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UNIFORM PROCEDURES IN UNCOUNTABLE STRUCTURES

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Abstract. This article contributes to the general program of extending techniques and ideas of effective algebra to computable metric space theory. It is well-known that relative computable categoricity (to be defined) of a computable algebraic structure is equivalent to having a c.e. Scott family with finitely many parameters (e.g., [1]). The first main result of the article extends this characterisation to computable Polish metric spaces. The second main result illustrates that just a slight change of the definitions will give us a new notion of categoricity unseen in the countable case (to be stated formally). The second result also shows that the characterisation of computably categorical closed subspaces of $\mathbb{R}^n$ contained in [17] cannot be improved. The third main result extends the characterisation to not necessarily separable structures of cardinality κ using κ-computability.

§1. Introduction. This article contributes to the general program (e.g., [14, 16–20]) which aims to extend the key ideas and techniques of *countable* effective algebra to the study of *uncountable* structures. We follow the standard terminology of effective algebra [1, 5] and say that a computable presentation (a computable copy, a constructivization) of a countably infinite algebraic structure $\mathcal{A}$ is an isomorphic copy $\mathcal{C}$ of $\mathcal{A}$ upon the domain ω in which all predicates and functions are (uniformly) Turing computable.

There is a large body of research focused on describing *computable categoricity* in standard algebraic classes [1, 5]. An algebraic structure $\mathcal{A}$ is computably categorical (also known as autostable) if it has a unique computable copy, up to computable isomorphism [1, 5]. In many algebraically natural classes computable categoricity is equivalent to the stronger notion of relative computable categoricitiy. Recall that a (countable) computable algebraic structure $\mathcal{A}$ is *relatively computably categorical* (r.c.c.) if for any (not necessarily computable) presentation $\mathcal{B}$ of $\mathcal{A}$ there exists an isomorphism $f : \mathcal{B} \to \mathcal{A}$ computable relative to $D(\mathcal{B})$, where $D(\mathcal{B})$ indicates the open diagram of $\mathcal{B}$. Clearly, relative computable categoricity implies computable categoricity, and it is well-known that computable categoricity does not imply relative computable categoricity in general (Ventsov was the first to separate these two notions, see [4] for a detailed survey). The subtle difference between computable categoricity and relative computable categoricity is captured by the syntactical notion of a Scott family. A Scott family [1] is a collection of formulae that describe automorphism orbits of tuples in the structure (we will give a more detailed definition

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shortly). It is well-known (e.g., [4]) that a computable algebraic structure $\mathcal{A}$ is relatively computably categorical if and only if $\mathcal{A}$ possesses a computably enumerable Scott family consisting of first-order existential formulae with finitely many parameters (from $\mathcal{A}$). This characterization has proved to be very useful in the study of computable categoricity. In this article we attack the following general problem:

Find a syntactical characterization of relative computable categoricity for uncountable structures.

Note that the notion of a relatively computably categorical uncountable structure has not been discussed yet. In fact, there is no standard and generally excepted definition of a computable uncountable structure (let alone a computably categorical uncountable structure).

Nonetheless, in the special important case of a Polish metric space we do have a standard and widely accepted notion of computability [21, 24] and a natural notion of (relative) computable categoricity [17] (to be clarified). This approach and its variations covers only separable structures. In the case of inseparable structures there is no general common notion of computability that would be regarded as standard. We follow the approach introduced in [10] and use admissible computability [22, 23] to naturally extend all notions of effective algebra to structures of cardinality $\kappa > \omega$. In particular, we have a naturally induced notion of κ-computable categoricity.

In each of the two frameworks, we obtain the desired syntactical characterization. Furthermore, we discover some unusual features that do not appear in the countable world of effective algebra. We now discuss the results in more detail.

1.1. Computable Polish spaces. A *structure* on a Polish metric space $\mathcal{M}$ is any sequence of points $(\alpha_i)_{i\in\omega}$ which is dense in $\mathcal{M}$. A structure $(\alpha_i)_{i\in\omega}$ is computable if $(d(\alpha_i, \alpha_j))_{i,j\in\omega}$ is a uniformly computable sequence of reals.[1] We can extend many standard notions of effective algebra to Polish metric spaces. For example, a Polish metric space is *computably categorical* if the space possesses a unique computable structure up to computable surjective isometry (between completions) [17]. Examples of computably categorical metric spaces include Cantor space, the Urysohn space, and separable Hilbert spaces [17]. Furthermore, l_p is computably categorical iff $p = 2$ [14]. Also, $(C[0, 1], \sup)$ and some closed subspaces of Cantor space are not computably categorical [17]. See [16] for further results. As noted in [17], there is also a natural notion of relative computable categoricity for Polish metric spaces. A computable Polish metric space $(M, d, (\alpha_i)_{i\in\omega})$ is relatively computably categorical (r.c.c.) if we require the surjective isometries to be computable with respect to the structures under consideration (which themselves do not have to be effective). We note that any countable structure can be effectively transformed into a Polish space preserving all reasonable effective properties of the structure (in the sense of [11]).[2]

[1] Note we do not require $\nu : i \to \alpha_i$ to be 1-1, but one can effectively and uniformly transform ν into an injective μ (we leave the easy verification to the reader). So we may assume all our structures are injective.

[2] First, effectively encode the structure A into a countable undirected irreflexive graph [11], denote it $G(A)$. For any $x, y \in G(A)$, let $d(x, y) = 1$ if there is an edge between x and y, and let $d(x, y) = 2$ otherwise. The resulting metric space $M(A)$ has a computable presentation (in the above sense) iff A has a computable copy. Many other properties (including all variations of computable categoricity) are preserved. In particular, there exists a Polish metric space having exactly two computable presentations,

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We will give a syntactical description of r.c.c. Polish metric spaces shortly. We first address the questions: Which formal language should we use? And if we need to use parameters, what would a parameter mean in this context? What does a Scott family mean for a Polish space?

1.1.1. The language. One would expect that continuous logic [2] should be the right choice, but this approach does not seem to agree very well with computability (at least not in the sense that we need). We view separable metric spaces as structures in the first-order language

$$\mathcal{L} = \{d_{<r}(\cdot,\cdot), d_{>r}(\cdot,\cdot) : r \in \mathbb{Q}\},$$

where $\mathcal{M} \models d_{<r}(x,y)$ iff $d_{\mathcal{M}}(x,y) < r$, and similarly $\mathcal{M} \models d_{>r}(x,y)$ iff $d_{\mathcal{M}}(x,y) > r$. We do not allow $=$ and $\neg$ when we form $\mathcal{L}$-formulae; thus, we restrict ourselves to *positive* atomic $\mathcal{L}$-formulae with no equality. If X is a countable metric space, we write $D^+(X)$ for the positive open diagram of X in the language $\mathcal{L}$ which consists of conjunctions and disjunctions of positive atomic $\mathcal{L}$-formulae. Clearly, $(\alpha_i)_{i\in\mathbb{N}}$ is a computable structure on $\mathcal{M}$ iff $D^+((\alpha_i)_{i\in\mathbb{N}})$ is a computably enumerable set, under the standard Gödel numbering (we identify α_i with its index i).

1.1.2. An approximate Scott family. In countable structure theory, a Scott family of a structure $\mathcal{A}$ is a collection of formulae $\mathbb{S}$ such that

(1) any finite tuple $\bar{a}$ in $\mathcal{A}$ satisfies some $\theta \in \mathbb{S}$ and
(2) if $\theta \in \mathbb{S}$ holds on both $\bar{a}$ and $\bar{b}$ in $\mathcal{A}$, then there exists an automorphism of $\mathcal{A}$ taking $\bar{a}$ to $\bar{b}$.

Since perfect Polish metric spaces are uncountable, we cannot hope to just take the same definition literally. Instead, we allow both (1) and (2) to be true "up to ϵ". This can be viewed as a sequence of families, one family for each positive rational ϵ. Equivalently, we may assume that the family is indexed by rational numbers, so that Θ_ϵ is true with precision ϵ. We will give a rigorous definition in the preliminaries.

1.1.3. Uniform categoricity. Before we state the first main result, we extend another standard notion from effective algebra to separable spaces. A computable space $\mathcal{M} = (M, d, (\alpha_i)_{i\in\omega})$ is *uniformly computably categorical* if there exists a uniform procedure that, given any *computable* structure $(\beta_i)_{i\in\omega}$ on $\mathcal{M}$ produces a surjective isometry between the completions of $(\beta_i)_{i\in\omega}$ and $(\alpha_i)_{i\in\omega}$. If we do not restrict ourselves to computable structures and allow $(\beta_i)_{i\in\omega}$ be any dense set on $\mathcal{M}$, then $\mathcal{M}$ is *uniformly relatively computably categorical*.

We will allow the uniform procedure to use finitely many parameters, and there are (at least) two ways to interpret what a parameter means in a metric space.

1.1.4. Parameters. In a presentation of the structure, a list of parameters typically corresponds to a finite sequence of elements of ω. As a finite object, this list is trivially computable relative to the presentation.

The situation is different in Polish metric spaces. If (M, d) is a Polish metric space, $(\alpha_i)_{i\in\omega}$ is a structure on $\mathcal{M}$, and $x \in \mathcal{M}$ is some point, then x might not correspond

up to computable isometry (follows from [6]). Our approach to metric spaces thus generalises effective algebra. This generalization is proper since some of the new features of computable Polish spaces that we discover in this article would not be possible in the countable world.

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to any of the α_i. However, since $(\alpha_i)_{i\in\omega}$ is dense in $\mathcal{M}$, x will correspond to an equivalence class of Cauchy sequences from $(\alpha_i)_{i\in\omega}$. To specify x, it will suffice to specify one of these Cauchy sequences; for practical reasons, we will require that this Cauchy sequence converges quickly. Thus we will say that x is computable with respect to a structure $(\alpha_i)_{i\in\omega}$ if there is a computable sequence $(i_n)_{n\in\omega}$ such that $d(x, \alpha_{i_n}) < 2^{-n}$ for all n.

Now, consider the interval $[0, \beta]$, with the usual distance metric inherited from $\mathbb{R}$, where β is left-c.e. It is easy to see that the automorphism-invariant point $\beta/2$ is computable w.r.t. some dense structures on the space, while it is not computable w.r.t. some other structures, see [17] for a detailed proof. What would a "parameter" mean in Polish metric spaces? We suggest two different answers. Remarkably, both notions will allow us to describe relative categoricity.

The first approach to parameters uses points that are computable w.r.t. any given *computable* structure on the space. We say that a finite tuple of points $x_1, \dots, x_k$ is *intrinsically computable* (i.c.) if any structure on $\mathcal{M}$ computes an automorphic image of the tuple. This definition is supported by examples of closed subspaces of $\mathbb{R}^n$, see [17]. If we do not restrict ourselves to computable structures and consider arbitrary dense subsets of $\mathcal{M}$, we get the notion of a *relatively intrinsically computable* (r.i.c.) point.

The second approach allows a whole neighbourhood of a point to be a parameter. That is, any point from the neighbourhood can be put as a parameter, and this variation will not affect the desired properties. In this case we say that the parameter c is *stable*. (To be clarified in the preliminaries.)

1.2. Results on Polish Spaces.

THEOREM 1.1. *Let $\mathcal{M}$ be a computable Polish metric space. The following are equivalent*:

(1) *$\mathcal{M}$ is relatively computably categorical.*
(2) *$\mathcal{M}$ is uniformly relatively computably categorical with stable parameters.*
(3) *$\mathcal{M}$ is uniformly computably categorical with stable parameters.*
(4) *$\mathcal{M}$ possesses a c.e. approximate Scott family with stable parameters.*

We note that the stable parameters in the theorem above can be chosen r.i.c. To see why, recall that the structure is relatively c.c., so any tuple of special points from the computable structure will be r.i.c. We now investigate the case when parameters are merely *intrinsically computable*. That is, we require a parameter to be computable only in all *computable* structures of the space. If a space is uniformly categorical after fixing a finite tuple of such parameters, we say that the space is *weakly uniformly computably categorical* (weakly u.c.c.). It is known that any c.c. closed subspace of $\mathbb{R}^n$ is in fact weakly u.c.c. [17]. Interestingly enough, weak uniform categoricity does not imply relative categoricity even for closed subspaces of $\mathbb{R}^2$.

THEOREM 1.2. *There exists a weakly u.c.c. closed subspace of $\mathbb{R}^2$ that is not r.c.c.*

The theorem does not resemble any known result in effective algebra where parameters are finite objects and thus can be nonuniformly fixed in any given copy. Furthermore, the space $\mathcal{M}$ witnessing Theorem 1.2 has a single point a so that after adjoining a to the signature of $\mathcal{M}$, the space becomes relatively computably

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categorical. Theorem 1.2 implies that the mentioned above characterisation of c.c. subspaces of $\mathbb{R}^n$ [17] cannot be improved, this answers a question left open in [17].

In countable effective algebra, the use of Scott families rather than uniform procedures is usually a matter of taste. It is rather interesting that in the context of metric spaces approximate Scott families seem to significantly simplify some proofs. Any proof that uses an approximate Scott family will merely concentrate on local properties of the formulae, thus skipping the unpleasant analytic back-and-forth argument. To demonstrate this feature, in Proposition 5.1, we produce an approximate Scott family for the Urysohn space. The proof can be compared with the original brute-force argument that takes several pages [17].

1.3. Inseparable structures. Our goal is to study computable presentations of uncountable structures. In Polish spaces, we heavily relied on the dense countable sequence and the metric to define what a computable presentation means. In absence of a metric, one can imagine using some sort of effective topology. But if a structure is also not separable then we need to come up with some other, new notion of computability. One possible approach to applying computability to such uncountable structures uses admissible computability [22, 23]. The notions of computability are extended. For an admissible ordinal, in particular, for an uncountable regular cardinal κ, we could define κ-computability using Turing machines with tapes and computation lengths bounded by κ. However, the original definition was syntactic, using Σ_1-definability over L_κ. This approach has been successfully used to investigate uncountable linear orderings [8, 9] and free abelian groups [7]. Carson, Johnson, Knight, Lange, McCoy, and Wallbaum (see [3]) obtained the syntactic characterisation of relatively Σ_α relations in this context, for all $\alpha < \kappa$. Here we consider relative computable categoricity.

There is some work on relative computable categoricity in the context of κ-computability. Johnson [12, 13] showed that for $\kappa = \aleph_1$, the Zil′ber field of cardinality $\aleph_1$ is not computably categorical, but it is relatively Δ^0_2-categorical. By contrast, the Zil′ber cover is relatively computably categorical. Johnson [12, 13] proved a transfer theorem for categoricity of "abstract excellent classes". For these classes, there is a unique model in each uncountable power. For a successor cardinal κ, the structure of cardinality κ has a computable copy. If $\kappa < \lambda$, and the structure of cardinality κ is relatively κ-computably categorical, then the structure of cardinality λ is relatively λ-categorical.

It turns out that the natural setting allows for infinite arities. For a regular cardinal κ, a *κ-structure* $\mathcal{M}$ is a set of size κ equipped with functions and relations which are allowed to have arities of any ordinal below κ. Such a structure is *κ-computable* if its universe is κ, and uniformly, its functions and relations are κ-computable, that is $\Delta_1(L_\kappa)$-definable. And as expected, such a structure is *relatively κ-computably categorical* if for all $\mathcal{N} \cong \mathcal{M}$ there is an $\mathcal{N}$-computable isomorphism between $\mathcal{N}$ and $\mathcal{M}$, where again a κ-structure is identified with its atomic diagram. We prove the following:

THEOREM 1.3. *A κ-computable κ-structure $\mathcal{M}$ is relatively κ-computably categorical if and only if there is some $A \subset \mathcal{M}$ of size less than κ such that the structure $(\mathcal{M}, a)_{a \in A}$ has an effective, continuous Scott family.*

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All undefined notions will be clarified later in the article. We note that as far as relative computable categoricity is concerned, this approach is orthogonal to the one discussed above: every separable structure is relatively $2^{\aleph_0}$-computably categorical because countably much information can be given as a parameter.

§2. Preliminaries. Recall that we view separable metric spaces as structures in the first-order language $\mathcal{L} = \{d_{<r}(\cdot,\cdot), d_{>r}(\cdot,\cdot) : r \in \mathbb{Q}\}$, where $\mathcal{M} \models d_{<r}(x,y)$ iff $d_{\mathcal{M}}(x,y) < r$, and similarly $\mathcal{M} \models d_{>r}(x,y)$ iff $d_{\mathcal{M}}(x,y) > r$, and we restrict ourselves to *positive* atomic $\mathcal{L}$-formulae with no equality. If X is a countable metric space, we write $D^+(X)$ for the positive open diagram of X in the language $\mathcal{L}$. A sequence of points $(\alpha_i)_{i\in\mathbb{N}}$ dense in $\mathcal{M}$ is a *structure* on $\mathcal{M}$, and the structure is *computable* iff $D^+((\alpha_i)_{i\in\mathbb{N}})$ is a computably enumerable set, under the standard Gödel numbering (we identify α_i with its index i).

We use $X, Y, Z \ldots$ and $(\alpha_i)_{i\in\omega}$, $(\beta_j)_{j\in\omega}$, $(\gamma_k)_{k\in\omega}$ $\ldots$ to denote structures on a metric space. The structures are not assumed to be computable unless we specifically add that assumption. Points in a structure will be typically identified with the respective indices (i.e., α_i is identified with i). Let x be a point in (M,d) equipped with a structure $(\alpha_i)_{i\in\omega}$. We say that a sequence $(x_i)_{i\in\omega}$ is a fast Cauchy name of x w.r.t. $(\alpha_i)_{i\in\omega}$ if x_i are points from the sequence $(\alpha_i)_{i\in\omega}$ with the property $d(x,x_i) < 2^{-i}$. A point is computable w.r.t. $(\alpha_i)_{i\in\omega}$ if it has a computable sequence which is also a fast Cauchy name w.r.t. this structure.

2.1. Computable isometries. Let X and Y be structures on $\mathcal{M}$, where Y is computable and X is not necessarily computable. Recall that $D^+(X)$ stands for the positive open diagram of X. We say that a surjective isometry

$$U : \bar{X} \to \bar{Y}$$

is computable w.r.t. X and Y if there exists an enumeration operator Φ such that for any point $x \in X$, $\Phi^{D^+(X)}(x)$ lists a fast Cauchy name w.r.t. Y that converges to $U(x)$. We write $\Phi_j^{D^+(X)}(x)$ for the jth element of the sequence. This approach is equivalent to the standard definition from [21]. The above definition can be used to formalise the notion of relative computable categoricity that was stated informally in the introduction.

2.2. Uniform categoricity. We defined uniform categoricity in the introduction, but we did not clarify the notion of uniform computable categoricity with (finitely many) parameters. We say that $(\mathcal{M}, d, (\alpha_i)_{i\in\omega})$ is *uniformly computably categorical* with parameters points $\bar{c} = (c_1, \ldots, c_k)$ if there exists a uniform procedure that, given any computable structure $(\beta_i)_{i\in\omega}$ on $(\mathcal{M}, d)$ and fast Cauchy names for $c_1, \ldots, c_k$ (w.r.t. $(\beta_i)_{i\in\omega}$), produces a surjective isometry between the completions of $(\beta_i)_{i\in\omega}$ and $(\alpha_i)_{i\in\omega}$. If $\bar{c}$ are (relatively) i.c., then we can take computable fast Cauchy names of the points.

Similarly, $(M, d, (\alpha_i)_{i\in\omega})$ is *uniformly computably categorical* with stable parameters $\bar{c} = (c_1, \ldots, c_k)$ if there exist balls $B_1, \ldots, B_k$ around $c_1, \ldots, c_k$ and a uniform procedure that given any computable structure $(\beta_i)_{i\in\omega}$ and fast Cauchy names of points $\bar{d} \in \overline{B}$ (w.r.t. $(\beta_i)_{i\in\omega}$) produces a surjective isometry between the completions of $(\beta_i)_{i\in\omega}$ and $(\alpha_i)_{i\in\omega}$. Clearly, we can choose $\bar{d}$ from the dense sequence $(\beta_i)_{i\in\omega}$, in which case there will be computable fast Cauchy names for the points.

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2.3. Approximate Scott families. Recall that the language $\mathcal{L}$ is built up using binary predicates $d_{<r}$ and $d_{>r}$ where r ranges over $\mathbb{Q}^+$, with neither equality nor negation. A $\mathbb{Q}$-*indexing* of a family $\mathbb{F}$ of L-formulae is a map $\mu : \mathbb{F} \to \mathbb{Q}^+$. If $\mathbb{F}$ admits a $\mathbb{Q}$-indexing μ, and there is no other indexing of $\mathbb{F}$ that is of any importance to us, we write Θ_ϵ for $\Theta \in \mathbb{F}$ to express that $\mu(\Theta) = \epsilon$.

In the following, tuples of $\mathcal{M}$-points are viewed as elements of the corresponding direct product of $\mathcal{M}$ with the sup-metric inherited from d_M. We let $\mathrm{nbh}_\epsilon(\bar{x})$ denote the ϵ-ball about $\bar{x}$ in this metric.

Definition 2.1. Let $\mathcal{M}$ be a Polish space. We say that a nonempty collection $\mathbb{S}$ of $\mathbb{Q}$-indexed first-order $\mathcal{L}$-formulae is an *approximate Scott family* if the following two conditions hold:

(1) For all tuples $\bar{x}$ and $\bar{x}'$ in $\mathcal{M}$, if $M \models \Theta_\epsilon(\bar{x})$ and $M \models \Theta_\epsilon(\bar{x}')$ for some $\Theta \in \mathbb{S}$ and $\epsilon \in \mathbb{Q}^+$, then there exists an automorphism of $\mathcal{M}$ taking $\bar{x}$ into $\mathrm{nbh}_\epsilon(\bar{x}')$.
(2) For every tuple $\bar{x}$ in $\mathcal{M}$ and $\epsilon_1 \in \mathbb{Q}^+$, there exists a tuple $\bar{x}' \in \mathrm{nbh}_{\epsilon_1}\bar{x}$ and a formula Θ_{ϵ_2} with $\epsilon_2 < \epsilon_1$ in $\mathbb{S}$ such that $M \models \Theta_{\epsilon_2}(\bar{x}')$.

An approximate Scott family $\mathbb{S}$ is *c.e.* if there exists an effective listing of the Gödel numbers of $\mathbb{S}$, and the $\mathbb{Q}$-indexing is given by a (partial) computable function that halts on each member of $\mathbb{S}$. Without loss of generality, we may assume that $\mathbb{S}$ is actually indexed by rationals of the form 2^{-i}.

We now formally define the notion of a Scott family with stable parameters. We say that parameters $\bar{c} = (c_1, \ldots, c_n)$ of an approximate Scott family $\mathbb{S}$ are *stable* if there exists an open neighborhood $\overline{B} = B_1 \times \cdots \times B_k$ of $\bar{c}$ such that for any $\bar{c}' \in \overline{B}$, replacing $\bar{c}$ with $\bar{c}'$ in $\mathbb{S}$ gives an approximate Scott family of the space with parameters $\bar{c}'$ and furthermore,

(1)* If $M \models \Theta_\epsilon(\bar{c}, \bar{x})$ and $M \models \Theta_\epsilon(\bar{c}', \bar{x}')$, then there exists an automorphism of $\mathcal{M}$ taking $\bar{c}\bar{x}$ into $\mathrm{nbh}_\epsilon(\bar{c}'\bar{x}')$.

2.4. Notation and conventions. For notational convenience, we will sometimes write $X, Y, Z, \ldots$ instead of the longer $(\alpha_i)_{i\in\omega}, (\beta_i)_{i\in\omega}, (\gamma_i)_{i\in\omega}$ to denote structures on a space. We also often say X-computable instead of computable w.r.t. X, if X is a structure on a space. If X is a structure on $\mathcal{M}$, we call the points of X *special in* X or simply *special* if it is clear which X we refer to.

§3. Proof of Theorem 1.1. (1) $\to$ (2) Roughly, the idea is to mimic the usual forcing construction from [1] using finite $\mathcal{L}$-substructures of the space. The metric on each finite structure will not be completely defined, but instead will only be defined to some precision ϵ. We need to explicitly make sure subsequent finite structures are both more precise and involve a larger number of elements. In other words, we need the distance between the finitely many points to be decided with a better precision as we extend a finite substructure to a larger finite substructure. Then we can proceed more or less as in the standard proof (take a sufficiently generic structure etc.). The proof is more notationally heavy than the standard one and there are a few further subtleties to worry about, but overall there are no surprises in the argument. (An experienced reader may safely skip the formal argument below.)

Fix a computable structure X on $\mathcal{M}$ (see the beginning of Section 2 for the definition). Without loss of generality, assume X has no repetition. We identify

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elements of X with their indices from ω. Consider the collection $\mathbb{P}$ of all quadruples (D, p, ℓ), where

(p1) D is a finite partial positive diagram upon an initial segment $0, \dots, i$ of ω and in the language $\mathcal{L}$.
(p2) $p : \{0, \dots, i\} \to X$ is a finite partial embedding, i.e., D is true on the respective p-images of $1, \dots, i$.
(p3) $\ell < i$ is such that each of the first ℓ-many elements of X are within 2^{-i} of some element of $rng(p)$.
(p4) For each $k, j \leq i$ with $k \neq j$, there are $r, q \in \mathbb{Q}^+$ with $|r - q| < 2^{-i}$ such that D contains $d_{<r}(k, j)$ and $d_{>q}(k, j)$.

We write $\sigma, \tau, \rho \dots$ to denote elements of $\mathbb{P}$, we assume $\sigma = (D_\sigma, p_\sigma, \ell_\sigma)$ and we write i_σ for $\sup dom(p_\sigma)$. (Recall $dom(p_\sigma)$ is an initial segment of ω.)

If ℓ_1, ℓ_2 are two finite strings in X (that is, functions from finite initial segments of ω to X), perhaps of different lengths, then let m be the length of the shortest of the two. Define $d(\ell_1, \ell_2) = \sup_{j<m} d(\ell_1(j), \ell_2(j))$. We say that τ is a refinement of σ and write $\tau \Subset \sigma$ if:

(e1) $dom(p_\tau) \supset dom(p_\sigma)$ and $D_\tau \supset D_\sigma$;
(e2) $\ell_\tau > \ell_\sigma$;
(e3) $d(p_\sigma, p_\tau) < 2^{-i_\tau}$.

Define $\Subset$ accordingly. The relation $\Subset$ is a strict partial order on $\mathbb{D}$. Furthermore, each infinite $\Subset$-ascending chain $(D_i, p_i, s_i)_{i\in\omega}$ corresponds to an isomorphic image $\mathcal{N}$ of $\mathcal{M}$ under the isometric surjective map $p = \lim_i p_i$. The positive diagram of $\mathcal{N}$ can be effectively enumerated given only positive information about $\cup_i D_i$, and without loss of generality we may identify it with $\cup_i D_i$. (A minor adjustment of our definitions will allow this set to be exactly the positive diagram, while in general $\cup_i D_i$ it is only e-equivalent to it. We leave the easy but tedious details to the reader.) Conversely, given any structure Y on $\mathcal{M}$ and a finite partial diagram of any finite tuple $\bar{y}$ of special elements from Y, we can extend $\bar{y}$ to a $\bar{z}$ whose partial diagram σ satisfies $(p1)$–$(p4)$. We furthermore can represent Y as a sequence through $\mathbb{P}$ extending this σ.

Suppose $\sigma \in \mathbb{D}$. For any pair i, j in the domain of p_σ, pick r least and q largest such that $d_{<r}(i, j)$ and $d_{>q}(i, j)$ are mentioned in D_σ. Define $\overline{d}_\sigma(i, j) = r$ and $\underline{d}_\sigma(i, j) = q$.

Say that $f : \sigma \to \mathcal{M}$ is σ-admissible if for all $i, j \in dom(p_\sigma)$

$$\underline{d}_\sigma(i, j) < d_{\mathcal{M}}(f(i), f(j)) < \overline{d}_\sigma(i, j).$$

We also say that all extensions of f to maps with larger domains are σ-admissible.

Let Φ be an enumeration operator and $D = \bigcup_{\sigma_i} D_{\sigma_i}$ for some $\Subset$-ascending chain $(\sigma_i)_i$. For $\tau \in \mathbb{P}$ and D, $\lim_i \Phi_i^D$ is τ-admissible iff already for some finite subset D_{σ_i} and some i we can see this is the case (unless Φ^D lists sequences that are not fast Cauchy), which is a c.e. event. In this case, we say simply that $\Phi^{D_{\sigma_i}}$ is τ-admissible. Similarly, if Φ^D defines a collection of fast Cauchy names, then for every $x \in X$ and any $m \in \omega$, a condition of the form $d_{\mathcal{M}}(\lim_i \Phi_i^D(m), x) < 2^{-s}$ will be witnessed by some D_{σ_i} unless $(\Phi_k^D(m))_k$ is not fast Cauchy. In this case we write $d_{\mathcal{M}}(\lim_k \Phi_k^{D_\sigma}(m), x) < 2^{-s}$.

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We say that σ *forces* Φ *to be a computable isometry onto* $\bar{X}$ and write $\sigma \Vdash \Phi$ if for all $\tau \Supseteq \sigma$ there exist $\rho \Supseteq \tau$ and i with the properties:

(1) Φ^{D_ρ} is τ-admissible;
(2) for each $k \leq i_\tau$ and $j \leq i$, we have $d_{\mathcal{M}}(\Phi_j^{D_\rho}(k), \Phi_{j+1}^{D_\rho}(k)) < 2^{-j-1}$;
(3) for each $j \leq \ell_\tau$, there exists $v \in dom(p_\rho)$ such that $d_{\mathcal{M}}(\lim_k \Phi_k^{D_\rho}(v), j) < 2^{i_\tau}$.

In (1) and (3) above we require all mentioned computations to halt. Notice that each of (1)–(3) above are naturally c.e. events.

According to our definitions, if (1) or (2) or (3) fails for a σ, then $\lim_i \Phi_i$ cannot possibly be a surjective isometry of the completion of any $D \in [\mathbb{P}]$ extending D_σ. If this is the case for σ, we write $\sigma \nVdash \Phi$. On the other hand, (1)–(3) are clearly closed upwards under $\Subseteq$. Also, if $\sigma \Vdash \Phi$, then (2) ensures that Φ defines a sequence of D-computable points for any [infinite] $\Subseteq$-extension of σ. Also, if $\sigma \Vdash \Phi$, then (1) implies that $\lim_k \Phi_k$ is an isometric embedding, and (3) witnesses its completion contains X. Thus, the completion is a surjective isometry onto $\mathcal{M}$.

In the construction, we list all enumeration operators Φ and build a path through $\mathbb{P}$ by stages. We try to ensure $\sigma \nVdash \Phi$ (if we can) for each Φ, one-by-one. Since in any case we build an isometric copy D of $\mathcal{M}$ and $\mathcal{M}$ is r.c.c., some finite σ must force that Φ is a surjective isometry of D to $\overline{X}$.

This means that for any extension of the finite partial diagram σ to a (positive) diagram of a structure on $\mathcal{M}$, the operator Φ is a surjective computable isometry between the completion of the structure and $\overline{X}$. The desired stable tuple of parameters, really, a neighborhood of p_σ in $\mathcal{M}$ within which all choices are good.

(2) $\rightarrow$ (3). This is obvious.

(3) $\rightarrow$ (4). Recall the definition of an approximate Scott family with stable parameters (Section 2.3). Suppose uniform categoricity with stable parameters is witnessed by Φ, a tuple of balls $\bar{B}$, and $\bar{b}$ in $\bar{B}$. Let Y be a computable structure on $\mathcal{M}$. We define a c.e. approximate Scott family as follows. We may assume $\bar{b}$ is special. Given a tuple $\bar{y}$ and a positive rational $\epsilon = 2^{-i} < 1$, compute $\Phi_{i+2}^{D(Y),\bar{b}}(\bar{y})$. Suppose the use of the computation is $D_{\bar{y},\epsilon}$, and let $\bar{z}$ be all points of Y mentioned in $D_{\bar{y},\epsilon}$ unequal to those among $\bar{y}$. Take the conjunction $\phi(\bar{b}\bar{z}\bar{y})$ of all formulae in $D_{\bar{y},\epsilon}$, and set $\Theta_{\epsilon,\bar{y}} = \exists \bar{z} \phi(\bar{B}\bar{z}\bar{w})$, where $\bar{w}$ are free variables. Define $\mathbb{S}$ to be the collection of $\Theta_{\epsilon,\bar{y}}$ where $\bar{y}$ and ϵ range over finite tuples of Y and rationals of the form 2^{-i}, respectively. Clearly, the definition of $\mathbb{S}$ is effective.

The density of Y in $\mathcal{M}$ implies that $\mathbb{S}$ satisfies (2) of Definition 2.1. We assume (for simplicity) that there are no parameters and show that $\mathbb{S}$ satisfies (1) of Definition 2.1. The proof of $(1)^*$ is almost literally the same. Suppose $\bar{x}'$ and $\bar{x}''$ both satisfy $\Theta_{2^{-i}}$. Since $\Theta_{2^{-i}}$ isolates an open set in $\mathcal{M}$, there will be tuples $\bar{y}'$ and $\bar{y}''$ of Y-special points in the 2^{-i-2}-nbhds of $\bar{x}'$ and $\bar{x}''$, respectively, that satisfy $\Theta_{2^{-i}}$. Define new computable structures Y' and Y'' replacing $\bar{y}$ by $\bar{y}'$ and $\bar{y}''$, respectively, and also by renaming the existential witnesses $\bar{z}'$ and $\bar{z}''$ corresponding to $\Theta_{2^{-i}}(\bar{y}')$ and $\Theta_{2^{-i}}(\bar{y}'')$ (taken to be Y-special) by the tuple $\bar{z}$ of Y-special points witnessing the existential quantifier in $\Theta_{2^{-i}}(\bar{y})$. Note the points stay "where they are" in $\mathcal{M}$, but they get new names.

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By the assumption, Φ_{i+2} is a 2^{i+2}-isometry from both Y' and Y'' onto a dense subset of $\bar{Y}$, and furthermore the computations of Φ_{i+2} are identical on $\bar{y}$ viewed either as a tuple in Y' or in Y'' or Y. Suppose this computation outputs a tuple $\bar{v}$ in Y. It follows that here exist automorphic images $\bar{w}$, $\bar{w}'$ and $\bar{w}''$ of $\bar{y}$, $\bar{y}'$ and $\bar{y}''$ (all viewed as elements of Y) which are at most 2^{-i-2}-far from $\bar{v}$ and thus are at most 2^{-i-1}-far apart. Recall that $\bar{y}'$ and $\bar{y}''$ were 2^{-i-2}-close to $\bar{x}'$ and $\bar{x}''$. Thus, 2^{-i-2}-nbhds of $\bar{w}'$ and $\bar{w}''$ must contain some automorphic images $\bar{a}'$ and $\bar{a}''$ of $\bar{x}'$ and $\bar{x}''$. It follows that $\bar{a}'$ and $\bar{a}''$ are at most 2^{-i}-far apart, as desired.

(4) $\to$ (1). Let $\mathbb{S}$ be an effective approximate Scott family with stable parameters $\bar{d}$ in $\overline{B}$ (see Section 2.3 for the definitions). We fix a computable structure Y on $\mathcal{M}$, and fix any X dense in $\mathcal{M}$. We may assume that $\bar{d}$ are special in Y. In what will follow, elements of Y (and of X) are identified with the respective natural numbers.

At the beginning of each step $i > 0$, we will have a rational $\epsilon_i = 2^{-i}$ as a precision parameter, a finite tuple $\bar{a}_i$ from Y and $\bar{b}_i$ from X, and a partial map $\bar{b}_i \to \bar{a}_i$. We may assume that $\bar{b}_i$ is an initial segment of X, and that $\bar{b}_{i+1}$ extends $\bar{b}_i$ by one extra point.

At the end of step i, we will define $\epsilon_{i+1} = \epsilon_i/2$, choose $\bar{b}_{i+1}$ extending $\bar{b}_i$, and define $\bar{a}_{i+1}$ so that

(a.) the initial segment of $\bar{a}_{i+1}$ of length the same as $\bar{a}_i$ is within $\mathrm{nbh}_{\epsilon_i}(\bar{a}_i)$;
(b.) $\mathrm{nbh}_{\epsilon_{i+1}/4}(\bar{a}_{i+1})$ contains an isomorphic image of $\bar{b}_{i+1}$;
(c.) If i is odd, then $Y\restriction_i$ is within $\mathrm{nbh}_{\epsilon_{i-1}}$ of some substring of $\bar{a}_{i+1}$.

In (b.), an isomorphism is any surjective isometry of $\bar{X}$ onto $\bar{Y}$.

It is clear that a successful maintenance of (a.) implies that, for each n, the n-prefixes of $\bar{a}_i$ form a (uniformly) rapidly converging sequence in $\bar{Y}^n$. If we are successful in satisfying (b.), then the limit of these n-tuples will have the same distance matrix as the first n points of X. Therefore, we can naturally define an isometric embedding U as the limit of partial maps $\bar{a}_i \to \bar{b}_i$. Finally, if (c.) is satisfied at the end of each stage, then each special point of Y is in the closure of the image $U(X)$, and thus (the completion of) U is surjective. Indeed, (c.) implies that for each y and for infinitely many i, there exists an x such that $U(x)$ is within $\mathrm{nbh}_{\epsilon_{i-1}+2\epsilon_{i+1}}(y)$. Since the sequence $(\epsilon_{i-1} + 2\epsilon_{i+1})_i$ converges to 0, we conclude that $U(X)$ is dense in $\bar{Y}$.

Step 0. Declare $\bar{a}_0 = \bar{d}$. We may assume that $\bar{d}$ lie in the centres of the respective balls in $\bar{B}$. Let r be the smallest radius among the balls witnessing stability of the parameters. Set the precision parameter $\epsilon_0 < r/16$. (This choice of ϵ_0 will imply that the initial segment of the sequence $\bar{a}_i$ will never leave $\bar{B}$.) Choose a formula $\Theta_{\epsilon_0/16}$ that holds on $\bar{d}$, and fix a special X-tuple $\bar{c}$ in the $\epsilon_0/16$-nbhd of an isomorphic image of $\bar{d}$ in the completion of X such that $X \models \Theta_{\epsilon_0/16}(\bar{c})$.

Step i alternates between two basic modules:

Extension (i *is even*).

(1) Choose $z \in X$ least that is not yet among $\bar{b}_i$.
(2) Let $\bar{x} \in \mathrm{nbh}_{\epsilon_i/16}(\bar{b}_i z)$ be the first found tuple of special points such that $\bar{x}$ satisfies some $\Theta_\delta \in \mathbb{S}$ labeled by $\delta < \epsilon_i/16$.
(3) Pick some $\bar{y}$ in Y whose prefix is within $\epsilon_i/2$ of $\bar{a}_i$ and so that $\bar{y}$ satisfies Θ_δ.
(4) Set $\bar{a}_{i+1} = \bar{y}$ and $\bar{b}_{i+1} = \bar{b}_i z$, define $\epsilon_{i+1} = \epsilon_i/2$, and proceed to the next step.

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Onto (i *is odd*). For every point y among the first i special points of Y, do the following:

(1) Pick a first found tuple v in X, a tuple $\bar{w}u$ in Y, and a formula Θ_δ in $\mathbb{S}$ labeled by $\delta < \epsilon_i/16$ such that
   – $\bar{w}u \in \mathrm{nbh}_{\epsilon_i/4}(\bar{a}_i y)$;
   – $X \models \Theta_\delta(\bar{b}_i v)$ and $Y \models \Theta_\delta(\bar{w}u)$.
(2) Define $\bar{b}_{i+1} = \bar{b}_i v$ and $\bar{a}_{i+1} = \bar{w}u$, and set $\epsilon_{i+1} = \epsilon_i/2$.

Verification. Recall that each $\Theta \in \mathbb{S}$ is an existential projection of an open positive formula. The witnesses of the latter form a (nonempty) open set in the corresponding power of $\mathcal{M}$. From this, we obtain the following:

Fact 3.1. Suppose Θ is a formula in $\mathbb{S}$, and let $\bar{m}$ be a tuple of points in $\mathcal{M}$ such that $M \models \Theta(\bar{m})$. Then for each $\epsilon > 0$, there exists a tuple of X-special points $\bar{x} \in \mathrm{nbh}_\epsilon \bar{m}$ such that $X \models \Theta(\bar{x})$.

As we explained above, it is sufficient to verify that the conditions (a.)–(c.) for every i, and that the substeps use only positive information about $D(X)$.

Extension. It follows from the definition of an approximate Scott family and Fact 3.1 that there exist Θ_δ and $\bar{x}$ with the desired properties. Whence, we will eventually find an atomic fact in $D(X)$ implying Θ_δ on such a tuple. By the I.H., there exists an isomorphic image of $\bar{b}_i$ within $\mathrm{nbh}_{\epsilon_i/4}(\bar{a}_i)$. In particular, the choice of the precision parameter guarantees that the initial segment of this isomorphic image will be within the nbhd witnessing stability of $\bar{d}$. Thus, we can apply property $(1)^*$ of an approximate Scott family with stable parameters. Consequently, by the choice of δ, we can find a $\bar{y} \in \mathrm{nbh}_{\epsilon_i/2}(\bar{a}_i)$ such that $\Theta_\delta(\bar{y})$ holds (this will be the true image of $\bar{b}_i$ or any tuple of special points sufficiently close to it). At some stage we will see a proof of that fact from $D(Y)$. It follows from the triangle inequality that $\mathrm{nbh}_{\epsilon_i/8}(\bar{y})$ contains an isomorphic image of $\bar{b}_{i+1} = \bar{b}_i z$. Both (a.) and (b.) hold for $i+1$.

Onto. The verification of this step is similar. Using the properties of $\mathbb{S}$, Fact 3.1, and the I.H., we argue that such $\bar{w}u$ and $\bar{x}v$ can always be found. Indeed, $\mathrm{nbh}_{\epsilon_i/4}(\bar{a}_i)$ contains some isomorphic image of $\bar{b}_i$. Each neighborhood of the latter contains a tuple satisfying some Θ_δ labeled by a $\delta < \epsilon_i/16$, and whence its isomorphic image satisfies the formula as well. Note the initial segment of the tuple stays in $\bar{B}$. It is sufficient for us to wait for $D(X)$ and $D(Y)$ to enumerate facts implying what we need. It is clear that (a.)–(c.) hold for i.

§4. Proof of Theorem 1.2.

We first prove a proposition that was initially designed as a technical tool for the proof of the theorem. Nonetheless, the result gives a slightly different perspective on why r.i.c. parameters can be replaced by stable parameters in a Scott family (Theorem 1.1). We also believe the result has some independent interest.

Definition 4.1. Let c be a point in a Polish metric space $\mathcal{M}$. A *formal name* of the point c is a sequence $(\Theta_i)_{i\in\omega}$ of existential formulae in one free variable such that $U_i = \{x \in \mathcal{M} : \mathcal{M} \models \Theta_i(x)\}$ is an open set of diameter at most 2^{-i-1} that converges to c; i.e., $\bigcap_i U_i = \{c\}$.

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Recall that our language allows neither equality nor negation. It follows that $\{x \in \mathcal{M} : \mathcal{M} \models \Theta(x)\}$ will be an open set for any existential formula Θ. We can relax the definition above and allow finitely many stable parameters $\bar{B}$ in the formulae $(\Theta_i)_{i\in\omega}$. In this case we require that for any tuple $\bar{b}$ in $\bar{B}$ the formulae $\Theta_i(\bar{b}, x)$ should determine sets $U_i^{\bar{b}}$ of diameter at most 2^{-i-1} that converge to c, but different $\bar{b}$ may correspond to different sequences of open sets.

PROPOSITION 4.2. *Let $\mathcal{M}$ be a computable Polish metric space. Then an automorphism-invariant point $a \in \mathcal{M}$ is r.i.c. iff it admits a c.e. formal name with stable parameters.*

PROOF. First observe that computability of a point a w.r.t. X is equivalent to having an enumeration operator Φ such that $(\Phi_i^{D(X)})_{i\in\omega}$ converges to the point. The first part of the proof is similar to the proof of (1) $\to$ (4) of Theorem 1.1. The definition of $\mathbb{P}$ is the same. Then take the first σ that forces some Φ to list open neighbourhoods in all extensions of σ and conclude that on all such extensions the procedure is uniform after taking stable parameters that come from σ (just as in (1) $\to$ (4) in Theorem 1.1).

Now, on any extension Y of σ the sequence $(\Phi_i^{D(Y)})_{i\in\omega}$ computes exactly the same point on $\mathcal{M}$, since the point is automorphism-stable. There is at least one computable extension Y of σ that can be obtained by a finite adjustment of some other computable structure upon $\mathcal{M}$. For each i, let D_i be the use of the computation $\Phi_i^{D(Y)} = m$. Quantify over the points mentioned in D_i that are not the stable parameters and not $\mathcal{M}$. A rather straightforward argument (similar to the proof of (4) $\to$ (2) in Theorem 1.1) shows that the constructed family of existential formulae is a formal name for x. ⊣

We now prove Theorem 1.2. To prove the theorem, it is sufficient to construct a computable Polish metric space $\mathcal{M} = (M, d, X)$ and a point $a \in M$ that satisfy the following:

(1) $\mathcal{M}$ is rigid.
(2) a is intrinsically computable but not relatively intrinsically computable.
(3) There exists a uniform effective procedure that, given $D(Y)$ of a dense Y and a fast Cauchy name of a w.r.t. Y, produces a surjective isometry $\bar{Y} \to \bar{X}$.

The rigidity requirement will ensure that a is automorphism invariant, allowing us to use Proposition 4.2. Conditions (2) and (1) together imply that the space is not relatively categorical. Finally, (3) implies that the space is weakly uniformly categorical with parameter a.

A crude description of the space. The space will be a closed subset of the unit square $\mathbb{B}_2 = [0, 1] \times [0, 1]$. We describe the space by enumerating a set X of (rational) points in $\mathbb{B}_2$, and then we set $\mathcal{M} = \overline{X}$. The set $X = \bigcup_s X_s$ will be enumerated by stages in the construction that will be described later. We explain several important features that X will have.

We will initially put $(0, 1)$ into X_0. We will then set $a = (0, 0)$; a will not be in X, but it will be an accumulation point of X and thus an element of M. We also fix a sequence of rationals $(\delta_n)_{n\in\omega}$ that converges to 0 "very fast":

$$\sum_{i>n} \delta_i < \frac{1}{100}\delta_n$$

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for every n. We also assume $\delta_0 < 1/10$. At the beginning of every stage s, we will choose a number n_s larger than any number previously mentioned in the construction and put $(0, \delta_{n_s})$ into X_s, thus making $(0,0)$ an accumulation point. In the construction, we will add more points to X. This will always be done by taking a finite initial segment $X_{s'}$ of the currently defined X_s and a (small enough) rational r and enumerating

$$X_{s'} + (r,0) = \{(x,y) + (r,0) : (x,y) \in X_{s'}\}$$

into X_{s+1}. We will call this operation the *r-shift* of $X_{s'}$. We may omit r and $X_{s'}$ and say simply *shift* if it is clear from the context which r and s' we use. As we will see, our r's will be chosen small enough to ensure $X \subseteq \mathbb{B}_2$. Indeed, if we number the r's in the order in which they are chosen, we will have

$$\sum_{i>n} r_i < \frac{1}{100} r_n$$

for every n, just as with the δ_n.

Basic properties of $\mathcal{M}$. Already the crude description above allows us to make several conclusions about $\mathcal{M} = \overline{X}$. All we need to know is that in the construction there will be infinitely many stages at which new points are introduced to X by shifting.

CLAIM 4.3. *$\mathcal{M}$ becomes uniformly categorical after fixing $a = (0,0)$.*

PROOF OF CLAIM. First, note that the pair $\{(0,0),(0,1)\}$ is an automorphism base of $\mathcal{M}$. Indeed, every point in $\mathcal{M}$ is completely described by its distances to $a = (0,0)$ and $(0,1)$. An easy exercise in Euclidean geometry shows that a better approximation of the $\{(0,0),(0,1)\}$-coordinates of a point x gives a better approximation to the point, with all possible effective uniformity.

Furthermore, since $\delta_0 < 1/10$, we see that $(0,1)$ is the unique point of $\mathcal{M}$ that satisfies $d(a,x) > 1/2$, and so a computes $(0,1)$ w.r.t. any structure on $\mathcal{M}$. Therefore, the space $\mathcal{M}$ becomes uniformly computably categorical after adjoining a to its signature. Indeed, an algorithm for an isometry between $(\mathcal{M}, a, X)$ and any $(\mathcal{M}, a, Y)$ can be sketched as follows:

(1) Search for the element $b \in Y$ with $d(a,b) > 1/2$.
(2) On input x, compute $d(x,a)$ and $d(x,(0,1))$ to a sufficiently high degree of precision.
(3) Search for an element $y \in Y$ such that $d(y,a)$ and $d(y,b)$ are sufficiently close to $d(x,a)$ and $d(x,(0,1))$.
(4) Output y.

Determining the required levels of precision is an exercise in Euclidean geometry and the triangle inequality. ⊣

CLAIM 4.4. *$\mathcal{M}$ is rigid.*

PROOF OF CLAIM. Note that there exists an $m < 1$ such that

$$(x,1) \in \mathcal{M} \Rightarrow x \leq m.$$

Indeed, we have $m = \sum_s r_s$, where the r_s range over all q such that q-shifts were ever performed in the construction. Since the r_i are small (in the same sense as δ_n), we will have $m < 1$. Furthermore, if we define m_y to be least such that

$$(x,y) \in \mathcal{M} \Rightarrow x \leq m_y,$$

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we see that $(m_{\delta_{n_s}})_{s\in\omega}$ is nonincreasing (recall that $(0,\delta_{n_s})$ is enumerated at stage s, and so $(0,\delta_{n_s})$ will not partake in shifts that occur before stage s). Finally,

$$(x,0)\in\mathcal{M}\Rightarrow x=0.$$

We conclude that the point a is the unique point satisfying

$$(\exists z,w)\ \left(d(a,z)=\sqrt{1+m^2}\ \wedge\ d(z,w)=m\ \wedge\ d(a,w)=1\right),$$

and is thus automorphism invariant. As we have already noted above, $(0,1)$ is the unique point at distance 1 of $(0,0)$. Since $\{(0,0),(0,1)\}$ is an automorphism base of $\mathcal{M}$, we conclude that $\mathcal{M}$ is rigid. ⊣

The two claims above together with the analysis preceding the claims imply that it remains to make a intrinsically computable but not relatively intrinsically computable. Since $\mathcal{M}$ will be rigid, Proposition 4.2 implies that making a not r.i.c. is equivalent to diagonalising against all potential formal names for a. Fix an effective list $(Y_e)_{e\in\omega}$ of all computably enumerable structures in the language $\mathcal{L}$. Fix also an effective listing of all c.e. sequences of existential $\mathcal{L}$-formulae $(\Theta_\epsilon(x,\bar{c}))_\epsilon$ effectively labeled by rationals, where the finitely many parameters $\bar{c}$ are special points from the computable structure X. Note that Proposition 4.2 ensures that parameters are stable, and since X is dense in $\mathcal{M}$ it will be sufficient to diagonalise against all families with parameters from X. Thus, we need to satisfy that

$$\mathcal{D}_{\Theta,\bar{c}}:(\Theta_\epsilon(x,\bar{c}))_\epsilon \text{ is not a formal name for } a=(0,0),$$

where Θ ranges over all c.e. families of existential $\mathcal{L}$-formulae effectively labeled by $\mathbb{Q}$ and $\bar{c}$ over X (equivalently, over ω). Also, for every $e\in\omega$ we need to meet

$$\mathcal{I}_e:\text{If } \overline{Y}_e\cong\mathcal{M} \text{ then } a=(0,0) \text{ is computable w.r.t. } Y_e.$$

Note that seeing that Y_e fails to determine a structure on a metric space is a c.e. event: Y_e fails to determine a structure on some metric space if it fails the triangle inequality—that is, if there are $i,j,k\in Y_e$ and $q_1,q_2\in\mathbb{Q}^+$ with $Y_e\models d_{<q_1}(i,j)\wedge d_{<q_2}(j,k)\wedge d_{>(q_1+q_2)}(i,k)$.

The basic strategy for $\mathcal{D}_{\Theta,\bar{c}}$. At the first stage this strategy is visited, it chooses an n larger than any number previously mentioned in the construction and defines $\delta=\delta_n$. Let $\Theta_{\delta/4}(x,\bar{c})=\exists\bar{z}\psi(x,\bar{c},\bar{z})$. The strategy searches for a special point b and a tuple of special points $\bar{y}$ such that

$$X_s\models\psi(b,\bar{c},\bar{y}) \text{ and } d(a,b)<\delta/4.$$

Since every positive atomic $\mathcal{L}$-formula isolates an open set in a metric space, we know that if $\exists\bar{z}\psi(a,\bar{c},\bar{z})$ is satisfied, there are such b and $\bar{y}$. If we ever see this happen at stage s, we perform the *δ-shift* of X_s by enumerating

$$X_s+(\delta,0)=\{(x,y)+(\delta,0):(x,y)\in X_s\}$$

into X. After this is done, we will have $b'=b+(\delta,0)$ also satisfies $\Theta(x,\bar{c})$, since the witnesses $\bar{y}$ will also get shifted. But $d(a,b')\geq\delta$ contradicts $d(a,b')<\delta/4$ which must be the case if $(\Theta_\epsilon)_\epsilon$ is to be a formal name for a.

The strategy will perform only one shift; once it has enumerated points into X, it never performs another shift. The strategy has two outcomes, *win* and *wait*; it

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takes outcome *wait* while searching for the b and $\bar{y}$, and then takes outcome *win* after performing the δ-shift.

The full strategy for $\mathcal{D}_{\Theta,\bar{c}}$.

The full strategy is only a small modification of the basic strategy. Suppose the strategy first sees the desired b and $\bar{y}$ at some stage s. It does not immediately perform the δ-shift. Instead, it takes outcome *wait* for one more stage. Then, at the next stage $t > s$ at which the strategy is visited on the priority tree, the strategy performs the δ-shift of X_s. That is, it enumerates all of $X_s + (\delta, 0)$ into X_{t+1}. From this stage on, the strategy takes outcome *win*.

Note that, assuming the strategy is visited infinitely many times during the construction, this will suffice to show that the requirement is met.

The basic strategy for $\mathcal{I}_e$. This strategy will define a sequence $(w_k)_{k\in\omega}$, which is intended to be a fast Cauchy name for a w.r.t. Y_e. It will also define an auxiliary sequence $(v_k)_{k\in\omega}$.

The basic strategy in isolation is quite simple. Recall that $(\delta_n)_{n\in\omega}$ denotes the sequence converging to 0 "very fast" (see the beginning of the proof). The idea is as follows: for any s, we know that there is a point $(0, \delta_{n_s})$ which is precisely distance δ_{n_s} from a (recall n_s from the crude description of the space). We will search for two points in Y_e which are approximately distance δ_{n_s} apart, and when we find such points, we will believe that they are close to a. In the absence of any r-shifts, we will be correct: one of the points must be δ_{n_s}, and the other must be either a or $(0, \delta_{n_t})$ for $t > s$.

Let k be least such that w_k is not yet defined. Let $n = n_k$ and $\epsilon = \delta_n$. If $k > 0$, let $m = n_{k-1}$ and $\tau = \delta_m$. The strategy searches for a pair of special points w, v in Y_e such that

$$Y_e \models .9 \cdot \epsilon < d(w, v) < 1.1 \cdot \epsilon.$$

If τ is defined, we also require that at least one of w and v is within $.1\tau$ of either w_{k-1} or v_{k-1}. If such a pair is ever found in Y_e, the strategy sets $w_k = w$ and $v_k = v_{k-1}$ and then repeats the process.

The outcomes are ∞ and $wait_k$ for each $k \in \omega$. It takes outcome $wait_k$ while k is least with w_k undefined; when w_k is defined, the strategy takes outcome ∞ for a single stage and then begins taking outcome $wait_{k+1}$.

The full strategy for $\mathcal{I}_e$.

We must primarily modify the choice of ϵ and τ. If s is the first stage at which σ is visited, σ takes no action at this stage. To simplify the later description, we declare that σ has outcome ∞ at stage s, even though the construction will not allow $\sigma^\frown\infty$ to be visited at this stage.

If σ is visited at stage s, and this is not the first stage at which σ is visited, let $t_0 < s$ be the last stage at which σ took outcome ∞. Let $n = n_{t_0+1}$ and $\epsilon = \delta_n$ (recall n_{t_0+1} from the crude description of the space). If t_0 was also not the first stage at which σ was visited, let $t_1 < t_0$ be the last stage before t_0 at which σ took outcome ∞, let $m = n_{t_1+1}$ and $\tau = \delta_m$. The strategy now proceeds as the basic strategy.

Construction. The priority ordering and the tree of strategies are typical for infinite injury, so we skip the usual definitions.

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At stage 0 we set $M_0 = \{(0,1)\}$ and do nothing else. At every stage after 0, we begin by choosing a number n_s larger than any number previously mentioned in the construction and enumerating $(0,\delta_{n_s})$ into X_s. We then visit the root of the priority tree. We then proceed inductively, visiting strategies according to the outcome of the previous strategy, until some never before visited strategy is reached. Once that strategy has acted, we end the stage. This completes the description of the construction.

Finalising the proof. Much of the verification has already been done above. The infinite injury technique used in the construction is standard and contains no surprises, so we leave the usual inductive argument to the reader. It is clear, for instance, that all $\mathcal{D}$-requirements are met by strategies along the true path. It remains to argue that all $\mathcal{I}$-requirements are met along the true path.

CLAIM 4.5. *Suppose σ is an $\mathcal{I}_e$ strategy, Y_e is a structure on a metric space and σ takes outcome ∞ infinitely many times (and thus $(w_k)_{k\in\omega}$ is fully defined). For any $k_0 < k_1 < k_2$, let t be the stage at which w_{k_0} is defined, let $m = n_{t+1}$, and let $\delta = \delta_m$. Then $Y_e \models d_{<2\delta}(w_{k_1}, w_{k_2}) \wedge d_{<2\delta}(w_{k_1}, v_{k_2})$.*

PROOF. Induction on the stage at which w_{k_2} is defined, using the fast convergence of $(\delta_n)_{n\in\omega}$. Note that the τ of that stage is no larger than δ. ⊣

It follows that the sequence $(w_k)_{k\in\omega}$ defined by σ is a (possibly partial) fast Cauchy sequence.

CLAIM 4.6. *Suppose σ is an $\mathcal{I}_e$-strategy along the true path, and Y_e is a structure on $\mathcal{M}$. Then $d(w_k, a) < 2^{-k}$.*

Indeed, if ϵ was the value such that σ was searching for w and v satisfying $Y_e \models .9\cdot\epsilon < d(w,v) < 1.1\cdot\epsilon$ in order to define w_k, then one of w_k or v_k is within $.1\cdot\epsilon$ of a.

PROOF. Let s_0 be the stage at which σ defined w_k. Let $t < s_0$ be the last stage at which σ took outcome ∞, and let $m = n_{t+1}$. Then $(0,\delta_m)$ was enumerated into X by our construction at the beginning of stage $t+1$, and $\epsilon = \delta_m$. Since w_{k-1} was defined at stage t, or $k = 0$, our construction will choose that $n_{t+1} > k$, and so $\delta_m < 2^{-k-1}$.

By construction, no $\mathcal{D}_{\Theta,\bar{c}}$-strategy will choose δ_m, as m was already chosen by the construction at the beginning of a stage. Thus, since the $(\delta_n)_{n\in\omega}$ tend to 0 quickly, one of w_k or v_k must correspond to a point of the form $(x,0)$ or (x,δ_ℓ), where $x = \sum_{n\in F}\delta_n$ for some possibly empty $F \subset \omega$ and $\ell < m$, while the other of w_k or v_k must correspond to a point of the form (x',δ_m), where $x' = \sum_{n\in F'}\delta_n$ for some possibly empty $F' \subset \omega$.

Since $Y_e \models d_{<1.1\cdot\delta}(w_k, v_k)$ and $(\delta_n)_{n\in\omega}$ tends to 0 quickly, it must be that for every $n \le m$, $n \in F \iff n \in F'$. Note that $\sum_{n>m}\delta_n$ is small compared to δ_m, so to complete the proof of the claim, it will suffice to show that there is no $n \in F$ with $n \le m$. We do that now.

By construction, every $n \in F$ was chosen by some $\mathcal{D}_{\Theta,\bar{c}}$-strategy that performed its shift, so in particular $m \notin F$. Note also that any shift performed at or before stage t will not enumerate a translate of $(0,\delta_m)$, and so cannot contribute to F'. Since the strategy performing this shift has necessarily chosen an $n < m$, it also cannot contribute to F. Indeed, any $\mathcal{D}_{\Theta,\bar{c}}$ which sees its desired b and $\bar{y}$ at or before stage t will not enumerate a translate of $(0,\delta_m)$, and thus will not contribute to F or F'.

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By the assumption that σ is along the true path, it follows that no $\mathcal{D}_{\Theta,\bar{c}}$-strategy which is an ancestor of σ contributes to F. Furthermore, any $\mathcal{D}_{\Theta,\bar{c}}$-strategy which contributes to F and is incomparable with $\sigma^\frown\infty$ must have been first visited after stage t. Thus the n such a strategy contributes was chosen larger than m.

Thus, it suffices to consider whether $n < m$ can be contributed by some $\mathcal{D}_{\Theta,\bar{c}}$-strategy $\rho \supseteq \sigma^\frown\infty$ which is first visited at or before stage t and which performs its shift at some stage $s_1 > t$. Therefore $s_1 \geq s_0$. Let $t_1 < s_1$ be the last stage before s_1 at which ρ was visited, so $t_1 \geq t$. Then when ρ acts at stage s_1, it enumerates $X_{t_1} + (\delta_n, 0)$ into X. Let $m_1 = n_{t_1+1}$.

Note that if $s_1 = s_0$, then $t_1 = t$ and $\delta_{m_1} = \delta_m$. Since X_{t_1} contains no pair of points which are distance δ_{m_1} apart, if $n \in F$, it cannot be that $s_1 = s_0$. So $s_1 > s_0$.

Take k_1 such that σ defined w_{k_1} at stage s_1. Since $s_1 > s$, $k_1 > k$. Since σ has outcome ∞ at stage s_1, $.9\delta_{m_1} < d(w_{k_1}, v_{k_1}) < 1.1\delta_{m_1}$. Since X_{t_1} contains no pair of points which are distance δ_{m_1} apart, if $n \in F$, then no pair of points within $\delta_n/2$ of w_k can satisfy this inequality. Since $\delta_n/2 > \delta_m$, this contradicts Claim 4.5. ⊣

It follows that if the sequence $(w_k)_{k\in\omega}$ is total, it converges to a.

Claim 4.7. *If σ is an $\mathcal{I}_e$-strategy along the true path, and Y_e is a structure on $\mathcal{M}$, then σ infinitely often takes outcome ∞, and so $(w_k)_{k\in\omega}$ is fully defined.*

Proof. Suppose not. Let s_0 be the last stage at which σ takes outcome ∞. Let $n = n_{s_0+1}$. Eventually σ will find points w and v in Y_e which are arbitrarily close to a and the $(0, \delta_n)$, respectively. If s_0 is also the first stage at which σ is visited, this existence of such w and v contradicts our choice of s_0.

If s_0 is not the first stage at which σ is visited, let $t < s_0$ be the last stage before s_0 at which σ takes outcome ∞. Let $m = n_{t+1}$ and let k be such that σ defined w_k at stage s_0. Since one of w_k or v_k is strictly within $.1\delta_m$ of a by Claim 4.6, we can find w, v as above with w within $.1\delta_m$ of one of w_k, v_k, contradicting our choice of s_0. ⊣

This completes the proof of Theorem 1.2.

§5. The Urysohn space. We assume that the reader is familiar with the basic properties of the Urysohn space, see [15] for a detailed exposition. Otherwise, if the reader is not interested in the Urysohn space, they can skip this proof and go to the next subsection. This subsection will not be used in the rest of the article.

Approximate Scott families allow us to separate a technical approximate back-and-forth construction from the rest of the argument. As an illustration, we give a significantly simpler proof of the following known fact:

Proposition 5.1 ([17]). The Urysohn space is uniformly computably categorical (without parameters).

Proof. Using the rational Urysohn space $\mathbb{UQ}$, we can produce an effective approximate Scott family for the Urysohn space $\mathbb{U}$. More specifically, for every tuple of points $\bar{u}$ in $\mathbb{UQ}$ and a rational $\epsilon > 0$, we effectively produce the distance matrix $D(\bar{u})$ that has rational entries. Using this matrix, for each u_1 and u_2 in $\bar{u}$ and $r = d(u_1, u_2)$, we take

$$\psi_{u_1,u_2}(x, y) \leftrightharpoons d_{<r+\epsilon/4}(x, y) \,\&\, d_{>r-\epsilon/4}(x, y),$$

and syntactically define the $\epsilon/2$-nbhd of the tuple

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$$\Theta^{\bar{u}}_{\epsilon}(\bar{x}) \leftrightharpoons \bigwedge_{(u_1,u_2)\subseteq\bar{u}} \psi_{u_1,u_2}(x,y).$$

Each such formula describes the distance matrix with recision $\epsilon/4$. We then define $\mathbb{S}$ to be the collection of all such $\Theta^{\bar{u}}_{\epsilon}(\bar{x})$, where ϵ ranges over the positive rationals and $\bar{u}$ over finite tuples in $\mathbb{UQ}$.

Since distance in $\mathbb{UQ}$ is a recursive function of two arguments ranging over positive rationals, the family and the labeling are effective. Definition 2.1(2) for $\mathbb{S}$ follows at once from density of $\mathbb{UQ}$ in $\mathbb{U}$, and Definition 2.1(1) is a re-formulation of the *approximate extension property* (Definition 3.1 [15]) that holds on $\mathbb{U}$ (and indeed characterizes it). ⊣

Along these lines, approximate Scott families can be also used to simplify several known proofs including computable categoricity of separable Hilbert spaces and Cantor space [17].

§6. κ-computable categoricity. We cite [3, 10] for formal background on higher recursion theory in algebraic context, and we give only an informal explanation of this approach here. As in other contexts, the syntactical investigation of relative κ-computable categoricity is the formalisation of the question "what does it take to make the back-and-forth argument work?". Here such a construction has to have length κ, that is, has to be transfinite, and the main new ingredient is ensuring that we do not get stuck in limit stages. In particular, we need to define orbits of α-tuples of elements of our structures (which is further evidence for such arities being natural in this context). We work with formulas from $L_{\kappa^+,\kappa}$ which allow fewer than κ many variables (a precise generalisation of the familiar case $\kappa = \aleph_0$). The following definition incorporates the fact that sometimes our constructions have to deal with many, possibly infinitely many elements, at one step.

For a κ-structure $\mathcal{M}$, a *continuous Scott family* consists of

- a list $\langle x_i\rangle_{i<\kappa}$ of distinct variables;
- a closed unbounded set $C \subseteq \kappa$; and
- a sequence $\langle\varphi_\alpha\rangle_{\alpha\in C}$ of $L_{\kappa^+,\kappa}$-formulas φ in the language of $\mathcal{M}$,

such that

(1) for all $\alpha \in C$, the free variables of φ_α are $\bar{x}_\alpha = \langle x_i\rangle_{i<\alpha}$;
(2) each φ_α determines an orbit of an α-tuple of elements of $\mathcal{M}$;
(3) for every set $B \subset \mathcal{M}$ of size smaller than κ there is some $\alpha \in C$ and an α-tuple $\bar{a}$, containing all the elements of B, which satisfies φ_α;
(4) if $\beta < \alpha$ are in C, then φ_α implies φ_β; and
(5) for any limit point α of C, φ_α is equivalent to the conjunction $\bigwedge_{\beta\in C\cap\alpha}\varphi_\beta$.

Such a Scott family is *effective* (or *formally* Σ_1) if C is κ-computable, each φ_α is Σ_1 (a κ-computable disjunction of κ-finite existential formulas), and the list $\langle\varphi_\alpha\rangle_{\alpha\in C}$ is κ-computable.

Recall that Theorem 1.3 says that a κ-computable κ-structure $\mathcal{M}$ is relatively κ-computably categorical if and only if there is some $A \subset \mathcal{M}$ of size less than κ such that the structure $(\mathcal{M}, a)_{a\in A}$ has an effective, continuous Scott family.

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PROOF OF THEOREM 1.3. We generalise the countable proof. We force to add a generic permutation of κ. The forcing conditions are injective sequences from some $\alpha < \kappa$ to κ. The ordering is by extension. The important point is that this notion of forcing is κ-closed. This implies that in L we can find a "sufficiently generic" filter G, i.e., one meeting a prescribed collection of κ-many dense sets. This in turn allows us to prove the usual "forcing equals truth" lemma.

For a condition q let $q^{-1}\mathcal{M}$ be the structure on $\operatorname{dom} q$ defined so that q becomes an isomorphism from $q^{-1}\mathcal{M}$ to the restriction $\mathcal{M}\restriction_{\operatorname{range} q}$ of $\mathcal{M}$ to range q. Similarly we can define $g^{-1}\mathcal{M}$ for functions from κ to κ.

Letting g be the name for the permutation of κ obtained by a sufficiently generic, we find a condition p which forces that $\Phi(g^{-1}\mathcal{M})$ is an isomorphism from $g^{-1}\mathcal{M}$ to $\mathcal{M}$ (for some κ-Turing functional Φ).

In more down-to-earth terms, for all q extending p, $\Phi(q^{-1}\mathcal{M})$ preserves the structure, and we cannot force divergence: for every q extending p and every $\alpha < \kappa$ there is some r extending q such that $\alpha \in \operatorname{dom}\Phi(r^{-1}\mathcal{M})$. So every q extending p can be extended to some $g\colon \kappa \to \kappa$ such that $\Phi(g^{-1}\mathcal{M})$ is an isomorphism from $g^{-1}\mathcal{M}$ to $\mathcal{M}$. A direct diagonalisation construction shows that if such p does not exist then $\mathcal{M}$ cannot be relatively κ-computably categorical.

Let A be the range of p.

Recursively, we define a κ-computable club C and for each $\alpha \in C$, a condition $q_\alpha\colon \alpha \to \alpha$ such that

(1) If $\alpha < \beta$ then $q_\alpha \prec q_\beta$; if α is a limit point of C then $q_\alpha = \bigcup_{\beta\in\alpha\cap C} q_\beta$.
(2) $p \preceq q_0$.
(3) For all $\alpha \in C$, $\Phi(q_\alpha^{-1}\mathcal{M})$ is an isomorphism from $q_\alpha^{-1}\mathcal{M}$ to $\mathcal{M}\restriction_\alpha$.

Now by induction we define *quantifier-free*, κ-finite formulas φ_α for all $\alpha \in C$. Let $\bar{a}_\alpha$ be the enumeration of the first α many elements of $\mathcal{M}$ (the elements of $\mathcal{M}\restriction_\alpha$).[3]

We choose the formulas φ_α to be a κ-finite conjunction of atomic facts about $\bar{a}_\alpha$, such that stating these facts about $q^{-1}\bar{a}_\alpha$ is a sufficient oracle to make $\Phi(q_\alpha^{-1}\mathcal{M})$ converge on all of α. Furthermore, we do this recursively so that the monotonicity and continuity conditions stated above hold. Note that if α is a limit point of C then $\bigwedge_{\beta\in C\cap\alpha}$ is indeed a sufficient oracle to ensure convergence of Φ on all of α.

The usual argument shows that each φ_α defines the orbit of $\bar{a}_\alpha$: suppose that $(\mathcal{M}, a)_{a\in A} \models \varphi_\alpha(\bar{b})$ for some α-tuple $\bar{b}$ from $\mathcal{M}$. Let $r\colon \alpha \to \bar{b}$ be a bijection so that $r^{-1}\mathcal{M}$ agrees with $q_\alpha^{-1}\mathcal{M}$ on φ_α, and $p \preceq r$. Then we can extend r to a sufficiently generic $h\colon \kappa \to \kappa$. Then $h\colon h^{-1}\mathcal{M} \to \mathcal{M}$ is an isomorphism mapping α to $\bar{b}$, whereas $\Phi(h^{-1}\mathcal{M})$ is an isomorphism from $h^{-1}\mathcal{M}$ to $\mathcal{M}$ mapping α to $\bar{a}_\alpha$, whence $\bar{a}_\alpha$ and $\bar{b}$ are automorphic. ⊣

Is the club necessary? In other words, could we perform an effective back-and-forth construction one element at a time, rather than having to match infinite tuples each step? An infinitary example shows that sometimes the club is needed.

EXAMPLE 6.1. Let $\kappa = \aleph_1$, and let $\mathcal{M}$ consist of an equivalence relation that partitions the universe of $\mathcal{M}$ into $\aleph_1$ many equivalence classes, each of countably

[3]Of course technically $\bar{a}_\alpha = \langle i\rangle_{i<\alpha} = \mathrm{id}\restriction_\alpha$ but that doesn't look right typographically.

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infinite size, and a single relation R of arity ω, which holds of an ω-tuple $\bar{a}$ if and only if $\bar{a}$ enumerates the entirety of a single equivalence class.

The structure $\mathcal{M}$ is relatively $\aleph_1$-computably categorical by a simple back-and-forth argument. However, let $\bar{a}$ be an ω-tuple which injectively enumerates the entirety of a single equivalence class, and let $\bar{b}$ be an ω-tuple which injectively enumerates a proper subset of a single equivalence class. Then $\bar{a}$ and $\bar{b}$ do not have the same quantifier-free type, but for all $n < \omega$, $\bar{a} \restriction_n$ and $\bar{b} \restriction_n$ are in the same orbit.

It is not surprising though that an infinitary relation prevents us from performing a back-and-forth construction one element at a time. For $\kappa = \aleph_1$, this is the only obstacle.

THEOREM 6.2. *Let $\mathcal{M}$ be an $\aleph_1$-computable, relatively $\aleph_1$-computable categorical structure, all of whose relations and functions have finite arity.*

Then there is some countable $A \subset \mathcal{M}$ such that $(\mathcal{M}, a)_{a \in A}$ has an effective, continuous Scott family $\langle \varphi_\alpha \rangle_{\alpha < \omega_1}$, i.e., with $C = \omega_1$.

PROOF. By Theorem 1.3, let $(C, \langle \varphi_\alpha \rangle)$ be an effective, continuous Scott family for $\mathcal{M}$.

The first step is renumbering. Without loss of generality (by thinning out C), we may assume that C contains only limit ordinals. We then rearrange each C-interval in order type ω: there is a bijection $f : \omega_1 \to \omega_1$ such that $f[C]$ is the set of limit ordinals, and such that f is order-preserving on C. Translating by f, we may assume that C is the collection of limit ordinals. Furthermore, observing the proof of the theorem, we may assume that each φ_α is quantifier-free and countable.

Now we "fill in". For each limit ordinal α (or $\alpha = 0$) and each $n > 0$ we let

$$\varphi_{\alpha+n}(\bar{x} \restriction_{\alpha+n}) = \exists x_n, x_{n+1}, x_{n+2}, \ldots \left[\varphi_{\alpha+\omega}(\bar{x} \restriction_{\alpha+\omega})\right].$$

Observe that for $n > 0$, $\varphi_{\alpha+n}$ indeed defines an orbit, and that if $\alpha < \beta$ then φ_β implies φ_α. It remains to show that the extended family is still continuous. Here of course we use the assumption that the relations of $\mathcal{M}$ are finitary. The only interesting case is showing that for all limit α,

$$\varphi_{\alpha+\omega} \equiv \bigwedge_{n<\omega} \varphi_{\alpha+n}.$$

The point is that every atomic fact implied by $\varphi_{\alpha+\omega}$ mentions variables from $\bar{x} \restriction_{\alpha+n}$ for some n, and so is already implied by $\varphi_{\alpha+n}$. ⊣

§7. Further comments. We briefly discuss further directions. As noted in the introduction, the theory of computable Polish metric spaces is a generalisation of effective algebra. It is well-known that there is no Borel (let alone effective) way to encode a Polish space into a countable structure. Nonetheless, it is not clear whether some natural extensions of the standard effective-algebraic definitions to Polish spaces lead to more general notions.

For example, define the degree spectrum of a Polish metric space M to be the collection of all X that can enumerate a copy of the space (in the sense as above). Assuming that the Turing degree spectrum of a countable structure is the collection of all X that can compute a copy of the structure, it follows that any degree spectrum of a structure is a degree spectrum of some space.

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QUESTION 7.1. *Is there a Polish metric space whose degree spectrum cannot be realised as a degree spectrum of a countable structure*?

(For the comparison to make more sense we might want to look at e-spectra of countable structures rather than at Turing spectra.) Our methods seem to imply that two (or at most countably many) incomparable cones cannot be a spectrum of a Polish space, but we don't know much beyond that.

We strongly conjecture that the main forcing construction can be adjusted to show that each invariant relatively intrinsically c.e. open subset of a computable Polish metric space has a formal name with stable parameters. Nonetheless, it is much more natural to ask whether such results can be extended to definability of operations rather than of open subsets (e.g., + on a normed space that makes it a Banach space). See [16, 17] for various natural examples of this sort.

We also suspect that our results can be extended to produce Σ^c_α Scott families of relatively Δ^0_α-categorical spaces, but to attack the problem we first need to agree on what relatively Δ^0_α-categoricity means for Polish spaces. (See [18] for some partial results on Δ^0_α-categorical spaces.) We also leave open whether Theorem 6.2 can be extended to $\kappa \geq \aleph_2$. It would be nice to support our notion of relative κ-categoricity by several interesting examples (e.g., coming from functional analysis), but we leave this outside the scope of this article.

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