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Nonlinear force-free configurations in cylindrical geometry

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We find a new family of solutions for force-free magnetic structures in cylindrical geometry. These solutions have radial power-law dependence and are periodic but non-harmonic in the azimuthal direction; they generalize the vacuum z-independent potential fields to current-carrying configurations.

Key words: astrophysical plasmas

1. Introduction

Force-free magnetic configurations satisfying the condition

$$\boldsymbol{B} = \kappa \boldsymbol{J}, \tag{1.1}$$

where $\boldsymbol{B}$ is magnetic field and $\boldsymbol{J}$ is current density, are examples of magnetic structures that may represent the final stages of magnetic relaxation, or can be used as building blocks of plasma models (Lundquist 1951; Woltier 1958; Taylor 1974; Priest & Forbes 2000).

Particular linear examples of force-free equilibria, with spatially constant κ, were considered by Chandrasekhar & Kendall (1957). The most often-used configurations are Lundquist fields in cylindrical geometry (Lundquist 1951) and spheromaks in spherical geometry (Bellan 2000).

Using the self-similar assumption, Lynden-Bell & Boily (1994) (see also Aly 1994) found nonlinear self-similar solutions in spherical geometry. Their model of axially symmetric twisted configurations has been widely used in astrophysical and space applications (e.g. Thompson, Lyutikov & Kulkarni 2002; Shibata & Magara 2011). In the spirit of Lynden-Bell & Boily (1994), in this paper we construct similar nonlinear magnetic configurations in cylindrical geometry.

2. Self-similar configuration in cylindrical geometry

Shafranov (1966) and Grad (1967) formulated what is known as the Grad–Shafranov equation, separating a complicated magnetic configuration in the set of nested/foliated flux surfaces, given by the condition that the flux function P is constant on the

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surface, and the encompassed current flow. Let us look for force-free equilibria that are independent of the coordinate z. The two Euler potentials α and β (or, equivalently, the related Clebsh variables) are

$$\left.\begin{aligned}\alpha &= z,\\ \beta &= P(r,\phi),\end{aligned}\right\} \tag{2.1}$$

while the magnetic field can be written as

$$\boldsymbol{B} = \boldsymbol{\nabla} P \times \boldsymbol{\nabla} z + g(P)\boldsymbol{\nabla} z, \tag{2.2}$$

where g is some function.

Next we introduce a self-similar ansatz

$$\left.\begin{aligned} P(r,\phi) &= r^{-l} f(\phi),\\ g(P) &= \mathcal{C}|P|^p,\\ \boldsymbol{B} = \boldsymbol{\nabla} P \times \boldsymbol{\nabla} z + \mathcal{C}|P|^p \boldsymbol{\nabla} z &= \{f', lf, \mathcal{C}|f|^p\} r^{-(l+1)}.\end{aligned}\right\} \tag{2.3}$$

The absolute value of $|P|$ in the nonlinear term ensures that magnetic field is real ($f(\phi)$ can become negative). Below, any appearance of f to a non-integer power is to be understood to involve $\sqrt{f^2} = |f|$.

By dimensionality (equating radial powers in different terms in (1.1) with magnetic field given by (2.3)),

$$p = 1 + 1/l. \tag{2.4}$$

The equation for f becomes

$$lf'' + l^3 f + \mathcal{C}^2 (1+l) f^{(2+l)/l} = 0 \tag{2.5}$$

(note that the component B_z enters here as B_z^2. This justifies the use of $|P|$).

For vacuum fields $\mathcal{C} = 0$, the above relations reproduce

$$\left.\begin{aligned} P &\propto r^{-m}\sin(m\phi),\\ B_r &\propto r^{-(1+m)} f',\\ B_\phi &\propto m r^{-(1+m)} f,\end{aligned}\right\} \tag{2.6}$$

with integer m.

The first integral is

$$f'^{,2} + l^2 f^2 + \mathcal{C}^2 |f|^{2(1+l)/l} = H_0. \tag{2.7}$$

By redefining $f \to \sqrt{H_0} f$ and $\mathcal{C} \to \mathcal{C} H_0^{-1/(2l)}$, the parameter H_0 can be set to unity,

$$f'^{,2} + l^2 f^2 + \mathcal{C}^2 |f|^{2(1+l)/l} = 1. \tag{2.8}$$

Equation (2.8) is the main equation describing nonlinear force-free structures in cylindrical geometry. It depends on one parameter – the current strength $\mathcal{C}$. For a given $\mathcal{C}$, the value of l is then determined as an eigenvalue problem by requiring periodicity in ϕ, as we describe next.

We can solve for f in quadratures:

$$\phi = \int (\sqrt{1 - l^2 f - \mathcal{C}^2 |f|^{2(1+l)/l}})^{-1}\, \mathrm{d}f \tag{2.9}$$

(so that the integration constant in (2.8) is just a phase ϕ where $f = 0$).

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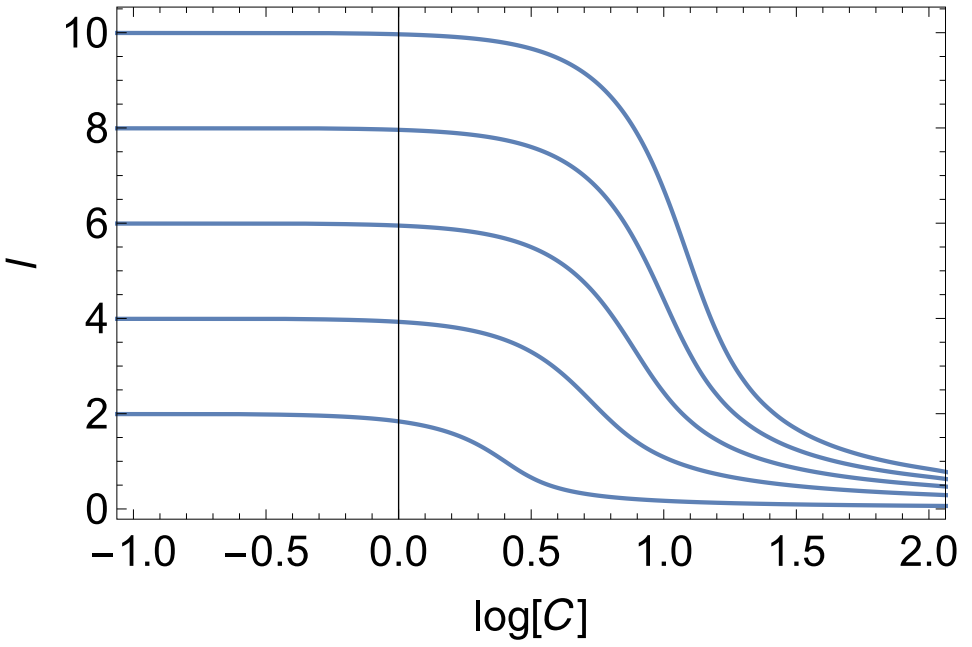


FIGURE 1. Dependence of the radial index l on the current parameter $\mathcal{C}$ for various harmonics $2\tilde{m} = 2, 4, 6, 8, 10$.

Periodicity in ϕ requires

$$\int_0^{f_{\max}} (\sqrt{1 - l^2 f - \mathcal{C}^2 |f|^{2(1+l)/l}})^{-1} \, \mathrm{d}f = \frac{\pi}{2\tilde{m}}, \tag{2.10}$$

where $\tilde{m} = 1, 2 \ldots$ is an integer azimuthal number (see a comment after (3.5) as to why odd solutions, $\propto 2\tilde{m} + 1$, in the denominator, are discarded). The value of $f_{\max}$ satisfies

$$1 - \tilde{m}^2 f_{\max} - \mathcal{C}^2 f_{\max}^{2(1+l)/l} = 0. \tag{2.11}$$

For given $\mathcal{C}$, the relations (2.10)–(2.11) constitute an eigenvalue problem on l. (For the vacuum no-current case $\mathcal{C} = 0$, this reduces to $l = 2\tilde{m}$, an integer – as it should.) In practice, we follow the following procedure: for each $\tilde{m} = 1, 2, \ldots$, we assume some l and find $\mathcal{C}$ using the relations (2.10)–(2.11). Thus, for each m, there is a continuous relation $\mathcal{C}(l)$. (Physically, of course, it is the current $\mathcal{C}$ that determines the radial index l.)

Results are plotted in figures 2 and 3. In figure 2 we plot a particular solution for $l = 1$ and $\tilde{m} = 2$. The flux functions form a 'petal' pattern in azimuthal angle with the number of 'petals' equal to $2\tilde{m}$. There is a corresponding axial, unidirectional magnetic field B_z.

In figure 1 we plot the curves $l(\mathcal{C})$ for various $2\tilde{m} = 2, 4, 6, 8, 10$. Each curve starts at a point $\{\mathcal{C} = 0, l = 2\tilde{m}\}$. For non-zero current $\mathcal{C} > 0$, the radial dependence becomes more shallow, $l < 2\tilde{m}$.

In figure 3 we plot values of $\mathcal{C}$ as a function of azimuthal number m for different values of $l = 0.25, \ldots, 2$. Dashed lines are for convenience only; they connect points corresponding to the same radial parameter l.

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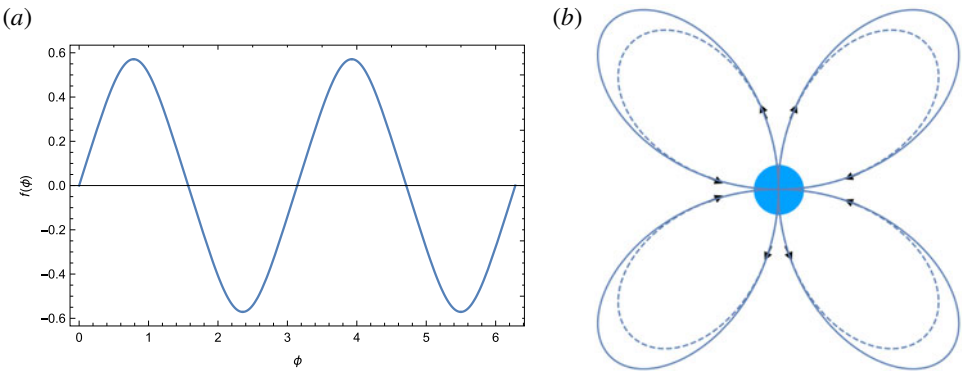


FIGURE 2. (*a*) Example of solution $f(\phi)$ for $l = 1$, $2\tilde{m} = 2$. In this case $\mathcal{C} = 2.517$. (*b*) Structure of poloidal field. (Due to the assumed self-similar radial structure, the solutions do not extend to $r = 0$.) Dashed line is the corresponding vacuum case, equation (2.6).

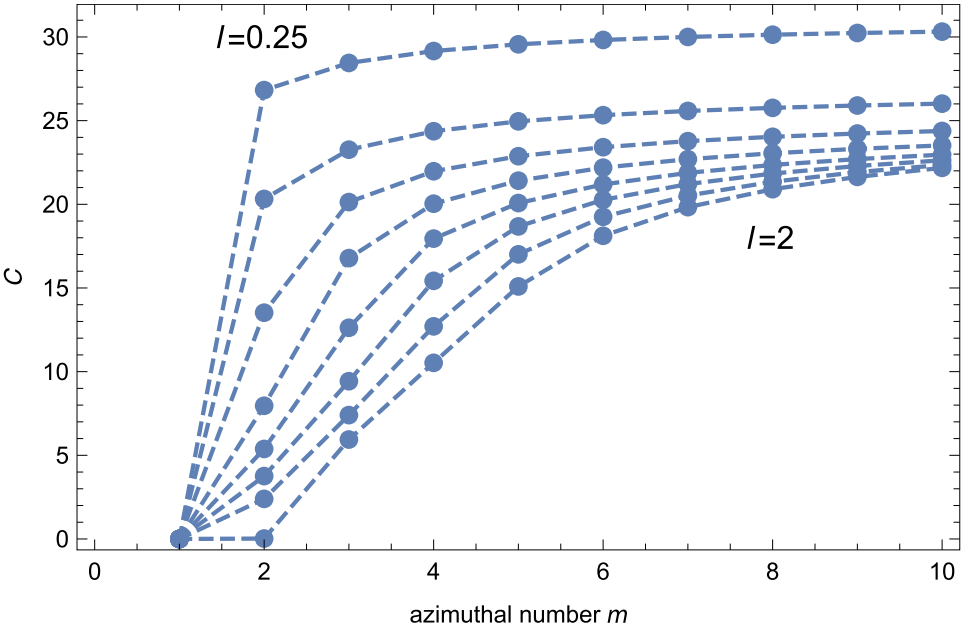


FIGURE 3. Values of $\mathcal{C}$ as a function of azimuthal number m for different values of $l = 0.25, \ldots, 2$ (in steps of 0.25). Top curves correspond to smaller l.

3. Analysis of the solutions

In a formulation of force-free fields in the form

$$\operatorname{curl} \boldsymbol{B} = \kappa \boldsymbol{B}, \tag{3.1}$$

the value of κ is

$$\kappa = \mathcal{C} \frac{1+l}{lr} f^{1/l}. \tag{3.2}$$

It is constant on flux surfaces P, (2.3).

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The current density (we incorporate factors of $4\pi/c$ into the definition of magnetic field) is

$$\left.\begin{aligned} j_r &= \mathcal{C}\frac{1+l}{l} r^{-(l+2)} \partial_\phi (|f|)^{(1+l)/l}, \\ j_\phi &= \mathcal{C}(l+1) r^{-(l+2)} (|f|)^{(1+l)/l}, \\ j_z &= r^{-(l+2)}(f'' + l^2) = -\mathcal{C}^2 \frac{1+l}{l} r^{-(l+2)} (|f|)^{(2+l)/l}. \end{aligned}\right\} \tag{3.3}$$

The total axial current is

$$I_z = \int_{r_0}^{\infty} r\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\phi j_z = -\frac{r_0^{-l}}{l} \int_0^{2\pi} \mathrm{d}\phi (f'' + l^2 f), \tag{3.4}$$

where r_0 is the inner boundary. The total axial current vanishes if the following two conditions are satisfied:

$$\left.\begin{aligned} \int_0^{2\pi} f\,\mathrm{d}\phi = 0, \\ f'(2\pi) = f'(0). \end{aligned}\right\} \tag{3.5}$$

All the solutions considered here satisfy these conditions: the second one requires even azimuthal numbers, $2\tilde{m}$. Generally, there is a larger family of self-similar force-free equilibria with non-zero total axial current.

There is a non-zero toroidal current

$$\left.\begin{aligned} j_\phi = \mathcal{C}(l+1) r^{-2-l} |f|^{(1+l)/l}, \\ \int_0^{2\pi} \mathrm{d}\phi j_\phi \neq 0. \end{aligned}\right\} \tag{3.6}$$

The radial current density integrated over ϕ satisfies

$$j_r \propto \int_0^{2\pi} \mathrm{d}\phi \partial_\phi (|f|)^{(1+l)/l} = (|f|)^{(1+l)/l}|_0^{2\pi} = 0. \tag{3.7}$$

4. Discussion

In this paper we make analytical progress with the highly nonlinear problem(s) of magnetohydrodynamics (Lynden-Bell & Boily 1994). We find a class of nonlinear self-similar force-free equilibria in cylindrical geometry. The solutions we find all connect to the vacuum case, in which case the flux function is $P_{vac} \propto r^{-m} \sin(m\phi)$. Structures with vanishing total axial current require even values of m (hence $m \to 2\tilde{m}$). For non-zero distributed current with the current parameter $\mathcal{C}$ the radial dependence changes to r^{-l}, with $l < 2\tilde{m}$, while remaining periodic in ϕ at $2\tilde{m}$. Solutions for a given m resemble vacuum solutions $\propto \sin(2m\phi)$, but they are not exactly harmonic in the nonlinear case.

For very large currents the solutions asymptote to $l \approx 0$, but never reach this limit. The case $l = 0$ corresponds to $B_r \propto 1/r$. Mathematically, this is the analogue of the split monopole case in spherical geometry – the split monopole case can be achieved in spherical geometry (with a corresponding anti-monopole in the opposite hemisphere), but is not possible in cylindrical geometry.

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