Math. Proc. Camb. Phil. Soc. (1986), **99**, 79
Printed in Great Britain

A note on the normal subgroups of mapping class groups

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(*Received* 7 *January* 1985; *revised* 17 *May* 1985)

0. If F_g is a closed, orientable surface of genus g, then the *mapping class group* of F_g is the group whose elements are orientation preserving self homeomorphisms of F_g modulo isotopy. We shall denote this group by M_g. Recall that a group is said to be *linear* if it admits a faithful representation as a group of matrices (where the entries for this purpose will be in some field).

One of the outstanding problems in the theory of surface automorphisms is the question of whether the groups M_g are linear. For $g = 1$, we have an affirmative answer, namely $M_1 \cong \mathrm{SL}_2\mathbb{Z}$. One of the difficulties in answering this question resides in the problems of handling matrices which are larger than 2×2. However, since the work of Thurston has given great insight into the nature of the elements of M_g, it is perhaps now more logical to look into the obstructions which reside in the mapping class groups themselves.

An example of this approach is the following question, which has been independently posed by R. Penner, J. McCarthy and the author:

Question 1. Can a normal subgroup of a mapping class group have all its non-identity elements (pseudo-)Anosov?

A negative answer to this question would put grave constraints on any non-faithful representation and in particular, when interpreted in the context of braid groups (which although not mapping class groups are intimately related to them) has been claimed to show the faithfulness of the Burau representation [9].

Our theme in this note will be that purely algebraic information about the groups M_g may be obtained from its action on the projective lamination space, and we apply this in particular to derive some results about normal subgroups. This falls naturally into two parts. § 1 shows that perhaps the (hoped for) negative answer to the above question is too optimistic. We show:

Theorem 1. *Let $\theta \in M_1$ be an Anosov map. Then there is a K, such that the normal closure of θ^K has all its non-identity elements Anosov.*

In particular, the bad behaviour the above question seeks to rule out is (in some sense) generic on the torus. Moreover, a trivial extension of Theorem 1 shows that any 3-string braid corresponding to a pseudo-Anosov map of the thrice punctured disc also has this bad property. In contrast, we also show that the normal closure of an Anosov map can be all of M_1. The corresponding statement for $g \geqslant 2$ is again unknown. The methods here are special to the torus and rely on results from Fuchsian groups and small cancellation theory.

§ 2 contributes some positive results about normal subgroups in braid and mapping class groups and applies to all genera. We prove the following results. Let $\mathscr{T}$ be a mapping class or braid group.

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THEOREM 2·1. *Suppose $i: \mathscr{T} \to A \times B$ is an injective homomorphism for which the centre of $\mathscr{T}$ injects on each factor. Let π_A, π_B denote the obvious projections. Then at least one of $\pi_A \circ i$ or $\pi_B \circ i$ is an injection.*

Hence a faithful representation of $\mathscr{T}$ cannot be manufactured from two non-faithful representations. Along similar lines we have:

THEOREM 2·2. *Suppose $\rho: \mathscr{T} \to A$ is a representation for which ρ restricted to the centre of $\mathscr{T}$ is faithful.*

Then ρ is faithful on $\mathscr{T}$ if and only if ρ is faithful when restricted to some nontrivial non-central normal subgroup of $\mathscr{T}$.

We conclude in § 3 with an application to the normal structure of mapping class and braid groups which exhibits a new property shared by these groups and linear groups, namely that they have nilpotent Frattini subgroups.

1. In order to save words we introduce the following notation. For $g \in G$, let $\langle g \rangle_N$ denote the normal closure of g inside G, that is, the smallest normal subgroup of G which contains g. A subgroup will be said to be *purely (pseudo-)Anosov* if all its non-identity elements are (pseudo-)Anosov. The force of Question 1 lies in the normality assumption - given a pair of (pseudo-)Anosov maps θ, ψ which do not generate an abelian-by-finite group, it is an easy exercise in the theory of surface automorphisms to show that, for large K, the subgroup generated by θ^K, ψ^K is purely (pseudo-)Anosov. (See the proof of Lemma 2·5.)

Before proceeding with the proof of Theorem 1, we give some examples.

Example 1. Suppose we may find a 1-submanifold C on F_g and maps $\theta, t \in M_g$ such that

(i) $tC = C$.

(ii) $t(\theta^r C) = \theta^{-s} C$.

Then $\langle \theta \rangle_N$ contains $\theta^s t \theta^r t^{-1}$ which fixes C. Thus $\langle \theta \rangle_N$ is not purely (pseudo-)Anosov.

We may apply this criterion to $\mathrm{SL}_2\mathbb{Z}$ to produce examples of Anosov maps whose normal closure is not purely Anosov.

Suppose $\theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2\mathbb{Z}$. Then we seek an element t fixing $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ say, (up to sign is sufficient) and carrying $\begin{pmatrix} a \\ c \end{pmatrix}$ to $\begin{pmatrix} d \\ -c \end{pmatrix}$. (This is the case $r = s = 1$ of the above).

That is, we seek an integer x with

$$\begin{pmatrix} -1 & -x \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} d \\ -c \end{pmatrix}$$

Hence $-a - xc = d$, or $x = -\operatorname{trace}(\theta)/c$.

So if c divides $\operatorname{trace}(\theta)$, then $\langle \theta \rangle_N$ contains a power of a Dehn twist. Thus the maps $\begin{pmatrix} r+1 & r \\ 1 & 1 \end{pmatrix}$ do not have purely Anosov normal closure. Observe that this gives examples with arbitrarily large dilatation.

Example 2. Given a closed orientable surface of genus $\geqslant 2$, we leave the reader to check that one may find a pair of essential simple curves C and D which fill the surface, with the further property that there is an essential simple closed curve D^* disjoint from D, so that $C \cup D^*$ does *not* fill. (In fact, almost every pair C, D one writes down has this property.) Let C^* be an essential simple curve missing $C \cup D^*$.

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By the construction of exposé 13 of [3], it is seen that $\theta = T_C T_D^{-1}$ is isotopic to a pseudo-Anosov map, where T_C denotes the Dehn twist about C.

Then $\theta D^* = T_C T_D^{-1} D^* = T_C D^*$, which is a curve disjoint from C^*.

Consider the map $T_{D^*}^{-1} \theta T_{D^*} \theta^{-1} \in \langle \theta \rangle_N$.

Observe that $T_{D^*}^{-1} \theta T_{D^*} \theta^{-1} = T_{D^*}^{-1} T_{\theta D^*}$. The right hand side of this equation is clearly a map fixing C^*, hence not pseudo-Anosov. Thus, $\langle \theta \rangle_N$ is not purely pseudo-Anosov. Examples of this nature have also been found by R. Penner.

In fact, the technique used in the second example may be strengthened to place severe restrictions on which simple closed curves C and D can have $\langle T_C T_D^{-1} \rangle_N$ purely pseudo-Anosov. However, these seem to be insufficient to show that this behaviour is impossible.

We now embark on our proof of Theorem 1. For reasons that will become apparent, it is initially more convenient to work with $\mathrm{PSL}_2\mathbb{Z}$. That this is sufficient for our purposes will be shown below. The advantage of working with the projectivised group is that this group acts in a well-known way on the hyperbolic plane $\mathbb{H}^2$. This action enables us to recognise 'Dehn twists' as maps with parabolic action.

We have an exact sequence

$$1 \to F_2 \to \mathrm{PSL}_2\mathbb{Z} \to \mathrm{PSL}_2\mathbb{Z}_2 \to 1,$$

where F_2 denotes the free group of rank 2. Then $\mathbb{H}^2/F_2$ is homeomorphic to a 2-sphere with 3 puncture points, which are cusps in the hyperbolic surface. Identifying F_2 with π_1 of this surface in a natural way, standard Fuchsian group theory shows that the elements of F_2 which act parabolically on $\mathbb{H}^2$ are exactly those elements of π_1 which may be freely homotoped into the cusps. These parabolics correspond exactly to the non-Anosov elements of $F_2 \subset \mathrm{PSL}_2\mathbb{Z}$.

If we regard $\mathrm{PSL}_2\mathbb{Z}$ as generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = a$ and $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = b$, then F_2 is generated by a^2 and b^2. The third cusp corresponds to the element a^2b^2.

Given an Anosov map $\theta \in \mathrm{PSL}_2\mathbb{Z}$, there is K so that $\theta^K \in F_2$, since this subgroup is of finite index. Set $\psi = \theta^K$.

Our claim will be that we may iterate ψ so that its normal closure in $\mathrm{PSL}_2\mathbb{Z}$ is purely Anosov. Notice that since $F_2 \lhd \mathrm{PSL}_2\mathbb{Z}$, $\langle \psi \rangle_N$ lies inside F_2.

$\mathrm{PSL}_2\mathbb{Z}_2$ acts on F_2 by conjugation – this action arises geometrically as the orientation preserving symmetry group of the two-sphere with 3 cusps, hence is isomorphic to the symmetric group on 3 letters.

Thus we may identify the normal closure of ψ in $\mathrm{PSL}_2\mathbb{Z}$,with the normal closure of the elements ψ, $g_1\psi g_1^{-1}, \ldots, g_5\psi g_5^{-1}$ in F_2 where the g_i's are some generators of the $\mathrm{PSL}_2\mathbb{Z}_2$ action.

Putting together the above remarks, we see that the normal closure in $\mathrm{PSL}_2\mathbb{Z}$ of ψ is purely Anosov if and only if the normal closure in F_2 of the elements

$$\psi, g_1\psi g_1^{-1}, \ldots, g_5\psi g_5^{-1}$$

is purely Anosov. This latter condition holds if and only if none of the elements a^2, b^2, a^2b^2, or their powers die in the mapping

$$p\colon F_2 \to F_2/\langle \psi, \ldots, g_5\psi g_5^{-1} \rangle_N.$$

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It may be necessary to raise ψ to some power, as the techniques of example 1 applied to $\begin{pmatrix} 17 & 8 \\ 2 & 1 \end{pmatrix}$ show.

We now appeal to the ideas of small cancellation theory. (See [5], p. 240, 250, 281.) For large T, the relators $\{\psi^T, \ldots, g_5\psi^T g_5^{-1}\}$ become a set which satisfy the hypotheses of small cancellation. Thus the quotient $F_2/\langle\psi^T, \ldots, g_5\psi^T g_5^{-1}\rangle_N$ may be arranged to have only the obvious torsion elements and the property that one of the elements a^2, b^2, a^2b^2 die. Thus we have excluded all possible parabolics from $\langle\psi^T\rangle_N$, which is therefore purely Anosov.

Proof of Theorem 1. Given $\theta \in \mathrm{SL}_2\mathbb{Z}$, the above argument shows we may find K so that $\langle(\pi\theta)^K\rangle_N$ is purely Anosov. (Recall $\pi\colon \mathrm{SL}_2\mathbb{Z} \to \mathrm{PSL}_2\mathbb{Z}$ is the projection map.) Further, we may suppose that $(\pi\theta)^K \in F_2$.

Observe that $\langle(\pi\theta)^K\rangle_N$ is freely generated as a group by some set of conjugates $\{g(\pi\theta)^K g^{-1}\}$. Since conjugation in $\mathrm{PSL}_2\mathbb{Z}$ and $\mathrm{SL}_2\mathbb{Z}$ coincide, to show $\langle\theta^K\rangle_N$ is purely Anosov, it suffices to show that this subgroup avoids $-I$.

However, if $-I = \Pi_j g_j \theta^{Kn_j} g_j^{-1}$, π-projection gives a relation in F_2, a contradiction. This completes the proof. ∎

We now turn our attention to the second question:

Question. Can the normal closure of a (pseudo-)Anosov map ever be all of M_g?

Again, observe that the force of this statement lies in considering a (pseudo-)Anosov map, for the normal closure of a Dehn twist contains the Lickorish generators for M_g, hence certainly generates.

Our example in genus 1 comes from considering the well known presentation $\mathrm{SL}_2\mathbb{Z} \cong \{x, y | x^2 = y^3 = \alpha, \alpha^2 = 1\}$. For $g \in SL_2\mathbb{Z}$, consider the word $gxg^{-1}y = \theta$. Then $\langle\theta\rangle_N = \mathrm{SL}_2\mathbb{Z}$ if and only if adjoining the relation corresponding to the word θ kills the whole group. That this is so is verified by an elementary piece of combinatorial group theory:

$$gxg^{-1}y = 1 \Rightarrow g^{-1}y^2g = x^{-2} = \alpha^{-1} = \alpha.$$

Hence $y^2 = \alpha$, since α is clearly central. So

$$y^2 = y^3 \Rightarrow y = 1 \Rightarrow \alpha = 1 \Rightarrow x = 1.$$

We thus obtain a veritable plethora of examples.

We conclude this section by observing that since there is a map from the 3-string braid group onto $\mathrm{SL}_2\mathbb{Z}$, a trivial extension of the above proofs shows that there are 3-braids whose normal closure is purely pseudo-Anosov, and indeed the generic behaviour of Theorem 1 holds. Perhaps this is evidence that to hope for a negative answer to Question 1 is overly optimistic.

2. In this section we derive somewhat more positive results. Throughout, $\mathscr{T}$ will denote a braid or mapping class group and PL will denote the projective measured lamination space for the relevant surface. Explicit definitions of PL and rudiments of the natural action of $\mathscr{T}$ on PL may be found in [2] or [3]. For the convenience of the reader unfamiliar with these notions it will be sufficient to use the following definitions. An element $\theta \in \mathscr{T}$ is *pseudo-Anosov* if it has precisely two fixed points on PL, one source and one sink. An element $R \in \mathscr{T}$ is *reducible* if it has a fixed point in PL corresponding to an invariant 1-manifold $C \subset F_g$. In fact, $R(C)$ is isotopic to C.

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Since the case that $\mathscr{T}$ is a braid group is less familiar, we digress temporarily to provide some clarification. In the usual theory of surface automorphisms when the surface has boundary, one works modulo isotopies which are allowed to move boundary points within each boundary component. However, in identifying the braid group with the automorphism group of a punctured disc, it is necessary to modify this and work modulo isotopies which fix pointwise both the punctures and the boundary of the disc. Thus the Dehn twist about the boundary, trivial in Thurston's theory, becomes essential here and corresponds to the central full twist. The proof we shall give below is arranged to work for both braid and mapping class groups, provided we adopt the convention that this central twist is not considered to be a reducible. Note that PL is the same in both theories.

We now commence with the proof proper. It is elementary to see that the results 2·1 and 2·2 of § 0 are both implied by the following lemma:

LEMMA 2·1. *Let $N_1, N_2 \lhd \mathscr{T}$ be nontrivial and non-central. Then $N_1 \cap N_2 \neq \{e\}$.*

Of course, M_g, $g \geqslant 3$, has trivial centre so the centre restriction is vacuous. Moreover, all obvious representations of braid groups and M_2 satisfy this requirement. The point of avoiding the centre is that we shall show that this is the only normal subgroup which does not contain very many pseudo-Anosov maps.

We prove Lemma 2·1 via a series of elementary lemmata.

LEMMA 2·2. *Suppose $N \lhd \mathscr{T}$ is torsion. Then N is central.*

Proof. Braid groups are torsion free, so the case $\mathscr{T} = M_g$ is the only relevant one here. In this case, it is well known that torsion subgroups are finite.

Let C be any simple closed curve and T_C the Dehn twist about C. Then conjugation by T_C induces an automorphism of N, by normality, and since N is finite, there is a k, so that conjugation by T_C^k is the identity automorphism of N. Hence for each $\alpha \in N$ we have

$$T_C^{-k} \alpha T_C^k = \alpha.$$

Thus $T_{\alpha C}^k = T_C^k$, implying $\alpha C = C$, and so $T_C \alpha T_C^{-1} \alpha^{-1} =$ identity. Since Dehn twists generate $\mathscr{T}$, this proves the lemma. |

LEMMA 2·3. *A reducible map of infinite order cannot centralize a pseudo-Anosov map.*

Proof. Suppose there is such a reducible R (with invariant 1-manifold C) which centralizes a pseudo-Anosov θ. Observe that for each k, $\theta^k C$ is also a reducing 1-manifold for R. For large k, $C \cup \theta^k C$ fills the surface. However, this graph is left invariant by R. A standard argument now shows that some power of R is isotopic to the identity, that is, R has finite order, a contradiction. |

LEMMA 2·4. *Let θ be a pseudo-Anosov map with invariant laminations $L^+, L^- \in$ PL. Suppose R is reducible and of infinite order. Then $R(L^+)$ cannot be L^+ or L^-.*

Proof. Suppose $R(L^+) = L^-$. Then the map $R^{-1}\theta R$ is a pseudo-Anosov map fixing L^+, and since pseudo-Anosov maps having one lamination in common have both laminations in common, we see that $R^{-1}\theta R(L^-) = L^-$, implying $R(L^-) = L^+$. Thus, squaring enables us to assume that $R(L^+) = L^+$.

Consider the pseudo-Anosov maps $\{R^k \theta R^{-k} | k \in \mathbb{Z}\}$. These all fix L^+ and L^- and have

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the same dilatation, so there must be some repetitions, that is $R^k\theta R^{-k} = \theta$ for some k. But this contradicts Lemma 2·3. |

LEMMA 2·5. *Let N be a nontrivial non-central normal subgroup of $\mathscr{T}$. Then N contains a pseudo-Anosov map.*

Proof. By Lemma 2·2, we may suppose that N contains a map of infinite order, R, which we may assume reducible, or we are finished.

By Lemma 2·4, given any pseudo-Anosov θ with invariant laminations $L^+, L^- \in PL$, $R(L^+), R(L^-)$ are not L^+, L^-.

The map $R\theta R^{-1}$ is pseudo-Anosov, with invariant laminations $R(L^+)$, $R(L^-)$.

Choose disjoint disc neighbourhoods A_+, A_-, B_+, B_- of the points L^+, L^-, $R(L^+)$, $R(L^-)$ respectively. Then there is a K, so that $\theta^K(PL\backslash A_-) \subset A_+$, $\theta^{-K}(PL\backslash A_+) \subset A_-$, and so on. Moreover, we may choose K so large that the maps $\theta^K\colon A_+ \to A_+$, etc. are all contractions, by [8], proposition VIII.

One may now explicitly compute that if we set $x = \theta^\kappa$ and $y = R\theta^\kappa R^{-1}$, any freely reduced word in the group generated by x and y has all its fixed points on PL lying inside either closure $(A_+ \cup A_-)$ or closure $(R(A_+) \cup R(A_-))$. The contraction property guarantees that each such word has exactly two fixed points on PL. Hence, every freely reduced element of $\langle x, y\rangle$ is pseudo-Anosov; in particular $x^{-1}y$ is. But this is the element $\theta^{-\kappa}R\theta^\kappa R^{-1}$, which lies in N, completing the proof. |

Remark. This proof also shows that two commuting pseudo-Anosovs must have the same invariant laminations.

LEMMA 2·6. *Let $N \lhd \mathscr{T}$ be a nontrivial, non-central normal subgroup. Then the set $axis(N) = \{L^+_\theta, L^-_\theta \,|\, L^\pm_\theta$ are the invariant laminations of pseudo-Anosov $\theta \in N\}$ is dense in* PL.

Proof. By Lemma 2·5, we have at least one pseudo-Anosov map, ψ say, in N. Given any simple closed curve C, there is a K so that C and $\psi^K C$ fill, so by [3], the maps $T^r_C\psi^K T^{-r}_C\psi^{-K} = T^r_C T^{-r}_{\psi^K C}$ are pseudo-Anosov for all r. As $r \to \infty$ it is easy to see that the stable invariant lamination of this map converges in PL to C.

Hence, the closure of the above set includes every simple closed curve and thus is all of PL. |

Proof of Lemma 2·1. Suppose that N_1, N_2 are as in the statement with $N_1 \cap N_2 = \{e\}$. Then since $[N_1, N_2] \leqslant N_1 \cap N_2$, we have that every element of N_1 centralizes every element of N_2.

Fix any pseudo-Anosov $\theta \in N_1$, with invariant laminations L^+, $L^- \in$ PL. Since each pseudo-Anosov $\psi \in N_2$ commutes with θ, the remark following 2·5 shows that axis $(N_2) = \{L^+, L^-\}$. This contradicts 2.6. |

Remark. Throughout, it was not really essential to work with the full mapping class of braid groups but only with 'sufficiently large' subgroups – where the meaning of this is that there should be 'enough' pseudo-Anosov maps in the subgroup. For example, our results continue to hold for $\mathscr{T}$ = Torelli subgroup or the pure braid group.

We may use Theorem 2·2 to obtain results about the Burau and Gassner representations. (See [1] for an exposition of these representations.) Let $M_n(\mathbb{Z}[t, t^{-1}])$ denote the set of $n \times n$ matrices with coefficients in $\mathbb{Z}[t, t^{-1}]$. B_n will denote the n-string braid group.

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PROPOSITION 2·7. (Magnus and Peluso [7]). *The Burau representation*

$$\psi: B_3 \to M_2(\mathbb{Z}[t, t^{-1}])$$

is faithful.

Proof. Consider the subgroup $N = \langle \sigma_1^{-1}\sigma_2, \sigma_1\sigma_2^{-1} \rangle$. It is an elementary check that this is normal.

One finds that $\psi(\sigma_1^{-1}\sigma_2) = \begin{pmatrix} 1-\bar{t} & -1 \\ t & -t \end{pmatrix}$ and $\psi(\sigma_1\sigma_2^{-1}) = \begin{pmatrix} 1-t & -\bar{t} \\ 1 & -\bar{t} \end{pmatrix}$.

Putting $t = -1$ gives a matrix group which is free of rank 2. Using the Hopfian property of free groups twice, we see that the map ψ is an isomorphism on N. By 2·2, ψ must be faithful on B_3. |

PROPOSITION 2·8. (Birman [1]). *To show* $\psi: B_4 \to M_3(\mathbb{Z}[t, t^{-1}])$ *is faithful it suffices to show that the matrices* $\{\psi(\sigma_1\sigma_3^{-1}), \psi(\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1})\}$ *generate a free group of rank* 2.

Proof. An elementary calculation from the presentation of B_4 gives that the subgroup $\langle \sigma_1\sigma_3^{-1}, \sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1} \rangle$ is normal in B_4. Thus the result follows from 2·2. |

Remarks. (1) In fact one can show that $\sigma_1\sigma_3^{-1}$ and $\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}$ generate a free group of rank 2 in B_4, so the condition is necessary.

(2) In [1], various calculations are made to show that the kernel of ψ lies inside certain subgroups on B_n. Most of these subgroups are normal in B_n, so if one is content to know that $\ker \psi$ intersects these subgroups nontrivially (which is, after all, sufficient for faithfulness questions) then Theorem 2·2 provides all these calculations simultaneously.

In a similar vein, there is a natural map $\pi: P_k \to P_{k-1}$ where P_k denotes the pure braid group on k strings. One finds (see [1]) that $\ker \pi$ is free of rank $k-1$. This appears to give a new criterion for the faithfulness of the Gassner representation on P_4.

We conclude by remarking that 2·2 continues to hold if N is replaced by a subnormal subgroup of $\mathscr{T}$. This does not seem to be a significant improvement in practice.

3. The groups M_g share many properties with linear groups; they are both residually finite [4] and satisfy the so-called Tits alternative [6]. This is one of the motivations for studying the question of linearity. Using the methods and results of § 2 we give a proof of a new common property. In order to state this, we recall a definition:

Definition. If G is a group, we define the *Frattini subgroup* of G denoted $\Phi(G)$ to be the intersection of all the maximal subgroups of G.

THEOREM 3·1.(Platonov [10]). *Let G be a finitely generated linear group. Then $\Phi(G)$ is nilpotent.*

Despite the inaccessibility of the Frattini subgroup in general, our methods show fairly quickly:

THEOREM 3·2. (*a*) $\Phi(M_g) = \mathbb{Z}_2 \quad (g = 2)$

$$= (e) \quad (g \geqslant 3).$$

(*b*) $\Phi(B_n) = (e)$ *for all* n.

Thus 3·2 shows that the obstruction to linearity provided by the Platonov theorem vanishes in these cases. Before proceeding with the proof, we recall an elementary lemma of group theory. Let $\alpha(G)$ denote the centre of G and $\gamma_2(G)$ its commutator subgroup.

LEMMA 3·3. (*a*) $\Phi(G) \supseteq \alpha(G) \cap \gamma_2(G)$.

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(*b*) *If G is finitely generated and $A \leqslant G$ with $\langle A, \Phi(G)\rangle = G$, then $A = G$.*

Proof. (*a*) Let M be a maximal subgroup of G which does not contain $\alpha(G)$. Then $\langle M, \alpha(G)\rangle = G$ by maximality. Hence M is normal in G. But

$$G/M \cong \langle M, \alpha(G)\rangle/M \cong \alpha(G)/M \cap \alpha(G)$$

is abelian. So $M \geqslant \gamma_2(G)$, proving the result.

(*b*) Suppose that $A < G$. Then by Zorn's lemma we may find a maximal M with $A \leqslant M < G$. Then $\langle A, \Phi(G)\rangle \leqslant \langle M, \Phi(G)\rangle \leqslant M$, a proper subgroup of G. |

In particular, this lemma explains the apparently anomalous case $g = 2$. In [1] a presentation of M_2 is given, and by abelianizing this it may be computed that $\gamma_2(M_2) \supset \alpha(M_2) \cong \mathbb{Z}_2$. Thus $\Phi(M_2) \supset \mathbb{Z}_2$.

Proof of 3·2. (*a*) The Frattini subgroup is normal. In the light of Lemma 2·2 it is therefore sufficient to show that $\Phi(M_g)$ is always torsion to prove the required result, since it is well known that the formula of 3·2(*a*) describes the centre of M_g.

We suppose otherwise and derive a contradiction. If $\Phi(M_g)$ is not a torsion subgroup, then by Lemma 2·6 the set axis (ΦM_g) is dense in PL. Let $T_1, \ldots, T_n$ be any collection of Dehn twist generators for M_g. Since axis (ΦM_g) is dense in PL, we may find pseudo-Anosov maps $\theta_1, \ldots, \theta_n \in \Phi(M_g)$ with the property that if L_i^+ and L_i^- denote the invariant laminations of θ_i, then $(L_i^+, L_i^-) \neq (L_j^+, L_j^-)$ for $i \neq j$. Each T_i is reducible and of infinite order, and Lemma 2·4 shows that the four points $L_i^+, L_i^-, T_i(L_i^+), T_i^{-1}(L_i^-)$ are distinct. Hence we may find neighbourhoods A_i^+ and A_i^- of L_i^+ and L_i^- so that the sets $A_i^+, A_i^-, T_i(A_i^+)$ and $T_i^{-1}(A_i^-)$ are pairwise disjoint. By shrinking if necessary, we may arrange that $A_i^+ \cup A_i^-$ is disjoint from $A_j^+ \cup A_j^-$ if $i \neq j$.

By [8], there is an integer $K > 0$, so that the maps $\psi_i = \theta_i^K . T_i . \theta_i^K$ are all pseudo-Anosov. Further, by choosing a larger K if necessary, we may suppose

$$\theta_i^{-K})\mathrm{PL}\backslash A_i^+) \subset A_i^- \quad \text{and} \quad \theta_i^K(\mathrm{PL}\backslash A_i^-) \subset A_i^+,$$

for each i. We next claim that $\psi_i^{-1}(\mathrm{PL}\backslash A_i^+) \subset A_i^-$ and $\psi_i(\mathrm{PL}\backslash A_i^-) \subset A_i^+$ for each i. We verify the latter: if $p \in \mathrm{PL}\backslash A_i^-$, then $\theta_i^K(p) \in A_i^+$. Hence $T_i . \theta_i^K(p) \in T_i(A_i^+) \subset \mathrm{PL}\backslash A_i^-$. Thus $\psi_i(p) \in A_i^+$ as required.

We can now show that the group $\langle \psi_1, \ldots, \psi_n\rangle = H$ is free. This follows the well known pattern for showing that a Schottky group is free. We show that there is no relation between ψ_1 and ψ_2, the general case being similar. If we choose

$$p \in \mathrm{PL}\backslash(A_1^+ \cup A_1^- \cup A_2^+ \cup A_2^-)$$

then any reduced word in ψ_1 and ψ_2 moves p into one of the A-sets. Thus the word does not act as the identity on PL, and so the word is not the identity. Since the groups M_g are not free, H must be a proper subgroup. However, it is clear that $\langle H, \Phi M_g\rangle = M_g$, since the left hand side contains a generating set. This contradicts 3·3(*b*).

(*b*) An identical proof shows that $\Phi(B_n)$ lies in the centre of B_n, which is infinite cyclic and generated by the full twist Δ. Thus for some r we have $\Phi(B_n) = \langle \Delta^r\rangle$. Abelianizing B_n gives a surjection to $\mathbb{Z}$ for which Δ maps nontrivially. If now p is any prime not dividing r, the kernel of the composition $B_n \to \mathbb{Z} \to \mathbb{Z}_p$ exhibits a maximal subgroup not containing Δ^r and the result is proved. |

Remark. As in the remark following the proof of Lemma 2·1, this result can be

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extended to show that for sufficiently large subgroups the Frattini subgroup is torsion. Applying this to the Torelli subgroup, for example, we see that

$$\Phi \text{ (Torelli subgroup)} = (e).$$

The author is indebted to R. Fenn and A. Papadopoulos, who both made useful suggestions.

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