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BLOWN-UP BOUNDARIES ASSOCIATED WITH AMPLE CONES OF K3 SURFACES

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Abstract. In this article, we apply the Bestvina–Mess type formula for relatively hyperbolic groups, which is established by Tomohiro Fukaya, to automorphism groups of K3 surfaces, and we show that the virtual cohomological dimension of automorphism groups of K3 surfaces is determined by the covering dimension of the blown-up boundaries associated with their ample cones.

§1. Introduction.

In this article, we discuss the classification problem of automorphism groups of K3 surfaces over $\mathbb{C}$.

There are two approaches to automorphisms of K3 surfaces. The first one is the classification under some finiteness conditions, such as the classification of finite subgroups, and that of possible finite order of automorphisms. For example, a well-known and important result in this approach is Mukai's result [15] Theorem (0.3), which says that symplectic finite subgroups of the automorphism groups of K3 surfaces are isomorphic to subgroups of the Mathieu group M_{23}, one of the sporadic simple groups.

The other approach is the application of Borcherds' Lattice Theory (often called Borcherds' method) to compute a generating set of infinite automorphism groups. The problem of finding a generating set of the automorphism groups of Kummer surfaces was first proposed by Klein. This problem took more than 100 years to be solved by Kondo [13] using Borcherds' method. Although this method is very useful and, in fact, the automorphism groups of some specific K3 surfaces have been calculated, this method is not applicable to the general case.

Shigeru Mukai recently proposed at the occasion of the Algebra Symposium in Tokyo in 2018, the following conjecture on discrete group theory and automorphism groups, from which one may hopefully find a general framework for dealing with infinite automorphism groups of elliptic K3 surfaces.

Conjecture 1. *Let X be an elliptic K3, Enriques or Coble surface, and let* $\mathrm{Aut}(X)$ *be the automorphism group of X. Then, the virtual cohomological dimension* $\mathrm{vcd}(\mathrm{Aut}(X))$ *of* $\mathrm{Aut}(X)$ *is equal to the maximum of the Mordell–Weil rank induced by all elliptic surface structures on X (for the definition of the virtual cohomological dimension of a group, see Definitions 4.3 and 4.5).*

Remark 1.1. The virtual cohomological dimension of $\mathrm{Aut}(X)$ is easily seen to be larger than the rank of the Mordell–Weil group of any elliptic fibration. Hence, the main point of Mukai's conjecture lies in finding the upper bound of $\mathrm{vcd}(\mathrm{Aut}(X))$ as stated therein.

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This conjecture could be a turning point from the conventional study of automorphism groups of elliptic K3, Enriques and Coble surfaces by computing their generators to that by computing their cohomological invariants. Our main result is a formula which determines the virtual cohomological dimension of the automorphism group of a K3 surface by the *covering dimension* of the *blown-up boundary* associated with the ample cone of the K3 surface (for the definitions of covering dimension and blown-up boundary, see Definitions 4.6 and 5.27, respectively).

THEOREM 1. *Let X be an algebraic K3 surface with its Picard rank $\rho(X) \geq 3$. If the symplectic automorphism group $\mathrm{Aut}_s(X)$ of X is non-elementary, then*

$$\mathrm{vcd}(\mathrm{Aut}(X)) = \dim \partial A_X^{bl} + 1,$$

where ∂A_X^{bl} is the blown-up boundary associated with the ample cone of X (see Theorem 5.35 for the detailed discussion).

To show this result, we focus on the natural action of automorphism groups on hyperbolic spaces, which comes from the action on the cohomology of degree 2. Then, we can consider the symplectic automorphism group of a K3 surface as a discrete subgroup of Möbius transformations of a hyperbolic space. Moreover, we can show the following theorem.

THEOREM 2. *Let X be an algebraic K3 surface with $\rho(X) \geq 3$, and let $\mathrm{Aut}_s(X)$ be the symplectic automorphism group of X. Then, $\mathrm{Aut}_s(X)$ is geometrically finite (for the definition of geometrically finite groups, see Definition 3.29).*

Bestvina and Mess [5] showed that the cohomological dimension of a torsion-free hyperbolic group is determined by the covering dimension of its Gromov boundary. Recently, Tomohiro Fukaya [10] generalized the formula for relatively hyperbolic groups satisfying a certain condition. Geometrically finite groups are typical examples of relative hyperbolic groups. We can immediately obtain Theorem 1 by applying Fukaya's result [10, Corollary B].

On the other hand, the paper of Fukaya is still unpublished at the time when the first draft of this article was written, and his theorem and his proof are stated using geometric group theory. For readability to algebraic geometers, this article also contains the result and the proof of Fukaya, which are simplified and modified to fit in with our situation, in §5.2, using the geometry of ample cones of K3 surfaces and hyperbolic geometry.

REMARK 1.2. After the author finished the first draft of this article, Tomohiro Fukaya informed the author that Theorem 2 has been independently obtained and proved by Kohei Kikuta. He showed geometrical finiteness for the automorphism groups of irreducible symplectic varieties over a field of characteristic zero, and his proof is different in some points from our proof given in this article. For more details, the reader is advised to refer to his paper [12].

An application of Theorem 1 is the case that the ample cones of K3 surfaces are *sphere packings* (see Definition 5.39 for sphere packings).

THEOREM 3. *Let X be an algebraic elliptic K3 surface with $\rho(X) \geq 3$. If ∂A_X is a connected sphere packing, and every elliptic divisor of X corresponds to the tangent point of some two spheres associated with (-2)-curves, then*

$$\mathrm{vcd}(\mathrm{Aut}(X)) = \max\{\mathrm{rk}\,\mathrm{MW}(f)\} = \rho(X) - 3,$$

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where f runs all elliptic fibrations of X and $\mathrm{rk}\,\mathrm{MW}(f)$ *is the rank of the Mordell–Weil group* $\mathrm{MW}(f)$ *associated with f.*

Baragar [2] constructed K3 surfaces whose ample cones are sphere packings. Then, we can compute the virtual cohomological dimension of their automorphism groups. Note that these K3 surfaces provide examples that affirm Conjecture 1, and, to the best of our knowledge, they are the first non-trivial examples in the sense that the verification of the conjecture does not rely on explicit calculation of the group structure of the automorphism groups.

Finally, using hyperbolic geometry, we construct counterexamples to Mukai's conjecture for elliptic K3 surfaces. These examples have the property that the virtual cohomological dimension of their automorphism group is one, while the rank of the Mordell–Weil group is zero for every elliptic fibration. For details, see the §6.2.

§2. Lattices and K3 surfaces.

In this section, we recall some definitions and facts about lattices and K3 surfaces.

2.1 Lattices.

Let L be a free abelian group of rank r, and let $\langle\ ,\ \rangle : L \times L \to \mathbb{Z}$ be a symmetric bilinear form. A symmetric bilinear form $\langle\ ,\ \rangle$ is said to be *non-degenerate* if $\langle x, y\rangle = 0$, for any $y \in L$ implies $x = 0$. We call a pair $(L, \langle\ ,\ \rangle)$ of L and a non-degenerate symmetric bilinear form a *lattice* of rank r. The dual $\mathrm{Hom}(L, \mathbb{Z})$ is denoted by L^*. Since L is non-degenerate, the natural map

$$L \to L^*, x \mapsto f_x \tag{2.1}$$

is injective, where f_x is the linear function on L defined by $f_x(y) \coloneqq \langle x, y\rangle$. A lattice L is said to be *unimodular* if the natural map (2.1) is an isomorphism. A lattice L is said to be *even* if $\langle x, x\rangle$ is an even integer for any $x \in L$. For an even lattice L, the quotient group L^*/L is denoted by A_L. An element x of L with $\langle x, x\rangle = -2$ is called a *root* of L.

A subgroup S of a lattice L is called a *sublattice* of L if S with the restriction of the bilinear form of L is a lattice. A sublattice S of L is said to be *primitive* if the quotient group L/S is torsion-free. The orthogonal complement of a sublattice S, denoted by $S^\perp$, is a primitive sublattice of L.

An isomorphism from L to itself is called an *automorphism* of L. The set of automorphisms of L is called the *orthogonal group* of L, and it is denoted by $O(L)$.

2.2 K3 surfaces.

Theorem 2.1 [14, Theorem 4.5]. *For a K3 surface X, the second cohomology group $H^2(X, \mathbb{Z})$ with the cup product is a unimodular even lattice with the signature $(3, 19)$. Then, $H^2(X, \mathbb{Z})$ is isomorphic to $U^{\oplus 3} \oplus E_8^{\oplus 2}$, where U is the hyperbolic lattice of rank 2.*

Definition 2.2. Let ω_X be a nowhere vanishing holomorphic 2-form of X. The *Néron–Severi lattice* $\mathrm{NS}(X)$ is defined by

$$\mathrm{NS}(X) \coloneqq \{x \in H^2(X, \mathbb{Z}) \ :\ \langle x, \omega_X\rangle = 0\} = H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{R}).$$

The Néron–Severi lattice $\mathrm{NS}(X)$ is a lattice by the cup product of signature $(1, \rho(X) - 1)$, where $\rho(X)$ is the Picard number of X. The *transcendental lattice* T_X of X is defined by $\mathrm{NS}(X)^\perp$.

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DEFINITION 2.3. Let X be a K3 surface. We define

$$\Delta(X) \coloneqq \{\delta \in \mathrm{NS}(X) : \delta \text{ is a root of } \mathrm{NS}(X)\}$$

and

$$\Delta^+(X) \coloneqq \{\delta \in \Delta(X) : \delta \text{ is effective}\}.$$

For $\delta \in \Delta^+(X)$, the *reflection* s_δ in $H^{1,1}(X,\mathbb{R})$ is defined by

$$s_\delta(x) = x + \langle x, \delta\rangle \delta \quad for \quad x \in H^{1,1}(X,\mathbb{R})$$

The *reflection group* $W(X)$ is the subgroup of $O(H^{1,1}(X,\mathbb{R}))$ generated by the set of reflections $\{s_\delta : \delta \in \Delta(X)\}$. The cone $P(X) = \{x \in H^{1,1}(X,\mathbb{R}) : \langle x, x\rangle > 0\}$ has two connected components and the one containing the Kähler class is denoted by $P^+(X)$ and called the *positive cone*.

DEFINITION 2.4. The group $W(X)$ acts on the space $P^+(X)$ in a canonical way. The *Kähler cone* of X is the one of the fundamental domains of $W(X)$ given by

$$D(X) = \{x \in P^+(X) : \langle x, \delta\rangle > 0 \text{ for any } \delta \in \Delta^+(X)\}.$$

DEFINITION 2.5. Let X be an algebraic K3 surface. The *ample cone* $A(X)$ is defined to be

$$A(X) = \{x \in \mathrm{NS}(X) \otimes \mathbb{R} \cap P^+(X) : \langle x, \delta\rangle > 0 \text{ for any } \delta \in \Delta^+(X)\}.$$

Let $f : X \to \mathbb{P}^1$ be an elliptic fibration of a K3 surface X, and let $j : J \to \mathbb{P}^1$ be the jacobian fibration of f. The sections of j generate a group $MW(f)$ that is called the Mordell–Weil group of the elliptic fibration $f : X \to \mathbb{P}^1$. The Mordell–Weil group $MW(f)$ is a finitely generated abelian group. We denote by $r(f)$ the rank of $MW(f)$.

The following formula is called Shioda–Tate formula.

LEMMA 2.6. *[[18], Formula (5.10); 9, Formula (4.3.2)] Let $f : X \to \mathbb{P}^1$ be an elliptic fibration of a K3 surface, and let $F_v = f^{-1}(t_v)$ $(1 \le v \le k)$ be the singular fibres of f. We denote by m_v the number of irreducible components of F_v. Then, the $r(f)$ is given by the following formula:*

$$r(f) = \rho(X) - 2 - \sum_{v=1}^{k} (m_v - 1).$$

§3. Hyperbolic geometry.

In this section, we recall some basic facts of n-dimensional hyperbolic spaces and discrete groups of isometries of n-dimensional hyperbolic spaces, which are contained in [17].

Notation

Let (X,d) be a metric space, and let r be a positive real number. Throughout this section, we denote by:

$B(a,r)$ the open ball of radius r centered at a point a of X;
$C(a,r)$ the closed ball of radius r centered at a point a of X;
$S(a,r)$ the sphere of radius r centerd at a point a of X;
$N(S,r)$ the r-neighborhood of a subset S of X;
$I(X)$ the group of isometries of X.

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We denote by S^n the n-dimensional unit sphere of $\mathbb{R}^{n+1}$. A vector e_i in $\mathbb{R}^n$ is the vector whose i-th entry is 1 and the other entries are 0.

In hyperbolic geometry, it is useful to use some models of hyperbolic spaces for different situations. Therefore, we begin with recalling the *hyperboloid model* of hyperbolic spaces. In §3.3 and §3.4 we recall the *conformal ball model*, and the *upper half-space model* of hyperbolic spaces.

3.1 The hyperboloid model.

DEFINITION 3.1. Let n be a non-negative integer. Let $x=(x_1,\dots,x_n,x_{n+1})$ and $y=(y_1,\dots,y_n,y_{n+1})$ be vectors in the $(n+1)$-dimensional real vector space $\mathbb{R}^{n+1}$. The *Lorentzian inner product* of x and y is defined by

$$x\circ y=x_1y_1+\cdots+x_ny_n-x_{n+1}y_{n+1}.$$

The *Lorentzian (n+1)-space* is the vector space $\mathbb{R}^{n+1}$ together with the Lorentzian inner product. A real $(n+1)\times(n+1)$ matrix A is said to be *Lorentzian* if for any $x,y\in\mathbb{R}^{n+1}$, $x\circ y=Ax\circ Ay$.

DEFINITION 3.2. The *hyperboloid model* H^n of hyperbolic n-spaces is defined to be the set $\{x\in\mathbb{R}^{n+1}:x\circ x=-1,\ x_{n+1}>0\}$. The *hyperbolic distance* $d_H(x,y)$ between x and y is defined by the equation

$$\cosh d_H(x,y)=-x\circ y.$$

By [17, Theorem 3.2.2], the hyperbolic distance d_H is a metric on H^n. By the definition of d_H, Lorentzian $(n+1)\times(n+1)$ matrices naturally act isometrically on H^n. We denote the group of isometries by $I(H^n)$.

DEFINITION 3.3. Let H^n be the hyperboloid model of n-dimensional hyperbolic spaces. The set of all Lorentzian $(n+1)\times(n+1)$ matrices with matrix multiplication is called the *Lorentz group* of $(n+1)\times(n+1)$ matrices, and is denoted by $O(n,1)$. The *positive Lorentz group* $O^+(n,1)$ is defined by

$$O^+(n,1)=\{A\in O(n,1):AH^n=H^n\}.$$

By [17, Theorem 3.2.3], there is an isomorphism from the positive Lorentz group $O^+(n,1)$ to the group of hyperbolic isometries $I(H^n)$.

DEFINITION 3.4. A *hyperbolic m-plane* of H^n is the nonempty intersection of H^n with an $(m+1)$-dimensional vector subspace of $\mathbb{R}^{n+1}$.

3.2 Möbius transformations.

To define the conformal ball model and the upper half-space model of hyperbolic n-spaces, we use the hyperboloid model and *Möbius transformation*. In this section, we recall the definition and some basic facts of Möbius transformation.

DEFINITION 3.5. Let E^n be an n-dimensional Euclidean space. Let a be a unit vector in E^n, and let t be a positive real number. Consider the hyperplane of E^n is defined by

$$P(a,t)=\{x\in E^n:a\cdot x=t\},$$

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where $a \cdot x$ is the Euclidean inner product of a and x. The *reflection* ρ of E^n in the plane $P(a,t)$ is defined by

$$\rho(x) = x + 2(t - a \cdot x)a.$$

Let a be a point in E^n, and let r be a positive real number. The *sphere* of E^n of radius r centered at a is defined by

$$S(a,r) = \{x \in E^n : |x-a| = r\},$$

where $|x-a|$ is the Euclidean norm of $x-a$. The *reflection* (or *inversion*) σ of E^n in the sphere $S(a,r)$ is defined by

$$\sigma(x) = a + \left(\frac{r}{|x-a|}\right)^2 (x-a).$$

Let ∞ be a point not in E^n. We define $\hat{E}^n = E^n \cup \{\infty\}$ whose topology is given by the one-point compactification of E^n.

DEFINITION 3.6. Let a be a unit vector in E^n, and let t be a positive real number. The *extended hyperplane* of $\hat{E}^n$ is defined by $\hat{P}(a,t) = P(a,t) \cup \{\infty\}$. Let ρ be the reflection of the hyperplane $P(a,t)$. Extend ρ to a bijection $\hat{\rho} : \hat{E}^n \to \hat{E}^n$ by setting $\hat{\rho}(\infty) = \infty$. In a similar way, the inversion σ of a Euclidean sphere $S(a,r)$ in E^n is extended to $\hat{\sigma} : \hat{E}^n \to \hat{E}^n$ by setting $\hat{\sigma}(\infty) = a$ and $\hat{\sigma}(a) = \infty$. The extended map $\hat{\rho}$ (resp. $\hat{\sigma}$) is called the reflection of $\hat{P}(a,t)$ (resp. $\hat{S}(a,r)$), which is a homeomorphism of $\hat{E}^n$.

DEFINITION 3.7. A *sphere* of $\hat{E}^n$ is either an extended hyperplane or a Euclidean sphere. A *Möbius transformation* of $\hat{E}^n$ is a finite composition of reflections of $\hat{E}^n$ in spheres.

3.3 The conformal ball model.

In this section, we recall the conformal ball model of hyperbolic space. Identify $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\}$ in $\mathbb{R}^{n+1}$. Let B^n be the open unit ball of $\mathbb{R}^n$, and let $|x|$ be the *Euclidean norm* of a vector x in $\mathbb{R}^n$: that is, $|x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$.

The *stereographic projection* ζ from B^n to the hyperbolic space H^n is defined by the formula

$$\zeta(x) = \left(\frac{2x_1}{1-|x|^2}, \ldots, \frac{2x_n}{1-|x|^2}, \frac{1+|x|^2}{1-|x|^2}\right).$$

This map ζ is bijective.

DEFINITION 3.8. A metric d_B on B^n is defined by the formula $d_B(x,y) = d_H(\zeta(x), \zeta(y))$. The metric space (B^n, d_B) is called the conformal ball model of hyperbolic n-spaces.

DEFINITION 3.9. A *Möbius transformation of* B^n is a Möbius transformation ϕ of $\hat{E}^n$ that leaves B^n invariant: that is, ϕ satisfies $\phi(B^n) = B^n$. The group of Möbius transformations of B^n is denoted by $M(B^n)$.

It is well-known that every Möbius transformation of B^n restricts to an isometry of the conformal ball model B^n and the restriction induces an isomorphism from $M(B^n)$ to $I(B^n)$.

The boundary of B^n in $\mathbb{R}^n$, denoted by ∂B^n, is the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$. A *hyperbolic m-plane* of B^n is the image of a hyperbolic m-plane of H^n by ζ^{-1}. By [17, Theorem 4.5.3], a hyperbolic m-plane of B^n is the intersection of B^n with either an m-dimensional vector subspace of $\mathbb{R}^n$ or an m-dimensional sphere of $\mathbb{R}^n$ orthogonal to ∂B^n.

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DEFINITION 3.10. Let c be a point in ∂B^n. Let S be an $(n-1)$-dimensional inscribed sphere of B^n tangent at c, and let T be the one of connected components of $\mathbb{R}^n \backslash S$ which is bounded. We call the intersection of B^n with S (resp. T) a *horosphere* of B^n (resp. a *horoball* of B^n).

3.4 The upper half-space model.

DEFINITION 3.11. Let $x = (x_1, \dots, x_n)$ be a vector in $\mathbb{R}^n$, and let e_n be the vector in $\mathbb{R}^n$ whose entries are all zero except for the n-th, which is one. Let ρ (resp. σ) be the reflection of $\hat{E}^n$ in $P(e_n, 0)$ (resp. $S(e_n, \sqrt{2})$). The maps ρ and σ is given by the formula

$$\rho(x) = (x_1, \dots, x_{n-1}, -x_n),$$
$$\sigma(x) = e_n + \frac{2(x - e_n)}{|x - e_n|^2} \text{ for } x \in \mathbb{R}^n \backslash \{e_n\}.$$

The n-dimensional *upper half-space* U^n is defined to be

$$U^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}.$$

Then, the map $\eta = \sigma\rho$ is a map from the upper half-space U^n to the open unit ball B^n, and is called the *standard transformation* from U^n to B^n. The *Poincaré metric* on U^n is defined by

$$d_U(x, y) = d_B(\eta(x), \eta(y)).$$

The pair (U^n, d_U) is a metric space, and is called the *upper half-model* of hyperbolic n-spaces. By the definition, this is isomorphic to the conformal ball model B^n of hyperbolic n-spaces. The group of isometries of U^n is denoted by $I(U^n)$.

DEFINITION 3.12. A *Möbius transformation of* U^n is a Möbius transformation ϕ that leaves U^n invariant: that is, ϕ satisfies $\phi(U^n) = U^n$. The group of Möbius transformations of U^n is denoted by $M(U^n)$.

Every Möbius transformation of the upper half-space model U^n restricts to an isometry of U^n and the restriction induces an isomorphism from $M(U^n)$ to $I(U^n)$.

DEFINITION 3.13. Identify E^{n-1} with $E^{n-1} \times \{0\}$ in E^n. Let ϕ be a Möbius transformation of $\hat{E}^{n-1}$, and let $\overline{a} = (a, 0)$. The extension $\overline{\phi}$ of ϕ is defined as follows:

(1) If ϕ is the reflection of $\hat{E}^{n-1}$ in $P(a, t)$, then $\overline{\phi}$ is the reflection of $\hat{E}^n$ in $P(\overline{a}, t)$.
(2) If ϕ is the reflection of $\hat{E}^{n-1}$ in $S(a, t)$, then $\overline{\phi}$ is the reflection of $\hat{E}^n$ in $S(\overline{a}, t)$.
(3) In the general case, ϕ is the composition $\phi = \sigma_1 \cdots \sigma_m$ of reflections. Then, we define the map $\overline{\phi}$ by the composition $\overline{\phi} = \overline{\sigma_1} \cdots \overline{\sigma_m}$.

We call the extension of ϕ the *Poincaré extension.*

Poincaré extension is well-defined. Moreover, the following holds.

THEOREM 3.14 [17, Theorem 4.4.1]. *A Möbius transformation ϕ of $\hat{E}^n$ leaves U^n invariant if and only if ϕ is the Poincaré extension of a Möbius transformation of $\hat{E}^{n-1}$. In particular, Poincaré extension induces the isomorphism from $M(\hat{E}^{n-1})$ to $M(U^n)$.*

DEFINITION 3.15. Let η be the standard transformation from U^n to B^n. A subset P is a *hyperbolic m-plane* (resp. *horosphere*, *horoball*) of U^n if the image $\eta(P)$ of P by η is a hyperbolic m-plane (resp. horosphere, horoball) in B^n.

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THEOREM 3.16 [17, Theorems 4.6.3. and 4.6.5.].

(1) *A subset P of U^n is a hyperbolic m-plane of U^n if and only if P is the intersection of U^n with either an m-plane of E^n orthogonal to E^{n-1} or an m-sphere of E^n orthogonal to E^{n-1}.*
(2) *A subset Σ of U^n is a horosphere of U^n based at a point b of $\hat{E}^{n-1}$ if and only if Σ is either a Euclidean hyperplane in U^n parallel to E^{n-1} if $b = \infty$, or the intersection of U^n with a Euclidean sphere in $\overline{U}^n$ tangent to E^{n-1} at b if $b \neq \infty$.*

3.5 Types of transformations and elementary groups.

Let B^n be the conformal ball model of n-dimensional hyperbolic spaces. We write $\overline{B^n}$ for the closed unit ball $B^n \cup \partial B^n$.

DEFINITION 3.17. Let ϕ be an element of $M(B^n)$. Then, ϕ maps the closed unit ball $\overline{B^n}$ to itself. By the Brouwer Fixed-Point Theorem, ϕ has a fixed point in $\overline{B^n}$.

(1) ϕ is *elliptic* if ϕ has a fixed point in B^n;
(2) ϕ is *parabolic* if ϕ has no fixed point in B^n and has a unique fixed point in ∂B^n;
(3) ϕ is *hyperbolic* if ϕ has no fixed point in B^n and fixes two points in ∂B^n.

Let ϕ be an isometry of U^n, and let η be the standard transformation from U^n to B^n. Then ϕ is said to be *elliptic* (resp. *parabolic*, *hyperbolic*) if $\eta\phi\eta^{-1}$ is elliptic (resp. parabolic, hyperbolic).

DEFINITION 3.18. A subgroup G of $M(B^n)$ is *elementary* if G has a finite orbit in the closed ball $\overline{B^n}$. Elementary groups can also be classified into three types as follows:

(1) The group G is of *elliptic type* if G has a finite orbit in B^n.
(2) The group G is of *parabolic type* if G fixes a point of ∂B^n and has no other finite orbit in $\overline{B}^n$
(3) The group G is of *hyperbolic type* if G is neither of elliptic type nor of parabolic type.

THEOREM 3.19 [17, Theorem 5.5.2.]. *Let Γ be a subgroup of $M(B^n)$, and let $O(n)$ be the orthogonal group of the n-dimensional Euclidean space. Then, Γ is an elementary discrete subgroup of elliptic type if and only if Γ is conjugate in $M(B^n)$ to a finite subgroup of the orthogonal group $O(n)$.*

THEOREM 3.20 [17, Theorem 5.5.5]. *Let Γ be a subgroup of $M(U^n)$. The group Γ is an elementary discrete subgroup of parabolic type if and only if Γ is conjugate in $M(U^n)$ to an infinite discrete subgroup of $I(E^{n-1})$.*

3.6 Dirichlet domains.

DEFINITION 3.21. Let X be a metric space. A subset D of X is a *fundamental domain* for a group Γ of isometries of X if:

(1) the set D is open and connected in X;
(2) the elements of $\{gD : g \in \Gamma\}$ are mutually disjoint;
(3) $X = \bigcup_{g\in\Gamma} g\overline{D}$,
where $\overline{D}$ is the closure of D in X.

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A fundamental domain D is *locally finite* if the set $\{g\overline{D} : g \in \Gamma\}$ is locally finite: that is, for each $x \in X$, there is an $\epsilon > 0$ such that the open ball $B(x,\epsilon)$ of radius ϵ centered at x meets only finitely many members of $\{g\overline{D} : g \in \Gamma\}$.

DEFINITION 3.22. Let Γ be a group of homeomorphisms of a metric space X. Γ is said to be *discontinuous* if for any compact subset K of X, the set $K \cap gK$ is empty except for only finitely many $g \in \Gamma$.

Let Γ be a discontinuous group of isometries of a hyperbolic n-space H^n. By [17, Theorems 6.6.10 and 6.6.12], there is a point a such that the stabilizer group Γ_a of a is trivial. For each $g \neq 1$ in Γ, we set $H_g^+(a) = \{x \in H^n : d_H(x,a) < d_H(x,ga)\}$.

DEFINITION 3.23. The *Dirichlet domain* $D_\Gamma(a)$ for Γ, with center a, is either H^n if Γ is trivial or

$$D_\Gamma(a) = \bigcap_{g \in \Gamma \setminus \{1\}} H_g^+(a).$$

Let $D_\Gamma(a)$ be the Dirichlet domain for Γ with center a as above. By [17, Theorem 6.6.13], $D_\Gamma(a)$ is a locally finite fundamental domain for Γ.

3.7 Limit sets.

DEFINITION 3.24. A point a of ∂B^n is a *limit point* of a subgroup G of $M(B^n)$ if there is a point $x \in B^n$ and a sequence $\{g_i\}$ of elements of G such that the sequence $\{g_i x\}_{i \in \mathbb{N}}$ converges to a. The set of limit points of G is called the *limit set* of G, and is denoted by $\Lambda(G)$. The complement of $\Lambda(G)$ in ∂B^n is called the *ordinary set* of G, and is denoted by $O(G)$.

DEFINITION 3.25. A point a of ∂B^n is a *bounded parabolic limit point* of a discrete subgroup group Γ of $M(B^n)$ if a is a fixed point of a parabolic element of Γ and the orbit space $(\Lambda(\Gamma)\setminus\{a\})/\Gamma$ is compact.

REMARK 3.26. The terms bounded parabolic limit point and cusped limit point are used to refer to the same notion in this article. Their equivalence is proved in [17]. For the definitions, see [17, p. 618], and for the equivalence, see [17, Theorem 12.6.4].

3.8 Geometrically finite discrete groups.

DEFINITION 3.27. Let Γ be a discrete subgroup of $M(B^n)$ with a parabolic element that fixes a point a of ∂B^n. Let Γ_a be the stabilizer group of a. A horoball $B(a)$ based at a is said to be a *horocusped region* for Γ based at a if for each $g \in \Gamma \setminus \Gamma_a$, we have

$$B(a) \cap gB(a) = \emptyset.$$

If a horocusped region $B(a)$ based at a is not maximal with respect to the inclusion relation, $B(a)$ is said to be *proper*.

Let $M = B^n/\Gamma$, and let $\pi : B^n \to M$ be the quotient map. Let $B(a)$ be a (proper) horocusped region based at a. The open subset $V = \pi(B(a))$ of M is called the *(proper) horocusp of M corresponding to $B(a)$*.

DEFINITION 3.28. Let $C(\Gamma)$ be the hyperbolic convex hull of the limit set $\Lambda(\Gamma)$ of Γ. The *convex core* of $M = B^n/\Gamma$ is the set

$$C(M) = (C(\Gamma) \cap B^n)/\Gamma.$$

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DEFINITION 3.29. Let Γ be a discrete subgroup of $M(B^n)$, let $M = B^n/\Gamma$, and let $C(M)$ be the convex core of M. The group Γ is said to be *geometrically finite* if there is a (possibly empty) finite union V of proper horocusps of M, with disjoint closures, such that $C(M)\backslash V$ is compact.

THEOREM 3.30 [17, Theorem 12.4.10]. *Let Γ be a discrete subgroup of $M(B^n)$ with an integer $n > 1$. Then, the following are equivalent:*

(1) *The orbit space B^n/Γ has finite volume.*
(2) *The group Γ is geometrically finite and $\Lambda(\Gamma) = \partial B^n$.*
(3) *There are finite points $p_1, \cdots p_k$ of $\overline{B^n}$ such that $P \cap B^n$ is a fundamental domain with finite volume, where P is the hyperbolic convex hull of $\{p_1, \cdots, p_k\}$ in $\overline{B^n}$.*

By [17, Theorems 12.6.4 and 12.6.5], we have the following theorem.

THEOREM 3.31. *Let Γ be a discrete subgroup of $M(B^n)$, and let C be the set of bounded parabolic limit points of Γ. For each $c \in C$, there exists a horocusped region $U(c)$ based at c for Γ such that:*

(1) *the members of $\{\overline{U}(c) : c \in C\}$ are mutually disjoint;*
(2) *$\gamma U(c) = U(\gamma c)$ for any $\gamma \in \Gamma$ and $c \in C$;*
(3) *$\{\overline{U}(c)\backslash\{c\}\}$ is a locally finite family of closed subsets of $B^n \cup O(\Gamma)$.*

§4. Cohomological dimension of groups.

4.1 Cohomology of groups.

In this section, we review some definitions and basic facts about cohomology of groups, which are contained in [8].

DEFINITION 4.1. Let R be a ring and M be a left R-module. Let n be a non-negative integer. Then, $\operatorname{projdim}_R M \leq n$ if there is a left projective resolution of length n:

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

The following lemma is well-known.

LEMMA 4.2 [8, Chapter VIII, Lemma (2.1)]. *The following conditions are equivalent:*

(1) $\operatorname{projdim}_R M \leq n$.
(2) $\operatorname{Ext}^i_R(M, -) = 0$ *for all* $i > n$.
(3) $\operatorname{Ext}^{n+1}_R(M, -) = 0$.
(4) *For any left R-module exact sequence $0 \to K \to P_{n-1} \to \cdots \to P_0 \to 0$ with P_i projective, K is projective.*

DEFINITION 4.3. We define *the cohomological dimension* $\operatorname{cd}\Gamma$ as following:

$$\begin{aligned}\operatorname{cd}\Gamma &= \operatorname{projdim}_{\mathbb{Z}\Gamma}\mathbb{Z} \\ &= \inf\{n : \mathbb{Z} \text{ admits a projective resolution of length } n\} \\ &= \inf\{n : H^n(\Gamma, -) = 0 \text{ for all } i > n\} \\ &= \sup\{n : H^n(\Gamma, M) \neq 0 \text{ for some } \Gamma\text{-module } M\}.\end{aligned}$$

Recall that the functor $\operatorname{Ext}^i_R(M, -) = H^i(\operatorname{Hom}_R(M, -))$. Then, $\operatorname{Ext}^i_{\mathbb{Z}\Gamma}(\mathbb{Z}, -) = H^i(\Gamma, -)$. By the previous Lemma, the above definition is well-defined.

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The following theorem is called Serre's theorem that claims well-definedness of the *virtual cohomological dimension*.

THEOREM 4.4 [8, Chapter VIII, Theorem (3.1)]. *If* Γ *is a torsion-free group and* Γ' *is a subgroup of finite index, then* $\mathrm{cd}\,\Gamma = \mathrm{cd}\,\Gamma'$.

DEFINITION 4.5. Let Γ be a virtually torsion-free group. By Theorem 4.4, any torsion-free subgroup of finite index has the same cohomological dimension. The dimension is called the *virtual cohomological dimension*, and is denoted by $\mathrm{vcd}\,\Gamma$.

4.2 Bestvina–Mess formula.

DEFINITION 4.6. Let X be a topological space. The *order of an open covering* $\mathcal{U}$ of X is defined to be

$$\mathrm{ord}\,\mathcal{U} = \sup\{k : \text{there exist } U_1,\dots,U_k \in \mathcal{U} \text{ such that } \cap_{i=1}^{k} U_i \neq \emptyset\}.$$

The *covering dimension* $\dim X$ of X is defined as follows: $\dim X \leq n$ if for any finite open covering $\mathcal{U}$ of X, there exists a finite open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{ord}\,\mathcal{V} \leq n+1$, and $\dim X = n$ if $\dim X \leq n$ and $\dim X \not\leq n$.

Bestvina and Mess [5] proved the following formula for any virtually torsion-free hyperbolic group G:

$$\mathrm{vcd}\,G = \dim \partial G + 1,$$

where ∂G is the Gromov boundary of G and $\dim \partial G$ is the covering dimension of ∂G. In [6], this formula, called the Bestvina–Mess formula, is generalized for groups that admit $\mathcal{Z}$*-structures.* Recently, Fukaya [10] proved the Bestvina–Mess formula for relatively hyperbolic groups.

DEFINITION 4.7. A compact metrizable space X is a *Euclidean retract* (or ER) if X is finite-dimensional, contractible, and locally contractible.

DEFINITION 4.8. Let X be an ER. A closed subset Z of X is called a $\mathcal{Z}$-set if $X \backslash Z$ is homotopy dense in X, that is, there exists a homotopy $H : X \times [0,1] \to X$ such that $H_0 = id$, and $H_t(X) \subset X \backslash Z$ for any $t > 0$.

DEFINITION 4.9. Let $\overline{X}$ be an ER, and let Z be a closed subset of $\overline{X}$. A pair $(\overline{X}, Z)$ is a $\mathcal{Z}$-structure for a group G if:

(1) Z is a $\mathcal{Z}$-set of $\overline{X}$.
(2) $\overline{X} \backslash Z$ admits a covering space action of G with compact quotient.
(3) For any compact subset K of X and any open covering $\mathcal{U}$ of $\overline{X}$, for all but finitely many $g \in G$, there exists $U_g \in \mathcal{U}$ such that $gK \subset U_g$.

THEOREM 4.10 [6, Theorem 1.7]. *Let* G *be a virtually torsion-free group. If* $(\overline{X}, Z)$ *be a* $\mathcal{Z}$*-structure on some torsion-free subgroup of* G *of finite index, then* $\mathrm{vcd}\,G = \dim Z + 1$, *where* $\dim Z$ *is the covering dimension of* Z.

EXAMPLE 4.11. Let B^n be the n-dimensional conformal model of hyperbolic spaces. and let Γ be a cocompact discrete subgroup of $M(B^n)$. Then, the pair $(\overline{B^n}, S^{n-1})$ is a $\mathcal{Z}$-structure of a group Γ, and so $\mathrm{vcd}\,\Gamma = \dim S^{n-1} + 1 = n$.

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§5. Automorphism groups of K3 surfaces and virtual cohomological dimension.

In this section, we consider the natural actions of the automorphism groups of K3 surfaces on hyperbolic spaces. First, we recall the constructions of hyperbolic spaces induced by hyperbolic lattices, and discuss the actions of the automorphism groups of K3 surfaces on certain hyperbolic spaces, and moreover, some properties of automorphism groups. Second, we discuss some invariants of automorphism groups, for example, the virtual cohomological dimension.

5.1 Geometrical finiteness of $\mathbf{Aut}_s(X)$.

Let X be an algebraic K3 surface with $\rho(X) \geq 3$ defined over $\mathbb{C}$, and let f be the quadratic form defined by $f(x) = \langle x, x\rangle$ for $x \in NS(X) \otimes \mathbb{R}$, where $\langle , \rangle$ is the inner product on $NS(X)$ induced by the cup product on $H^2(X, \mathbb{Z})$. Since X is algebraic, f has the signature $(1, n)$, where $n = \rho(X) - 1$. Then, there exists $M \in \mathrm{GL}(n+1, \mathbb{R})$ such that $f_{n+1}(Mx) = f(x)$, where $f_{n+1}(x) = x_0^2 - (x_1^2 + x_2^2 + \cdots + x_n^2)$. Therefore, we can identify the connected component of the set

$$\{x \in NS(X) \otimes \mathbb{R} : \langle x, x\rangle = 1\},$$

containing ample classes of X with the hyperboloid model of hyperbolic n-spaces. We denote this hyperboloid model of hyperbolic n-spaces by $\mathbb{H}^n_X$. The hyperbolic distance function d on $\mathbb{H}^n_X$ satisfies $\cosh d(x, y) = \langle x, y\rangle$.

Remark 5.1. To construct the hyperboloid model of hyperbolic space, it is necessary that the Néron–Severi lattice $\mathrm{NS}(X)$ is of hyperbolic type as a lattice. Moreover, in order for the resulting hyperboloid model to have dimension at least two, its Picard rank must be at least three. Then, unless otherwise stated, we assume that X is algebraic and its Picard rank is at least three.

Definition 5.2. We call $\mathbb{H}^n_X$ the *hyperbolic space associated with* X. Let $\overline{P^+}(X)$ be the closure of the positive cone $P^+(X)$ in $H^{1,1}(X, \mathbb{R})$. We define $\mathbb{P}(\overline{P^+}(X))$ by

$$\mathbb{P}(\overline{P^+}(X)) = (\overline{P^+}(X) \backslash \{0\})/\mathbb{R}_{>0}.$$

For a point x in $\overline{P^+}(X)\backslash\{0\}$, we denote by $[x]$ the class in $\mathbb{P}(\overline{P^+}(X))$ corresponding to x. The inclusion from $\mathbb{H}^n_X$ to $\overline{P^+}(X)$ induces a continuous injective map $i : \mathbb{H}^n_X \to \mathbb{P}(\overline{P^+}(X))$. Since i is a homeomorphism from $\mathbb{H}^n_X$ to $i(\mathbb{H}^n_X)$, we identify $\mathbb{H}^n_X$ with $i(\mathbb{H}^n_X)$.

The *boundary of* $\mathbb{H}^n_X$ *at infinity*, denoted by $\partial\mathbb{H}^n_X$, is the boundary of $i(\mathbb{H}^n_X)$ in $\mathbb{P}(\overline{P^+}(X))$, which is equal to the set $\{x \in (\mathrm{NS}(X) \otimes \mathbb{R}) \cap \overline{P^+}(X) \backslash \{0\} : \langle x, x\rangle = 0\}/\mathbb{R}_{>0}$. We set $\overline{\mathbb{H}}^n_X = \mathbb{H}^n_X \cup \partial\mathbb{H}^n_X$.

Definition 5.3. Let $A(X)$ be the ample cone of X. We define

$$\mathrm{Aut}(A(X)) \coloneqq \{\phi \in O(NS(X)) : \phi A(X) = A(X)\} \text{ and } A_X \coloneqq A(X) \cap \mathbb{H}^n_X.$$

The closure of A_X in $\overline{\mathbb{H}^n_X}$ is denoted by $\overline{A_X}$. We denote $\overline{A_X} \backslash A_X$ by ∂A_X, called *the boundary associated with the ample cone of* X.

By the unimodulality of $H^2(X, \mathbb{Z})$, the discriminant groups of $NS(X)$ and T_X are isomorphic:

$$A_{NS(X)} = NS(X)^*/NS(X) \simeq T_X^*/T_X = A_{T_X}.$$

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Each isomorphism ϕ of $NS(X)$ or T_X gives rise to an isomorphism of the group $A_{NS(X)} \simeq A_{T_X}$, denoted by ϕ^*. Since the automorphism group $\mathrm{Aut}(X)$ of X naturally acts on the ample cone of X, there is a natural homomorphism $\rho : \mathrm{Aut}(X) \to \mathrm{Aut}(A(X))$. As for this homomorphism, the following theorem is well-known.

THEOREM 5.4 [14, Theorem 8.1]. *The kernel and the cokernel of ρ are finite groups.*

Since $\mathrm{Aut}(X)$ is finitely generated [19, Proposition 2.2] and Selberg's Lemma [1], $\mathrm{Aut}(X)$ is virtually torsion-free.

DEFINITION 5.5. Let X be a K3 surface, and let ω_X be a nowhere vanishing holomorphic 2-form of X. An automorphism ϕ of X is said to be *symplectic* if ϕ acts trivially on ω_x. The *symplectic automorphism group* of X, denoted by $\mathrm{Aut}_s(X)$, is the group of symplectic automorphisms of X.

The following lemma is well-known.

LEMMA 5.6 [14, Lemma 8.11]. *The symplectic automorphism of X acts trivially on the transcendental lattice T_X.*

Let ϕ be the symplectic automorphism of X. Then, by Lemma 5.6, ϕ acts trivially on $A_{NS(X)} \simeq A_{T_X}$. By the Torelli Theorem [7] and [16], we can identify $\mathrm{Aut}_s(X)$ with the subgroup
$\{\phi \in O^+(NS(X)) : \phi A(X) = A(X), \phi^*|_{A_{NS(X)}} = \mathrm{id}_{A_{NS(X)}}\}$. Therefore, $\mathrm{Aut}_s(X)$ is a subgroup of finite index in $\mathrm{Aut}(A(X))$. Since any element of $O^+(NS(X))$ is an isometry of the hyperbolic space $\mathbb{H}^n_X$ associated with X, the symplectic automorphism group $\mathrm{Aut}_s(X)$ of X is a discrete subgroup of isometries of $\mathbb{H}^n_X$.

PROPOSITION 5.7. *Let X be an algebraic K3 surface with $\rho(X) \geq 3$, and let $a \in A_X$. Then, the Dirichlet domain $D(a)$ of $\mathbb{H}^n_X$, with center a, for the reflection group $W(X)$ is equal to $A_X = A(X) \cap \mathbb{H}^n_X$.*

Proof. Recall that $H^+_w(a) = \{x \in \mathbb{H}^n_X : d(x,a) < d(x,wa)\}$ for $w \in W(X)$. If w is the reflection s_δ for some root δ,

$$\begin{aligned} & d(x,a) < d(x, s_\delta a) \\ \Leftrightarrow\ & \cosh d(x,a) < \cosh d(x, s_\delta a) \\ \Leftrightarrow\ & \langle x, a\rangle < \langle x, s_\delta a\rangle \\ \Leftrightarrow\ & 0 < \langle a, \delta\rangle \langle x, \delta\rangle \\ \Leftrightarrow\ & 0 < \langle x, \delta\rangle. \end{aligned}$$

Then, we have that $H^+_{s_\delta}(a) = \{x \in \mathbb{H}^n_X : \langle x, \delta\rangle > 0\}$. Since A_X is a fundamental domain for $W(X)$ on $\mathbb{H}^n_X$ whose sides are bounded by roots of $\mathrm{NS}(X)$, we have $A_X = \cap_{\delta \in \Delta^+(X)} H^+_{s_\delta}$.

By the definition of $D(a)$, we have $D(a) \subset A_X$. Since $D(a)$ and A_X are convex fundamental domains for $W(X)$, we have $D(a) = A_X$. □

By [17, Theorem 12.8.4], the orbit space $\mathbb{H}^n_X / O^+(\mathrm{NS}(X))$ has finite volume. By Theorem 3.30, we have the following.

LEMMA 5.8. *Let X be an algebraic K3 surface with $\rho(X) \geq 3$. Then, $O^+(NS(X))$ is geometrically finite and of the first kind (i.e., its limit set is equal to S^{n-1}). Moreover, a fundamental domain of $\mathbb{H}^n_X$ for $O^+(NS(X))$ is a generalized polytope; that is, the*

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fundamental domain is the convex hull of finitely many vertices, which may lie on the boundary at infinity.

LEMMA 5.9. *Let $a \in A_X$, and let P be the Dirichlet domain, centered at a, of $\mathbb{H}^n_X$ for $O^+(NS(X))$. Then,* $\mathrm{Aut}(A(X))\cdot\overline{P} = \overline{A_X}^H$, *where $\overline{P}$ (resp. $\overline{A_X}^H$) are the closure of P (resp. A_X) in $\mathbb{H}^n_X$.*

Proof. By Lemma 5.7 and the definition of P, we have

$$P = \{x \in \mathbb{H}^n_X : d(x,a) < d(x,ga) \text{ for any } g \neq 1 \text{ in } O^+(NS(X))\},$$
$$A_X = \{x \in \mathbb{H}^n_X : d(x,a) < d(x,wa) \text{ for any } w \neq 1 \text{ in } W(X)\}.$$

Then, we have $P \subset A_X$ and so $\overline{P} \subset \overline{A_X}^H$. Hence, we obtain $\mathrm{Aut}(A(X))\cdot\overline{P} \subset \overline{A_X}^H$.

Conversely, because the tessellation $\{g\overline{P} : g \in O^+(NS(X))\}$ of $\mathbb{H}^n_X$ is locally finite, the union $\mathrm{Aut}(A(X))\cdot\overline{P}$ is a closed subset. Then, it is enough to show that $A_X \subset \mathrm{Aut}(A(X))\cdot\overline{P}$. For any $x \in A_X$, there exist $w \in W(X)$ and $\gamma \in \mathrm{Aut}(A(X))$ such that $x \in w\gamma\overline{P}$. Then, we have $x \in w\gamma\overline{P} \cap A_X \subset w\overline{A_X}^H \cap A_X$. By [17, Theorem 6.6.4], we have $w = 1$. In particular, we have $x \in \gamma\overline{P}$. Hence, we have $A_X \subset \mathrm{Aut}(A(X))\cdot\overline{P}$. □

DEFINITION 5.10. Let P be the same as in Lemma 5.9. We define sets of bounded parabolic limit points of $O^+(NS(X))$ as follows:

- I_P = the set of ideal vertices of P;
- $V = Aut(A(X))\cdot I_P$;
- $V_{NS} = O^+(NS(X))\cdot I_P$;
- $V_X = \{c \in V : \mathrm{vcd}\, W(X)_c < n-1\}$,

where $\mathrm{vcd}\, W(X)_c$ is the stabilizer group of $W(X)$ at a point c.

DEFINITION 5.11. Let X be an algebraic K3 surface with $\rho(X) \geq 3$. Since $\mathrm{Aut}_s(X)$ is of finite index in $\mathrm{Aut}(A(X))$, the limit set of $\mathrm{Aut}_s(X)$ is equal to that of $\mathrm{Aut}(A(X))$. We denote this limit set by Λ_X, and is called the *limit set of the K3 surface X*. We denote the hyperbolic convex hull of Λ_X by $C(X)$, and put $B(X) = C(X) \cap \mathbb{H}^n_X$.

LEMMA 5.12. *For any $c \in V \backslash V_X$, there exists a horoball $U(c)$ based at c such that $B(X) \cap U(c) = \varnothing$. Moreover, the point c is isolated in the boundary ∂A_X associated with the ample cone.*

Proof. If $\mathrm{Aut}(A(X))$ is elementary, this lemma is obvious.

Suppose that $\mathrm{Aut}(A(X))$ is non-elementary. Let $\phi : \mathbb{H}^n_X \to U^n$ be an isometry from $\mathbb{H}^n_X$ to the upper half-space model U^n such that $\phi(c) = \infty$. The isometry ϕ induces the natural homeomorphisms from $\partial\mathbb{H}^n_X \backslash \{c\}$ to ∂U^n. We also denote this homeomorphism by ϕ. Let b be a point of U^n corresponding to a point a in A_X. We can consider the stabilizer group $W(X)_\infty$ as the Poincaré extension of a subgroup of $I(E^{n-1})$. Since $\mathrm{vcd}\, W(X)_c = n-1$, $W(X)_\infty$ is a crystallographic group of E^{n-1}. Then, the Dirichlet domain K of E^{n-1} for $W(X)_\infty$, centered at b, is compact. Proposition 5.9 implies that $\Lambda_X \subset \partial A_X$. Then, we have $\phi(\Lambda_X) \subset K \cup \{\infty\}$. Since $\mathrm{Aut}(A(X))$ is non-elementary, the limit set Λ_X is perfect: that is, every point of Λ_X is not isolated. Hence, the infinity ∞ is not in $\phi(\Lambda_X)$. By the compactaness of K, there exists a horoball B_∞ based at ∞ such that $C(K) \cap B_\infty = \varnothing$, where $C(K)$ is the hyperbolic convex hull of K. In particular, $B(X) \cap B_\infty = \varnothing$. The inverse image $\phi^{-1}(B_\infty)$ of B_∞ by ϕ is the desired horoball. □

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LEMMA 5.13. *Let e be an elliptic divisor of X, let f_e be an elliptic fibration associated to e, and let $\mathrm{Aut}(X)_e$ be the stabilizer group of e. Then,*

$$\mathrm{vcdAut(X)_e} = \mathrm{vcd}\,\mathrm{MW}(f_e), \text{ and } \mathrm{vcd}\,W(X)_e = \sum_{v\in\mathbb{P}^1}(m_v-1),$$

where $\mathrm{MW}(f_e)$ is the Mordell-Weil group of f_e, and m_v be the number of connected components of the fiber $f^{-1}(v)$ for $v\in\mathbb{P}^1$.

Proof. Let v be a point in $\mathbb{P}^1$, and let m_v be the number of connected components of the fiber $f^{-1}(v)$. By Shioda–Tate formula,

$$\rho(X)-2 = \mathrm{vcd}\,MW(f_e)+\sum_{v\in\mathbb{P}^1}(m_v-1).$$

If $m_v\neq 1$, we denote by W_v the Weyl group generated by the connected components of $f^{-1}(v)$, which is of type $\tilde{A}$, $\tilde{D}$, or $\tilde{E}$. Since W_v fixes the elliptic divisor e, we have $\oplus W_v\subset W(X)_e$. Here, for each v with $m_v\neq 1$, the Weyl group W_v acts on the (m_v-1)-dimensional Euclidean space, and admits an (m_v-1)-dimensional simplex Δ_v as a fundamental domain for the action of W_v. Hence, $\oplus W_v$ acts on the $\big(\sum_{v\in\mathbb{P}^1}(m_v-1)\big)$-dimensional Euclidean space, and has the fundamental domain $\prod_v\Delta_v$. Then, we have

$$\sum_{v\in\mathbb{P}^1}(m_v-1) = \mathrm{vcd}(\oplus W_v)\leq \mathrm{vcd}\,W(X)_e.$$

Since the Mordell–Weil group of f_e is a subgroup of $\mathrm{Aut}(X)_e$, then $\mathrm{rk}\,\mathrm{MW}\,f_e\leq \mathrm{vcd}\,\mathrm{Aut}(X)_e$, where $\mathrm{rk}\,\mathrm{MW}\,f_e$ is the rank of Mordell–Weil group of f_e.

Since both $W(X)_e$ and $\mathrm{Aut}(X)_e$ fix the elliptic divisor e, which is a cusped limit points of $O^+(NS(X))$, these two groups are elementary of either elliptic or parabolic type. Therefore, there exists an integer l (resp. m) such that $W(X)_e$ (resp. $\mathrm{Aut}(X)_e$) has a subgroup that is isomorphic to $\mathbb{Z}^l$ (resp. $\mathbb{Z}^m$) of finite index in $W(X)_e$ (resp. $\mathrm{Aut}(X)_e$). Hence, we have

$$\sum_{v\in\mathbb{P}^1}(m_v-1)\leq l, \tag{5.1}$$

$$\mathrm{rk}\,\mathrm{MW}\,f_e\leq m, \tag{5.2}$$

so that $\rho(X)-2\leq l+m$. On the other hand, $O^+(NS(X))_e$ has a subgroup that is isomorphic to $\mathbb{Z}^{l+m}$. Then, $l+m=\rho(X)-2$. Inequalities (5.1) and (5.2) imply that

$$l=\sum_{v\in\mathbb{P}^1}(m_v-1),$$
$$m=\mathrm{rk}\,\mathrm{MW}\,f_e.$$

These are namely the desired equalities. □

By Lemma 5.13, we have $V_X=\{c\in V:\mathrm{vcd}\,\mathrm{Aut}(X)_c\geq 1\}=\{c\in V:\mathrm{vcd}\,\mathrm{Aut}(A(X))_c\geq 1\}$. Then, each element of V_X is fixed by some parabolic transformation of $\mathrm{Aut}(A(X))_c$.

As noted in the introduction, the following theorem is also found by Kohei Kikuta at about the same time (see the introduction for details).

THEOREM 5.14. *Let X be an algebraic K3 surface with $\rho(X)\geq 3$, and let $\mathrm{Aut}_s(X)$ be the symplectic automorphism group of X. Then, $\mathrm{Aut}_s(X)$ is geometrically finite.*

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Proof. By Theorem 5.4, the symplectic automorphism group $\mathrm{Aut}_s(X)$ is a subgroup of finite index in $\mathrm{Aut}(A(X))$. Then, it is enough to show that $\mathrm{Aut}(A(X))$ is geometrically finite. Recall the notation of P, V, V_{NS}, and V_X in Definition 5.10

By Theorem 3.31, there exist proper horoballs at $c \in V_{NS}$ such that:

(1) $\{\overline{U(c)} : c \in V_{NS}\}$ are mutually disjoint;
(2) $gU(c) = U(gc)$ for any element $g \in O^+(NS(X))$;
(3) $\{\overline{U(c)} \backslash c\}$ is locally finite in $\mathbb{H}^n_X$.

By Lemma 5.12, shrinking the horoballs $U(c)$ sufficiently small, we obtain that

$$B(X) \backslash \bigcup_{c \in V_X} U(c) = B(X) \backslash \bigcup_{c \in V} U(c).$$

By applying Lemma 5.9,

$$\begin{aligned} B(X) \backslash \bigcup_{c \in V_X} U(c) &\subset \overline{A_X}^H \backslash \bigcup_{c \in V} U(c) \\ &= \Big(\bigcup_{g \in \mathrm{Aut}(A(X))} g\overline{P} \Big) \backslash \bigcup_{c \in V} U(c) \\ &= \bigcup_{g \in \mathrm{Aut}(A(X))} \big(g\overline{P} \setminus \bigcup_{c \in V} U(c) \big) \\ &= \bigcup_{g \in \mathrm{Aut}(A(X))} g(\overline{P} \backslash \bigcup_{c \in V} U(c)). \end{aligned}$$

Hence, we have

$$B(X) \backslash \bigcup_{c \in V_X} U(c) \subset \bigcup_{g \in \mathrm{Aut}(A(X))} g(\overline{P} \backslash \bigcup_{c \in V} U(c)). \tag{5.3}$$

Let $\pi : \mathbb{H}^n_X \to \mathbb{H}^n_X / \mathrm{Aut}(A(X))$ be the natural quotient map. We define $V_c = U(c) / \mathrm{Aut}(A(X))$. By the inclusion 5.3, we obtain

$$(B(X) / \mathrm{Aut}(A(X))) \backslash \bigcup_{c \in I_P \cap V_X} V_c \subset \pi(\overline{P} \backslash \bigcup_{c \in V} U(c)).$$

By Lemma 5.8, P is a generalized polytope. For each vertex c of P in the boundary at infinity, we have the corresponding horoball $U(c)$. Therefore, the set $\overline{P} \backslash \bigcup_{c \in V} U(c))$ is obtained by removing the parts at infinity from $\overline{P}$, and is compact. Then, $(B(X) / \mathrm{Aut}(A(X))) \backslash \bigcup_{c \in I_P \cap V_X} V_c$ is compact and $\{V_c\}_{c \in I_P \cap V_X}$ is a set of finitely many horocusps with disjoint closures. By Definition 3.29, $\mathrm{Aut}(A(X))$ is geometrically finite. □

PROPOSITION 5.15. *Let p be a cusped limit point of $\mathrm{Aut}(A(X))$. Then, p is a horopoint of $\overline{A_X}$, that is, there exists a horoball U based at p such that U meets just the sides of $\overline{A_X}$ incident with p.*

Proof. We now pass to the upper half-space model U^n and conjugate $\mathrm{Aut}(A(X))$ so that $p = \infty$. Let P be a Dirichlet polyhedron P, centered at $a \in A_X$, for $O^+(NS(X))$. From the proof of [17, Theorem 12.4.4], there exist a horoball Σ based at ∞ and elements $g_1, \ldots, g_k$ in $O^+(NS(X))$ such that

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$$\Sigma \subset \bigcup_{f\in O^+(\mathrm{NS}(X))_\infty} f(\bigcup_{i=1}^{k} g_i\overline{P})$$

and Σ meets just the vertical sides of fg_iP.

Let S be a side of A_X with $S\cap\Sigma\neq\varnothing$. There exists an element $x\in S$ and $r>0$ such that $B(x,r)\cap S$ is homeomorphic to B^{n-1}. Since P is locally finite, shrinking r, there exists $h_1,\ldots,h_l\in O^+(NS(X))$ such that $x\in h_j\overline{P}$ and $B(x,r)\subset\bigcup_{j=1}^{l} h_j\overline{P}$. In particular, we have

$$B^{n-1}\cong B(x,r)\cap S\subset\bigcup\{h_jT : h_j\in O^+(\mathrm{NS}(X)), T \text{ is a side of } \overline{P}, \text{and } x\in h_jT\}.$$

Hence, there exists a side T of $g\overline{P}$ such that $T^\circ\cap S\neq\varnothing$. By [17, Theorem 6.3.8], we have $T\subset S$. By $x\in T\cap\Sigma$, then T is a vertical side of $g\overline{P}$. Therefore, we have $\infty\in T\subset S$. □

There is a natural one-to-one correspondence between the set of maximal parabolic subgroups of $\mathrm{Aut}_s(X)$ and the set of rational rays in $\partial A(X)$ whose stabilizer subgroup in $\mathrm{Aut}_s(X)$ has rank at least one. By multiplying a positive integer, we have a primitive element in $NS(X)$. From this argument and Lemma 5.13, we have the following proposition.

PROPOSITION 5.16. *Let P be a maximal parabolic subgroup of $\mathrm{Aut}_s(X)$. Then, there exists an element $w\in W(X)$ such that the fixed point of wPw^{-1} is an elliptic divisor e. Moreover, the rank of P is equal to the rank of the Mordell-Weil group of the elliptic fibration f_e associated with e.*

5.2 The blown-up boundary of ample cones of K3 surfaces.

The notion of the blown-up coronae for relatively hyperbolic groups (a generalization of geometrically finite groups) is defined in the paper of Fukaya–Oguni [11], and Fukaya [10] established Bestvina–Mess type formula for relatively hyperbolic groups. We can immediately obtain Theorem 1 by applying Fukaya's result [10, Corollary B].

For readability to algebraic geometers, in this section, we give a simplified version of Fukaya's result, using the geometry of ample cones of K3 surfaces and hyperbolic geometry. Note that the proof method and some definitions in this section are not itself new.

DEFINITION 5.17. Let U be an open subset of $\mathbb{R}^n$, and let p be a point in U. We denote the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$ by S^{n-1}. We put

$$\mathrm{Bl}_p(U)=\{(x,t)\in U\times S^{n-1} : |x-p|t=(x-p)\},$$

where $|\cdot|$ is the Euclidean norm. We define a function $\pi_p:\mathrm{Bl}_p(U)\to U$ by setting $\pi_p(x,t)=x$. We call the function π_p the *blowing-up of U at a point p (to S^{n-1})*. For a subset A of U, the closure of $\pi_p^{-1}(A\backslash\{p\})$ in $\mathrm{Bl}_p(U)$ is called the *strict transform of A at a point p*, and is denoted by $\mathrm{St}_p(A)$.

Since π_p restricts to a homeomorphism from $\mathrm{Bl}_p(U)\backslash\pi_p^{-1}(p)$ to $U\backslash\{p\}$, we identify these two spaces.

DEFINITION 5.18. Let $\mathcal{P}=\{p_1,\ldots,p_m\}$ be a subset of U. For each $p\in\mathcal{P}$, we take an open neighborhood U_p of p such that $U_p\cap\mathcal{P}=\{p\}$. Let $\pi_{p_i}:\mathrm{Bl}_{p_i}(U_{p_i})\to U_{p_i}$ be the blowing-up of U_{p_i} at a point p_i. We define $\mathrm{Bl}_{\mathcal{P}}(U)$ by

$$\mathrm{Bl}_{\mathcal{P}}(U)=(U\backslash\mathcal{P})\amalg(\coprod_{p\in\mathcal{P}}\mathrm{Bl}_p(U_p))/\sim,$$

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where the equivalence relation $\sim$ is generated by $x \sim \pi_{p_i}(x,t)$ for all $(x,t) \in \coprod_{p\in\mathcal{P}} \pi_p^{-1}(U_p\backslash\{p\})$.

We denote by $[x]$ (resp. $[x,t]$) the class in $\mathrm{Bl}_{\mathcal{P}}(U)$ corresponding to x (resp. (x,t)). We define the function $\pi_{\mathcal{P}} : \mathrm{Bl}_{\mathcal{P}}(U) \to U$ by setting

$$\begin{cases} \pi_{\mathcal{P}}([x]) = x & (x \in U\backslash\mathcal{P}), \\ \pi_{\mathcal{P}}([x,t]) = x & ((x,t) \in \mathrm{Bl}_p(U_p), p \in \mathcal{P}). \end{cases}$$

We call the function $\pi_{\mathcal{P}}$ the *blowing-up of U at a subset $\mathcal{P}$*. For a subset A of U, the closure of $\pi_{\mathcal{P}}^{-1}(A\backslash\mathcal{P})$ in $\mathrm{Bl}_{\mathcal{P}}(U)$ is called the *strict transform of A at a subset $\mathcal{P}$*, and is denoted by $\mathrm{St}_{\mathcal{P}}(A)$.

Definition 5.19. Let Γ be a geometrically finite subgroup of $M(B^n)$, and let C be the set of cusped limit points of Γ. For finite subsets $\mathcal{P}$ and $\mathcal{Q}$ of C with $\mathcal{P} \subset \mathcal{Q}$, there exists the natural projection $\pi_{\mathcal{P},\mathcal{Q}} : \mathrm{Bl}_{\mathcal{Q}}(\mathbb{R}^n) \to \mathrm{Bl}_{\mathcal{P}}(\mathbb{R}^n)$ such that $\pi_{\mathcal{P}} \circ \pi_{\mathcal{P},\mathcal{Q}} = \pi_{\mathcal{Q}}$.

We denote the projective limit of $(\mathrm{Bl}_{\mathcal{P}}(\mathbb{R}^n),\ \pi_{\mathcal{P},\mathcal{Q}})$ by $\pi_C = \varprojlim \pi_{\mathcal{P}} : \mathrm{Bl}_C(\mathbb{R}^n) \to \mathbb{R}^n$. We call the function π_C the *blowing-up of $\mathbb{R}^n$ at C*. For a subset A of $\mathbb{R}^n$, the closure of $\pi_C^{-1}(A\backslash\mathcal{P})$ in $\mathrm{Bl}_C(\mathbb{R}^n)$ is called the *strict transform of A at C*, and is denoted by $\mathrm{St}_C(A)$. The strict transform of $\overline{B^n}$ (resp. ∂B^n) is denoted by $\overline{B}^{bl}$ (resp. ∂B^{bl}).

Lemma 5.20. *The action of a geometrically finite subgroup Γ of $M(B^n)$ on $\mathbb{R}^n$ (resp. B^n, ∂B^n) is extended to a natural action on $\mathrm{Bl}_C(\mathbb{R}^n)$ (resp. $\overline{B}^{bl}$, ∂B^{bl}).*

Proof. Since Γ preserves the cusped limit point of Γ, the action of Γ on $\mathbb{R}^n$ can be extend to that on $\mathrm{Bl}_C(\mathbb{R}^n)$. As Γ preserves $\overline{B^n}$ and ∂B^n, Γ acts on $\overline{B}^{bl}$ (resp. ∂B^{bl}). □

Definition 5.21. Let Γ be a geometrically finite subgroup of $M(B^n)$. A point a of ∂B^{bl} is a *limit point of Γ in ∂B^{bl}* if there is a point $x \in B^n$ and there exists a sequence $\{g_i\}_{i\in\mathbb{N}}$ of Γ such that $a = \lim g_i x$. The set of limit points of Γ in ∂B^{bl} is called *the blown-up limit set of* Γ, and is denoted by $\Lambda^{bl}(\Gamma)$.

Lemma 5.22. *Let $\eta : U^n \to B^n$ be the standard transformation from U^n to B^n. For non-zero vector $v \in \partial U = E^{n-1}$ and $a \in \overline{U^n}$, we define a half line $l : [0,\infty) \to \overline{U^n}$ by $l(t) = tv + a$, and put $L(t) = (\eta \circ l)(t)$. Then,*

$$\mathrm{St}_{e_n}(\mathrm{Im}(L)) \cap \pi_{e_n}^{-1}(e_n) = \left\{ (e_n, \frac{v}{|v|}) \right\} \subset \mathrm{Bl}_{e_n}(\mathbb{R}^n).$$

In particular, the intersection of $\mathrm{St}_{e_n}(\mathrm{Im}(L))$ with $\pi_{e_n}^{-1}(e_n)$ is determined by the direction of v.

Proof. By direct calculation, we obtain

$$L(t) - e_n = \frac{2(tv + \rho(a) - e_n)}{|tv + \rho(a) - e_n|^2}.$$

Then,

$$\lim_{t\to\infty} \frac{L(t) - e_n}{|L(t) - e_n|} = \lim_{t\to\infty} \frac{tv + \rho(a) - e_n}{|tv + \rho(a) - e_n|} = \frac{v}{|v|}.$$

Hence, $\mathrm{St}_{e_n}(\mathrm{Im}(L)) = \mathrm{Im}(L) \cup \{(e_n, \frac{v}{|v|})\}$. □

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COROLLARY 5.23. *Let V be a vector subspace of $\partial U^n = E^{n-1}$, and let a be a point of $\overline{U^n}$. Let A be an open ϵ-neighborhood of $V+a$ in U^n for a positive number ϵ. Then,*

$$\mathrm{St}_{e_n}(\eta(A)) \cap \pi_{e_n}^{-1}(e_n) = \left\{ (e_n, \frac{v}{|v|}) \in \mathrm{Bl}_{e_n}(\mathbb{R}^n) : v \in V \backslash \{0\} \right\} \cong S^{r-1},$$

where r is the dimension of the vector space V.

LEMMA 5.24. *Let Γ be a geometrically finite subgroup of $M(B^n)$. For every cusped limit point p of Γ, we have*

$$\Lambda^{bl}(\Gamma) \cap \pi_C^{-1}(p) = \Lambda^{bl}(\Gamma_p) \cong S^{r-1},$$

where r is the rank of the stabilizer group Γ_p, that is, the rank of maximal abelian subgroup of Γ_p.

Proof. By conjugating Γ in $M(B^n)$, we can suppose $p = e_n$. Let $\eta : U^n \to B^n$ be the standard transformation. The group Γ acts on U^n by setting $\gamma(x) = (\eta^{-1}\gamma\eta)(x)$ for any $\gamma \in \Gamma$. Here, the point $p = e_n$ corresponds to the point ∞ at infinity of U^n under the standard transformation η.

First, we show that $\Lambda^{bl}(\Gamma_p) \cong S^{r-1}$. There is a Γ_∞-invariant r-plane Q of E^{n-1} such that Q/Γ_∞ is compact. For any $r > 0$, the r-neighborhood $N(Q,r)$ of Q in U^n is Γ_∞-invariant. The r-plane Q has a description of the form $Q = V + a$, where V is the r-dimensional vector subspace of E^{n-1} and a is a vector in Q. By Corollary 5.23, we have

$$\Lambda^{bl}(\Gamma_p) \subset \{ (p, \frac{v}{|v|}) \in \mathrm{Bl}_p(\mathbb{R}^n) : v \in V \backslash \{0\} \}.$$

Conversely, for $v \in V \backslash \{0\}$ and any integer $n > 0$, there is an element γ_n of Γ_∞ such that $\gamma_n a \in N(nv + a, R)$, where R is the Euclidean diameter of a fundamental domain of Q for Γ_∞. By Lemma 5.22,

$$\lim_{n\to\infty} \gamma_n a = (p, \frac{v}{|v|}) \in \Lambda^{bl}(\Gamma_p).$$

Hence, we have

$$\Lambda^{bl}(\Gamma_p) = \{ (p, \frac{v}{|v|}) \in \mathrm{Bl}_p(\mathbb{R}^n) : v \in V \backslash \{0\} \} \cong S^{r-1}.$$

Second, we show that $\Lambda^{bl}(\Gamma) \cap \pi_C^{-1}(p) = \Lambda^{bl}(\Gamma_p)$. Let a be a point in $\Lambda^{bl}(\Gamma) \cap \pi_C^{-1}(p)$. There exist a point $x \in B^n$ and a sequence $\{g_i\}$ of Γ such that $a = \lim_{n\to\infty} g_i x$. If the sequence $\{g_i\}$ has infinitely many elements of Γ_p, by the definition of $\Lambda^{bl}(\Gamma_p)$, we have $a \in \Lambda^{bl}(\Gamma_p)$. Otherwise, there exists a proper cusped region $U(Q,r)$ such that $g_i x \in N(Q,r)$ for infinitely many $i \in \mathbb{N}$. By Corollary 5.23, a is in $\Lambda^{bl}(\Gamma_p)$. Hence, we have $\Lambda^{bl}(\Gamma) \cap \pi_C^{-1}(p) \subset \Lambda^{bl}(\Gamma_p)$. The converse inclusion is obvious. □

LEMMA 5.25. *For any $x \in B^n$,*

$$\Lambda^{bl}(\Gamma) = \overline{\Gamma x} \cap \partial B^{bl}.$$

In particular, the blown-up limit set of Γ is a Γ-invariant closed subset of B^{bl}.

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Proof. By the definition of $\Lambda^{bl}(\Gamma)$, it is obvious that $\Lambda^{bl}(\Gamma) \supset \overline{\Gamma x} \cap \partial B^{bl}$.

Conversely, for any $a \in \Lambda^{bl}(\Gamma)$, we put $\overline{a} = \pi_C(a) \in \Lambda(\Gamma)$. If $\overline{a}$ does not lie in the set C of cusped limit points of Γ, we can show that a is a limit point of Γ in ∂B^{bl} in the same way as a limit point of Γ in ∂B^{n-1}. If $a \in C$, applying Lemma 5.24 to the cusped limit point a,

$$a \in \Lambda^{bl}(\Gamma) \cap \pi_C^{-1}(\overline{a}) = \Lambda^{bl}(\Gamma_{\overline{a}}) = \overline{\Gamma_{\overline{a}} x} \cap \partial B^{bl} \subset \overline{\Gamma x} \cap \partial B^{bl}.$$

Therefore, $\Lambda^{bl}(\Gamma) \subset \overline{\Gamma x} \cap \partial B^{bl}$. □

LEMMA 5.26. *Let Γ be a non-elementary geometrically finite subgroup of $M(B^n)$. Then,*

$$\Lambda^{bl}(\Gamma) = \mathrm{St}_C(\Lambda(\Gamma)).$$

Proof. By Lemma 5.25, we have $\Lambda^{bl}(\Gamma) \subset \overline{\pi_C^{-1}(\Lambda(\Gamma) \backslash C)} = \mathrm{St}_C(\Lambda(\Gamma))$.

Conversely, it is enough to show that $\mathrm{St}_C(\Lambda(\Gamma)) \cap \pi_C^{-1}(p) \subset \Lambda^{bl}(\Gamma_p)$ for each $p \in C$. By conjugating Γ in $M(B^n)$, we can suppose $p = e_n$. As a similar way in Lemma 5.24, we pass the upper half-space model U^n, and Γ acts on U^n. Let $U(Q,r)$ be a proper cusped region for Γ_∞ based at ∞, where Q is a Γ_∞-invariant r-plane of E^{n-1} such that Q/Γ_∞ is compact. Then, $\Lambda(\Gamma) \subset N(Q,r) \cup \{\infty\}$. There exist a vector subspace V of E^{n-1} and a vector a of Q such that $Q = V + a$. By Corollary 5.23 and Lemma 5.24, we have

$$\mathrm{St}_C(\Lambda(\Gamma)) \cap \pi_C^{-1}(p) \subset \left\{(e_n, \frac{v}{|v|}) : v \in V \backslash \{0\}\right\} = \Lambda^{bl}(\Gamma_p).$$ □

DEFINITION 5.27. Let X be an algebraic K3 surface with $\rho(X) \geq 3$, and let V (resp. V_X) be the same as in Definition 5.10. We denote the strict transform of ∂A_X (resp. $\overline{A_X}$) at V_X by ∂A_X^{bl} (resp. $\overline{A_X}^{bl}$). We call ∂A_X^{bl} the *blown-up boundary* associated with the ample cone of X.

The following result is given by Baragar [3]. By using the geometrical finiteness of $O^+(NS(X))$, we can give another proof of the result.

THEOREM 5.28 [3, Theorem 3.3]. *For an algebraic K3 surface X with $\rho(X) \geq 3$, the set $\partial A_X \backslash \Lambda_X$ is equal to $V \backslash V_X = \{c \in V : \mathrm{Aut}(A(X))_c$ is finite$\}$.*

Proof. Let x be a point of A_X. For any $y \in \partial A_X$, we denote the hyperbolic line connecting x to y by $l_{x,y}$.

Let P be the Dirichlet domain of $O^+(NS(X))$ centered at x. Then, there exists a horoball $U(c)$ based at c such that:

(1) $\{\overline{U(c)} : c \in V_{NS}\}$ are mutually disjoint;
(2) $gU(c) = U(gc)$ for any element $g \in O^+(NS(X))$;
(3) $\{\overline{U(c)} \backslash c\}$ is locally finite in $\mathbb{H}^n_X$,

where the set V_{NS} is the same as in Definition 5.10. Let V and V_X be as in Definition 5.10. We put $K = \overline{P} \backslash \bigcup_{c \in V} U(c)$. Then, by Lemma 5.9, $\overline{A_X}^H \backslash \bigcup_{c \in V} U(c) = \mathrm{Aut}(A(X)) \cdot K$.

If $y \notin V$, since cusped regions $\{U(c) : c \in V_{NS}\}$ are mutually disjoint, there exists a sequence $\{a_n\}_{n \in \mathbb{N}}$ and $\gamma_n \in \mathrm{Aut}(A(X))$ such that a_n converges to infinity as $n \to \infty$ and $l_{x,y}(a_n) \in \gamma_n K$. Since the Euclidean diameter of $\gamma_n K$ goes to zero as n goes to infinity, we have $y = \lim_{n \to \infty} l_{x,y}(a_n) = \lim_{n \to \infty} \gamma_n x \in \Lambda_X$.

If $y \in V \backslash V_X$ and the rank of the stabilizer group $\mathrm{Aut}(A(X))_y$ is 0, by Lemma 5.12, the point y is isolated in ∂A_X. In particular, y is not in the limit set Λ_X.

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Otherwise, y is in V_X. Since $V_X = \{c \in V : \mathrm{vcd}\,\mathrm{Aut}(X)_c \geq 1\}$, y is a bounded parabolic limit point. In particular, y is in the limit set Λ_X. □

Definition 5.29. Let X be an algebraic K3 surface with $\rho(X) \geq 3$. In Definition 5.11, we defined the limit set Λ_X of X. We denote the blown-up limit set of $\mathrm{Aut}_s(X)$ by Λ_X^{bl}.

Corollary 5.30. *If the symplectic automorphism group $\mathrm{Aut}_s(X)$ of an algebraic K3 surface X with $\rho(X) \geq 3$ is non-elementary, then $\partial A_X^{bl} \backslash \Lambda_X^{bl} = V \backslash V_X$.*

Proof. By Theorem 5.28, we have $\partial A_X \backslash V = \Lambda(\Gamma) \backslash V_X$. Then, by Lemma 5.26, we have the desired equality. □

Let C be the set of cusped limit points of $\mathrm{Aut}(A(X))$. Since $\mathrm{Aut}(A(X))$ is geometrically finite, C is equal to V_X. By Lemma 5.15, shrinking the horoballs $U(c)$ sufficiently small, there exist haroballs $U(c)$ based at $c \in C$ such that:

(1) $\{U(c) : c \in C\}$ are mutually disjoint;
(2) $gU(c) = U(gc)$ for any $g \in \mathrm{Aut}(A(X))$ and $c \in C$;
(3) $\{\overline{U(c)} \backslash \{c\}\}$ is locally finite in B^n;
(4) $U(c)$ meets just the sides of $\overline{A_X}$ incident with c.

Since a horoball $U(c)$ is homeomorphic to its strict transformation on $\overline{B}^{bl}$, we identify $U(c)$ with its strict transformation. To apply the Bestvina–Mess formula (Theorem 4.10) to automorphism groups of K3 surfaces, we construct a homotopy on $\overline{A_X}^{bl} \backslash \cup_{c \in C} U(c)$.

Lemma 5.31. *Let a be a point of A_X, and let $K(a,t) = C(a,t) \cap \overline{A_X} \backslash \cup_{c \in C} U(c)$, where $C(a,t)$ is the closed t-ball centered at a. For every $x \in \overline{A_X} \backslash (\cup_{c \in C} U(c) \cup C)$, let S_x be the smallest (horo)sphere (based) centered at x that meets $K(a,t)$. Then, $K(a,t) \cap S_x$ is a singleton.*

Proof. Suppose that there exists y_1, y_2 in $K(a,t) \cap S_x$ with $y_1 \neq y_2$. Then, the hyperbolic line $[y_1, y_2]$ connecting y_1 to y_2 is contained in $C(a,t) \cap \overline{A_X}$. For any $z \in (y_1, y_2)$, we have $d(x,z) < d(x,y_1) = d(x,y_2)$. Then, we have $z \in \cup_{c \in C} U(c)$. Since horoballs $U(c)$ are mutually disjoint, there exists $c \in C$ such that $(y_1, y_2) \subset U(c)$ and $y_1, y_2 \in K(a,t) \cap \partial U(c)$.

We now pass to the upper half-space model U^n and conjugate $\mathrm{Aut}(A(X))$ so that $c = \infty$. We denote by Σ (resp. w_1, w_2) the horoball (resp. the point) on U^n corresponding to $\partial U(c)$ (resp. y_1, y_2). By Lemma 5.15, the Euclidean segment l connecting y_1 to y_2 is contained in $K(a,t) \cap \Sigma$. This contradicts the minimality of S_x. □

Definition 5.32. Let $K(a,t)$ be the same as in Lemma 5.31. We put

$$U = \cup_{c \in C} U(c) \cup C.$$

For every $x \in \overline{A_X} \backslash U$ and $t \geq 0$, we define the function $\rho_a : \overline{A_X} \backslash U \times [0, \infty) \to \overline{A_X} \backslash U$ as follows: if $x \in K(a,t)$, we define $\rho_a(x,t) = x$, otherwise, we define $\rho_a(x,t)$ by the unique point of $K(a,t) \cap S_x$.

In the case $t = 0$, for any $x \in \overline{A_X} \backslash U$, we have $\rho_a(x,0) = a$. Then, the restriction of ρ_a at $t = 0$ is a constant map.

Proposition 5.33. *Let ρ_a be the same as above. Then, ρ_a is continuous.*

Proof. If ρ_a is not continuous, there exists a sequence $\{(x_n, t_n)\}$ such that (x_n, t_n) converges to (x,t) and $\rho_a(x_n, t_n)$ converges to $y \neq \rho_a(x,t)$.

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If x is in B^n, passing to a subsequence of $\{(x_n,t_n)\}$, we can assume that each x_n is in B^n. By the definition of $\rho_a(x_n,t_n)$ (resp. $\rho_a(x,t)$), there exists r_n (resp. r) such that $S(x_n,r_n)\cap K(a,t_n)=\{\rho_a(x_n,t_n)\}$ (resp. $S(x,r)\cap K(a,t)=\{\rho_a(x,t)\}$). Passing to a subsequence of (x_n,t_n), we can suppose that r_n converges to r_0. Since $d((x_n,\rho_a(x_n,t_n))=r_n$, we have $d(x,y)=r_0$. By the definition of $\rho_a(x,t)$, we obtain $r_0>r$. Hence, for a sufficiently large n, we have $r_n>d(x_n,\rho_a(x_n,t_n))$. This contradicts the definition of r_n.

If x is in ∂B^n, passing to the upper half-space model U^n, there exists a (horo)sphere S_k of U^n (based) centered at x_k such that $S_k\cap K(a,t_k)=\{\rho_a(x_k,t_k)\}$. We denote the Euclidean center (resp. radius) of S_k by $p_k=(p_{k,1},\cdots,p_{k,n})$ (resp. r_k). Then, we have that $|p_k-\rho_a(x_k,t_k)|=r_k$, $|x_k-p_k|\leq r_k$, and $r_k\leq p_{k,n}$, where $|\ |$ is the Euclidean norm. Passing to a subsequence of $\{(x_n,t_n)\}$, we can assume that p_k (resp. r_k) converges to some point $p=(p_1,\cdots,p_n)$ (resp. $r\geq 0$). As k goes to infinity, we have

$$|p-y|=r,\ \ |x-p|\leq r,\ \text{ and } r\leq p_n.$$

Since x is in ∂B^n, we have that $p=(x_1,\cdots,x_{n-1},r)$ and $|x-p|=r$. Since $|p-y|=r$, this contradicts the uniqueness of $\rho_a(x,t)$. □

PROPOSITION 5.34. *ρ_a extends to $\overline{\rho_a}:\overline{A_X}^{bl}\setminus\cup_{c\in C}U(c)\times[0,\infty)\to\overline{A_X}^{bl}\setminus\cup_{c\in C}U(c)$.*

Proof. Let x be a point in $\overline{A_X}^{bl}\setminus\cup_{c\in C}U(c)$ with $\pi_C(x)=c$. We now pass to the upper half-space model U^n so that $a=e_n$ and $c=\infty$ by an isometry $\phi:B^n\to U^n$. Let Σ be the horosphere $\{x\in U^n:x_n=r\}$ based at ∞ corresponding to $U(c)$.

If $K(a,t)\cap\Sigma=\varnothing$ or a singleton, we define $\overline{\rho_a}(x,t)$ by the point of $K(a,t)$ whose n-th element is maximal in $K(a,t)$.

Otherwise, by Lemma 5.15, $\overline{A_X}\cap\Sigma$ is a $(n-1)$-dimensional Euclidean polyhedron, and $K(a,t)\cap\Sigma$ is a convex subset of Σ. Here, there exists the half-line l_x on Σ starting at re_n such that $\mathrm{St}_C(\phi^{-1}\circ l)\cap\pi_C^{-1}(c)=\{x\}$. For any $t\geq 0$, there exists a sufficiently large $R_t>0$ such that $\rho_a(x,s)$ is in Σ for any $s\in(0,t]$ and any x with $|x|>R_t$. Then, if we define $\overline{\rho_a}(x,t)$ by the intersection of $K(a,t)$ with l_x, this is the desired extension of ρ_a. □

THEOREM 5.35. *Let X be an algebraic K3 surface with $\rho(X)\geq 3$. If $\mathrm{Aut}_s(X)$ is non-elementary, then $(\overline{A_X}^{bl}\setminus\cup_{c\in C}U(c),\partial A_X^{bl})$ is a $\mathcal{Z}$-structure for some torsion-free subgroup of $\mathrm{Aut}(X)$ of finite index. In particular,*

$$\mathrm{vcd}(\mathrm{Aut}(X))=\dim\partial A_X^{bl}+1.$$

Proof. See Definition 4.9 for a $\mathcal{Z}$-structure, and Theorem 4.10. By Selberg's Lemma, there exists a torsion-free subgroup Γ of $\mathrm{Aut}(X)$ of finite index. It suffices to verify that Γ satisfies the conditions (1), (2), and (3) in Definition 4.9. As for the condition (2), since Γ is torsion-free and its action on $\mathbb{H}^n_X$ is discontinuous, then this action is a covering space action. The compactness of $(\overline{A_X}\setminus\cup_{c\in C}U(c))/\Gamma$ can be concluded from that of $(\overline{A_X}\setminus\cup_{c\in C}U(c))/\mathrm{Aut}(X)$. The condition (3) immediately follows from the Lebesgue Covering Lemma. Therefore, the essential part of the proof is to show that ∂A_X^{bl} is a $\mathcal{Z}$-set of $\overline{A_X}^{bl}\setminus\cup_{c\in C}U(c)$.

We define the homotopy $H:(\overline{A_X}^{bl}\setminus\cup_{c\in C}U(c))\times[0,1]\to\overline{A_X}^{bl}\setminus\cup_{c\in C}U(c)$ setting

$$H(t,x)=\begin{cases}(x,t) & (t=1)\\ \rho_a(x,\log\frac{1+t}{1-t}) & (t\neq 1).\end{cases}$$

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By Proposition 5.34 and the definition of ρ_a, H is continuous. Hence, ∂A_X^{bl} is a $\mathcal{Z}$-set of $\overline{A}_X^{bl}\setminus\cup_{c\in C}U(c)$. Then, $(\overline{A}_X^{bl}\setminus\cup_{c\in C}U(c),\partial A_X^{bl})$ is a $\mathcal{Z}$-structure for Γ. By Theorem 4.10, we have the desired equality. □

COROLLARY 5.36. *If Λ_X is homeomorphic to a Cantor set and X has an elliptic fibration whose rank is larger than 0, then*

$$\mathrm{vcd}(\mathrm{Aut}(X)) = \max \mathrm{MW}(f : X \to \mathbb{P}^1),$$

where f runs all the elliptic fibrations of X.

Proof. For any $c \in C$, we have $\Lambda_X^{bl} \cap \pi_C^{-1}(c) \cong S^{r-1}$, where r is the rank of $\mathrm{Aut}(A(X))_c$. By Proposition 5.16, r is equal to the rank of the Mordell weil group corresponding to c. If $c \notin C$, since Λ_X is totally disconnected, $\{c\}$ itself is an open neighborhood. Then, the topological dimension of Λ_X^{bl} is equal to $\max \mathrm{MW}(f : X \to \mathbb{P}^1)$. □

EXAMPLE 5.37. Let X be the K3 surface whose Néron–Severi lattice $NS(X)$ is given by

$$NS(X) = \begin{pmatrix} 2 & 4 & 1 \\ 4 & 2 & 0 \\ 1 & 0 & -2 \end{pmatrix}.$$

Baragar [4] studied the automorphism group of this K3 surface and the growth of orbits of curves. He showed that Λ_X is a Cantor set and $\mathrm{Aut}(A(X))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}*\mathbb{Z}/2\mathbb{Z}*\mathbb{Z}/2\mathbb{Z}$.

Although the automorphism group of X is completely determined in [4] as above and then this is not a new example of computing the virtual cohomological dimensions, we can also apply Corollary 5.36 to this K3 surface X.

5.3 The ample cones of K3 surfaces and sphere packings.

Let X be a K3 surface, and let $\phi : \mathbb{H}_X^n \to B^n$ be an isometry from $\mathbb{H}_X^n$ to the conformal ball model B^n. The isometry ϕ induces the natural homeomorphism from $\partial\mathbb{H}_X^n$ to ∂B^n. For a root δ and an ample class a of X, we set

$$H_\delta = \{x \in \mathbb{H}_X^n : d(x,a) = d(x,s_\delta a)\},$$
$$H_\delta^+ = \{x \in \mathbb{H}_X^n : d(x,a) < d(x,s_\delta a)\}.$$

The closure of H_δ (resp. H_δ^+) in $\overline{\mathbb{H}_X^n}$ is denoted by $\overline{H_\delta}$ (resp. $\overline{H_\delta^+}$). We set

$$\mathcal{S}_X = \{\delta \in \Delta^+(X) : H_\delta \cap \overline{A_X}^H \text{ is a side of } \overline{A_X}^H\},$$

where $\overline{A_X}^H$ is the closure of A_X in $\mathbb{H}_X^n$.

DEFINITION 5.38. Let δ be an element of $\mathcal{S}_X$. We define the *sphere associated with* δ by $S_\delta = \phi(\overline{H_\delta}) \cap \partial B^n$, which is isomorphic to the $(n-2)$-sphere. We define $B_\delta = \phi(\overline{H_\delta^+}\setminus\overline{H_\delta}) \cap \partial B^n$. We call B_δ the *open ball associated with* δ.

By Proposition 5.7, we have $A_X = \cap_{\delta\in\mathcal{S}_X} H_\delta^+$ and $\partial A_X = \cap_{\delta\in\mathcal{S}_X}\overline{B_\delta}$, where $\overline{B_\delta}$ is the closure of B_δ in ∂B^n.

DEFINITION 5.39. The boundary ∂A_X associated with the ample cone of X is a *sphere packing* if for any two different δ_1, δ_2 in $\mathcal{S}_X$, their associated spheres S_{δ_1} and S_{δ_2} are either disjoint or tangent to each other.

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∂A_X is called a *connected sphere packing* if ∂A_X is a sphere packing, and for any two different δ, $\delta^{'}$ in $\mathcal{S}_X$, there exist roots $\delta_1,\dots,\delta_n$ in $\mathcal{S}_X$ such that $\delta=\delta_1$, $\delta^{'}=\delta_n$, and two spheres S_{δ_i} and $S_{\delta_{i+1}}$ are tangent for $i=1,\dots,n-1$.

THEOREM 5.40. *Let X be an algebraic elliptic K3 surface with $\rho(X)\geq 3$. If ∂A_X is a connected sphere packing, and every elliptic divisor of X corresponds to the tangent point of some two spheres associated with elements of $\mathcal{S}_X$, then*

$$\mathrm{vcd}(\mathrm{Aut}(X))=\max\{\mathrm{rk}\,\mathrm{MW}(f)\}=\rho(X)-3,$$

where f runs all elliptic fibrations of X.

Proof. As ∂A_X is connected, we can sort the set of the open balls associated with elements of $\mathcal{S}_X$ to $\{B_i\}_{i\in\mathbb{N}}$ such that for any $i\in\mathbb{N}$, the intersection of B_i with $\cup_{j=1}^{i-1}B_j$ is not empty. Let $\mathcal{P}_m$ be the set $\{p : p\in B_i\cap B_j \text{ for some two distinct } i,j\leq m\}$. We define the following sets:

- $U_m=\bigcap_{i=1}^{m}B_i$;
- $F_m=\bigcap_{i=1}^{m}\overline{B_i}$;
- $F_m^{bl}=\mathrm{St}_{\mathcal{P}_m}(F_m)=$ the strict transform of F_m at $\mathcal{P}_m$.

For any $l<m$, let $\pi_{\mathcal{P}_l,\mathcal{P}_m}$ be the same as in Definition 5.19. The map $\pi_{\mathcal{P}_l,\mathcal{P}_m}$ restricts to a map from F_m^{bl} to F_l^{bl}. These maps form a projective system. Since ∂A_X is a subset of F_m for any $m\in\mathbb{N}$, there exists the natural map $\phi_m:\partial A_X^{bl}\to F_m^{bl}$. Then, we have the natural map $\phi:\partial A_X^{bl}\to\varprojlim F_m^{bl}$. The map ϕ is a homeomorphism. To show this, we consider the following:

$$\psi_m:\varprojlim F_i^{bl}=\varprojlim \mathrm{St}_{\mathcal{P}_i}(F_i)\to\cap_{i\in\mathbb{N}}\mathrm{St}_{\mathcal{P}_m}(F_i)=\mathrm{St}_{\mathcal{P}_m}(\partial A_X).$$

Since $\partial A_X^{bl}=\varprojlim\mathrm{St}_{\mathcal{P}_m}(\partial A_X)$, the maps ψ_m induce the natural map $\psi:\varprojlim F_m^{bl}\to\partial A_X^{bl}$, which is the inverse of ϕ.

For each $m\in\mathbb{N}$, F_m^{bl} is a topological manifold with its interior U_m, and then F_m^{bl} and U_m are homotopy equivalent. Hence, we have

$$H^*(\partial A_X^{bl},\mathbb{Z})\cong\lim_{m\to\infty}H^*(F_m^{bl},\mathbb{Z})\cong\lim_{m\to\infty}H^*(U_m,\mathbb{Z}). \tag{5.4}$$

For each $m\in\mathbb{N}$ and a sufficiently small $\epsilon>0$, we define V_{m+1} by the ϵ-neighborhood of $\partial B^n\backslash B_{m+1}$ in U_m. Then, we can see as follows:

- V_{m+1} is contractible;
- $U_{m+1}\cup V_{m+1}=U_m$;
- $U_{m+1}\cap V_{m+1}$ is homotopy equivalent to the $(n-2)$-sphere with finite k punctures,

where k is the number of the set $\{p : p \text{ is a tangent point of } B_i \text{ and } B_{m+1} \text{ for some } i\leq m\}$. By Mayer–Vietoris exact sequence and the $(n-2)$-sphere with k punctures is homotopy equivalent to the wedge sum of $(k-1)$-copies of S^{n-3}, we have the following exact sequence:

$$\cdots\to H^*(U_m)\to H^*(U_{m+1})\oplus H^*(V_{m+1})\to H^*(\vee_{i=1}^{k-1}S^{n-3})\to H^{*+1}(U_m)\to\cdots.$$

Therefore, we can inductively conclude that

$$H^*(U_m)=0\quad(*\geq n-2).$$

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In particular, by Theorem 5.35 and the equation 5.4, we have

$$\begin{aligned}\operatorname{vcd}\operatorname{Aut}(X) &= \max\{m : H_c^m(\overline{A_X}^H \setminus \cup_{c\in C} U(c), \mathbb{Z}) \neq 0\}\\ &= \max\{m : H^m(\partial A_X^{bl}, \mathbb{Z}) \neq 0\} + 1\\ &\leq n-2 = \rho(X) - 3,\end{aligned}$$

where H_c^* is the cohomology with compact support. Lemma 5.13 implies that there exists an elliptic fibration f such that $\operatorname{rk}\operatorname{MW}(f) = \rho(X) - 3$. □

Example 5.41. The Apollonian circle packing is generated as follows: we begin with four mutually tangent circles. There is four curvlinear triangles. We can put the inscribed circle for each curvlinear triangle, so that the curvlinear triangle separated into new curvlinear triangles. Inductively, we put inscribed circles for each curvelinear triangle, and the closure of the union of the inscribed circles is the Apollonian circle packing, which is shown in the following Figure 1.

Baragar [2] constructed a K3 surface Y_2 with ample cone the Apollonian circle packing, whose Néron–Severi lattice is given by

$$\begin{pmatrix} -2 & 2 & 2 & 4\\ 2 & -2 & 2 & 4\\ 2 & 2 & -2 & 0\\ 4 & 4 & 0 & 0\end{pmatrix}.$$

By Theorem 5.40, we have $\operatorname{vcd}\operatorname{Aut}(Y_2) = 1$.

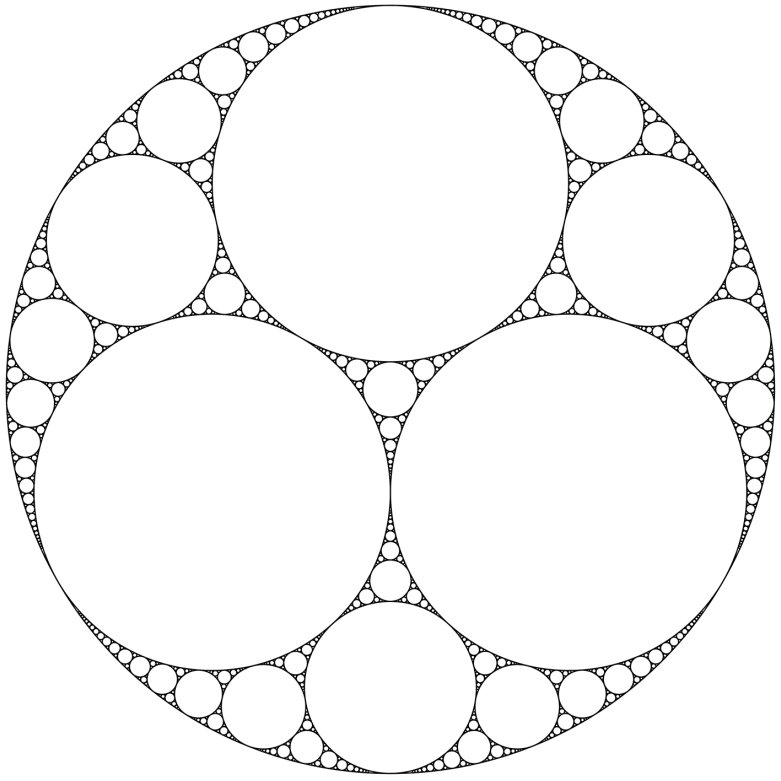


Figure 1.
The Apollonian circle packing.

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EXAMPLE 5.42. The Apollonian sphere packing is the three-dimensional analog of the Apollonian circle packing. We begin with five mutually tangent spheres, and in the space between any four of them, we put the inscribed sphere. As the same manner in the previous example, we can inductively put a sphere so that this sphere touch other four spheres. The closure of the union of the inscribed spheres is the Apollonian sphere packing. Baragar [2] also constructed a K3 surface Y_3 with ample cone the Apollonian sphere packing, whose Néron–Severi lattice is given by

$$\begin{pmatrix} -2 & 2 & 2 & 2 & 4 \\ 2 & -2 & 2 & 2 & 4 \\ 2 & 2 & -2 & 2 & 4 \\ 2 & 2 & 2 & -2 & 0 \\ 4 & 4 & 4 & 0 & 0 \end{pmatrix}.$$

By Theorem 5.40, we have $\operatorname{vcd}\operatorname{Aut}(Y_3) = 2$.

§6. Counterexamples to Mukai's conjecture for elliptic K3 surfaces.

In this section, we construct counterexamples to Mukai's conjecture for elliptic K3 surfaces. Namely, we use hyperbolic geometry to construct the Néron–Severi lattice of K3 surfaces such that the virtual cohomological dimension of its automorphism group is equal to 1, while the maximal rank of Mordell–Weil groups is equal to 0.

6.1 A Transformation from the upper half-plane to the hyperboloid model.

To study the action of the automorphism group of a K3 surface on the upper half-plane U^2, we construct a standard transform from U^2 to the hyperboloid model of the hyperbolic plane.

Let U^2 be the upper half-plane. It is well-known that it can be identified with the subset of $\mathbb{C}$ consisting of elements with a positive imaginary part, and the group of linear fractional transformations corresponds to the group of orientation-preserving isometries of U^2. Moreover, there exists an isometry from U^2 to the conformal ball model B^2 of the hyperbolic plane defined by

$$\sigma : U^2 \to B^2 : z \mapsto \frac{z-i}{z+i},$$

where i is the imaginary unit. Here σ is called the *Cayley transform*. Let $V^{1,2}$ be a 3-dimensional vector space with the inner product $\langle\ ,\ \rangle$, where $\langle x,x\rangle = x_0^2 - x_1^2 - x_2^2$ for $x = (x_0, x_1, x_2) \in V^{1,2}$. Then, the hyperboloid model H^2 associated with $V^{1,2}$ is defined by

$$H^2 = \{x \in V^{1,2} : \langle x,x\rangle = 1, x_0 > 0\}.$$

The stereographic projection ζ is defined by the formula

$$\zeta(x) = \left(\frac{1+\|x\|^2}{1-\|x\|^2}, \frac{2x_1}{1-\|x\|^2}, \frac{2x_2}{1-\|x\|^2}\right).$$

Here, the map $\zeta \circ \sigma : U^2 \to H^2$ is an isometry from U^2 to H^2. This map can be extended to $\overline{U^2}$ to $\overline{H^2}$, also denoted by $\zeta \circ \sigma$.

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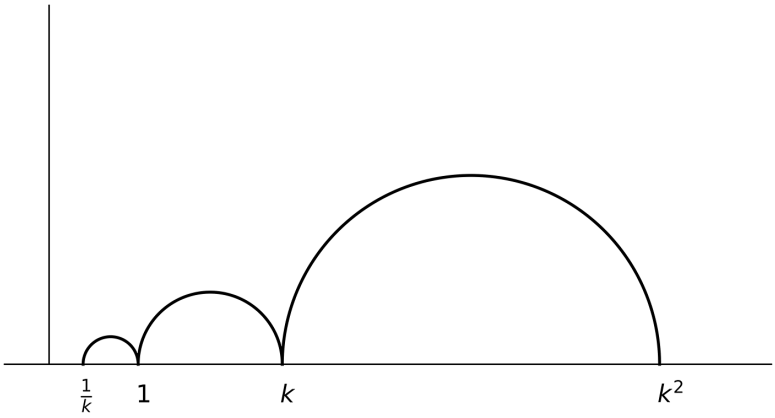


Figure 2.
Geodesics connectiong k^n to k^{n+1} $(k=-1,0,1)$.

Let us recall that $\partial U^2=\mathbb{R}$ and $\partial H^2=\{x\in V^{1,2}:\langle x,x\rangle=0,x_0>0\}/\mathbb{R}_{>0}$. Then, for every $x\in\mathbb{R}$, we have $\sigma(x)=(\frac{x^2-1}{x^2+1},\frac{-2x}{x^2+1})$, and we may take the point $(1,\frac{x^2-1}{x^2+1},\frac{-2x}{x^2+1})$ as a representative of the class of $\zeta\circ\sigma(x)$.

Here, we fix a positive real number k. Considering the set $\{k^n:n\in\mathbb{Z}\}$, and the geodesics of U^2 connecting k^n to k^{n+1}, we have the following Figure 2.

From now on, we will construct the Néron–Severi lattice of K3 surfaces whose elliptic divisors correspond to the points $k^n(n\in\mathbb{Z})$, and (-2)-curves correspond to geodesics connecting k^n to k^{n+1}. We define the isotropic vector

$$e_n=(1,\frac{k^{2n}-1}{k^{2n}+1},\frac{-2k^n}{k^{2n}+1})\in V^{1,2},$$

which corresponds to k^n. Next, let δ_n be the element of $V^{1,2}$ with $\delta_n^2=-2$ such that $\langle\delta_n,e_n\rangle=\langle\delta_n,e_{n+1}\rangle=0$. Then, by the direct computation, we have

$$\delta_n=\pm(\frac{k^{2n+1}+1}{k^n(k-1)}\sqrt{2},\frac{k^{2n+1}-1}{k^n(k-1)}\sqrt{2},-\frac{k+1}{k-1}\sqrt{2}).$$

Here, we want δ_n to be an effective (-2)-curve, so its intersection with $(1,0,0)\in V^{1,2}$ should be positive. Then, we define

$$\delta_n=(\frac{k^{2n+1}+1}{k^n(k-1)}\sqrt{2},\frac{k^{2n+1}-1}{k^n(k-1)}\sqrt{2},-\frac{k+1}{k-1}\sqrt{2}).$$

Let $L(m)$ be the group generated by δ_1, δ_2, and δ_3, where $m=k+\frac{1}{k}$. Then, by the direct computation, the Gram matrix of $L(m)$ is given by

$$\begin{pmatrix}-2&2&6+4m\\2&-2&2\\6+4m&2&-2\end{pmatrix}.$$

In particular, $L(m)$ is a lattice when $m=k+\frac{1}{k}$ is an integer, so we assume $m\in\mathbb{Z}$ from now on.

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PROPOSITION 6.1. *Let $m = k + \frac{1}{k}$ be a positive integer, and let $L(m)$ be the same as above. Then, we have*

$$\delta_{n+3} = \delta_n - (m+1)\delta_{n+1} + (m+1)\delta_{n+2},$$

for any $n \in \mathbb{Z}$. In particular, for any $n \in \mathbb{Z}$, δ_n is an element of the lattice $L(m)$.

Proof. This equality can be confirmed by a straightforward computation. However, we will provide a proof by using hyperbolic geometry.

Let $v_1 = -\delta_1 + \delta_3$, $v_2 = -\delta_2 + \delta_3$, $v_3 = \delta_1 - m\delta_2 + \delta_3$. Then, the reflection s_i associated with v_i is defined by

$$s_i(u) = u - 2\frac{\langle v_i, u\rangle}{\langle v_i, v_i\rangle} v_i.$$

As for the reflection s_1, we have

$$\begin{aligned}\langle v_1, v_1\rangle &= (-\delta_1 + \delta_2)^2 = -4 - 2(6+4m) = -8(2+m),\\ \langle v_1, u\rangle &= \langle -\delta_1 + \delta_2, x\delta_1 + y\delta_2 + z\delta_3\rangle = 4(2+m)(x-z),\end{aligned}$$

where $u = x\delta_1 + y\delta_2 + z\delta_3$. Then, $s_1(u) = z\delta_1 + y\delta_2 + x\delta_3$, and so the matrix representation A_1 of s_1 with the basis δ_1, δ_2 and δ_3 is given by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let A_2, A_3 be the matrix representations of s_2, s_3 with the same basis as above, respectively. Then, we have

$$A_2 = \begin{pmatrix} 1 & 0 & 0 \\ -(1+m) & 0 & 1 \\ -(1+m) & 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1+\frac{2}{m-2} & \frac{2}{m-2} & \frac{2}{m-2} \\ -\frac{2m}{m-2} & 1-\frac{2m}{m-2} & -\frac{2m}{m-2} \\ \frac{2}{m-2} & \frac{2}{m-2} & 1+\frac{2}{m-2} \end{pmatrix}.$$

Here, v_3 is orthogonal to the vectors $(1,0,0), (0,1,0) \in V^{1,2}$. Since the vectors $(1,1,0), (1,-1,0) \in V^{1,2}$ are representatives of $0, \infty \in U^2$, then the set of fixed points of s_3 is equal to the positive part of the imaginary axis of U^2. Therefore, s_3 acts on U^2 as the reflection with respect to the imaginary axis. In the same way, because $\langle v_2, \delta_2 + \delta_3\rangle = \langle v_2, v_3\rangle = 0$, s_2 is the reflection with respect to the geodesic which is orthogonal to the imaginary axis and passes through k^3. Because $\langle v_1, \delta_2\rangle = \langle v_1, v_3\rangle = 0$, s_1 is the reflection with respect to the geodesic which is orthogonal to the imaginary axis and the geodesic defined by δ_2.

We set $\phi = s_2 \circ s_1$. Since s_1, s_2 are reflections with respect to semicircles centered at the origin, ϕ acts on U^2 as a multiplication by a scalar.

Since $\phi(\delta_2) = s_2(\delta_2) = \delta_3$, comparing the endpoints of the geodesic defined by δ_2, we find $\phi(\{k^2, k^3\}) = \{k^3, k^4\}$. Therefore, the action of ϕ on U^2 is equal to the multiplication by k.

Let R be the matrix representation of ϕ. Then, we have

$$R = A_2 A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -(1+m) \\ 0 & 1 & (1+m) \end{pmatrix}.$$

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Hence, we get

$$\delta_4 = \delta_1 - (m+1)\delta_2 + (m+1)\delta_3.$$

Applying ϕ^n to this equation, we have the desired equation. □

From the above proof, we obtain the following proposition.

PROPOSITION 6.2. *Let A_1, A_2, and A_3 be as above. If $m = 3$ or 4, then the matrices A_1, A_2, and A_3 are isometries of the lattice $L(m)$. Moreover, they induce isometries of the upper half-space U^2.*

6.2 Counterexamples.

We will show that K3 surfaces whose Néron–Severi lattice is isomorphic to $L(3)$, or $L(4)$ are counterexamples to Mukai's conjecture.

THEOREM 6.3. *Let X be a K3 surface whose Néron–Severi lattice is isomorphic to $L(3)$ or $L(4)$. Then, the virtual cohomological dimension of its automorphism group is equal to 1, whereas every Mordell–Weil group has rank 0.*

Proof. In the case $m = 3$, we have

$$v_3 = (1, -3, 1), \quad L(3) = \begin{pmatrix} -2 & 2 & 18 \\ 2 & -2 & 2 \\ -18 & 1 & -2 \end{pmatrix}.$$

First, we show that any geodesics defined by (-2)-elements do not intersect with the imaginary axis. Let $\delta = x\delta_1 + y\delta_2 + z\delta_3$ be such an element. Then, since $\langle \delta, v_3 \rangle = 0$ and $\delta^2 = -2$, we have $x + y + z = 0$, $x^2 + y^2 + z^2 - 2xy - 2yz - 18xz = 1$. By eliminating y, we obtain $4(x^2 + z^2 - 3xz) = 1$. This is a contradiction.

Let D be the fundamental domain for $W(X)$ on U^2 that contains the positive part of the imaginary axis. Since A_1, A_2, and A_3 preserve it, we have $A_1, A_2, A_3 \in \mathrm{Aut}(D)$. Then, some power of ϕ is an automorphism of X, and so $\mathrm{vcd}(\mathrm{Aut}(X)) \geq 1$.

Here, the set of facets of D naturally corresponds to the set of (-2)-elements δ_n and $s_3(\delta_n)$ $(n \in \mathbb{Z})$. Suppose the contrary. There exists a facet of D that separates some k^m from D. Then, applying ϕ^n and $s_3 \circ \phi^n$ $(n \in \mathbb{Z})$ to the facet, we can conclude that D does not contain any isotropic element of $L(3)$. Since $\delta_2 + \delta_3$ is isotropic, this is the contradiction that there exists $w \in W(X)$ such that $w(\delta_2 + \delta_3) \in D$. Therefore, the facets of D are defined by (-2)-element δ_n and $s_3(\delta_n)$ $(n \in \mathbb{Z})$, and each $\pm k^n$ corresponds to an elliptic fibration. Since such an elliptic fibration has rank 0 by Shioda–Tate formula, the maximum rank of Mordell–Weil groups of X is equal to 0. The same holds for $m = 4$. □

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References

[1] R. C. Alperlin, *An elementary account of Selberg's lemma*, L'Enseign. Math. **33** (1987), 269–273.
[2] A. Baragar, *The Apollonian circle packing and ample cones for K3 surfaces*, preprint, arXiv:1708.06061.
[3] A. Baragar, *The ample cone and orbits of curves on K3 surfaces*, preprint.
[4] A. Baragar, *Orbits of curves on certain K3 surfaces*, Compos. Math. **137** (2003), 115–134.
[5] M. Bestvina and G. Mess, *The boundary of negatively curved groups*, J. Am. Math. Soc. **4** (1991), 469–481.
[6] M. Bestvina, *Local homology properties of boundaries of groups*, Michigan Math. J. **43** (1996), no. 1, 123–139.
[7] D. Burns and M. Rapoport, *On the Torelli problem for Kählerian K3-surfaces*, Ann. Sci. Éc. Norm. Supér **8** (1975), 235–273.
[8] K. S. Brown, *Cohomology of Groups*, Springer, New York, NY, 1982.
[9] F. Cossec, I. Dolgachev, and C. Liedtke, *Enriques Surfaces I*, 2nd edition, Springer, Singapore, 2025.
[10] T. Fukaya, *Blown-up corona of relatively hyperbolic groups*, preprint, arXiv:2402.11501.
[11] T. Fukaya and S. Oguni, *Coronae of relatively hyperbolic groups and coarse cohomologies*, J. Topol. Anal. **8** (2016), 431–474.
[12] K. Kikuta, *Geometrical finiteness for automorphism groups via cone conjecture*, preprint, arXiv:2406.18438.
[13] S. Kondo, *The automorphism group of a generic Jacobian Kummer surface*, J. Algebraic Geom. **7** (1998), 589–609.
[14] S. Kondo, *K3 Surfaces (Japanese)*, Kyoritu Shuppan, Tokyo, 2015.
[15] S. Mukai, *Finite groups of automorphisms of K3 surfaces and the Mathieu group*, Invent. Math. **94** (1988), 183–221.
[16] I. Piatetski-Shapiro and I. R. Shafarevich, *A Torelli theorem for algebraic surfaces of type K3*, Math. USSR. **5** (1971), 547–587.
[17] J. G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, 3rd ed., Springer, Cham, 2019.
[18] T. Shioda, *On the Mordell-Weil lattices*, Math. Univ. St. Paul **39** (1990), 211–240.
[19] H. Sterk, *Finiteness results for algebraic K3 surfaces*, Math. Z. **189** (1985), 507–513.

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