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ARTICLES

ON THE DYNAMIC EFFICIENCY OF BALANCED GROWTH PATHS IN AN ENDOGENOUS GROWTH SETTING

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In overlapping-generations economies with life-cycle saving and exogenous growth, the laissez-faire equilibrium balanced growth path fails in general to achieve optimality, but is dynamically efficient if the marginal product of physical capital is greater than the growth rate of the economy. In this paper, we accommodate the concept of dynamic (in)efficiency in an overlapping-generations economy with endogenous growth due to human capital accumulation. We show that the condition that the marginal product of physical capital is larger than the growth rate of the economy is necessary but no longer sufficient for the dynamic efficiency of the laissez-faire equilibrium balanced growth path.

Keywords: Endogenous Growth, Human Capital, Dynamic Efficiency

1. INTRODUCTION

It is well known that in overlapping-generations growth models with only physical capital à la Diamond (1965), the laissez-faire equilibrium balanced growth path will not generally coincide with the social optimum. The reason is that the capital–labor ratio associated with private saving will in general be different from the Golden Rule one, either in its standard form, or modified to deal with discounting of future utility. Focusing on the standard approach, the possibility that the laissez-faire capital–labor ratio is higher than the Golden Rule one is particularly appealing from a policy making point of view, as it implies that a Pareto improvement can be achieved: by transforming the excess capital into consumption, both the generations alive in the new balanced growth path and those living along the transition will be better off. Such a dynamically inefficient

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equilibrium is unambiguously associated with a marginal product of capital lower than the growth rate of the economy. Symmetrically, when the marginal product of capital is larger than the growth rate, the laissez-faire capital–labor ratio is lower than the Golden Rule one and the equilibrium is dynamically efficient, as there is no way to attain the optimal balanced growth path without sacrificing the welfare level of some generations.

In a model with endogenous accumulation of human capital, a second potential departure from optimality arises, as individuals may not choose the correct amount of education investment. This suggests that, to assess the dynamic efficiency or inefficiency of the laissez-faire equilibrium, it may not be enough to compare the marginal product of physical capital and the growth rate of the economy. There are two reasons for this. First, the relationship between the marginal product of physical capital and the growth rate of the economy may no longer imply a precise relationship between the laissez-faire and the Golden Rule physical capital-to-efficient labor ratio. In effect, because the growth rate of the laissez-faire economy is no longer exogenously given, it may also itself differ from the Golden Rule one. Thus, it could well be the case that the marginal product of physical capital is larger than the growth rate because the growth rate (and not the physical capital-to-labor ratio) is too small. Second, even if there were an unambiguous relationship concerning the divergence between the marginal product of physical capital and the growth rate of the economy, on one hand, and the laissez-faire accumulation of physical capital and the Golden Rule accumulation, on the other, the potential disparity between the laissez-faire human capital and the Golden Rule human capital should be also taken into account. It is the purpose of this paper to explore whether the relationship between the rate of return to physical capital and the growth rate of output is enough to assess the efficiency of the competitive equilibrium balanced growth path in an endogenous growth framework.

In order to do that, we first need to define the concept of dynamic efficiency in this setting, starting with some notion of social optimum. Diamond (1965) defines the Golden Age Path as the balanced growth path along which a representative individual would have the highest utility level, subject to the feasibility conditions and the additional constraint that all individuals have the same level. Later, Samuelson (1968, 1975a, and 1975b) renamed this the Two Part Golden Rule, encompassing Phelps (1961)'s rule for capital accumulation and the so-called "biological" interest rate [Samuelson (1958)]. This approach yields the well-known results mentioned at the beginning of this introduction and is the root of the analyses of dynamic (in)efficiency that have been carried out in this framework by Galor and Ryder (1991) and Lang (1992). A connected strand of literature involves introducing uncertainty into Diamond's model. Abel et al. (1989) allow both the rate of population growth and the production technology to be stochastic. In their model, the test for dynamic efficiency entails a comparison between capital income and investment. Zilcha (1990, 1991) introduces uncertainty about production and defines two types of efficiency for feasible production–consumption allocations, using first and second degree stochastic dominance. This has given rise to a

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subsequent exchange between Rangazas and Russell (2005) and Barbie and Kaul (2009).

The insights arising from the contributions reviewed in the preceding paragraph cannot easily be translated in terms of the dynamic efficiency or inefficiency of balanced growth paths in an endogenous growth setting. Indeed, in the presence of endogenous productivity growth that translates into consumption growth, it is impossible to identify the highest individual utility level along a balanced growth path, because consumption levels and thus (for reasonable utility functions) utility grow without limit. Arrow and Kurz (1970) sidestep this problem by assuming that a social planner maximizes a discounted sum of individual utilities defined over consumption levels per unit of efficient labor. In an overlapping-generations model with exogenously given labor-augmenting technical change, Buiter (1979) also notes the need to distinguish between consumption expressed in terms of output per unit of natural and of efficient labor in order to properly characterize the Golden Rule. He argues that, provided that the individual utility function is homogeneous, an affine transformation of this function can be obtained by replacing per capita consumptions by consumption per unit of efficient labor. Doing this, one can still search for the balanced growth path along which each individual would have the highest utility level, subject to the constraint that all individuals have the same level. The result of this exercise is a generalization of Diamond's results: whenever the marginal product of capital is larger (resp. smaller) than the growth rate of productivity plus the growth rate of natural labor, the physical capital-to-efficient labor ratio is lower (resp. higher) than the Golden Rule one, and the laissez-faire economy is dynamically efficient (resp. inefficient).

In Del Rey and Lopez-Garcia (2013) we take this approach further. Using the model with endogenous productivity growth in Boldrin and Montes (2005) and Docquier et al. (2007), we posit that the objective of the social planner is to maximize the utility of a representative individual with consumptions measured in terms of output per unit of efficient labor. In that framework, we identify the balanced growth path along which each individual would have the highest utility level thus defined, subject to the constraint that all individuals have the same level. In the present paper, building on this Golden Rule concept, we explore the relationship between the laissez-faire equilibrium and the Golden Rule optimum and define the concept of dynamic (in)efficiency in this endogenous growth setting. We are thus able to show that when the rate of return to physical capital is lower than the growth rate of output, the competitive equilibrium is dynamically inefficient, as in the Diamond economy. Importantly, under these conditions, a permanent reduction of *either* physical capital *or* the amount of output devoted to education will be Pareto improving: both the welfare of those individuals alive along the new balanced growth path and that of those alive in the transient periods will be higher. Moreover, when the rate of return to physical capital happens to be equal to the growth rate of output, the competitive equilibrium is also dynamically inefficient: this situation is necessarily associated with a physical capital–labor ratio that falls short of the Golden Rule level, but the laissez-faire amount of output devoted to

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education is greater than its Golden Rule counterpart. Then, a permanent reduction of the amount of output devoted to education will be Pareto-improving. More interestingly, we also show that a rate of return to physical capital higher than the growth rate of output is *necessary* but not sufficient for the dynamic efficiency of the competitive equilibrium in an endogenous growth framework. Indeed, under these circumstances, there are still instances where Pareto improvements can be obtained through a permanent reduction of the investment in *human* capital. This is a consequence of the fact that, in this situation, in spite of the physical capital–labor ratio falling short of the Golden Rule level, the laissez-faire amount of output devoted to education can be less or greater than its Golden Rule counterpart. We illustrate our results with some simulations obtained with Cobb–Douglas production functions and logarithmic preferences.

The rest of the paper is organized as follows. Section 2 introduces the endogenous growth model and characterizes the laissez-faire balanced growth path. Section 3 presents the social optimum and the definition of dynamic efficiency. Section 4 compares the laissez-faire and the optimal balanced growth paths. Section 5 shows what can be inferred from the relationship between the rate of return to physical capital and the growth rate of output in terms of the dynamic efficiency or inefficiency of the laissez-faire equilibrium in our endogenous growth framework. Section 6 concludes. All proofs are in the Appendix.

2. THE MODEL AND THE LAISSEZ-FAIRE EQUILIBRIUM

We consider the overlapping-generations model with both human and physical capital as presented in Boldrin and Montes (2005), Docquier et al. (2007), and Del Rey and Lopez-Garcia (2013). At period t, L_{t+1} individuals are born, and coexist with L_t middle-aged and L_{t-1} old. Population grows at the exogenous rate n so that $L_t = (1+n)L_{t-1}$ with $n > -1$. Agents are born with some level of human capital h_{t-1}, measured in units of efficient labor per unit of natural labor. Human capital in period t results from the interaction of the amount of output invested in education d_{t-1} and the inherited human capital h_{t-1} according to the production function $h_t = E(d_{t-1}, h_{t-1})$. Assuming constant returns to scale, the production of human capital can be written in intensive terms as $h_t/h_{t-1} = e(\tilde{d}_{t-1})$, where $e(.)$ satisfies the Inada conditions and $\tilde{d}_{t-1} = d_{t-1}/h_{t-1}$ is the amount of output devoted to education per unit of inherited human capital.[1]

There is a single good in the economy, Y_t, which is produced by means of physical capital K_t and human capital H_t (i.e., labor in efficiency units), according to a constant-returns-to-scale production function $Y_t = F(K_t, H_t)$. Only the middle-aged work, supplying inelastically one unit of natural labor, so that $H_t = h_t L_t$. Physical capital depreciates fully each period. Letting $k_t = K_t/L_t$ be the physical capital per unit of natural labor and $\tilde{k}_t = K_t/H_t = k_t/h_t$ the physical capital per unit of efficient labor, the technology can be described as $Y_t/H_t = f(\tilde{k}_t)$, where $f(.)$ also satisfies the Inada conditions.

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Individuals only consume in their second and third periods, so that, for an individual born at $t-1$, c_t^m and c_{t+1}^o denote her consumption levels when middle-aged and old, respectively. The aggregate feasibility constraint is written

$$F(K_t, H_t) = c_t^m L_t + c_t^o L_{t-1} + d_t L_{t+1} + K_{t+1}, \tag{1}$$

or, expressed in terms of output per unit of efficient labor,

$$f(\tilde{k}_t) = \tilde{c}_t^m + \frac{\tilde{c}_t^o}{e(\tilde{d}_{t-1})(1+n)} + (1+n)\tilde{d}_t + e(\tilde{d}_t)(1+n)\tilde{k}_{t+1}, \tag{2}$$

where $\tilde{c}_t^m = c_t^m / h_t$ and $\tilde{c}_t^o = c_t^o / h_{t-1}$ denote consumption of the middle-aged and old living at period t per unit of (their respective) efficient labor. Note that $h_t / h_{t-1} = e(\tilde{d}_{t-1}) = 1 + g_t$, where g_t is the growth rate of productivity from period $t-1$ to period t.

Factor prices are determined under perfect competition by their marginal products, so that, if $1 + r_t$ and w_t are respectively the interest factor and the wage rate per unit of efficient labor,

$$1 + r_t = f'(\tilde{k}_t); \qquad w_t = f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t). \tag{3}$$

Individuals behave as pure life-cyclers. The lifetime utility function of an individual born at period $t-1$ is $U_t = U(c_t^m, c_{t+1}^o)$, and it is assumed to be strictly quasi-concave and homogeneous of degree $j > 0$.[2] Individuals not only decide about the allocation of their resources along their life cycles, but also decide how much to invest in education. In their first period, individuals born at $t-1$ borrow in perfect credit markets the amount they need to pay for the amount of education that maximizes their lifetime resources. In their second period they work, consume, and save to finance consumption in their third period. Letting s_t stand for savings of a middle-aged individual,

$$c_t^m = w_t h_t - (1 + r_t) d_{t-1} - s_t; \qquad c_{t+1}^o = (1 + r_{t+1}) s_t. \tag{4}$$

The first-order conditions associated with the individual decision variables, d_{t-1}, c_t^m, and c_{t+1}^o, are

$$w_t e'(d_{t-1} / h_{t-1}) = (1 + r_t), \tag{5}$$

$$\frac{\partial U(c_t^m, c_{t+1}^o)/\partial c_t^m}{\partial U(c_t^m, c_{t+1}^o)/\partial c_{t+1}^o} = (1 + r_{t+1}), \tag{6}$$

where use has been made of the homogeneity of degree one of the $E(.)$ function. Note that (5), using (3), implicitly characterizes the ratio $\tilde{d}_{t-1}$ as a function of $\tilde{k}_t$; i.e., $\tilde{d}_{t-1} = \phi(\tilde{k}_t)$.

The homogeneity assumption on preferences implies that the ratio c_{t+1}^o / c_t^m is a function of r_{t+1} only. This allows one to write consumption in the second period as $c_t^m = \pi(r_{t+1})\omega_t$, where $\omega_t = w_t h_t - (1 + r_t) d_{t-1}$ denotes income in middle age and the function $\pi(.)$ depends on the interest rate only. Equilibrium in the market

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for physical capital is achieved when the (physical) capital stock available at $t+1$, K_{t+1}, equals gross savings made by the middle-aged at t, $s_t L_t$, minus the amount of output devoted to human capital investment by the young at t, $(1+n)d_t L_t$, i.e., when $K_{t+1} = s_t L_t - (1+n)d_t L_t$. This equilibrium condition can be expressed as

$$\tilde{k}_{t+1} = \frac{[1-\pi(r_{t+1})]\left\{w_t - \frac{(1+r_t)\phi(\tilde{k}_t)}{e[\phi(\tilde{k}_t)]}\right\}}{e[\phi(\tilde{k}_{t+1})](1+n)} - \frac{\phi(\tilde{k}_{t+1})}{e[\phi(\tilde{k}_{t+1})]}. \tag{7}$$

Using (3), (7) implicitly provides a first-order difference equation, $\tilde{k}_{t+1} = \Psi(\tilde{k}_t)$, which, given an initial condition $\tilde{k}_0$, characterizes the time path of $\tilde{k}$. The amount invested in education follows from $\tilde{d}_t = \phi(\tilde{k}_{t+1})$ in (5).

The laissez-faire ratio of physical capital to labor in efficiency units, $\tilde{k}_{\mathrm{LF}}$, is a fixed point of the Ψ function, i.e. $\tilde{k}_{\mathrm{LF}} = \Psi(\tilde{k}_{\mathrm{LF}})$. The associated ratio of output devoted to education per unit of inherited human capital, $\tilde{d}_{\mathrm{LF}}$, follows from $\tilde{d}_{\mathrm{LF}} = \phi(\tilde{k}_{\mathrm{LF}})$. Letting g_{LF} be the laissez-faire growth rate of any variable expressed in terms of output per unit of natural labor, we have $1 + g_{\mathrm{LF}} = e(\phi(\tilde{k}_{\mathrm{LF}}))$ and all variables measured in terms of output will then grow at a rate $(1 + g_{\mathrm{LF}})(1 + n)$. As for individual consumption behaviour, the homogeneity property of the utility function implies:

$$\frac{\partial U(\tilde{c}^m_{\mathrm{LF}}, \tilde{c}^o_{\mathrm{LF}})/\partial \tilde{c}^m}{\partial U(\tilde{c}^m_{\mathrm{LF}}, \tilde{c}^o_{\mathrm{LF}})/\partial \tilde{c}^o} = (1 + r_{\mathrm{LF}}) \tag{8}$$

where $\tilde{c}^m_{\mathrm{LF}}$ and $\tilde{c}^o_{\mathrm{LF}}$ are consumptions when middle-aged and old-aged measured in terms of output per unit of efficient labor along the laissez-faire balanced growth path, and $1 + r_{\mathrm{LF}} = f'(\tilde{k}_{\mathrm{LF}})$.

We will assume that the laissez-faire equilibrium exists and is (locally) stable.[3] A particularly appealing case, which will provide the framework for the simulation results discussed in Section 4, is that where the production and utility functions are of the Cobb–Douglas type, i.e., $f(\tilde{k}) = A\tilde{k}^{\alpha}$, $e(\tilde{d}) = B\tilde{d}^{\lambda}$, and $U = \log c^m + \beta \log c^o$.[4] In this case, as shown in Appendix A, (7) has a particular simple form, and the balanced growth path (with a nonzero value) of $\tilde{k}_{\mathrm{LF}}$ is unique and globally stable. It is well known that in overlapping-generation models with exogenous growth, the interest rate can be either greater or less than the economy's growth rate [Diamond (1965)]. The conjecture that this fact carries over to the current model is illustrated in the inner cells of Table 1, and is confirmed with the simulations reported in the fourth column of Table 2 for a variety of values of the parameters α, β, and λ in the triple Cobb–Douglas case. These tables will be used in Section 4 to analyze the relationship between the laissez-faire and optimal balanced growth paths fully.

3. SOCIAL OPTIMUM AND DYNAMIC EFFICIENCY

The presence of productivity growth entails that, along a balanced growth path, consumption per unit of natural labor, c^m_t and c^o_{t+1}, will tend to infinity. As a

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consequence, for reasonable forms of the utility function, a social planner cannot possibly choose the balanced growth path that maximizes $U_t = U(c_t^m, c_{t+1}^o)$, because U_t will also grow without limit. Just as a balanced growth path is characterized by consumption (and all other variables) expressed in terms of outputs per unit of efficient labor, $\tilde{c}_t^m = c_t^m / h_t$ and $\tilde{c}_{t+1}^o = c_{t+1}^o / h_t$, that remain constant over time, so it is natural for the policy maker to consider a valuation function defined over $\tilde{c}^m$ and $\tilde{c}^o$. Moreover, if the social planner wants to treat all individuals equally while being respectful of their preferences, using this valuation function is the only possible alternative. Indeed, because the individual utility function is homogeneous of degree j, we can take the monotonic transformation given by $\tilde{U}_t = U\left(\tilde{c}_t^m, \tilde{c}_{t+1}^o\right) = (1/h_t^j)U(c_t^m, c_{t+1}^o)$. Notice that this social valuation function not only respects individual ordinal preferences, but also preserves the same functional form.

We can therefore postulate that the social planner's objective is to choose the values $(\tilde{c}_*^m, \tilde{c}_*^o, \tilde{k}_*, \tilde{d}_*)$ that maximize $\tilde{U} = U(\tilde{c}^m, \tilde{c}^o)$ subject to the balanced growth version of the feasibility constraint (2) and the technological relationship $1 + g = e(\tilde{d})$. This characterization of the social optimum is clearly reminiscent of the "Two-Part Golden Rule" in the exogenous growth model of Diamond (1965) and Samuelson (1968, 1975a, and 1975b). In the present model, where the accumulation of human capital is the engine of growth, the Golden Rule balanced growth path will then be the one that provides the maximum level of welfare, $\tilde{U}_* = U(\tilde{c}_*^m, \tilde{c}_*^o)$, that can be achieved by a representative individual, subject to the feasibility constraint and the additional constraint that everyone else attains the same level of utility. This is the approach adopted by Del Rey and Lopez-Garcia (2013), and the optimality conditions, derived therein, are

$$f'(\tilde{k}_*) = e(\tilde{d}_*)(1+n), \tag{9}$$

$$e'(\tilde{d}_*)\left\{\frac{\tilde{c}_*^o}{\left[e(\tilde{d}_*)(1+n)\right]^2} - \tilde{k}_*\right\} = 1, \tag{10}$$

$$\frac{\partial U(\tilde{c}_*^m, \tilde{c}_*^o)/\partial \tilde{c}^m}{\partial U(\tilde{c}_*^m, \tilde{c}_*^o)/\partial \tilde{c}^o} = e(\tilde{d}_*)(1+n). \tag{11}$$

Equation (9) is the equality of the marginal product of physical capital and the growth rate of labor in efficiency units. Equation (10) is the condition that equates marginal benefit and marginal cost associated with changes in $\tilde{d}$. The meaning of this condition becomes clearer when one considers the balanced growth path version of the feasibility constraint (2), i.e.,

$$\tilde{c}^m + \frac{\tilde{c}^o}{e(\tilde{d})(1+n)} = f(\tilde{k}) - (1+n)\tilde{d} - e(\tilde{d})(1+n)\tilde{k}. \tag{12}$$

A higher value of $\tilde{d}$ increases the marginal rate of transformation between middle-aged and old consumption on the left-hand side of (12), $e(\tilde{d})(1+n)$. This implies

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that giving up one unit of consumption in middle age results in a greater amount of consumption in old age. On the other hand, a higher value of $\tilde{d}$ also increases the costs on the right-hand side of (12): the direct cost through the increase of the amount of resources that must be set aside, $(1+n)\tilde{d}$, and the indirect cost through the impact on $e(\tilde{d})(1+n)\tilde{k}$, as it makes it more costly to maintain the balanced growth path level of $\tilde{k}$. At the optimum, benefits and costs must be equal at the margin, giving rise to (10). Finally, Equation (11) is the equality of the marginal rate of substitution between second and third period consumption and the counterpart in the current model of the so-called "biological" interest rate [Samuelson (1958)].

The definition of dynamic (in)efficiency that follows from this approach is the following:[5]

DEFINITION 1. *A balanced growth path is dynamically inefficient if a reduction in either $\tilde{k}$ or $\tilde{d}$ expands the consumption possibilities, and thus the welfare as measured by $\tilde{U} = U(\tilde{c}^m, \tilde{c}^o)$, of both the generations living on the new balanced growth path and those generations living in transient periods. Otherwise, the balanced growth path is dynamically efficient.*

Note that conditions (9) and (10) can be obtained readily by differentiation of (12). Importantly, this exercise will allow us to interpret these conditions when, evaluated at the laissez-faire equilibrium, they are not satisfied with equality. Consider, first, the effects that a marginal change in $\tilde{k}$ will have on $\tilde{c}^m$ in (12) for arbitrarily given values of $\tilde{c}^o$ and $\tilde{d}$:

$$\left(\partial \tilde{c}^m / \partial \tilde{k}\right)_{\tilde{c}^o, \tilde{d}} = f'(\tilde{k}) - e(\tilde{d})(1+n). \tag{13}$$

If the marginal benefit of increasing $\tilde{k}$, $f'(\tilde{k})$, is higher (resp. lower) than its marginal cost, $e(\tilde{d})(1+n)$, an increase (resp. decrease) in $\tilde{k}$ will locally expand the feasibility constraint. This will clearly have a positive effect on $\tilde{U}$. Similarly, the effect of a marginal change in $\tilde{d}$ on $\tilde{c}^m$ for arbitrarily given values of $\tilde{c}^o$ and $\tilde{k}$ in (12) will be

$$\left(\partial \tilde{c}^m / \partial \tilde{d}\right)_{\tilde{c}^o, \tilde{k}} = (1+n)\left(e'(\tilde{d})\left\{\frac{\tilde{c}^o}{\left[e(\tilde{d})(1+n)\right]^2} - \tilde{k}\right\} - 1\right). \tag{14}$$

As before, if the marginal benefit exceeds (falls short of) the marginal cost, an increase (resp. decrease) in $\tilde{d}$ will locally expand the feasibility constraint and have a positive effect on $\tilde{U}$.

To illustrate the role of the feasibility constraint (12) and its interaction with the individual budget constraint, consider the solid line in Figure 1 (disregarding, for the time being, the dashed one). This solid line represents the feasibility constraint (12) evaluated under laissez-faire. Its slope is given by $(1+g_{\mathrm{LF}})(1+n)$. Along this line, two possible laissez-faire balanced growth paths are represented. At point A the slope of the individual indifference curve equals the interest factor

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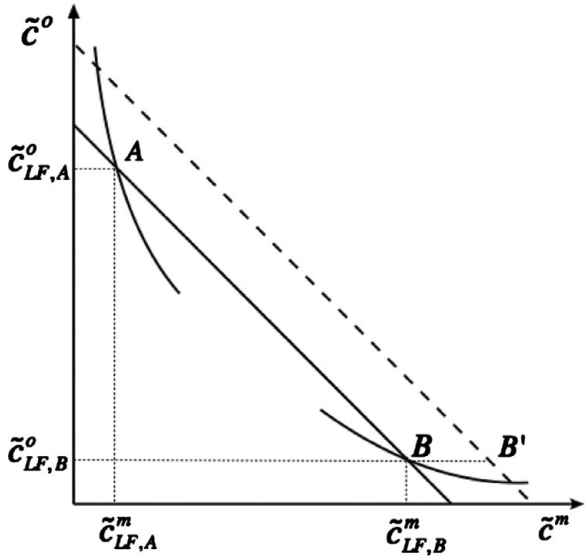


FIGURE 1. Feasibility constraint and laissez-faire equilibria.

$(1 + r_{\mathrm{LF},A})$, and this is greater than the economy's growth rate, $(1 + g_{\mathrm{LF}})(1 + n)$. At point B, however, the relationship between these rates is the reverse, as shown by the slope of the indifference curve through B.

A social planner can now decide to modify the level of $\tilde{k}$ and leave $\tilde{d}$ fixed or vice versa, provided that the change fulfills the economy's feasibility constraint.[6] A negative sign of (13) evaluated at $\tilde{k}_{\mathrm{LF}}$ will imply that, for given levels of $\tilde{c}^o_{\mathrm{LF}}$ and $\tilde{d}_{\mathrm{LF}}$, we can increase $\tilde{c}^m$ by reducing $\tilde{k}$. The same reasoning applies when the sign of (14) evaluated at $\tilde{d}_{\mathrm{LF}}$ is negative, because then, for given levels of $\tilde{c}^o_{\mathrm{LF}}$ and $\tilde{k}_{\mathrm{LF}}$, the social planner will be able to increase $\tilde{c}^m$ by reducing $\tilde{d}$. In both cases, comparing balanced growth paths, the feasibility constraint (12) will be pushed outward in a neighborhood of laissez-faire. Given the nature of the social planner utility function, $U(\tilde{c}^m, \tilde{c}^o)$, such a local expansion of the consumption possibility locus will necessarily increase the welfare level of all generations along the new balanced growth path. The sign of (13) is governed by the divergence between the interest factor and the economy's growth rate, and, as argued in the preceding and illustrated in Figure 1, any relationship between them is feasible under laissez-faire. The sign of (14), however, has yet to be elucidated. As will be shown in the next section, it is also the case that this derivative can have any sign, and under some circumstances an unambiguous relationship exists between (13) and (14).

A related point now emerges as important. In Diamond (1965)'s model, as well as its extension to allow for exogenous productivity growth, the dynamic (in)efficiency of the laissez-faire balanced growth path can be directly ascertained by looking at the relationship between the rate of return to capital and the

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TABLE 1. Different laissez-faire balanced growth paths and the sign of (13)

	$\tilde{d}_{\text{LF}} > \tilde{d}_*$	$\tilde{d}_{\text{LF}} \leqslant \tilde{d}_*$
$\tilde{k}_{\text{LF}} \geqslant \tilde{k}_*$	$f'(\tilde{k}_{\text{LF}}) < (1+g_{\text{LF}})(1+n)$	Not feasible
$\tilde{k}_{\text{LF}} < \tilde{k}_*$	$f'(\tilde{k}_{\text{LF}}) \lesseqgtr (1+g_{\text{LF}})(1+n)$	$f'(\tilde{k}_{\text{LF}}) > (1+g_{\text{LF}})(1+n)$

economy's growth rate. In turn, this relationship can be determined, on a one-to-one basis, by the relationship between the capital–labor ratios $\tilde{k}_*$ and $\tilde{k}_{\text{LF}}$. In our model, however, there are *two* key variables, $\tilde{k}$ and $\tilde{d}$, and their interaction determines both $(1 + r_{\text{LF}})$ and $(1 + g_{\text{LF}})(1 + n)$. It is unclear whether a, say negative, sign of (13) can be associated with a too high level of physical capital, because it may well be the case that it is $\tilde{d}$ that is too high. We therefore have to identify the feasible combinations of over- and underaccumulation of physical and human capital and explore the consequences of these for the relationship between $(1 + r_{\text{LF}})$ and $(1 + g_{\text{LF}})(1 + n)$.

4. COMPARING LAISSEZ-FAIRE AND OPTIMAL BALANCED GROWTH PATHS

Because in Diamond (1965)'s model $\tilde{k}_{\text{LF}}$ may either be greater or less than $\tilde{k}_*$ (and both can, by chance, coincide), a natural conjecture is that this might be possible in the current framework as well, and, by symmetry, that $\tilde{d}_{\text{LF}}$ might also be higher/lower than (or equal to) $\tilde{d}_*$. This conjecture immediately translates into the possibilities illustrated by the four cells in Table 1 (neglecting, for the moment, the content of these cells). However, as shown in Del Rey and Lopez-Garcia (2013, pp. 1745–1746), cell a_{12} is not feasible, thus implying that laissez-faire cannot possibly decentralize the Golden Rule balanced growth path.[7]

The next step, of course, focuses on the content of the feasible cells in Table 1, i.e., the relationship between $f'(\tilde{k}_{\text{LF}})$ and $(1+g_{\text{LF}})(1+n)$ for each of them. To this end, and without assuming any particular form for the functions involved, we can use the fact that $f'(.)$ [resp. $e(.)$] is monotonically decreasing (resp. increasing) in its argument. Thus, when $\tilde{k}_{\text{LF}} \geqslant \tilde{k}_*$ and $\tilde{d}_{\text{LF}} > \tilde{d}_*$, one has the following chain of inequalities: $f'(\tilde{k}_{\text{LF}}) \leqslant f'(\tilde{k}_*) = (1 + g_*)(1 + n) < (1 + g_{\text{LF}})(1 + n)$. From this, the sign in cell a_{11} in Table 1 arises. A similar argument applies to the sign in cell a_{22}. However, this approach implies an ambiguous sign in a_{21}, because, when $\tilde{k}_{\text{LF}} < \tilde{k}_*$ and $\tilde{d}_{\text{LF}} > \tilde{d}_*$, we have $f'(\tilde{k}_{\text{LF}}) > f'(\tilde{k}_*) = (1+g_*)(1+n) < (1+g_{\text{LF}})$ $(1 + n)$.

The feasibility of the cells in Table 1, as well as the indeterminacy in cell a_{12}, can be verified with the help of the Cobb–Douglas case that was introduced in Section 2 and is discussed in Appendix A. The results of the numerical exercises reported in Table 2 all assume that $n = 0$, $A = 1$, $B = 1$. The second and third columns

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TABLE 2. Simulations in the Cobb–Douglas case, with $n = 0$, $A = 1$, $B = 1$: (i) $\alpha = 0.2, \beta = 0.9, \lambda = 0.1$; (ii) $\alpha = 0.2, \beta = 0.9, \lambda = 0.13$; (iii) $\alpha = 0.27, \beta = 0.9, \lambda = 0.13$; (iv) $\alpha = 0.35, \beta = 0.8, \lambda = 0.3$; (v) $\alpha = 0.1, \beta = 0.7, \lambda = 0.6$.

(i)	$\tilde{k}_{LF} > \tilde{k}_*$	$\tilde{d}_{LF} > \tilde{d}_*$	$f'(\tilde{k}_{LF}) < (1+g_{LF})(1+n)$	$e'(\tilde{d}_{LF})\left\{\frac{\tilde{c}^o_{LF}}{[(1+g_{LF})(1+n)]^2} - \tilde{k}_{LF}\right\} < 1$	
(ii)	$\tilde{k}_{LF} < \tilde{k}_*$	$\tilde{d}_{LF} > \tilde{d}_*$	$f'(\tilde{k}_{LF}) < (1+g_{LF})(1+n)$	$e'(\tilde{d}_{LF})\left\{\frac{\tilde{c}^o_{LF}}{[(1+g_{LF})(1+n)]^2} - \tilde{k}_{LF}\right\} < 1$	
(iii)	$\tilde{k}_{LF} < \tilde{k}_*$	$\tilde{d}_{LF} > \tilde{d}_*$	$f'(\tilde{k}_{LF}) > (1+g_{LF})(1+n)$	$e'(\tilde{d}_{LF})\left\{\frac{\tilde{c}^o_{LF}}{[(1+g_{LF})(1+n)]^2} - \tilde{k}_{LF}\right\} < 1$	
(iv)	$\tilde{k}_{LF} < \tilde{k}_*$	$\tilde{d}_{LF} > \tilde{d}_*$	$f'(\tilde{k}_{LF}) > (1+g_{LF})(1+n)$	$e'(\tilde{d}_{LF})\left\{\frac{\tilde{c}^o_{LF}}{[(1+g_{LF})(1+n)]^2} - \tilde{k}_{LF}\right\} > 1$	
(v)	$\tilde{k}_{LF} < \tilde{k}_*$	$\tilde{d}_{LF} < \tilde{d}_*$	$f'(\tilde{k}_{LF}) > (1+g_{LF})(1+n)$	$e'(\tilde{d}_{LF})\left\{\frac{\tilde{c}^o_{LF}}{[(1+g_{LF})(1+n)]^2} - \tilde{k}_{LF}\right\} > 1$	

show that the combinations between laissez-faire and optimal values of $\tilde{k}$ and $\tilde{d}$ represented by cells a_{11} [case (i)], a_{21} [cases (ii)–(iv)], and a_{22} [case (v)] in Table 1 are indeed feasible. Note also that, as portrayed by cases (ii)–(iv), the relationship between the interest rate and the growth rate in cell a_{12}, i.e., when $\tilde{k}_{LF} < \tilde{k}_*$ and $\tilde{d}_{LF} > \tilde{d}_*$, is ambiguous. This is clearly shown by row (ii) on one hand and rows (iii) and (iv) on the other. This seems particularly interesting, because, in spite of having *under*accumulation of physical capital with respect to the Golden Rule, the *over*accumulation of human capital may give rise to a marginal product $f'(\tilde{k}_{LF})$ *lower* than the growth rate of the economy $(1 + g_{LF})(1 + n)$.

This overaccumulation of human capital is characteristic of overlapping-generations models with endogenous formation of human capital. For example, in Docquier et al. (2007), where the social planner adopts the conventional criterion of maximizing a discounted sum of utilities, laissez-faire is characterized by overinvestment in education in Regime I. Incidentally, this case is also related to overinvestment in physical capital (see Figure 1 on p. 368).[8] In contrast, when the objective of the social planner is to attain the Golden Rule, the overaccumulation of human capital is compatible also with the optimal accumulation of physical capital or even with underaccumulation of physical capital, as shown in Del Rey and Lopez-Garcia (2013, p. 1747). To illustrate this point, consider first the case where $\tilde{k}_{LF} = \tilde{k}_*$. The condition that determines the optimal choice of education (10) can be rewritten, using (9) and (12), in a way that can be directly compared to the condition characterizing the individual's choice of education (5):

$$\left[f(\tilde{k}_*) - \tilde{k}_* f'(\tilde{k}_*)\right] e'(\tilde{d}_*) = f'(\tilde{k}_*)\left[1 + \frac{e'(\tilde{d}_*)\Lambda_*}{f'(\tilde{k}_*)}\right] > f'(\tilde{k}_*), \qquad \textbf{(15)}$$

where the last inequality follows from the fact that the term $\Lambda_* = (1 + g_*)(1 + n)\tilde{k}_* + (1 + n)\tilde{d}_* + \tilde{c}^m_*$ is positive. Thus, when choosing how much to invest in education, if confronted with the *optimal* wage and interest rates, $f(\tilde{k}_*) - \tilde{k}_* f'(\tilde{k}_*)$ and $f'(\tilde{k}_*)$, the individual would fail to take into account the terms collected by Λ_*. In other words, she would neglect the the fact that a higher investment

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in education makes it more costly to keep $\tilde{k}_*$, $\tilde{d}_*$, and $\tilde{c}^m_*$ constant. As a result, $\tilde{d}_{\mathrm{LF}} > \tilde{d}_*$. When $\tilde{k}_{\mathrm{LF}} > \tilde{k}_*$, the wage is higher and the interest rate lower under laissez-faire. This, together with the effect of the terms collected in Λ_*, leads, once more, to overinvestment in education. Last, when $\tilde{k}_{\mathrm{LF}} < \tilde{k}_*$, although the wage is lower and the interest higher than at the optimum, the terms collected in Λ_* lead to general undetermination in the comparison of education investment rates, as portrayed in cells a_{21} and a_{22} of Table 1: overinvestment in education is still possible, but does not result necessarily.

5. DYNAMIC EFFICIENCY AND PARETO IMPROVEMENTS

We now have all the required tools to investigate the conditions under which the laissez-faire balanced growth path is dynamically efficient or inefficient, in the light of Definition 1. The experiment to be performed can be described as follows. The first step consists in exploring whether an expansion of the consumption possibility locus in a neighborhood of the initial values $\tilde{c}^m_{\mathrm{LF}}$ and $\tilde{c}^o_{\mathrm{LF}}$ is achievable through a reduction in either $\tilde{k}$ or $\tilde{d}$. Clearly, if such an expansion is feasible, the utility level $\tilde{U}$ of those generations living on the new balanced growth path will be higher. If the answer is yes, we will turn to the second step, i.e., checking whether the welfare levels of those generations living in the transient periods have also increased (or, at least, have not decreased). If the answer is also yes, we can definitely claim that the initial balanced growth path is dynamically inefficient, as a feasible change has been devised where no one is worse off and at least someone is better off. If, however, the answer is no, the conclusion cannot be other than the recognition that no Pareto improvements can be found when the social planner alters either $\tilde{k}$ or $\tilde{d}$. In other words, the starting balanced growth path is dynamically efficient.

The information contained in Tables 1 and 2 shows that, in effect, (13) and (14) can be negative under laissez-faire, so that, by permanently reducing $\tilde{k}$ or $\tilde{d}$, the social planner can increase the welfare level of the generations living on the new balanced growth path. When the interest factor falls short of the economy's growth rate, as is the case in simulations (i) and (ii) in the fourth column of Table 2, (13) will be negative. In terms of Figure 1, as $f'(\tilde{k}_{\mathrm{LF}}) < (1 + g_{\mathrm{LF}})(1 + n)$ at B, a small reduction in $\tilde{k}$ would give rise to the feasibility constraint represented by the dashed line, which is parallel to the initial solid one. On the new balanced growth path, B' is feasible, so $\tilde{U}$ will be larger.

The following lemma, proven in the Appendix, establishes an unambiguous relationship between (13) and (14):

LEMMA 1. *If* $f'(\tilde{k}_{\mathrm{LF}}) \leqslant (1 + g_{\mathrm{LF}})(1 + n)$, *then* $\left(\partial \tilde{c}^m / \partial \tilde{d}\right)_{\tilde{c}^o, \tilde{k}} > 0$.

In words, when $f'(\tilde{k}_{\mathrm{LF}})$ is strictly less than $(1 + g_{\mathrm{LF}})(1 + n)$ and thus (13) is negative, so will (14) be. And when both rates are the same and (13) is zero, (14) will continue to be negative.[9] The consequence is that a reduction of $\tilde{d}$ will

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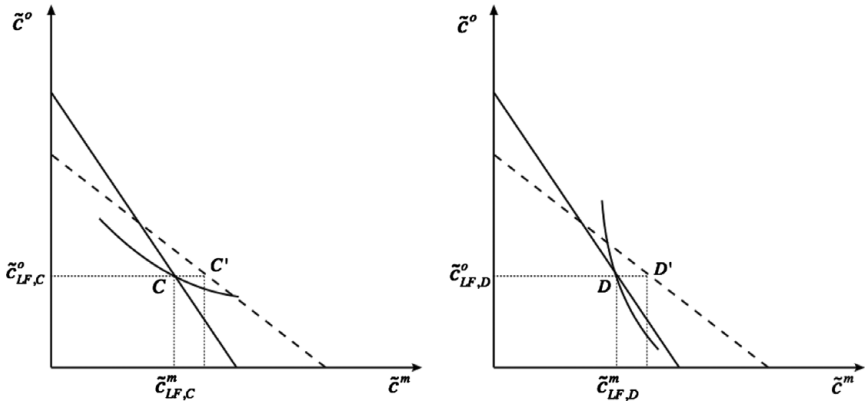


FIGURE 2. Two possible dynamically inefficient laissez-faire equilibria.

push the feasibility constraint (12) outward in a neighborhood of laissez-faire. The left-hand panel of Figure 2 depicts this case, where the interest factor at C is less than the growth rate. The new feasibility constraint after the reduction in $\tilde{d}$ has a greater intercept and a lower slope, but C' is located to the right of C and thus welfare at C' is higher. A first result can be stated:

PROPOSITION 1. *A laissez-faire equilibrium balanced growth path characterized by* $f'(\tilde{k}_{\mathrm{LF}}) \leqslant (1 + g_{\mathrm{LF}})(1 + n)$ *is dynamically inefficient.*

The proof of Proposition 1, provided in Appendix C, consists of showing the effects on welfare of the social planner enacting reductions of $\tilde{k}$ and/or $\tilde{d}$. These reductions expand the consumption possibilities (and thus the welfare) of both the generations living on the new balanced growth path and those living during the transient periods. However, as stated in Appendix C, the pattern of reductions is different according to whether $f'(\tilde{k}_{\mathrm{LF}})$ is strictly less than or just equal to $(1 + g_{\mathrm{LF}})(1 + n)$. This follows from the fact that although (14) is negative in both cases, (13) vanishes in the latter. We then obtain the following:

COROLLARY 1. *Given a laissez-faire equilibrium balanced growth path characterized by* $f'(\tilde{k}_{\mathrm{LF}}) < (1 + g_{\mathrm{LF}})(1 + n)$, *a Pareto improvement can be achieved through a permanent reduction in* $\tilde{k}$, $\tilde{d}$, *or both. If, however,* $f'(\tilde{k}_{\mathrm{LF}}) = (1 + g_{\mathrm{LF}})(1 + n)$, *such a Pareto improvement can only be achieved through a permanent reduction in* $\tilde{d}$.

Notice that, according to Table 1, an inefficient laissez-faire equilibrium balanced growth path characterized by $f'(\tilde{k}_{\mathrm{LF}}) < (1 + g_{\mathrm{LF}})(1 + n)$ is necessarily associated with $\tilde{d}_{\mathrm{LF}}$ greater than $\tilde{d}_*$, but $\tilde{k}_{\mathrm{LF}}$ may be greater or less than $\tilde{k}_*$. Moreover, (14) will also be negative if $f'(\tilde{k}_{\mathrm{LF}}) = (1 + g_{\mathrm{LF}})(1 + n)$. In this case, $\tilde{k}_{\mathrm{LF}} < \tilde{k}_*$ but $\tilde{d}_{\mathrm{LF}} > \tilde{d}_*$.

Turning now to the laissez-faire equilibria where $f'(\tilde{k}_{\mathrm{LF}}) > (1 + g_{\mathrm{LF}})\,(1 + n)$, it is clear that in this case (13) is positive, and thus the only way to expand

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consumption possibilities is to *increase*, rather than decrease, $\tilde{k}$. Starting from the resources available along a balanced growth path, accumulating more physical capital would entail a sacrifice for the generations living in the transient periods. As a consequence, there is no scope for Pareto improvements from modifying the amount of physical capital. In contrast, as will be shown immediately, there may be welfare improvements derived from changing the amount of *human* capital. The reason becomes apparent when one observes in Table 1 that a situation where the marginal product of physical capital exceeds the growth rate of the economy is always associated with $\tilde{k}_{\mathrm{LF}} < \tilde{k}_*$ but may be associated with either $\tilde{d}_{\mathrm{LF}} > \tilde{d}_*$ or $\tilde{d}_{\mathrm{LF}} \leqslant \tilde{d}_*$ (cells a_{21} and a_{22}). As proved in Appendix D, when $f'(\tilde{k}_{\mathrm{LF}}) > (1 + g_{\mathrm{LF}})\,(1+n)$, (14) may have any sign, with the consequence that nothing can be said with generality about the effects of modifying $\tilde{d}$ on consumption possibilities and welfare along a balanced growth path. This statement is illustrated in the third, fourth, and fifth rows in Table 2 for different constellations of the parameters of the production and utility functions in the Cobb–Douglas case. In all of them, the marginal product of physical capital is greater than the growth rate of the economy (and, of course, $\tilde{k}_{\mathrm{LF}} < \tilde{k}_*$), but in cases (iii) and (iv), it is true that $\tilde{d}_{\mathrm{LF}} > \tilde{d}_*$, whereas $\tilde{d}_{\mathrm{LF}} < \tilde{d}_*$ in (v). More importantly, (14) is positive in cases (iv) and (v), so that there is no scope for increasing welfare out of existing resources, but it is definitely *negative* in case (iii): a small reduction in $\tilde{d}$ would *enhance* welfare along a balanced growth path. Summarizing, even when $f'(\tilde{k}_{\mathrm{LF}}) > (1 + g_{\mathrm{LF}})(1 + n)$, if (14) is negative, there will be balanced-growth welfare gains associated with reducing $\tilde{d}$. This case is represented in the right-hand panel of Figure 2. The interest factor at D is greater than the growth rate, but nonetheless, by reducing $\tilde{d}$, the social planner can achieve a higher utility level at D' than at D. We can now reproduce the part of the proof of Proposition 1 in Appendix C concerning a *permanent reduction* in $\tilde{d}$ with no change whatsoever in $\tilde{k}$, and state the following proposition:

PROPOSITION 2. *Starting from a laissez-faire equilibrium balanced growth path characterized by* $f'(\tilde{k}_{\mathrm{LF}}) > (1 + g_{\mathrm{LF}})(1 + n)$*, there are times when a Pareto improvement can be achieved through a reduction in* $\tilde{d}$.

Having shown that a balanced growth path with a marginal product of physical capital greater that the growth rate is consistent with the existence of Pareto improvements resulting from a decrease in the amount of resources devoted to education, it is important to highlight the following:

COROLLARY 2. *A marginal product of physical capital greater than the growth rate of labor in efficiency units is a necessary but not a sufficient condition for the dynamic efficiency of the balanced growth path.*

The significance of Proposition 2 and Corollary 2 is better grasped by looking at the numerical examples. Cases (iii) and (iv) in Table 2 represent laissez-faire balanced growth paths where (a) the marginal product of capital is larger than the growth rate of the economy, (b) the physical capital–labor ratio is lower than the

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optimal one, and (c) the amount of resources devoted to education is greater than the optimal one. Still, although one can design Pareto improvements through a reduction in education expenditures in case (iii), such Pareto improvements are not feasible in simulation (iv). The key difference between the two examples is indeed the sign of the derivative (14).

As a consequence, in order to assess the dynamic efficiency of the laissez-faire equilibrium in this framework, neither the relationship between $\tilde{k}_{\mathrm{LF}}$ and $\tilde{k}_*$ nor the one between $f'(\tilde{k}_{\mathrm{LF}})$ and $(1 + g_{\mathrm{LF}})(1 + n)$ is enough; we also need to compare $\tilde{d}_{\mathrm{LF}}$ and $\tilde{d}_*$. This is in contrast to what is argued by Boldrin and Montes (2005, 2009). Indeed, they claim that "a sufficient condition for the equilibrium path to be dynamically efficient is that the gross rate of return on capital be larger than or equal to one plus the growth rate of output" (p. 656). Furthermore, this will be the case "for reasonable values of" the parameters involved in the production and utility functions (p. 657). On one hand, Proposition 1 shows that, even in the case where the two rates are equal, one can *always* achieve a Pareto improvement. On the other hand, Proposition 2 shows that, if we adopt the Golden Rule approach, the fact that $f'(\tilde{k}_{\mathrm{LF}}) > (1+g_{\mathrm{LF}})(1+n)$ is not a sufficient but a necessary condition for the dynamic efficiency of the laissez-faire balanced growth path. Moreover, we have found that, for (equally) reasonable values of the parameters, the laissez-faire equilibrium is dynamically inefficient in the Golden Rule sense.

6. CONCLUDING COMMENTS

This paper has analyzed the efficiency or inefficiency of the laissez-faire equilibrium balanced growth path in an endogenous growth setting where generations overlap and individuals are pure life-cyclers. Building on our previous characterization of the Golden Rule in such a framework, we have derived the implications of that Golden Rule concept for the notion of dynamic efficiency. We have shown that, in contrast to the case where growth is exogenous, in an economy where endogenous growth is the result of the accumulation of human capital, the relationship between the marginal return to physical capital and the growth rate of the economy is not sufficient to assess the dynamic efficiency of the laissez-faire equilibrium balanced growth path.

Specifically, when the rate of return to physical capital is lower than or equal to the growth rate, this condition is indeed sufficient to establish the dynamic inefficiency of the competitive equilibrium balanced growth path. And, interestingly, these potential welfare gains exist in spite of the fact that the physical capital–labor ratio may be lower than the optimal one, because the amount of output devoted to education is too large.

More importantly, there can be Pareto improvements even in balanced growth paths where the rate of return to physical capital is *higher* than the economy's growth rate. In this situation, the accumulation of physical capital falls short of its Golden Rule counterpart, but the amount of output devoted to education can be below or above the optimal one. In the latter case, welfare gains for both

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the generations living on the new balanced growth path and those living along the transient periods can result from the permanent *reduction* of the amount of output devoted to education. It then follows that a rate of return to physical capital higher than the growth rate of the economy is a necessary although not a sufficient condition for the dynamic efficiency of the laissez-faire equilibrium.

NOTES

1. In particular, the condition $e(0) = 0$ precludes anomalous situations where $h_t = 0$ in spite of $\tilde{d}_{t-1}$ being positive. The same applies to the production function of output in the next paragraph.

2. The reason that we only impose strict quasi-concavity instead of strict concavity of the individual utility function is that, as will be made clearer shortly, we are only interested in *ordinal* preferences. This is a less stringent assumption than the one made in Docquier et al. (2007), who need to assume that $0 < j < 1$ for the discounted sum of individual utilities to be well defined.

3. In overlapping-generations models with exogenous growth it is difficult to ensure existence, uniqueness, and local stability of balanced growth paths [see Galor and Ryder (1989), De la Croix and Michel (2002), and Li and Lin (2012)]. The same considerations apply in the current endogenous growth setting.

4. Although, strictly speaking, this utility function is not homogeneous, it can be taken to represent the limiting case of homogeneity of degree zero.

5. The reason that this definition focuses on *reductions* in either $\tilde{k}$ or $\tilde{d}$ is simple: increasing $\tilde{k}$ and/or $\tilde{d}$ necessarily harms those generations living at the time where the change takes place (see Section 5).

6. The social planner can, of course, change both $\tilde{k}$ and $\tilde{d}$. If this is the case, the results follow in an obvious way.

7. The intuition for this result lies in the fact that the individual decision to invest in human capital rests only on the comparison of benefits (future wage) and costs (interest rate to be paid for the loan). In contrast, the social planner takes also into account other costs associated with investing in education, i.e., the resources required to maintain the balanced growth path levels of consumption and physical and human capital measured in terms of efficient labor when $\tilde{d}$ increases (see equation (15)). Overaccumulation of physical capital under laissez-faire implies that the wage is higher and the interest rate is lower than their optimal counterparts. Because the individual also ignores the other costs of investing in education, she clearly overinvests in education under laissez-faire. Even if the wage and interest rate were optimal, the *other costs*, neglected by the individual, continue to imply overinvestment in education.

8. Overinvestment in education is also feasible in Bishnu (2013), who links consumption externalities and human capital accumulation. Nevertheless, the models are not easily comparable, as the balanced growth path in Bishnu (2013) displays no productivity growth, so that demography is the only source of long-run growth.

9. Notice that there is no contradiction between our claim in Section 4 that laissez-faire cannot decentralize the optimum and the possibility that the laissez-faire interest factor could be equal to the economy's growth rate. The second part says that $f'(\tilde{k}_{\mathrm{LF}}) = (1+g_{\mathrm{LF}})(1+n)$ is feasible, as illustrated in cell a_{21} in Table 1. The first part simply says that this equality cannot be fulfilled for $\tilde{k}_{\mathrm{LF}} = \tilde{k}_*$ and $\tilde{d}_{\mathrm{LF}} = \tilde{d}_*$.

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APPENDIX A: THE COBB–DOUGLAS CASE

Consider the functional forms $f(\tilde{k}_t) = A\tilde{k}_t^{\alpha}$, $e(\tilde{d}_t) = B\tilde{d}_t^{\lambda}$, and $U_t = \log c_t^m + \beta \log c_{t+1}^o$, introduced in Section 2. Starting with the laissez-faire equilibrium, the distribution of

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consumption entails $c_{t+1}^o/\beta c_t^m = (1+r_{t+1})$, so that savings are $s_t = [\beta/(1+\beta)]\,\omega_t$. From (5), the choice of education is $d_{t-1} = [\lambda(1-\alpha)/\alpha]\,k_t$. This allows one to write the physical capital market equilibrium condition (7) as a nonlinear first-order difference equation in $\tilde{k}$, i.e., $\tilde{k}_{t+1} = G(.)\tilde{k}_t^{\alpha(1-\lambda)}$, where $G(.)$ is a constant that depends on α, β, λ, n, A, and B. The balanced growth path with a positive value of $\tilde{k}$ is unique and globally stable, and is given by

$$\tilde{k}_{\mathrm{LF}} = \left\{ \frac{\left[\frac{1}{(1+n)}\frac{\beta}{(1+\beta)}\frac{\alpha(1-\alpha)(1-\lambda)A}{\alpha+\lambda(1-\alpha)}\right]^{1-\lambda}}{\left[\frac{\lambda(1-\alpha)}{\alpha}\right]^{\lambda} B} \right\}^{\frac{1}{1-\alpha(1-\lambda)}}. \tag{A.1}$$

The associated value of $\tilde{d}$ can be found from the optimal education choice and is written as

$$\tilde{d}_{\mathrm{LF}} = \left[\frac{B\lambda(1-\alpha)}{\alpha}\right]^{\frac{1}{1-\lambda}} \left(\tilde{k}_{\mathrm{LF}}\right)^{\frac{1}{1-\lambda}}. \tag{A.2}$$

Using (A.1) and (A.2), one can construct the simulation results reported in Table 2.

As for the social optimum, the distribution of consumption (11) is written $\tilde{c}_*^o/\beta\tilde{c}_*^m = (1+g_*)\,(1+n)$, and the productivity growth rate is $1+g_* = B\tilde{d}_*^{\lambda}$. We can then write (12) as

$$\tilde{c}_*^o = \frac{\beta}{1+\beta}\left[A\tilde{k}_*^{\alpha} - (1+n)B\tilde{d}_*^{\lambda}\tilde{k}_* - (1+n)\tilde{d}_*\right](1+n)B\tilde{d}_*^{\lambda}. \tag{A.3}$$

Also, $f'(\tilde{k}_*) = \alpha A\tilde{k}_*^{\alpha-1}$ and $e'(\tilde{d}_*) = \lambda B\tilde{d}_*^{\lambda-1}$, so that (10) becomes

$$\begin{aligned}&\lambda B\tilde{d}_*^{\lambda-1}\left\{\beta\left[A\tilde{k}_*^{\alpha} - (1+n)B\tilde{d}_*^{\lambda}\tilde{k}_* - (1+n)\tilde{d}_*\right] - (1+\beta)(1+n)B\tilde{d}_*^{\lambda}\tilde{k}_*\right\}\\ &\quad = (1+\beta)(1+n)B\tilde{d}_*^{\lambda},\end{aligned} \tag{A.4}$$

whereas (9) is $\alpha A\tilde{k}_*^{\alpha-1} = (1+n)B\tilde{d}_*^{\lambda}$. These two equations make it possible to characterize $(\tilde{k}_*, \tilde{d}_*)$.

APPENDIX B: PROOF OF LEMMA 1

Consider the sign of (14). Using (12) evaluated on the laissez-faire balanced growth path, we have

$$e'(\tilde{d}_{\mathrm{LF}})\left[f(\tilde{k}_{\mathrm{LF}}) - (1+g_{\mathrm{LF}})\,(1+n)\tilde{k}_{\mathrm{LF}} - \Lambda_{\mathrm{LF}}\right] \gtrless (1+g_{\mathrm{LF}})\,(1+n), \tag{B.1}$$

where $\Lambda_{\mathrm{LF}} = (1+g_{\mathrm{LF}})\,(1+n)\tilde{k}_{\mathrm{LF}} + (1+n)\tilde{d}_{\mathrm{LF}} + \tilde{c}_{\mathrm{LF}} > 0$. Because in general $f'(\tilde{k}_{\mathrm{LF}}) \gtrless (1+g_{\mathrm{LF}})\,(1+n)$, we write $(1+g_{\mathrm{LF}})\,(1+n) = f'(\tilde{k}_{\mathrm{LF}}) + \varepsilon$ for $\varepsilon \gtrless 0$. Then

$$e'(\tilde{d}_{\mathrm{LF}})\left\{f(\tilde{k}_{\mathrm{LF}}) - \left[f'(\tilde{k}_{\mathrm{LF}}) + \varepsilon\right]\tilde{k}_{\mathrm{LF}} - \Lambda_{\mathrm{LF}}\right\} \gtrless \left[f'(\tilde{k}_{\mathrm{LF}}) + \varepsilon\right], \tag{B.2}$$

which, using the balanced growth path version of (5), simplifies to

$$e'(\tilde{d}_{\mathrm{LF}})\left(-\varepsilon\tilde{k}_{\mathrm{LF}} - \Lambda_{\mathrm{LF}}\right) - \varepsilon \gtrless 0. \tag{B.3}$$

Therefore, if $\varepsilon \geqslant 0$ [i.e., when $f'(\tilde{k}_{\mathrm{LF}}) \leqslant (1+g_{\mathrm{LF}})\,(1+n)$], the sign of (B.3) is negative. This is the proof of Lemma 1. This result holds when $\tilde{k}_{\mathrm{LF}} \geqslant \tilde{k}_*$ in Table 1 but also applies

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when $\tilde{k}_{\mathrm{LF}} < \tilde{k}_*$ as long as $f'(\tilde{k}_{\mathrm{LF}}) \leqslant (1+g_{\mathrm{LF}})(1+n)$ [simulations (i) and (ii) in Table 2, respectively]. Notice, however, that if $\varepsilon < 0$ [i.e., when $f'(\tilde{k}_{\mathrm{LF}}) > (1+g_{\mathrm{LF}})(1+n)$], the sign of (B.3) is indeterminate, and along with it that of (14). This is the basis for the proof of Proposition 2 in Appendix D.

APPENDIX C: PROOF OF PROPOSITION 1

The proof of Proposition 1 proceeds in two steps, associated respectively with the effects of changes in $\tilde{k}$ and $\tilde{d}$.

We start by reducing $\tilde{k}_{\mathrm{LF}}$ in a situation where $f'(\tilde{k}_{\mathrm{LF}}) < (1+g_{\mathrm{LF}})(1+n)$. If $f'(\tilde{k}_{\mathrm{LF}}) = (1+g_{\mathrm{LF}})(1+n)$, (13) will be zero, and the proof proceeds from the next paragraph, referred to reductions in $\tilde{d}_{\mathrm{LF}}$ without modifying $\tilde{k}_{\mathrm{LF}}$. By virtue of (13), a small reduction in $\tilde{k}$ will expand consumption possibilities and welfare along a new balanced growth path. We thus have to focus on the effects related to the transition between balanced growth paths. Assume that in some time period, say T (where, by definition, $\tilde{k}_T = \tilde{k}_{\mathrm{LF}}$), a social planner decides on a *permanent reduction* of the amount of physical capital per unit of efficient labor, starting from the amount to be available in period $T+1$. Let $-\Delta\tilde{k}$ be this small reduction, so that $\tilde{k}_{T+1} = \tilde{k}_{\mathrm{LF}} - \Delta\tilde{k}$. This new value will be kept constant forever, so that a new balanced growth path is immediately implemented. Notice that $\tilde{d}_T$ has not been modified from its initial value, so that $\tilde{d}_T = \tilde{d}_{\mathrm{LF}}$. This allows one to write

$$f(\tilde{k}_T) - (1+n)\tilde{d}_T - e(\tilde{d}_T)(1+n)\tilde{k}_{T+1} = f(\tilde{k}_{\mathrm{LF}}) - (1+n)\tilde{d}_{\mathrm{LF}} - e(\tilde{d}_{\mathrm{LF}})(1+n)(\tilde{k}_{\mathrm{LF}} - \Delta\tilde{k}),$$

and, using the aggregate feasibility constraint,

$$\left[\tilde{c}^m_T + \frac{\tilde{c}^o_T}{e(\tilde{d}_{\mathrm{LF}})(1+n)}\right] > \left[\tilde{c}^m_{\mathrm{LF}} + \frac{\tilde{c}^o_{\mathrm{LF}}}{e(\tilde{d}_{\mathrm{LF}})(1+n)}\right]. \tag{C.1}$$

Equation (C.1) implies that in period T aggregate consumption per unit of efficient labor has increased, as a consequence of the reduction $-\Delta k$, by exactly the amount $e(\tilde{d}_{\mathrm{LF}})(1+n)\Delta\tilde{k}$. This, together with the fact that (13) is negative, implies that the initial laissez-faire balanced growth path is dynamically inefficient: consumption possibilities have been enhanced both for those individuals living in period T, when the reduction $-\Delta k$ takes place, and those living on the new balanced growth path.

Second, with respect to changes in $\tilde{d}_{\mathrm{LF}}$, Lemma 1 implies that, when $f'(\tilde{k}_{\mathrm{LF}}) \leqslant (1+g_{\mathrm{LF}})(1+n)$, a reduction in the amount of *human* capital (as measured by $\tilde{d}$) would *also* expand consumption possibilities and hence utility along the new balanced growth path. Therefore, as before, we need to track the effects related to the transition between balanced growth paths. Let $\tilde{d}_{\mathrm{LF}}$ be the amount of output per unit of efficient labor devoted to the production of human capital in a certain period, say $T-1$. We can assume that a social planner decides in period T on a *permanent reduction* of $\tilde{d}$, with no change whatsoever in $\tilde{k}$. Let $-\Delta\tilde{d}$ be this small reduction, so that $\tilde{d}_T = \tilde{d}_{\mathrm{LF}} - \Delta\tilde{d}$. This new value of $\tilde{d}$ will be kept constant forever, with the consequence that a new balanced growth path is immediately implemented. As $\tilde{k}_T = \tilde{k}_{T+1} = \tilde{k}_{\mathrm{LF}}$, it will be verified that

$$f(\tilde{k}_T) - (1+n)\tilde{d}_T - e(\tilde{d}_T)(1+n)\tilde{k}_{T+1} = f(\tilde{k}_{\mathrm{LF}}) - (1+n)(\tilde{d}_{\mathrm{LF}} - \Delta\tilde{d}) - e(\tilde{d}_{\mathrm{LF}} - \Delta\tilde{d})(1+n)\tilde{k}_{\mathrm{LF}}.$$

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Using the aggregate feasibility constraint and taking into account that $e(.)$ is an increasing function, (C.1) will then be satisfied. Therefore, the reduction $-\Delta\tilde{d}$ at time T allows an increase in aggregate consumption per unit of efficient labor available in period T by an amount $(1+n)\Delta\tilde{d} + [e(\tilde{d}_{\mathrm{LF}}) - e(\tilde{d}_{\mathrm{LF}} - \Delta\tilde{d}_{\mathrm{LF}})](1+n)\tilde{k}_{\mathrm{LF}}$. Together with the negativity of (14), this implies that the initial balanced growth path is dynamically inefficient.

APPENDIX D: PROOF OF PROPOSITION 2

To prove Proposition 2, consider the case where $f'(\tilde{k}_{\mathrm{LF}}) > (1+g_{\mathrm{LF}})(1+n)$ or, in terms of the notation introduced in Appendix B, $\varepsilon < 0$. In this case, the sign of (B.3), and therefore that of (14), is ambiguous in general. Thus $f'(\tilde{k}_{\mathrm{LF}}) > (1+g_{\mathrm{LF}})(1+n)$ is not sufficient to assess the dynamic efficiency of the laissez-faire equilibrium balanced growth path. This helps explain the different signs in the fifth column of simulations (iii)–(iv) in Table 2. Although, according to Table 1, this situation can only correspond to $\tilde{k}_{\mathrm{LF}} < \tilde{k}_*$, there can still be scope for a Pareto improvement provided that (B.3), and along with it (14), is negative [e.g., simulation (iii) in Table 2].

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