, as we have observed in the Remark made after Case 1, the argument still holds.) Thus, in order to show Lemma 4.2, it is enough to find in V_1' two vertex-disjoint blue paths P_1^b and P_2^b of lengths as given by (4.1).

Let $C_1' = v_1 v_2 \dots v_m v_1$ be a red cycle that has the minimum length among all red cycles longer than $n_1 - n_2$ contained in R_1'. In particular, R_1' contains no red cycles on $m - 1$ vertices. Note that, since we view a single edge as a cycle of length two, we must have $m \geqslant 4$. The choice of C_1' implies that, for every $i = 1, 2, \dots, m$, each edge $\{v_i, v_{i+2}\}$ is blue, where the addition is taken modulo m. Hence, vertices of C_1' can be covered by one (if m is odd) or two (if m is even) blue cycles. If $m \geqslant n_1 - n_2 + 2$, this fact already implies (4.1); thus let us assume that $m = n_1 - n_2 + 1$. Since $n_2 \geqslant 2$, there exists $v_1' \in V_1'$ which does not belong to C_1'. Furthermore, at least one edge joining v_1' and $V(C_1')$, say $\{v_1', v_{i_0}\}$, $1 \leqslant i_0 \leqslant m$, must be blue, since otherwise R_1' would contain a red cycle of length $m - 1$, contradicting the choice of C_1'. Now, to complete the proof of Subcase 2c and Lemma 4.2, observe that in the graph induced by the blue edges

$$\{v_1', v_{i_0}\} \cup \{\{v_i, v_{i+2}\} : i = 1, 2, \dots, m\}$$

there exist two vertex-disjoint paths P_1^b and P_2^b for which (4.1) holds. □

In the proof of Theorem 1.1 we employ the following fact, which in a way complements Lemma 4.2.

Fact 4.3. *For every $k \geqslant 1$, $n \geqslant 84k^3$ and every colouring of the edges of K_n with red and blue, the following holds. Either $V(K_n)$ can be partitioned into sets V_1, V_2, V_3 that fulfil assumptions* (i) *and* (ii) *of Lemma 4.2, or for each family of $\ell \geqslant 2$ disjoint sets $S_1, \dots, S_\ell$, $T_1, \dots, T_\ell \subseteq V(K_n)$, where $|S_i|, |T_i| \geqslant n/2k$ for $i = 1, \dots, \ell$, there exists a family of vertex-disjoint red paths $P_1, P_2, \dots, P_\ell$, each of length at most $10k$, such that, for each $i = 1, 2, \dots, \ell$, the path P_i has one end in S_i and the other in T_i.*

Remark. Note that, besides the ends, P_i may have more common points with the set $\bigcup_{j=1}^{\ell}(S_i \cup T_i)$.

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Proof. Let $S_1, \dots, S_\ell, T_1, \dots, T_\ell \subseteq V(K_n)$ be mutually disjoint large vertex-disjoint sets and suppose that for some $j \leqslant \ell$ we have already found short red paths $P_1, P_2, \dots, P_{j-1}$, with all required properties. In order to construct P_j, put $W = \bigcup_{i=1}^{j-1} V(P_i)$ and consider the graph R induced by the red edges in $V(K_n) \setminus W$. Furthermore, let N_r (or N'_r) denote the set of all vertices that lie at a distance r in R from the set $S_j \setminus W$ (or $T_j \setminus W$, respectively). Now, if

$$\sum_{r=1}^{5k} |N_r| > n/2 \qquad \text{and} \qquad \sum_{r=1}^{5k} |N'_r| > n/2,$$

then clearly sets $S_j \setminus W$ and $T_j \setminus W$ are joined by a red path P_j of length at most $10k$. Thus, assume that, say, $\sum_{r=1}^{5k} |N_r| \leqslant n/2$. In such a case, there exists r_0, $1 \leqslant r_0 \leqslant 5k$, for which

$$|N_{r_0}| \leqslant \frac{n}{10k} \leqslant \frac{1}{2}\left(\frac{n}{2k} - 5\right) - 10k^2 \leqslant \frac{1}{2}\Big(\min_{1 \leqslant i \leqslant \ell}\{|S_i \setminus W|, |T_i \setminus W|\} - 5\Big) - |W|,$$

and so sets $V_3 = N_{r_0} \cup W$, $V_1 = (S_j \setminus W) \cup \bigcup_{r=1}^{r_0-1} N_r$, and $V_2 = V(K_n) \setminus (V_1 \cup V_3)$ fulfil assumptions (i) and (ii) of Lemma 4.2. □

5. Proof of Theorem 1.1

Let us suppose that the edges of K_n, where n is a large natural number, are coloured with red and blue. Throughout the proof, we denote the graph induced by the red edges by R, and its blue counterpart by B. Clearly, without loss of generality we may and shall assume that at least half of the edges are coloured red, that is, $|E(R)| \geqslant \frac{1}{2}\binom{n}{2}$. Furthermore, because of Lemma 4.2, we shall assume that B contains no large bipartite graphs as described in assumptions (i) and (ii) of this lemma.

Let $\epsilon > 0$ be a very small constant, and $k_0 = 2/\epsilon$ (for our argument it is enough to take, say, $\epsilon = 10^{-60}$). We apply Lemma 2.1 to R to obtain an (ϵ, k, R)-regular partition $\hat{\Pi} = (\hat{V}_0, \hat{V}_1, \dots, \hat{V}_{\hat{k}})$, where $\hat{k} \geqslant k_0$, and $\hat{k}$ is bounded from above by a constant that depends only on ϵ and k_0. Note that, since B is the complement of R, a pair $(\hat{V}_i, \hat{V}_j)$, $1 \leqslant i < j \leqslant \hat{k}$, is either both (ϵ, R)- and (ϵ, B)-regular (or, for short, *regular*), or neither (ϵ, R)- nor (ϵ, B)-regular (then we simply call it *non-regular*).

Let us modify first $\hat{\Pi}$ by appending to $\hat{V}_0$ all sets from $\hat{\Pi}$ that belong to more than $2\epsilon^{1/2}\hat{k}$ non-regular pairs. Then, since among pairs $(\hat{V}_i, \hat{V}_j)$, $1 \leqslant i < j \leqslant \hat{k}$, at most $\epsilon\hat{k}^2$ are non-regular, the number of sets in the partition drops down by at most $\epsilon^{1/2}\hat{k}$. Thus, we arrive at a partition $\bar{\Pi} = (\bar{V}_0, \bar{V}_1, \dots, \bar{V}_k)$, where $1/\epsilon \leqslant k \leqslant \hat{k}$, $|\bar{V}_0| \leqslant 2\epsilon^{1/2}n$, $\bar{\Pi}$ contains at most $2\epsilon k^2$ non-regular pairs $(\bar{V}_i, \bar{V}_j)$, $1 \leqslant i < j \leqslant k$, and no set $\bar{V}_i$ belongs to more than $2\epsilon^{1/2}k$ of them. We classify pairs $\{i, j\}$, $1 \leqslant i < j \leqslant k$, into three groups, calling such a pair

- *black* if $(\bar{V}_i, \bar{V}_j)$ is non-regular,
- *red* if $(\bar{V}_i, \bar{V}_j)$ is regular and $e_R(\bar{V}_i, \bar{V}_j) \geqslant 0.2|\bar{V}_i||\bar{V}_j|$,
- *blue* if $(\bar{V}_i, \bar{V}_j)$ is regular and $e_B(\bar{V}_i, \bar{V}_j) > 0.8|\bar{V}_i||\bar{V}_j|$.

Let Γ_R denote the graph with the vertex set $\{1, 2, \dots, k\}$ whose set of edges consist of red pairs. Since the number of black pairs is less than $\epsilon k^2 < 0.001k^2$, elementary calculation reveals that the number of red pairs is at least $0.44\binom{k}{2}$ and so Γ_R contains a matching

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of size at least $0.12k$. Let us suppose that the largest matching of Γ_R consists of pairs $\{2i-1, 2i\}$, where $i = 1, 2, \ldots, s$, for some $s \geqslant 0.12k$.

We shall split vertices of K_n into two disjoint sets V^r and V^b and find red and blue cycles coming through all vertices of V^r and V^b, respectively. We shall construct V^r and V^b by modifying sets of the partition $\bar{\Pi}$; roughly speaking, V^r will consist of appropriately updated sets $\bar{V}_1, \ldots, \bar{V}_{2s}$, while all other vertices will be put into set V^b.

We begin with a partition of vertices of K_n into three sets $\tilde{V}^r$, $\tilde{V}^b$, and a small set $\tilde{V}_0$ of 'troublesome' vertices. Thus, let us choose subsets $V_i' \subseteq \bar{V}_i$ for $i = 1, \ldots, 2s$, such that, for every $i = 1, \ldots, s$, each vertex $v \in V_{2i-1}'$ is connected by red edges to at least $0.19|V_{2i}'|$ vertices of V_{2i}' and each $w \in V_{2i}'$ has at least $0.19|V_{2i-1}'|$ red neighbours in V_{2i-1}'. Fact 2.2 guarantees that we can do it in such a way that $|V_i'| \geqslant (1-\epsilon)|\bar{V}_i|$ for $i = 1, \ldots, 2s$. Now we apply Fact 4.3 and construct a family of red paths $\mathscr{P} = \{P_1, \ldots, P_s\}$ such that P_i joins sets V_{2i}' and V_{2i+1}' for $i = 1, \ldots, s-1$, P_s connects V_1' and V_{2s}', and each of the paths from $\mathscr{P}$ has length at most $10k$. We mark the vertices of $\bigcup_{i=1}^{s} V(P_i)$ as *inactive* and refer to all other remaining vertices as *active* ones. Now we 'balance' all s pairs choosing $V_i \subseteq V_i'$ in such a way that $|V_i| \geqslant |V_i'| - 10k^2$ and, for each $i = 1, \ldots, s$, both sets V_{2i-1} and V_{2i} have the same number of active vertices. Note that Lemma 3.2, applied with $r' = r'' = 0$, ensures the existence of a red cycle with vertex set $\tilde{V}^r = \bigcup_{i=1}^{2s} V_i$; it is enough to apply Lemma 3.2 to each of the pairs $(V_{2i-1} \setminus \bigcup_{i=1}^{2s} V(P_i), V_{2i} \setminus \bigcup_{i=1}^{2s} V(P_i))$ with $r' = r'' = 0$ to find paths joining the ends of consecutive paths from $\mathscr{P}$ and then merge them using paths from $\mathscr{P}$.

In order to specify the set $\tilde{V}^b$, we consider two cases. If $k - 2s \leqslant \epsilon^{1/3}k$, we put $\tilde{V}^b = \emptyset$ and $\tilde{V}_0 = \bar{V}_0 \cup \bigcup_{i=1}^{2s}(V_i' \setminus V_i) \cup \bigcup_{i=2s+1}^{k} V_i'$. For $k - 2s > \epsilon^{1/3}k$, denote by W' the set of all vertices from $W = \bigcup_{i=2s+1}^{k} V_i'$ which has in W less than $0.75|W|$ blue neighbours. Note that each pair $\{i, j\}$, $2s+1 \leqslant i < j \leqslant k$, is either black and blue, and no i, $2s+1 \leqslant i \leqslant k$, is contained in more than $2\epsilon^{1/2}k$ black pairs. Thus, because of regularity properties of the partition, for $i = 2s+1, \ldots, k$, we have $|W' \cap V_i'| \leqslant \epsilon|V_i'|$ and so $|W'| \leqslant \epsilon|W|$. Put $V_i = V_i' \setminus W'$ for $i = 2s+1, \ldots, k$, and let $\tilde{V}^b = \bigcup_{i=2s+1}^{k} V_i$ and $\tilde{V}_0 = \bar{V}_0 \cup \bigcup_{i=1}^{2s}(V_i' \setminus V_i) \cup W'$. Thus each vertex from $\tilde{V}^b$ has at least $0.7|\tilde{V}^b|$ blue neighbours in this set and so Dirac's theorem implies that $\tilde{V}^b$ spans a blue cycle.

Thus we arrive at the partition $\tilde{V}^r$, $\tilde{V}^b$, and $\tilde{V}_0$, where $\tilde{V}^r$ and $\tilde{V}^b$ span red and blue cycles, respectively, and

$$|\tilde{V}_0| \leqslant \begin{cases} 2\epsilon^{1/3}n & \text{if} \quad \tilde{V}^b = \emptyset, \\ 4\epsilon^{1/2}n & \text{if} \quad \tilde{V}^b \neq \emptyset. \end{cases}$$

Now we may start the main part of the proof, where we try to append vertices from $\tilde{V}_0$ to red and blue cycles on $\tilde{V}^r$ and $\tilde{V}^b$, respectively. We shall do it in a twofold way. First we shall create a family of red paths that can be included to the red cycle. More precisely, for some pairs (V_{2i-1}, V_{2i}), $i = 1, \ldots, s$, we construct families of red paths $\mathscr{P}_i'$ and $\mathscr{P}_i''$ which fulfil the assumptions of Lemma 3.2, so we can modify the argument we used to find the red spanning cycle in $\tilde{V}^r$ to prove the existence of the red cycle which, moreover, contains all 'additional' paths from $\bigcup_{i=1}^{s}(\mathscr{P}_i' \cup \mathscr{P}_i'')$ (the main difficulty here will be to adjust the number of vertices in both sets of a pair in order to preserve the parity condition (3.2)). All vertices included in such red paths, which eventually will be contained in the red cycle,

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we mark inactive, and, in order to make the argument more transparent, we call them *pink*. In the procedure we also create another type of inactive vertices, *azure* ones, which will be included in the blue cycle. Here our strategy will depend on the size of $\tilde{V}^b$. If $\tilde{V}^b \neq \emptyset$, we arrange azure vertices into blue paths, such that each of them has both ends with more than $0.6|\tilde{V}^b|$ blue neighbours in $\tilde{V}^b$. It is not hard to see that such paths can be easily added to the blue cycle. If $\tilde{V}^b = \emptyset$, then the blue cycle for which we are looking will be a spanning cycle of some complete balanced (or 'almost balanced') blue bipartite graph.

Before making the above description rigorous, we introduce one more definition. We say that a set V_i of $\Pi = (V_0, V_1, \ldots, V_k)$ is *saturated* if the number of inactive vertices in V_i is larger than $0.01|V_i| - 8$, and *unsaturated* otherwise. We remark that in the process of appending vertices of $\tilde{V}_0$ to blue and red cycles we shall 'lose' at most eight active vertices from $\tilde{V}^r \cup \tilde{V}^b$ for a single vertex from $\tilde{V}_0$. Thus, during the procedure of handling vertices from $\tilde{V}_0$ we create not more than $8|\tilde{V}_0|$ inactive vertices altogether. Consequently, if $\tilde{V}_b = \emptyset$, then

$$1000|\tilde{V}_0| \leqslant 2000\epsilon^{1/3}n \leqslant 0.0001k(n/k), \tag{5.1}$$

and at each stage of this process at most $0.02k$ among sets V_i, $i = 1, \ldots, k$, contain more than $0.009n/k \leqslant 0.01|V_i| - 8$ inactive vertices. For $\tilde{V}^b \neq \emptyset$, (5.1) can be strengthened to

$$1000|\tilde{V}_0| \leqslant 4000\epsilon^{1/2}n \leqslant \epsilon^{2/5}n/250, \tag{5.2}$$

and so in this case the number of sets V_i, $i = 1, \ldots, k$, with more than $0.01|V_i| - 8$ inactive vertices can be bounded from above by $\epsilon^{5/2}k/2$.

We first consider the case when $\tilde{V}^b \neq \emptyset$. We recall that then $|\tilde{V}_0| \leqslant 4\epsilon^{1/2}n$ while $|\tilde{V}^b| = \left|\bigcup_{i=2s+1}^{k} V_i\right| \geqslant \epsilon^{1/3}n$. We say that, for some i, $1 \leqslant i \leqslant 2s$, a set V_i is *red-dominated* if there exists at least $\epsilon^{2/5}k$ red pairs $\{i, j\}$, where $2s + 1 \leqslant j \leqslant k$, and call V_i *blue-dominated* otherwise.

Let us start with a couple of important observations on red- and blue-dominated sets. Note first that, because of (5.2), at each moment of the algorithm, for each red-dominated V_i, one can find a red pair $\{i, j\}$, $2s + 1 \leqslant j \leqslant k$, such that the set V_j is unsaturated. Moreover, there exist two active vertices x_1, x_2 in V_j such that, for $i = 1, 2$, the vertex x_i has two red active neighbours, say y_i and z_i, in V_i, and the red paths $y_1x_1z_1$ and $y_2x_2z_2$ are vertex-disjoint. On the other hand, if V_i is blue-dominated, then all except at most $2\epsilon^{2/5}k$ pairs $\{i, j\}$, where $2s + 1 \leqslant j \leqslant k$, are blue. Hence Fact 2.2 and an elementary averaging argument show that at least half of the vertices of V_i have in $\tilde{V}^b$ at least $0.7|\tilde{V}^b|$ blue neighbours. Finally, note that, since the pairs $\{2i - 1, 2i\}$, $i = 1, \ldots, s$, form the largest matching in Γ_R, for each $i = 1, \ldots, s$, at most one from the sets V_{2i-1}, V_{2i} is red-dominated.

Now we may describe the algorithm of dealing with vertices of $\tilde{V}_0$. We shall handle vertices of $\tilde{V}_0$ one by one, in each stage marking one of them either pink or azure. Let v be a vertex of $\tilde{V}_0$ with which we are dealing at some stage of the algorithm. Let us assume first that there exists a pair (V_{2i-1}, V_{2i}), $i = 1, \ldots, s$, where both sets V_{2i-1} and V_{2i} are unsaturated, such that v has two red active neighbours, say w and w', in one set of pair, say V_{2i-1}. Then there are two possibilities.

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Case 1: *V_{2i} is red-dominated.*
Let x be an active vertex of an unsaturated set $V_j \subseteq \tilde{V}^b$ with two red active neighbours y, z in V_{2i}. Then make the vertices v, w, w', x, y and z pink and, while applying Lemma 3.2 to the pair (V_{2i-1}, V_{2i}), use it for the families $\mathscr{P}_i'$ and $\mathscr{P}_i''$ enlarged by the paths wvw' and yxz, respectively.

Case 2: *V_{2i} is blue-dominated.*
Choose a vertex $x \in V_{2i}$ that has at least $0.7|\tilde{V}^b|$ blue neighbours in $\tilde{V}^b$. Mark the vertices of the red path wvw' pink and the vertex x azure.

Now suppose that v has at most one red neighbour in each of sets V_{2i-1}, V_{2i}, where $1 \leqslant i \leqslant s$ and both sets V_{2i-1} and V_{2i} remain unsaturated at this stage of the procedure. Then again we consider two cases.

Case 1: *both sets V_{2i-1} and V_{2i} are blue-dominated.*
Let $w \in V_{2i-1}$ and $w' \in V_{2i}$ be active blue neighbours of v such that both w and w' have at least $0.7|\tilde{V}^b|$ blue neighbours in $\tilde{V}^b$. Mark the vertices v, w and w' azure.

Case 2: *one of the sets V_{2i-1}, V_{2i}, say V_{2i-1}, is red-dominated while the other is blue-dominated.*
Let $w, w' \in V_{2i}$ be two active blue neighbours of v that have at least $0.7|\tilde{V}^b|$ blue neighbours in $\tilde{V}^b$ each. Furthermore, let x_1, x_2 be two active vertices of an unsaturated set $V_j \subseteq \tilde{V}^b$ for which there exist active vertices $y_1, y_2, z_1, z_2 \in V_{2i-1}$ such that the paths $y_1x_1z_1$ and $y_2x_2z_2$ are red and vertex-disjoint. Colour the vertices v, w and w' azure and the vertices x_1, x_2, y_1, y_2, z_1 and z_2 pink.

It is not hard to see that at the end of this procedure all vertices of $\tilde{V}^r$ that are not azure and all pink vertices from $\tilde{V}_0$ and $\tilde{V}^b$ lie at a red cycle, and there is a blue cycle consisting of all vertices of $\tilde{V}^b$ not marked pink and all vertices of $\tilde{V}_0$ and $\tilde{V}^r$ that are azure. Indeed, the existence of the red cycle follows from Lemma 3.2 applied to each pair (V_{2i-1}, V_{2i}), $i = 1, \ldots, s$, from which we removed all azure vertices and all internal vertices of the paths from $\mathscr{P}$. On the other hand, the number of pink vertices in $\tilde{V}^b$ is smaller than $|\tilde{V}_0| \leqslant 0.01\tilde{V}^b$, and so each vertex from $v \in \tilde{V}^b$ has more than $0.6|\tilde{V}^b|$ blue neighbours in $\tilde{V}^b$ that are not pink. Thus, from Dirac's theorem it follows that there exists a blue cycle $\tilde{C}^b$ which comes through all vertices of $\tilde{V}^b$ that are not coloured pink. Furthermore, during the procedure we have created at most $|\tilde{V}_0| \leqslant 0.01|\tilde{V}^b|$ blue paths of length either one or three, the ends of which are joined by blue edges to at least $0.6|\tilde{C}^b|$ of the vertices of $\tilde{C}^b$. One can easily see that all these paths can greedily be appended to $\tilde{C}^b$ one by one. This completes the proof of Theorem 1.1 in the case when $\tilde{V}^b \neq \emptyset$.

Now let us assume that $\tilde{V}^b = \emptyset$. In this case $\tilde{V}^b$ cannot be used as a reservoir of active vertices so, as we have already mentioned, we need to apply a slightly different strategy.

In order to simplify the description of the algorithm we assume first that the number of vertices in $\tilde{V}_0$ is even, and then say how to modify the argument in the case when $\tilde{V}_0$ contains an odd number of elements.

We split the process of dealing with vertices of $\tilde{V}_0$ into four phases.

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Phase 1: *appending vertices of $\tilde{V}_0$ in pairs.*
In the first period of the algorithm, instead of dealing with vertices of $\tilde{V}_0$ one by one, we append them to red paths in pairs. Thus, if there exists a pair of active vertices v_1, v_2 in $\tilde{V}_0$ and a pair (V_{2i-1}, V_{2i}), $i = 1, \ldots, s$, such that both sets V_{2i-1} and V_{2i} are unsaturated and, for $j = 1, 2$, the vertex v_j has two active red neighbours $w_j, z_j \in V_{2i+1-j}$, then mark the vertices of the red paths $w_1v_1z_1$ and $w_2v_2z_2$ pink. Examine unmarked vertices of $\tilde{V}_0$ and, if possible, repeat the above procedure.

Thus, Phase 1 ends when no pair of vertices $v_1, v_2 \in \tilde{V}_0$ can be appended to paths from families $\mathscr{P}_i'$ and $\mathscr{P}_i''$ associated with some pair (V_{2i-1}, V_{2i}), $i = 1, \ldots, s$. Let $V_0' \subseteq \tilde{V}_0$ denote the set of all vertices of $\tilde{V}_0$ that have not been marked pink. Note that, since we have appended two vertices from $\tilde{V}_0$ at a time, $|V_0'|$ and $|\tilde{V}_0|$ have the same parity, that is, $|V_0'| = 2\ell$ for some $\ell \geqslant 0$.

Phase 2: *creating unbalanced pairs of pink vertices.*
Let us suppose that there exists a vertex $v \in V_0'$ such that, for some unsaturated set V_i, where $1 \leqslant i \leqslant 2s$, v has two red active neighbours w and w' in V_i. Then colour pink the vertices of the red path wvw' and, to distinguish w, w' from the vertices we marked in the first phase of the algorithm, call $\{w, w'\}$ an *unbalanced* pink pair. Repeat this procedure until either ℓ vertices of V_0' are included into red paths, or each unmarked vertex from $\tilde{V}_0$ has at most one red active neighbour in each of the sets V_i, $i = 1, \ldots, 2s$, that is not saturated. Thus, during this phase ℓ' vertices from V_0' have been marked pink for some $0 \leqslant \ell' \leqslant \ell$ and, at the same time, we have created ℓ' unbalanced pink pairs in some of sets V_i, $i = 1, \ldots, 2s$. Furthermore, at the end of Phase 2, each of the $2\ell - \ell'$ unmarked vertices of V_0' has at most one red active neighbour in each of the set V_i, $i = 1, \ldots, 2s$, which has not yet been saturated.

Phase 3: *balancing unbalanced pink pairs.*
Let us denote the set of $2\ell - \ell'$ unmarked vertices of V_0' by V_0''. Let us also assume that all unbalanced pink pairs are contained in the sets $V_1, V_3, \ldots, V_{2s'-1}$, for some $0 \leqslant s' \leqslant s$, and that there are precisely r_i such unbalanced pairs in V_{2i-1}, for $i = 1, \ldots, s'$. Furthermore, let W_{2i}, $i = 1, \ldots, s'$, denote the set of all active vertices of V_{2i}. Since each W_{2i} contains at least $0.99|V_{2i}| > 98r_i$ active vertices and each vertex from V_0'' has at most one red neighbour among them, for each $i = 1, \ldots, s$, one can choose a set $W_{2i}' \subseteq W_{2i}$ such that

$$|W_{2i}'| \geqslant \max\{r_i, 2/\epsilon\} \tag{5.3}$$

and for each vertex v from V_0'' at most $0.02s'$ sets among $W_2', \ldots, W_{2s'}$ are connected to v by a red edge. Furthermore, from (5.3) it follows that in each W_{2i}' one can choose r_i vertices $w_1^{(i)}, \ldots, w_{r_i}^{(i)}$ such that each of them has less than $\epsilon|V_0''|$ red neighbours in V_0''. Mark azure all ℓ' vertices $w_j^{(i)}$, $i = 1, \ldots, s'$, $j = 1, \ldots, r_i$.

Phase 4: *the final stage.*
If $\ell' < \ell$, that is, when the number of the azure vertices generated in the previous phase is smaller than $|V_0''|$, we need to create additional $|V_0''| - \ell' = 2(\ell - \ell')$ azure vertices in $\bigcup_{i=1}^{2s} V_i$. Let V_{2i-1}, $i = 1, \ldots, s''$, denote those of the sets $V_1, \ldots, V_{2s-1}$ not yet

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saturated. Then, arguing as in the previous case, one can choose $2(\ell-\ell')$ vertices z_j, z'_j, $j=1,\dots,\ell-\ell'$, such that the following holds:

(i) for every $j=1,\dots,\ell-\ell'$, there exists i, $i=1,\dots,s''$, such that $z_j \in V_{2i-1}$ and $z'_j \in V_{2i}$,
(ii) every vertex z_j and z'_j, $j=1,\dots,\ell-\ell'$, has at most $\epsilon|V''_0|$ red neighbours in V''_0,
(iii) every vertex $v \in V''_0$ has at most $0.02|Z|$ red neighbours in the set $Z=\bigcup_{j=1}^{s''}(\{z_j,z'_j\})$,
(iv) after marking all vertices z_j and z'_j inactive, no set V_i, $i=1,\dots,2s''$, becomes saturated.

Now colour azure all vertices z_j, z'_j, $j=1,\dots,\ell-\ell'$.

One can easily see that there exists a red cycle which comes through all vertices of $\bigcup_{i=1}^{2s} V_i$ that are not marked azure and all vertices of $\tilde{V}_0$ that are coloured pink. Indeed, as before, apply Lemma 3.2 to each pair (V_{2i-1},V_{2i}) from which we removed the azure vertices and the internal vertices of the paths from $\mathscr{P}$ but added red paths created in Phases 1 and 2; then merge all paths obtained in this way by paths from family $\mathscr{P}$. Furthermore, all vertices of V''_0 and all vertices from $\bigcup_{i=1}^{2s} V_i$ marked azure span a balanced bipartite blue graph that has minimum degree $0.9|V''_0|$ and each azure vertex has at least $(1-\epsilon)|V''_0|$ neighbours in V''_0. Hence, Lemma 3.1 implies that there exists a blue cycle whose vertex set consists of V''_0 and all azure vertices of $\bigcup_{i=1}^{2s} V_i$. Consequently, if $\tilde{V}^b=\emptyset$, and $|\tilde{V}_0|$ is even, the assertion of Theorem 1.1 holds.

Thus, in order to complete the proof of Theorem 1.1 it is enough to consider the case when $|\tilde{V}_0|=2\ell-1$ for some natural $\ell \geqslant 1$. If one of the unsaturated sets V_{2i-1} (or V_{2i}), $i=1,\dots,s$, contains two active vertices v,v' joined by a red edge, then mark the vertices of the red path vv' pink, move any active vertex w from V_{2i} (or V_{2i-1}, respectively) to $\tilde{V}_0$ so that $|\tilde{V}_0 \cup \{w\}|$ is even and the previous argument applies. Hence, suppose that all edges between active vertices of the unsaturated set V_i, $i=1,\dots,2s$, are blue. Now apply the same algorithm as above in such a way that, during Phases 2 or 3, one of the sets V_{2i-1}, $i=1,\dots,s$, contains either at least two pairs of unbalanced pink vertices or at least two vertices marked as azure (one can easily do it by first handling sets V_{2i-1} which cannot be saturated in just one step of the algorithm, and, due to (5.1), there are a lot of such sets at each stage of the procedure). Thus, at least two of the vertices coloured azure in Phase 4 belong to the same set V_{2i} for some $i=1,\dots,s$. Consequently, although, unlike in the 'even' case, the bipartite blue graph consisting of V''_0 and ℓ azure vertices from $\bigcup_{i=1}^{s} V_i$ is not balanced (since $|V''_0|=\ell-1$), it contains at least one blue edge joining two azure vertices. Thus, to show that all vertices of V''_0 and all azure vertices lie at a blue cycle, instead of using Lemma 3.1 apply Lemma 3.2 with $r'=1$, $r''=0$, and x', y'' being the ends of a blue edge. □

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