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A counter-example to the three balls problem

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We work throughout in the p.l. category (see (8)) consisting of polyhedra and p.l. (piecewise linear) maps. We are concerned with the following problem which is of importance in the theory of p.l. transversality.

The three balls problem. Let B^{n+r}, $B^{n+s} \subset B^{n+r+s}$ denote standard transverse proper sub-balls. Suppose $\tilde{B}^{n+s} \subset B^{n+r+s}$ is another proper sub-ball such that

$$\tilde{B}^{n+s} \cap B^{n+r} = B^{n+s} \cap B^{n+r} = B^n$$

and such that the triples B^{n+r}, $\tilde{B}^{n+s}$, B^{n+r+s} and B^{n+r}, B^{n+s}, B^{n+r+s} are homeomorphic. Suppose further that there is a standard neighbourhood U of $\dot{B}^n$ in $\dot{B}^{n+r+s}$ such that $\tilde{B}^{n+s} \cap U = B^{n+s} \cap U$. Then, is there an isotopy of B^{n+r+s} carrying $\tilde{B}^{n+s}$ onto B^{n+s}, and which is fixed on $B^{n+r} \cup U$? (see Figure 1).

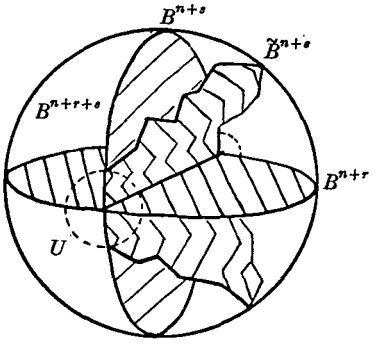


Fig. 1.

This problem was first investigated by Armstrong in connexion with transverse regular neighbourhoods (see (1)); it was restated in (6), where several equivalent problems were given, and in particular it follows from (6) that our counter-example implies that 'block transversality' is strictly stronger than 'transversality'. This seems to contradict an announcement of Morlet (4). The problem is famous for its apparent simplicity and for the firm initial belief of all workers in the field in their ability to solve it in under half-an-hour.

In this paper we give a counter-example for $n = 2$, $s = 4$, $r = 4$, but the techniques used can be employed to give many other such counter-examples. In our example, the

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inclusion of $\tilde{B}^{n+s}$ in B^{n+r+s} is standard inside U but the inclusion cannot even be carried by a homotopy to a homeomorphism onto B^{n+s}, keeping the required subsets fixed. We construct the example in two stages. In section 1 we use some fairly standard homotopy theory to find a homotopy class of maps of a solid torus to itself, modulo its boundary, which is not represented by a homeomorphism; in section 2 we use this and an embedding theorem to deduce the counter-example.

1. *A homotopy class not represented by a homoemorphism.* Let G_q denote the space of (continuous) maps of S^{q-1} to itself of degree ± 1, and let $\tilde{G}_q$ denote the semi-simplicial complex of which a k-simplex is a map $g: \Delta^k \times S^{q-1} \to \Delta^k \times S^{q-1}$ of degree ± 1 which is *block-preserving* over Δ^k, i.e. $g(\Delta^s \times S^{q-1}) = (\Delta^s \times S^{q-1})$ for each face Δ^s of the simplex Δ^k. $\tilde{G}_q$ was defined in ((5); section 5) where it was shown that $\tilde{G}_q$ has the same homotopy type as the (p.l.) singular complex of G_q and hence by Zeeman (7) the same homotopy type as G_q. The semi-simplicial complex $\tilde{P}L_q(\Sigma)$ was also defined in ((5); section 5) where it was shown that both $\tilde{G}_q$ and $\tilde{P}L_q(\Sigma)$ satisfy the extension condition. A k-simplex of $\tilde{P}L_q(\Sigma)$ is a block-preserving homeomorphism $f: \Delta^k \times S^{q-1} \to \Delta^k \times S^{q-1}$. Let

$$i: \tilde{P}L_q(\Sigma) \to \tilde{G}_q$$

be the inclusion.

Now a member of $\Pi_n(\tilde{G}_q)$ is represented by a map $g: \Delta^n \times S^{q-1} \to \Delta^n \times S^{q-1}$ such that $g|\dot{\Delta}^n \times S^{q-1}$ is the identity, and by the result quoted above the equivalence relation is the same as homotopy relative to $\dot{\Delta}^n \times S^{q-1}$. On the other hand, a member of

$$\Pi_n(\tilde{P}L_q(\Sigma))$$

is represented by a homeomorphism $h: \Delta^n \times S^{q-1} \to \Delta^n \times S^{q-1}$ such that $h|\dot{\Delta}^n \times S^{q-1}$ is the identity. Thus the following statements are equivalent.

(A) Every map $f: \Delta^n \times S^{q-1} \to \Delta^n \times S^{q-1}$, such that $f|\dot{\Delta}^n \times S^{q-1}$ is the identity, is homotopic, relative to $\dot{\Delta}^n \times S^{q-1}$, to a homeomorphism.

(B) $i_*: \Pi_n(\tilde{P}L_q(\Sigma)) \to \Pi_n(\tilde{G}_q)$ is onto.

$\tilde{P}L_q(\Sigma)$ is homotopy equivalent to the block bundle group $\tilde{P}L_q$ (see (5)) and by ((6); 5·4) $\Pi_n(\tilde{P}L_q)$ stabilizes for $q \geqslant n+2$. Denote the stable limit by $\tilde{P}L$. By ((5; 6)) and well-known results on Γ_n, $\Pi_n(\tilde{P}L) \cong \Pi_n(SO)$ for $n \leqslant 6$ (SO denotes the special orthogonal group). On the other hand, $\Pi_n(G_q)$ stabilizes for $q \geqslant n+2$ and the stable limit is Π^n, the stable n-stem of the homotopy groups of spheres.

Now $\Pi_2(SO) = 0$ but $\Pi^2 = Z_2$ and so statement (B) fails for $n = 2$, $q = 4$. Thus, immediately, we deduce the following.

PROPOSITION. *There is a map $f: \Delta^2 \times S^3 \to \Delta^2 \times S^3$ with $f|\dot{\Delta}^2 \times S^3$ the identity which is not homotopic relative to $\dot{\Delta}^2 \times S^3$ to a homeomorphism.*

2. *Construction of the counter-example.*

THEOREM. *Suppose there is a map $f: \Delta^t \times S^u \to \Delta^t \times S^u$ such that $f|\dot{\Delta}^t \times S^u$ is the identity, which is not homotopic, relative to $\dot{\Delta}^t \times S^u$, to a homoemorphism. Then, if $n = t$, $r > t+1 \geqslant 3$ and $s = u+1$, there exists a ball $\tilde{B}^{n+s} \subset B^{n+r+s}$ satisfying the hypotheses of the three balls problem, but for which the required isotopy does not exist.*

Thus using the Proposition, there is such a counter-example for $n = 2$, $r = 4$, $s = 4$.

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Proof of theorem. We identify the join $S^u * S^v$ with S^{u+v+1}. The inclusion of S^u in S^{u+v+1} induces an inclusion

$$j: \Delta^t \times S^u \to \Delta^t \times (S^{u+v+1} - S^v).$$

We take any v such that $v > t$ and $v > 1$. Then $jf | \dot{\Delta}^t \times S^u$ is the inclusion. By the embedding theorem of Irwin (2), there is an embedding

$$g: \Delta^t \times S^u \to \Delta^t \times (S^{u+v+1} - S^v)$$

such that g is homotopic to jf keeping $\dot{\Delta}^t \times S^u$ fixed. (Irwin's theorem requires that $v > t$ and $v > 1$.) Now there is a retraction

$$\rho: \Delta^t \times (S^{u+v+1} - S^v) \to \Delta^t \times S^u,$$

so, trivially, $\rho jf = f$. If g is homotopic, relative to $\dot{\Delta}^t \times S^u$, to an embedding $\tilde{g}$ with

$$\tilde{g}(\Delta^t \times S^u) = j(\Delta^t \times S^u),$$

then, denoting 'is homotopic relative to $\dot{\Delta}^t \times S^u$' by $\simeq$,

$$f = \rho jf \simeq \rho g \simeq \rho \tilde{g}: \Delta^t \times S^u \to \Delta^t \times S^u.$$

But $\rho \tilde{g}$ is a homeomorphism, and f has been specifically selected so that this cannot be true. Thus, keeping $\dot{\Delta}^t \times S^u$ fixed, g cannot be carried by a homotopy to an embedding onto $\Delta^t \times S^u$. Hence, regarding g now as an embedding into $\Delta^t \times S^{u+v+1}$, the image of g certainly cannot be carried by an isotopy of $\Delta^t \times S^{u+v+1}$, keeping

$$(\dot{\Delta}^t \times S^u) \cup (\Delta^t \times S^v) \quad \text{fixed, onto} \quad \Delta^t \times S^u. \tag{*}$$

Now the join of two spheres can be taken to be the union of two solid tori with their boundaries identified. Thus

$$S^{t-1} * S^{u+v+1} = (\Delta^t \times S^{u+v+1}) \cup (S^{t-1} \times \Delta^{u+v+2})$$

and this contains subsets

$$S^{t-1} * S^u = (\Delta^t \times S^u) \cup (S^{t-1} \times \Delta^{u+1}),$$

$$S^{t-1} * S^v = (\Delta^t \times S^v) \cup (S^{t-1} \times \Delta^{v+1}).$$

Taking the cone, with vertex c, on this triple, define

$$B^{n+r+s} = c * S^{t-1} * S^{u+v+1},$$

$$B^{n+r} = c * S^{t-1} * S^v,$$

$$B^{n+s} = c * S^{t-1} * S^u,$$

$$\tilde{B}^{n+s} = c * [(g(\Delta^t \times S^u)) \cup (S^{t-1} \times \Delta^{u+1})],$$

$$U = S^{t-1} \times \Delta^{u+v+2}.$$

Thus $B^n = c * S^{t-1}$, and $n = t$, $r = v + 1$, $s = u + 1$. If x is a point of $\dot{B}^n$, there is a ball neighbourhood N of x in $\dot{B}^{n+r+s}$ for which $\dot{B}^{n+s} \cap N$ and $\dot{\tilde{B}}^{n+s} \cap N$ are the same proper sub-ball, and such that $\dot{B}^{n+r} \cap N$ is another sub-ball. $\dot{B}^{n+r} \cup \dot{B}^{n+s}$ can be regarded as a suspension with x as one of the suspension points, and similarly so can $\dot{B}^{n+r} \cup \dot{\tilde{B}}^{n+s}$.

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Then, applying the cone unknotting theorem of (3) to $\dot{B}^{n+r+s} - \mathring{N}$, there is a homeomorphism between the triples $\dot{B}^{n+r}$, $\dot{B}^{n+s}$, $\dot{B}^{n+r+s}$ and $\dot{B}^{n+r}$, $\tilde{B}^{n+s}$, $\dot{B}^{n+r+s}$, and this extends conewise to give the homeomorphism between the two ball triples required in the statement of the problem.

If there exists an isotopy of B^{n+r+s}, fixed on $B^{n+r} \cup U$, carrying $\tilde{B}^{n+s}$ onto B^{n+s}, then, restricting this to $\dot{B}^{n+r+s} - \mathring{U}$, there would be an isotopy of $\Delta^t \times S^{u+v+1}$ fixed on

$$(\dot{\Delta}^t \times S^{u+v+1}) \cup (\Delta^t \times S^v),$$

carrying $g(\Delta^t \times S^u)$ onto $\Delta^t \times S^u$. This would contradict (*), so no such isotopy exists.

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