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Graphs with pre-assigned Salama entropies and optimal degrees for locally compact Markov shifts†

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Abstract. Suppose that S is a transitive locally compact Markov shift with Gurevic entropy less than $\log p$, where p is an integer. If the inequality is strict, then S admits a graph presentation with degree p. Additionally, we realize any pre-assigned values for the Salama entropies. This result is best possible, since there are examples of Gurevic entropy $\log p$ which do not admit a graph presentation with degree p. We also show that every non-compact locally compact Markov shift has presentations without synchronizing blocks.

1. *Results*

This section contains all the results of the paper except the proof of the main result, the Strong Degree Theorem, which is deferred to §2. The proof of the Strong Degree Theorem introduces a construction method for graph presentations, which can also be applied to SFTs, where it yields graph presentations which have a natural polynomial matrix. These polynomial matrices were recently used in symbolic dynamics to solve several famous problems, such as the shift equivalence problem [**KR**], or the FOG conjecture [**KRW**]. Very good basic texts for symbolic dynamics are [**K, LM**]. References on Markov shifts relevant for this paper are [**K, Se, FF1, FF2, Fi, F1, F2**].

A *subshift* is a shift invariant subset S of $\mathbb{N}^{\mathbb{Z}}$, endowed with the product topology of the discrete topology on $\mathbb{N} = \{1, 2, 3, \dots\}$, together with the left shift map. To reduce notation we use the same symbol for the space and the shift map. Two subshifts are *conjugate* if there is a homeomorphism $f : S \to T$ which commutes with the shift maps. Any subshift T of $\mathbb{N}^{\mathbb{Z}}$ which is conjugate to S is called a *presentation* of S. We denote the set of presentations of S by $\mathcal{P}res(S)$. A *locally compact subshift* is a subshift S where the space S is locally compact.

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A *Markov shift* S is by definition a subshift which is conjugate to the set S_G of bi-infinite walks along the edges of a countable directed graph G, with the left shift transformation acting on S_G. (We consider only non-degenerate graphs, and by renaming the edges we may assume that the edge set is a subset of $\mathbb{N}$.) The subshift S_G is called a graph presentation of S. The set of graph presentations of S is denoted by $\mathcal{G}(S)$. The Markov shift S is locally compact if and only if G has at most finitely many in-coming and out-going edges at every vertex of G. The Markov shift is compact if and only if G is a finite graph if and only if it is a shift of finite type (SFT). Transitivity of S means strong connectedness of G.

Note that for $T \in \mathcal{G}(S)$, $x = (x_i)_{i\in\mathbb{Z}} \in T$ and $n \leq m$ the block $x[n,m] := x_n x_{n+1} \dots x_m$ is *synchronizing*; that is for all $y, z \in T$ with $y[n,m] = x[n,m] = z[n,m]$ there is a point $w \in T$ with $w(-\infty, m] = y(-\infty, m]$ and $w[n,\infty) = z[n,\infty)$. We call a presentation T of S a *synchronizing presentation* if for all $x \in T$ there are $-\infty < n \leq m < \infty$ such that the block $x[n,m]$ is a synchronizing block. So graph presentations are synchronizing presentations, but not every synchronizing presentation is a graph presentation.

We study graph presentations under two aspects: degree and entropy. Let S be a locally compact transitive Markov shift. For a presentation $T \in \mathcal{P}res(S)$ we define outdegree, indegree and degree by

$$\begin{aligned}
\deg_+(T) &= \sup_{a\in\mathbb{N}} \#\{x_1 \mid x = (x_i)_{i\in\mathbb{Z}} \in T, x_0 = a\} \in \mathbb{N} \cup \{\infty\},\\
\deg_-(T) &= \sup_{a\in\mathbb{N}} \#\{x_{-1} \mid x \in T, x_0 = a\} \in \mathbb{N} \cup \{\infty\},\\
\deg(T) &= \sup(\deg_+(T), \deg_-(T)) \in \mathbb{N} \cup \{\infty\}.
\end{aligned}$$

If S is compact, then $\deg(T) < \infty$ and the topological entropy satisfies $h_{\text{top}}(S) \leq \log(\deg(T))$. Marcus [**M**] showed that if S is compact then there is a graph presentation $T \in \mathcal{G}(S)$ with $\deg(T) = p$, where p is the least integer such that $h_{\text{top}}(S) \leq \log p$. So in the compact case the least possible degree for a graph presentation is determined by the topological entropy.

If S is merely locally compact then there are presentations T of S with $\#\{x_1 \mid x \in T, x_0 = a\} = \infty$ for some symbol a. However, for graph presentations $T \in \mathcal{G}(S)$ we always have $\#\{x_1 \mid x \in T, x_0 = a\} < \infty$ and $\#\{x_{-1} \mid x \in T, x_0 = a\} < \infty$ for any $a \in \mathbb{N}$, nevertheless $\deg(T) = \infty$ is possible. The Gurevic entropy $h_G(S)$ [**G**] which is the topological entropy of the 1-point compactification of the Markov shift, controls whether there is a presentation with finite degree: there is $T \in \mathcal{P}res(S)$ with $\deg(T) < \infty$ if and only if $h_G(S) < \infty$, [**FF1**]. This result was strengthened by U. Fiebig [**Fi**] in two ways: if p is the least integer such that $h_G(S) < \log p$ then there is a synchronizing presentation $R \in \mathcal{P}res(S)$ with $\deg(R) \leq p$ and there is a graph presentation $T \in \mathcal{G}(S)$ with $\deg(T) \leq p^2$. We complete the result as follows.

DEGREE THEOREM. *Let S be a transitive locally compact Markov shift and $p \in \mathbb{N}$ such that $h_G(S) < \log p$. Then S has a graph presentation with degree less than or equal to p.*

Proof. This is a corollary of the Strong Degree Theorem below. □

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Since $T \in \mathcal{G}(S)$ with $\deg(T) = p$ implies $h_{\mathrm{G}}(S) \leq \log p$, the following lemma and example shows that the Degree Theorem is best possible in the non-compact case.

LEMMA 1.1. *For each integer $p \geq 2$ there is a mixing locally compact Markov shift S with $h_{\mathrm{G}}(S) = \log p$ and S has no graph presentation with outdegree or indegree less than or equal to p.*

Proof. Let $S_p = \{0, 1, 2, \ldots, p-1\}^{\mathbb{Z}}$ and $S = S_p - \{0^\infty\}$, where 0^∞ denotes the point $x \in S_p$ with $x_i = 0$ for all i. Then S is an open shift-invariant subset of S_p and thus a locally compact transitive Markov shift [**Fi, FF2**, Corollary 1.11]. Since S contains fixed points it is actually mixing. Since for every $n \geq 1$, S contains the SFT in S_p where just the block 0^n is forbidden, $h_{\mathrm{G}}(S) = \log p$. Assume S has a graph presentation S_{G} with $\deg_+(S_{\mathrm{G}}) \leq p$. If G has a vertex v which has outdegree less than or equal to $p-1$, then we add an edge e to the graph of G with initial vertex v and an arbitrary terminal vertex in the vertex set of G to obtain a new graph H. Then S_H is mixing and $\deg_+(S_H) \leq p$ and thus $h_{\mathrm{G}}(S_H) = \log p$. Thus S_{G} is a proper closed subshift of S_H with $h_{\mathrm{G}}(S_{\mathrm{G}}) = h_{\mathrm{G}}(S_H)$ and thus by [**F2**, Lemma 4.5] the 1-point compactification of S_H is not expansive. Since G differs from H just by the edge e, [**F2**, Lemma 4.1] implies that the 1-point compactification of S_{G} is not expansive either. This is a contradiction, since the 1-point compactification of S_{G} is conjugate to the full shift S_p. Thus, every vertex in G has outdegree exactly p.

Since the 1-point compactification of S is expansive, there is, by [**F2**, Lemma 4.1], a finite subset E of edges of G such that:

(1) $x \in S_{\mathrm{G}} \Rightarrow x_n \in E$ for some $n \in \mathbb{Z}$;

(2) $\#\{x(-\infty, 0] \mid x \in S_{\mathrm{G}}, x_0 = e, x_n \notin E \text{ for all } n < 0\} \leq 1$ for all $e \in E$.

Let $x \in S_{\mathrm{G}}$ with $x_n \notin E$ for all $n \leq 0$. For every $n \leq 0$ the initial vertex of x_n has outdegree $p \geq 2$, and thus there is a point $y^n \in S_{\mathrm{G}}$ such that $y^n(-\infty, n-1] = x(-\infty, n-1]$ and $(y^n)_n \neq x_n$. By (1) there is thus a smallest $m_n \geq n$ such that $(y^n)_{m_n} \in E$. Since E is finite there is thus $n < k \leq 0$ such that $(y^n)_{m_n} = (y^k)_{m_k}$. Then (2) implies $y^n(-\infty, m_n] = y^k(-\infty, m_k]$. Then (1) implies $(y^n)_i \neq (y^n)_j$ for all $i < j \leq m_n$ and $(y^k)_i \neq (y^k)_j$ for all $i < j \leq m_k$. Since $(y^n)_{n-1} = x_{n-1} = (y^k)_{n-1}$ thus $m_n = m_k \geq k > n$ and we obtain $(y^n)_n = (y^k)_n = x_n$; a contradiction. Thus the assumption $\deg_+(S_{\mathrm{G}}) \leq p$ is wrong. A symmetric argument shows that $\deg_-(S_{\mathrm{G}}) > p$. □

Example 1.2. Let $p \geq 2$ be an integer. We define a mixing non-compact Markov shift $S = S_{\mathrm{G}}$ with $h_{\mathrm{G}}(S) = \log p$ and $\deg(S_{\mathrm{G}}) = p$. Let G be the graph with vertex set $\mathbb{Z}$ and:

- p edges from vertex n to vertex $n+1$, $n \notin \{2^k \mid k \geq 1\} \cup \{-2^k - 1 \mid k \geq 1\}$;
- one edge from vertex n to vertex $n+1$, $n \in \{2^k \mid k \geq 1\} \cup \{-2^k - 1 \mid k \geq 1\}$;
- one edge from vertex 2^k to vertex -2^k, $k \geq 1$.

For each $k \geq 1$ the number of first return loops at vertex 0 of length $l(k) = 2 \cdot 2^k + 1$ is $p^{l(k)-2k+1}$. Thus $h_{\mathrm{G}}(S) \geq \log p$ and, since $\deg(S_{\mathrm{G}}) = p$, we get that $h_{\mathrm{G}}(S) = \log p$.

Thus, for non-compact Markov shifts with entropy $\log p$ there are two classes: one in which the elements do have a graph with degree p and the other in which they do not have such a graph. Since Marcus [**M**] showed that in the compact case there is always a graph with degree p, the compact and non-compact case are different.

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We refine the Degree Theorem by studying other entropies. For that we recall the different notions of entropies that we shall use.

For a locally compact subshift $S \subset \mathbb{N}^{\mathbb{Z}}$ let $B_n(S) = \{x[1,n] \mid x \in S\}$ denote the set of S-blocks of length n. The *transition entropy* of S is defined as [**FF1**]

$$h_{\mathrm{t}}(S) = \inf\{H_{\mathrm{t}}(T) \mid T \in \mathcal{P}res(S)\}$$

where

$$H_{\mathrm{t}}(T) = \sup_{a,b\in\mathbb{N}} \limsup_{n\to\infty} \frac{1}{n} \log \#\{w \in B_n(T) \mid w_1 = a, w_n = b\}.$$

For a general locally compact subshift S only $h_{\mathrm{t}}(S) \geq h_{\mathrm{G}}(S)$ holds, but for a transitive Markov shift S, $h_{\mathrm{t}}(S) = h_{\mathrm{G}}(S)$ always holds [**FF1**]. More importantly for us, $H_{\mathrm{t}}(T)$ is a conjugacy invariant: for all $T \in \mathcal{G}(S)$, $a, b \in \mathbb{N}$ it holds that

$$h_{\mathrm{t}}(S) = H_{\mathrm{t}}(T) = \limsup_{n\to\infty} \frac{1}{n} \log \#\{w \in B_n(T) \mid w_1 = a, w_n = b\}.$$

Salama introduced an entropy for a graph presentation $T \in \mathcal{G}(S)$ of a Markov shift S. The *outgoing Salama entropy* of T is defined as [**S1**]

$$H_+(T) = \limsup_{n\to\infty} \frac{1}{n} \log \#\{w \in B_n(T) \mid w_1 = a\}.$$

Similarly, the *incoming Salama entropy* is defined as [**S1**]

$$H_-(T) = \limsup_{n\to\infty} \frac{1}{n} \log \#\{w \in B_n(T) \mid w_n = a\}.$$

For transitive T, $H_+(T)$ and $H_-(T)$ do not depend on the particular choice of the symbol a. However, it is easy to give examples S with $R, T \in \mathcal{G}(S)$ with $H_+(T) \neq H_+(R)$. Thus in contrast to the transition entropy $H_{\mathrm{t}}(T)$, the outgoing Salama entropy $H_+(T)$ is not a conjugacy invariant for S. But the range of H_+ considered as a function on $\mathcal{G}(S)$ is a conjugacy invariant for S. In particular, the infimum over these values. Motivated by this, in [**FF1**] the *outgoing entropy* was defined for general locally compact subshifts S as

$$h_+(S) = \inf\{H_+(T) | T \in \mathcal{P}res(S)\}$$

and it was shown that $h_+(S) = h_{\mathrm{t}}(S)$ in general. However, it was left as an open problem if for a Markov shift S, $h_+(S)$ coincides with the infimum over $H_+(T)$ when considering only graph presentations T of S: is $\inf\{H_+(T) \mid T \in \mathcal{P}res(S)\} = \inf\{H_+(T) \mid T \in \mathcal{G}(S)\}$? The answer is yes and the infimum is actually a minimum; see Theorem 1.3 below.

To shed some light on the difference of outgoing Salama entropy and Gurevic entropy, Salama showed that for each $\lambda \leq \alpha$ there is a Markov shift S with $h_{\mathrm{G}}(S) = \log\lambda$ and a graph presentation $T \in \mathcal{G}(S)$ with $H_+(T) = \log\alpha$, [**S1**]. We strengthen this result by showing the following.

THEOREM 1.3. *Let S be a transitive locally compact non-compact Markov shift. Then the Salama entropy function*

$$H_+ \times H_- : \mathcal{G}(S) \to [h_{\mathrm{G}}(S), \infty] \times [h_{\mathrm{G}}(S), \infty]$$
$$T \to (H_+(T), H_-(T))$$

is surjective. In particular, there is a graph presentation T with $H_+(T) = H_-(T) = h_{\mathrm{G}}(S)$.

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Proof. This is a corollary of the Strong Degree Theorem. □

If S is compact then $H_+(T) = H_-(T) = h_G(S)$ for all $T \in \mathcal{P}res(S)$. So for non-compact Markov shifts, the range of the Salama entropy function cannot distinguish non-conjugate shifts with the same Gurevic entropy.

Before relating the Salama entropies to the degrees we want to give some important implications of the surjectivity of the Salama entropy function.

First of all let $T \in \mathcal{G}(S)$. Then there is a distinguished metric d_T on T generating the topology, namely for $x \neq y$, $d_T(x, y) := 2^{-k}$ where $k \geq 0$ is the least integer such that $x[-k, k] \neq y[-k, k]$. Now let $R, T \in \mathcal{G}(S)$. Then a factor map $f : (R, d_R) \to (T, d_T)$ is uniformly continuous if and only if there is some $k \geq 0$ such that for all $x \in R$, $(fx)_0$ depends only on $x[-k, k]$. Thus, we may call such f a block map. We say f is a block conjugacy if f is a conjugacy and f, f^{-1} are both block maps. Now for a graph presentation T of S, $H_+(T)$ is the Bowen entropy of T with respect to the metric d_T on T [**S1**]. Thus, if $R, T \in \mathcal{G}(S)$ and there is a block conjugacy from R to T then $H_+(R) = H_+(T)$. In other words, $H_+(T)$ is an invariant for block conjugacy of graph presentations. Therefore the surjectivity of the Salama entropy function implies the following.

COROLLARY 1.4. *Every transitive locally compact non-compact Markov shift S with finite Gurevic entropy has uncountably many block conjugacy classes in its graph presentations.*

Secondly, as mentioned above, the surjectivity of the Salama entropy function shows that $\min\{H_+(T) \mid T \in \mathcal{G}(S)\} = h_+(S)$, since $h_+(S) = h_G(S)$ [**FF1**]. Markov shifts with the same transition entropy cannot be distinguished by their Salama entropies for graph presentations.

Finally, the surjectivity of the Salama entropy function gives an entropy constraint for SFT factors which are block factors of every graph presentation of the Markov shift: there are examples for surjective, continuous, shift commuting maps $f : S \to T$ with $h_G(S) < h_{top}(T)$, where S is a locally compact transitive Markov shift and T a SFT [**P, F2**]. However, we have the following.

COROLLARY 1.5. *Let T be a SFT and let S be a locally compact transitive Markov shift. If for every $R \in \mathcal{G}(S)$ there is a surjective block map $f : R \to T$ then $h_G(S) \geq h_{top}(T)$.*

Proof. A surjective block map is a uniformly continuous map $(R, d_R) \to (T, d_T)$, thus $h_+(R) \geq h_+(T) = h_{top}(T)$ for every $R \in \mathcal{G}(S)$. Thus $h_G(S) \geq h_{top}(T)$ by Theorem 1.3. □

On the other hand it is not hard to see that if T has a fixed point and $h_G(S) > h_{top}(T)$ then for every $R \in \mathcal{G}(S)$ there is a block map $f : R \to T$.

Let T be a graph presentation of the Markov shift S. Then obviously $H_+(T) \leq \log(\deg_+(T))$ and $H_-(T) \leq \log(\deg_-(T))$. Using $h_G(S) = H_t(T)$, we also have $h_G(S) \leq H_+(T)$ and $h_G(S) \leq H_-(T)$. On the other hand the main result of this paper is (and its proof is given in §2) the following theorem.

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STRONG DEGREE THEOREM. *Let S be a locally compact non-compact transitive Markov shift with $h_G(S) = \log\lambda < \infty$. Let $\alpha, \beta \in [\lambda, \infty]$ and $p(\alpha)$, $p(\beta) \in \mathbb{N} \cup \{\infty\}$ such that $p(\alpha)$, $p(\beta) > \lambda$ and $p(\alpha) \geq \alpha$ and $p(\beta) \geq \beta$. Then there is a graph presentation $T \in \mathcal{G}(S)$ of S such that:*

- $\deg_+(T) = p(\alpha)$ *and* $H_+(T) = \log\alpha$;
- $\deg_-(T) = p(\beta)$ *and* $H_-(T) = \log\beta$.

Thus, in particular, if $\alpha, \beta > \lambda$ then there is always a graph presentation T with $H_+(T) = \log\alpha$, $H_-(T) = \log\beta$ and $\deg_+(T) = p(\alpha)$, $\deg_-(T) = p(\beta)$ where $p(\alpha)$, $p(\beta) \in \mathbb{N} \cup \{\infty\}$ are the least elements with $p(\alpha) \geq \alpha$ and $p(\beta) \geq \beta$. Thus T has least possible degrees among the graph presentations with $H_+(T) = \log\alpha$, $H_-(T) = \log\beta$.

Finally, we briefly study the whole set of presentations $\mathcal{P}res(S)$ of a Markov shift S. We define for a presentation T of S the synchronizing part of T as

$$SYN(T) = \{y \in T \mid y[-n, n] \text{ is a synchronizing block for some } n \geq 0\}.$$

Thus, if T is a graph presentation, or more generally a synchronizing presentation, then $SYN(T) = T$. If S is compact, thus a SFT, then $SYN(T) = T$ for all $T \in \mathcal{P}res(S)$, since there is some N such that all T-blocks of length greater than or equal to N are synchronizing. However, the following two theorems show that the situation in the non-compact case is different.

THEOREM 1.6. *Let S be a locally compact non-compact transitive Markov shift. Then there is a presentation T of S such that $SYN(T) = \emptyset$.*

Proof. We may assume that S is given in graph presentation. Since S is not compact we can fix a point $w \in S$ with $w_n \neq w_m$ for all $n > m \geq 0$. By shifting w if necessary we may assume that there is a point $q \in S$ such that $q_i \neq w_0$ for all i. Since S is transitive, there is $N \in \mathbb{N}$ and a point $p \in S$ with $p_0 = w_0$ and $S^N p = p$. We define a map $\Phi : S \to (\mathbb{N} \times \{0, 1\})^{\mathbb{Z}}$. For $s \in S$ and $i \in \mathbb{Z}$ let $(\Phi s)_i = (s_i, 1)$ if $s_i = w_n$ and $s_{i-j} = w_0$ for some $n \geq 0$ and some $-N + n^2 \leq j \leq n^2 + N$ and let $(\Phi s)_i = (s_i, 0)$ otherwise. Then Φ is a conjugacy where the inverse is a 1-block map which drops the second coordinate of a symbol. Thus $T = \Phi(S)$ is a presentation of S.

Now we show $SYN(T) = \emptyset$. Let $t \in T$ and $k \in \mathbb{N}$. Let $s \in S$ such that $\Phi s = t$. Then by continuity of Φ there is some L such that $x \in S$ with $x[-L, L] = s[-L, L]$ implies $(\Phi x)[-k, k] = t[-k, k]$. By the existence of q and by transitivity of S there are $M > L$ and $0 \leq r < N$ such that there are:

- $x \in S$ with $x[-L, L] = s[-L, L]$ and $x_i \neq w_0$ for all $i \leq -M$; and
- $y \in S$ with $y[-L, L] = s[-L, L]$, $y(-\infty, -M] = p(-\infty, r]$ and $y[M, \infty) = w[0, \infty)$.

By the choice of L, $(\Phi x)[-k, k] = (\Phi y)[-k, k] = t[-k, k]$ and thus there is $t' \in (\mathbb{N} \times \{0, 1\})^{\mathbb{Z}}$ with $t'(-\infty, k] = (\Phi x)(-\infty, k]$ and $t'[-k, \infty) = (\Phi y)[-k, \infty)$. Assume there is $z \in S$ with $\Phi z = t'$. Then let $n \geq 0$ be so large that $M + n - n^2 + N \leq -M$. Then $t'_{M+n} = (\Phi y)_{M+n} = (w_n, 1)$, since $y_{M+n} = w_n$ and $y_{M+n-j} = w_0$ for some $-N + n^2 \leq j \leq n^2 + N$. Then $(\Phi z)_{M+n} = (w_n, 1)$ implies $z_{M+n} = w_n$ and there is some $-N + n^2 \leq i \leq n^2 + N$ with $z_{M+n-i} = w_0$. Therefore $(\Phi x)_{M+n-i} = t'_{M+n-i} \in$

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$\{(w_0, 0), (w_0, 1)\}$ and thus $x_{M+n-i} = w_0$; a contradiction since $M+n-i \leq -M$. Thus the assumption $t' \in \Phi(S)$ was wrong, and so $t[-k, k]$ is not synchronizing. Since k, t were arbitrary this proves $SYN(T) = \emptyset$. □

THEOREM 1.7. *Let S be a locally compact non-compact transitive Markov shift. Then there is a presentation T of S such that $\emptyset \neq SYN(T) \neq T$.*

Proof. The proof is a slight modification of the proof of Theorem 1.6. We may assume that S is given in graph presentation. Since S is not compact we can fix a periodic point $q \in S$ and a point $w \in S$ with $w_n \neq w_m$ for all $n > m \geq 0$ such that $q_i \neq w_0$ for all $i \in \mathbb{Z}$ and $q_0 w_1$ is an S-block. We define a map $\Phi : S \to (\mathbb{N} \times \{0, 1\})^{\mathbb{Z}}$. For $s \in S$ and $i \in \mathbb{Z}$ let $(\Phi s)_i = (s_i, 1)$ if $s_i = w_n$ and $s_{i-j} = w_0$ for some $n \geq 0$ and some $0 \leq j \leq n^2$ and let $(\Phi s)_i = (s_i, 0)$ otherwise. Then Φ is a conjugacy and $T = \Phi(S)$ is a presentation of S.

We show that $\Phi q \notin SYN(T)$. Let $k \in \mathbb{N}$ and fix L such that $x \in S$ with $x[-L, L] = q[-L, L]$ implies $(\Phi x)[-k, k] = (\Phi q)[-k, k]$. By the choice of q, w and the transitivity of S there are M, $M' > L$ such that there are:

- $x \in S$ with $x[-L, L] = q[-L, L]$ and $x_{-M'} = w_0$; and
- $y \in S$ with $y(-\infty, M-1] = q(-\infty, 0]$ and $y[M+1, \infty) = w[1, \infty)$.

Then by the choice of L, $(\Phi x)[-k, k] = (\Phi y)[-k, k] = (\Phi q)[-k, k]$ and thus there is $t' \in (\mathbb{N} \times \{0, 1\})^{\mathbb{Z}}$ with $t'(-\infty, k] = (\Phi x)(-\infty, k]$ and $t'[-k, \infty) = (\Phi y)[-k, \infty)$. Assume there is $z \in S$ with $\Phi z = t'$. Then for $n \geq 0$ so large that $M + n - n^2 \leq -M'$ we get $t'_{M+n} = (\Phi y)_{M+n} = (w_n, 0)$, since $y_{M+n} = w_n$ and $y_{M+n-j} \neq w_0$ for all $0 \leq j \leq n^2$. Then $(\Phi z)_{M+n} = (w_n, 0)$ implies $z_{M+n} = w_n$ and $z_{M+n-i} \neq w_0$ for all $0 \leq i \leq n^2$. Therefore $(\Phi x)_{M+n-i} = t'_{M+n-i} \notin \{(w_0, 0), (w_0, 1)\}$ for all $0 \leq i \leq n^2$ and thus, since $M + n - n^2 \leq -M'$, $x_{-M'} \neq w_0$; a contradiction. Thus the assumption $t' \in \Phi(S)$ was wrong, and so $t[-k, k]$ is not synchronizing.

We show that $\Phi w \in SYN(T)$. Let x, $y \in S$ with $(\Phi x)_0 = (\Phi y)_0 = (\Phi w)_0 = (w_0, 1)$. Since $x_0 = y_0 = w_0$ there is $z \in S$ with $z(-\infty, 0] = x(-\infty, 0]$ and $z[0, \infty) = y[0, \infty)$. Since $(\Phi z)_i$ is determined by $z(-\infty, i]$ for all i we have $(\Phi z)(-\infty, 0] = (\Phi x)(-\infty, 0]$. Now let $i > 0$. If $z_i = w_n$ for some $n \geq 0$ then $(\Phi z)_i = (\Phi y)_i$ if $i - n^2 > 0$ and if $i - n^2 \leq 0$ then, since $z_0 = w_0$, $(\Phi z)_i = (\Phi y)_i = (w_n, 1)$. Thus $(\Phi z)[0, \infty) = (\Phi y)[0, \infty)$. This shows $\Phi w \in SYN(T)$. □

2. *Proof of the Strong Degree Theorem*

This section is devoted to the proof of the Strong Degree Theorem. Given a Markov shift S we shall construct a graph presentation T of S, by using the classical marker set approach, with the desired degrees and Salama entropies. The construction of the conjugacy is based on three concepts. The first basic tool is a new marker set which not only marks every point at least once, but also respects the Markov property. The second is a coding of marked rays into rays of Markov shifts which have a tree-like graph. The third, and the most important, are the combinatorial spindle lemmata, which are used to code short and long marked intervals into finite graphs which look like spindles.

The organization is as follows. In §2.1 we present the new marker set, show how it respects the Markov structure and show how to get the marker set with certain constraints on the lengths of the marked intervals. In §2.2 we present the construction of an embedding

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of the set $\{x \in S \mid x_0 = e\}$ into a Markov shift with a tree-like graph. This will be used to code marked rays. In §2.3, which is purely combinatorial, we introduce spindles and show how to choose spindles of certain pre-assigned lengths, numbers of maximal paths and degrees. These will be used to code marked intervals. In §2.4 we use these tools to define the desired conjugacy and finally in §2.5 we check the asserted properties.

2.1. *The marker set.* Throughout this section let S be a fixed transitive locally compact Markov shift with $0 < h_G(S) < \infty$ given in graph presentation with $S \subset \mathbb{N}^{\mathbb{Z}}$ and edge set either $\mathbb{N}$ or $\{1, 2, \ldots, N\}$ for some $N \in \mathbb{N}$. (So here we allow S to be compact.)

Definition 2.1. Let $L : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function. The marker set defined by L is the set $M := \{x \in S \mid x_0 \le x_i \text{ for all } -L(x_0) \le i \le L(x_0)\}$.

The marker set M is closed and open and if $x \in S$ with $x_0 = 1$ then $x \in M$. We say $x \in S$ is *marked at time n* if $S^n x \in M$. If x is not marked at time 0 then there is some $-L(x_0) \le i \le L(x_0)$ such that $x_i < x_0$. Again, if x is not marked at time i then there is some $-L(x_i) + i \le j \le i + L(x_i)$ such that $x_j < x_i$. Repeating this argument at most x_0 times, we find n such that x is marked at time n. Thus $\bigcup_{n\in\mathbb{Z}} S^n M = S$, which means that every point is marked at least once. In particular, if S is compact, then there is some N such that for every point x and every $i \in \mathbb{Z}$ there is some j with $i \le j \le i + N$ such that x is marked at time j.

For $n < m$ we say that $[n, m]$ is a *marked interval* for x if x is marked at time n and at time m, but not marked at times k with $n < k < m$. We say $[n, \infty)$ is a *marked ray* for x, if x is marked at n and x is not marked at $m > n$. Similarly, for $(-\infty, n]$. A marked interval $[m, m + n]$ is called a *long marked interval* if and only if $n > \max(L(x_m), L(x_{m+n}))$ and a *short marked interval* otherwise. Observe that if $[m, m + n]$ is a short marked interval for x, then $x_m = x_{m+n}$ and $x_{m+i} > x_m$ for all $0 < i < n$.

We shall use M to determine the coding lengths for the map $\Phi : S \to \Phi(S)$ we are going to construct. That is, we shall define Φ such that whenever $x \in S$ and $[n, m]$ is a marked interval for x, then $(\Phi x)[n, m]$ is determined by $x[n, m]$, if $[n, \infty)$ is a marked ray then $(\Phi x)[n, \infty)$ is determined by $x[n, \infty)$ and if $(-\infty, n]$ is a marked ray then $(\Phi x)(-\infty, n]$ is determined by $x(-\infty, n]$.

The next lemma says that the not marked coordinates in a marked interval will also be not marked for any other point that sees the same block.

LEMMA 2.2. *Let $x, y \in S$, $x[0, n] = y[0, n]$, $[0, n]$ a marked interval for x. Then for $0 < i < n$ the coordinate i is not marked for y.*

Proof. Fix $0 < i < n$. Assume y is marked at i. Then $y_{i+j} \ge y_i$ for all $-L(y_i) \le j \le L(y_i)$. Since x is not marked at i, we get $[i - L(y_i), i + L(y_i)]$ is not contained in $[0, n]$. If $i - L(y_i) < 0$ (the other case is symmetric), then $y_0 \ge y_i$ and thus by the monotonicity of L, $L(y_0) \ge L(y_i)$. Thus $i - L(y_i) \ge i - L(y_0) > -L(y_0)$. Thus, since x is marked at 0, $x_{i+j} \ge x_0 = y_0 \ge y_i = x_i$ for all $-L(y_i) \le j \le 0$. Since x is not marked at i, thus $i + L(y_i) > n$. Thus, since y is marked at i, $y_i \le y_n$ and, therefore, $L(y_i) \le L(y_n)$ and thus $i + L(y_i) < n + L(y_n)$. Since x is marked at n, we have $x_n \le x_{n+j}$ for all

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$0 \le j \le L(x_n)$. Thus, we conclude that $x_{i+j} \ge x_i$ for all $-L(x_i) \le j \le L(x_i)$ and thus x is marked at i; a contradiction. □

For $x \in \mathbb{N}^{\mathbb{Z}}$ let $x_- := x(-\infty, 0]$ and $x_+ := x[0, \infty)$.

LEMMA 2.3. *Let x, $y \in M$ with $x_0 = y_0$. Let $z \in \mathbb{N}^{\mathbb{Z}}$ with $z_- = x_-$ and $z_+ = y_+$. Then $z \in M$.*

Proof. We have $z \in S$, since S is given in graph presentation. Since x is marked at 0, we have for all $-L(x_0) \le i < 0$ that $x_i \ge x_0$ and thus, $z_i \ge z_0$. Since y is marked at 0, we have $y_i \ge y_0$ for all $0 < i \le L(y_0)$, and thus z is marked at 0. □

For $s \in S$ let $M(s) := \{n \in \mathbb{Z} \mid s \text{ is marked at time } n\}$.

LEMMA 2.4. *Let s, $t \in M$ such that $s_0 = t_0$. Let $x \in S$ with $x_- = s_-$ and $x_+ = t_+$. Then $M(x) = (M(s) \cap (-\infty, 0]) \cup (M(t) \cap [0, \infty))$.*

Proof. By Lemma 2.3, x is marked at time 0. Now let $i > 0$. If $i - L(x_i) \ge 0$ then $i \in M(x)$ if and only if $i \in M(t)$. If $-L(x_0) \le i - L(x_i) < 0$ then $x_0 \le x_j$ and $t_0 \le t_j$ for all $i - L(x_i) \le j \le 0$, since x, $t \in M$, and thus $i \in M(x)$ if and only if $i \in M(t)$. If $i - L(x_i) < -L(x_0)$ then, by monotonicity of L, $x_i > x_0$ and thus $i \notin M(x)$ and $i \notin M(t)$. Thus we obtain $M(x) \cap [0, \infty) = M(t) \cap [0, \infty)$. The argument for $M(x) \cap (-\infty, 0] = M(s) \cap (-\infty, 0]$ is symmetric. □

Thus these last three lemmata say that using this special marker set will have the advantage that the symbols assigned at the marked positions will be synchronizing. To provide some control over the number of marked intervals of certain lengths we use the following marker lemmata.

SIMPLE MARKER LEMMA 2.5. *Let S be a transitive locally compact Markov shift in graph presentation, let $p \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ be such that $h_G(S) < \log \lambda < \log p$. Let $(n_k)_{k \in \mathbb{N}}$ be a strictly increasing sequence of positive integers. Then there is a strictly increasing function $L : \mathbb{N} \to \mathbb{N}$ such that for all edges e, f, $k := \max(e, f)$ and $m > L(k)$ it holds that*

$$\#\{x[0, m] \mid x \in S, x_0 = e \text{ and } x_m = f\} < \lambda^{m - 7 \log_p m - 6n_k}.$$

Proof. Let $\nu < \mu < \lambda$ such that $h_G(S) < \log \nu$. For each i, j choose numbers $N(i, j)$ with $\#\{x[0, n] \mid x \in S, x_0 = i, x_n = j\} \le N(i, j)\nu^n$ for all n. Let $N(k) = \max\{N(i, j) \mid i, j \le k\}$. Since $\mu < \lambda$ there is some $M \ge p$ such that for all $m \ge M$ it holds that $m^7 \mu^m \le \lambda^m$. Now we choose a strictly increasing function L such that $L(k) > M + 6n_k$ and $N(k)\nu^{L(k)} \le \mu^{L(k) - 6n_k}$ for all k. Then for $m > L(k)$ it holds that $N(k)\nu^m \le \mu^{m - 6n_k}$. Thus for edges e, f in S and for $m > L(k)$, $k = \max(e, f)$, it holds that $m - 6n_k > M \ge p$ and thus

$$\begin{aligned} \#\{x[0, m] \mid x \in S, x_0 = e, x_m = f\} &\le N(e, f)\nu^m \le N(k)\nu^m \le \mu^{m - 6n_k} \\ &\le m^{-7} \lambda^{m - 6n_k} = p^{-7 \log_p m} \lambda^{m - 6n_k} \\ &< \lambda^{m - 7 \log_p m - 6n_k} \end{aligned}$$

since $p > \lambda$. □

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STRONG MARKER LEMMA 2.6. *Let S be a transitive locally compact Markov shift in graph presentation with $1 < h_{\mathrm{G}}(S) < \infty$. Let λ_n be a sequence with $h_{\mathrm{G}}(S) < \log \lambda_n$ and $\lim_{n\to\infty} \log \lambda_n = h_{\mathrm{G}}(S)$. Let $p \in \mathbb{N}$ with $p > \lambda_n$ for all n. Let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of positive integers. Then there is a strictly increasing function $L : \mathbb{N} \to \mathbb{N}$ such that for all edges e, f, $k = \max(e, f)$ it holds for all $m > L(i)$ and $i \geq k$ that*

$$\#\{x[0, m] \mid x \in S, x_0 = e \text{ and } x_m = f\} < (\lambda_i)^{m-7\log_p m-6n_k}.$$

Proof. Since $h_{\mathrm{G}}(S) < \log \lambda_i < \log p$, by the Simple Marker Lemma 2.5, there is for each $i \geq 1$ an increasing function $L_i : \mathbb{N} \to \mathbb{N}$ such that for all edges e, f, $k = \max(e, f)$ and $m > L_i(k)$ it holds that

$$\#\{x[0, m] \mid x \in S, x_0 = e \text{ and } x_m = f\} < (\lambda_i)^{m-7\log_p m-6n_k}.$$

For $L : \mathbb{N} \to \mathbb{N}$ choose a strictly increasing function such that $L(i) \geq L_i(i)$ for all i. Then for edges e, f, $k = \max(e, f)$ and for $i \geq k$ and $m > L(i)$ it holds that $m > L(i) \geq L_i(i) \geq L_i(k)$ and thus $\#\{x[0, m] \mid x \in S, x_0 = e \text{ and } x_m = f\} < (\lambda_i)^{m-7\log_p m-6n_k}$ as desired. □

2.2. *The trees.* We fix some notation. Let G be a finite or countable directed graph. Let $edge(G)$ denote the edge set of G. A vertex v of G is a *root* of G if v has no incoming edges. A vertex v of G is an *end* of G if v has no outgoing edges. We denote the set of roots of G by $root(G)$ and the set of ends by $end(G)$. A path in G is *maximal* if it is not a proper subpath of some path in G. The set of maximal paths of G is denoted by $\mathcal{M}P(G)$. In a finite graph every maximal path starts in a root and ends in an end.

A graph G is an *outgoing tree* if it has a unique root and for every vertex v which is not the root, there is a unique path from the root to v. Let v be a vertex of an outgoing tree G. We say v is a vertex of height h if the unique path starting at the root and ending in v has length h. So the root is the vertex at height 0. We say G is a tree of height h if every maximal path in G has length h. An *outgoing p-tree* is an infinite tree which has at each of its vertices outdegree exactly p. So the outgoing p-tree has p^n vertices at height n. The *outgoing p-tree of height h* is the finite subgraph of the outgoing p-tree consisting of all vertices of height less than or equal to h (and all the edges having their initial and terminal vertex in this subset of vertices). Thus, the outgoing p-tree of height h has p^h ends and p^h maximal paths each of these of length h.

We use the following two operations to create new graphs from a sample of graphs.

Let G be a graph. By the reversed graph of G, denoted by $\mathcal{R}(G)$, we mean the graph which has the same vertex set as G and the direction of the edges is reversed. So the roots in G are the ends in $\mathcal{R}(G)$ and the ends in G are the roots in $\mathcal{R}(G)$.

As an application of this we define that a graph G is an *incoming tree* if the reversed graph $\mathcal{R}(G)$ is an outgoing tree. Then, by definition, a vertex v in G is at height h if v in $\mathcal{R}(G)$ is at height h.

The next technique is gluing graphs. Let G, H be finite graphs such that G has an end and H has a unique root r. By GH we denote the finite graph obtained by gluing at each end v of G a copy H_v of H by identifying v with the root r_v of H_v. That is, GH is the

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disjoint union of G and the copies H_v of H, $v \in end(G)$, where the end v in G and the root r_v in H_v are identified.

Gluing graphs together is associative: $(GH)K = G(HK)$.

Let G denote a graph with an end and a unique root. Then let $G^1 := G$ and $G^{n+1} := G^n G$ for $n \geq 1$.

Let E denote the graph consisting of a single edge, that is E has two vertices, a root and an end and a unique edge from the root to the end. Let G be the outgoing p-tree of height d. Then the following finite tree will be used in many of the constructions. Let

$$T(c, p, d) := E^c G E^c.$$

Thus $T(c, p, d)$ is an outgoing tree of height $r = d + 2c$ with $R = p^d$ maximal path. The outdegree of $T(c, p, d)$ is p and the indegree is 1.

Finally we use $T(c, p, d)$ to construct an infinite graph as follows. Let

$$T^{c,p,d} = T_-^{c,p,d} T_+^{c,p,d} \quad \text{where } T_+^{c,p,d} := T(c, p, d)^\infty \text{ and } T_-^{c,p,d} := \mathcal{R}(E^{c-1} G E^c T_+).$$

We denote the unique edge in $T^{c,p,d}$ which begins at the root of T_+ by e^*.

The following lemma provides us with maps F_e which we shall use to map marked rays.

TREE LEMMA 2.7. *Let $S \subset \mathbb{N}^{\mathbb{Z}}$ be a transitive locally compact Markov shift given in graph presentation. Let $c, p, d \geq 2$ be integers and let $r = d+2c$. Then there is a sequence $1 < n_1 < n_2 < \cdots$ such that $n_k = 0 \bmod 3r$ for each k and for each edge e of S there is a subgraph T_e of $T^{c,p,d}$ and a bijection $F_e : \{x \in S \mid x_0 = e\} \to \{t \in S_{T_e} \mid t_0 = e^*\}$, such that for any $x, x' \in S$ with $x_0 = x'_0$ and $k \geq x_0$ it holds that*

$$x[0, k] = x'[0, k] \iff (F_{x_0} x)[0, n_k - 1] = (F_{x_0} x')[0, n_k - 1]$$
$$x[-k, 0] = x'[-k, 0] \iff (F_{x_0} x)[-n_k + 1, 0] = (F_{x_0} x')[-n_k + 1, 0].$$

Proof. Let $R = p^d$, $T_+ = T_+^{c,p,d}$ and $T_- = T_-^{c,p,d}$. For each $k \geq 1$ fix $m_k > 1$ such that $\#\{x[-k, k] \mid x \in S, x_0 \leq k\} < R^{m_k}$. Then let $n_1 := 3rm_1$ and for $k \geq 1$ let $n_{k+1} := 3rm_{k+1} + n_k$.

We now fix e and construct the subgraph T_e inductively as follows.

Beginning of induction. Since the subtree of height n_e of T_+ has more than $R^{m_e} > \#\{x[0, e] | x \in S, x_0 = e\}$ ends, we can fix a subtree $T_{e,e,+}$ of T_+ of height n_e with $\#\{x[0, e] \mid x \in S, x_0 = e\}$ ends and a bijection $F_{e,e,+} : \{x[0, e] \mid x \in S, x_0 = e\} \to \mathcal{MP}(T_{e,e,+})$. Similarly, we fix a subtree $T_{e,e,-}$ of T_- of height $n_e - 1$ with $\#\{x[-e, 0] \mid x \in S, x_0 = e\}$ roots and a bijection $F_{e,e,-} : \{x[-e, 0] \mid x \in S, x_0 = e\} \to \mathcal{MP}(T_{e,e,-})$.

Step of induction. Let $k \geq e$. Given an outgoing tree $T_{e,k,+}$ of height n_k and a bijection $F_{e,k,+} : \{x[0, k] \mid x \in S, x_0 = e\} \to \mathcal{MP}(T_{e,k,+})$ we construct $T_{e,k+1,+}$ and $F_{e,k+1,+}$ as follows. By the choice of n_{k+1} for each path $ex_1 \dots x_k$ in S we can glue at the terminal vertex of the path $F_{e,k,+}(ex_1 \dots x_k)$ a subtree of T_+ of height $n_{k+1} - n_k$ which has $\#\{y[0, k+1] \mid y \in S, y[0, k] = ex_1 \dots x_k\}$ ends. Doing this for all paths $ex_1 \dots x_k$ in S we obtain a tree $T_{e,k+1,+}$ of height n_{k+1} with $\#\{y[0, k+1] \mid y \in S, y_0 = e\}$ ends and there is a bijection $F_{e,k+1,+} : \{x[0, k+1] \mid x \in S, x_0 = e\} \to \mathcal{MP}(T_{e,k+1,+})$ such that $F_{e,k+1,+}(x[0, k+1]) = a_1 \dots a_{n_{k+1}}$ implies $F_{e,k,+}(x[0, k]) = a_1 \dots a_{n_k}$.

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Similarly, given an incoming tree $T_{e,k,-}$ of height $n_k - 1$ and a bijection $F_{e,k,-} : \{x[-k,0] \mid x \in S,\ x_0 = e\} \to \mathcal{M}P(T_{e,k,-})$, for each path $x_{-k}\dots x_{-1}e$ in S we glue at the initial vertex of the path $F_{e,k,-}(x_{-k}\dots x_{-1}e)$ a subtree of T_- of height $n_{k+1} - n_k$ which has $\#\{y[-k-1,0] \mid y \in S,\ y[-k,0] = x_{-k}\dots x_{-1}e\}$ roots. Doing this for all paths $x_{-k}\dots x_{-1}e$ in S we obtain a tree $T_{e,k+1,-}$ of height $n_{k+1} - 1$ with $\#\{y[-k-1,0] \mid y \in S,\ y_0 = e\}$ roots and there is a bijection $F_{e,k+1,-} : \{x[-k-1,0] \mid x \in S,\ x_0 = e\} \to \mathcal{M}P(T_{e,k+1,-})$ such that $F_{e,k+1,-}(x[-k-1,0]) = a_{-n_{k+1}+1}\dots a_1$ implies $F_{e,k,-}(x[-k,0]) = a_{-n_k+1}\dots a_1$.

Then let $T_{e,k} = T_{e,k,-}T_{e,k,+}$. Thus the graph $T_{e,k}$ is extended to a graph $T_{e,k+1}$ for each $k \geq e$. So we obtain in the limit a graph T_e which has all maximal paths of bi-infinite lengths. We define the bijection $F_e : \{x \in S \mid x_0 = e\} \to \{t \in S_{T_e} \mid t_0 = e^*\}$ by $(F_e x)[0, n_k - 1] = F_{e,k,+}(x[0,k])$ and $(F_e x)[-n_k - 1, 0] = F_{e,k,-}(x[-k,0]), k \geq e$. □

2.3. *The spindles.* A finite graph G is a *spindle* if it is a graph without loops, with a unique root, a unique end and such that all maximal paths have the same first edge and have the same last edge. Thus, in a spindle there is a unique edge starting from the root and a unique edge ending at the end. A spindle G has degree p if G has in- and outdegree less than or equal to p at each vertex. If all maximal paths in a spindle have the same length n, we say the spindle has length n. For a spindle G we denote by $\mathcal{P}_{+,h}(G)$ the set of paths of length h in G starting at the root, and by $\mathcal{P}_{-,h}(G)$ the set of paths of length h in G ending at the end.

Now we show how to choose spindles of degree p with pre-assigned lengths and number of maximal paths.

SPINDLE LEMMA 2.8. *Let $1 < \lambda < \infty$ and let $p \in \mathbb{N}$ with $p > \lambda$. Let $c, d \in \mathbb{N}$ such that $\lambda^{d+2c} < p^d$. Let $r = d + 2c$ and $R = p^d$. Then there is an integer $Q = Q(\lambda, c, p, d)$ such that for all $q \geq Q$ and all $1 \leq n \leq \lambda^q$ there is a spindle H of degree less than or equal to p and length q with n maximal paths and such that for all $1 \leq m \leq n$ it holds that*

$$\#\mathcal{P}_{+,m}(H) < R^{m/r} \quad \text{and} \quad \#\mathcal{P}_{-,m}(H) < R^{m/r}.$$

Proof. Since $\lambda^r < R$ there is an integer $N > 3$ such that $\lambda^r < R^{(1-3/N)}$. Choose $i_0 \geq 3N$ so large that for all $i \geq i_0$ it holds that $((N-2)i + 1)R \leq R^i$. Let $Q = (i_0 + 3)Nr$. Let $q \geq Q$ and $1 \leq n \leq \lambda^q$. Then write $q = iNr + j$ with $2Nr \leq j < 3Nr$. Since $q \geq Q$, we have that $i \geq i_0 \geq 3N > j/r$. Since

$$n \leq \lambda^q < R^{(1-3/N)(q/r)} \leq R^{(N-3)i + j/r} \leq R^{(N-3)i+i} = R^{(N-2)i}$$

we can write $n = \sum_{s=0}^{(N-2)i-1} x_s R^s$ for suitable $x_s \in \{0, 1, 2, \dots, R-1\}$. Since $\sum_{s=0}^{(N-2)i-1} x_s < R(N-2)i \leq R^i$ and $T(c,p,d)^i$ is a tree of height ri with R^i ends we can choose a subtree T_+ of $T(c,p,d)^i$ of height ri having $\sum x_s$ ends. Let $T_- = \mathcal{R}(T_+)$ be the reversed tree. Fix a bijection $g : end(T_+) \to root(T_-)$. Recall that E is the graph consisting of a single edge. Let F be the graph with two vertices, say v and w, and p edges from v to w. Then let $M := E^c F^d E^c$. Thus M is a spindle of degree p and length $r = d + 2c$ with $R = p^d$ maximal paths.

For each $x_s \neq 0$ take x_s copies of the graph $M^s E^{(N-2)ir - sr}$. In this way we get $\sum x_s$ spindles of length $(N-2)ir$, which we denote by M_v, $v \in end(T_+)$. Now identify

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$v \in end(T_+)$ with the root of M_v and identify the end of M_v with $g(v) \in root(T_-)$, for each $v \in end(T_+)$. Since M^s is a spindle with R^s maximal paths, this gives a spindle G of length iNr with $n = \sum_{s=0}^{(N-2)i-1} x_s R^s$ maximal paths. Thus $H := E^r G E^{j-r}$ is a spindle of degree p of length q with n maximal paths. Now let $1 \leq m \leq n$ and write $m = ar + b$ with $0 \leq b < r$. Then $\#\mathcal{P}_{+,m}(H) \leq \#\mathcal{P}_{+,(a+1)r}(H) \leq R^a \leq R^{m/r}$. Since $j - r \geq r$ a symmetric argument shows that H is the desired spindle. □

We actually need to have spindles with pre-assigned numbers of maximal paths of different lengths.

MAIN SPINDLE LEMMA 2.9. *Let $1 < \lambda < \infty$ and let $p \in \mathbb{N}$ with $p > \lambda$. Let $c, d \geq 2$ be integers such that $\lambda^{d+2c} < p^d$. Let $r = d + 2c$ and $R = p^d$. Then there is some integer $M = M(\lambda, c, p, d)$ such that for each $N \geq M$ and for each $(x_M, \ldots, x_N) \in \mathbb{Z}^{N-M+1}$ such that $0 \leq x_i \leq \lambda^i$ for all i and $\sum_{i=M}^N x_i > 0$ there is a spindle H of degree less than or equal to p with $\sum_{i=M}^N x_i$ maximal paths, for each $M \leq n \leq N$ the number of maximal path of length n is exactly x_n and for each $M \leq m \leq N$ it holds that*

$$\#\mathcal{P}_{+,m}(H) < R^{m/r} \quad \text{and} \quad \#\mathcal{P}_{-,m}(H) < R^{m/r}.$$

Proof. Since $p, d \geq 2$ we can fix a subtree T_0 of height r of $T(c, p, d)$ with $p^d - 1 = R - 1$ ends. Let $T_+ = E^r(T_0)^\infty$. For $a \in \mathbb{R}$ let $[a]$ denote the least integer greater than a.

Since $\lambda^r < R$, we can choose μ such that $\lambda^r < \mu^r < R$. Let $Q_1 \geq 1$ such that $q \geq Q_1$ implies $\lambda^q < \mu^{g(q)}$, where $g(q) = q - 2r[\log_{R-1} q] - 2r$. Let $Q_2 = Q(\mu, c, p, d)$ from the Spindle Lemma. Then there is $M = M(\lambda, c, p, d) \geq Q_1$ such that $q \geq M$ implies $g(q) \geq Q_2$.

Now we want to glue spindles at the vertices at heights rh, for some $h \geq 1$, of T_+, since those vertices have in- and outdegree 1. To describe this we use a map ρ on the vertices of T_+ which we define as follows.

For $h \geq 1$ let V_h be the vertices in T_+ at height h. Then $\#V_{rh} = (R-1)^{h-1}$. Fix a bijection $\rho : \bigcup_{h \geq 1} V_{rh} \to \mathbb{N}$ such that $\rho(v) < \rho(v')$ if $v \in V_{rh}$, $v' \in V_{rh'}$ and $h < h'$. Extend ρ to a map on the vertex set of T_+ by defining $\rho(v) = 0$ if $v \notin \bigcup_{h \geq 1} V_{rh}$. Observe that $\rho(v) \neq 0$ then v is a vertex with degree 1, since $c \geq 2$.

Let $N \geq M$ and $(x_M, \ldots, x_N) \in \mathbb{Z}^{N-M+1}$ be such that $0 \leq x_i \leq \lambda^i$, for all i and $\sum_{i=M}^N x_i > 0$. Let V be the finite subset of vertices v in T_+ with $x_{\rho(v)} \neq 0$. Let G_+ be the maximal finite subtree of T_+ which contains the root of T_+ and has its ends in the set V. Thus, G_+ is obtained from T_+ by deleting all vertices $w \notin V$ of T_+ such that there is no path in T_+ from w to a vertex $v \in V$. Let $G_- := \mathcal{R}(G_+)$, the reversed graph.

Let $M \leq n \leq N$ be such that $x_n \neq 0$. Then $x_n \leq \lambda^n < \mu^{g(n)}$. Thus by the Spindle Lemma and the choice of M we can choose a spindle $G(x_n)$ of length $g(n) = n - 2r[\log_{R-1} n] - 2r$ which has exactly x_n maximal paths and such that for each $1 \leq m \leq g(n)$ we have $\#\mathcal{P}_{+,m}(G(x_n)) < R^{m/r}$ and also $\#\mathcal{P}_{-,m}(G(x_n)) < R^{m/r}$. Let $rh(n)$ be the height of the vertex v_n in G_+ with $\rho(v_n) = n$. Since in T_+ at height $rh(n)$ there are $(R-1)^{h(n)-1}$ vertices, $h(n) \leq [\log_{R-1} n] + 1$. Let $k(n) := r[\log_{R-1} n] + r - rh(n)$ and let $G'(x_n) := E^{k(n)} G(x_n) E^{k(n)}$. Then $G'(x_n)$ is a spindle of length $n - 2rh(n)$ which has exactly x_n maximal paths. Now identify the root of $G'(x_n)$ with the vertex v_n in G_+ and identify the end of $G'(x_n)$ with the vertex v_n in G_-. Do this for all $M \leq n \leq N$

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such that $x_n \neq 0$. We obtain a spindle H with degree p, since the vertices in T_+ at heights $r \cdot n$ have degree 1 and $p \geq 2$. Since the vertex v_n is at height $rh(n)$, the maximal paths in H which use an edge from $G'(x_n)$ all have length n, and thus, for each $M \leq n \leq N$, H has x_n maximal paths of length n. Note further that for a vertex v in H which is a vertex in G_+ of height rh there are at most R paths of length r starting at v, namely at most $R-1$ of those are in G_+ and at most one is in $G'(x_n)$ where n is such that $v = v_n$. Thus $\#\mathcal{P}_{+,m}(H) < R^i \leq R^{m/r}$ where $m = ir + j$ with $0 \leq j < r$. The other statement is proved analogously. Thus H is a spindle with the desired properties. □

2.4. *The construction of the conjugacy.* We use the tools provided in §§2.1–2.3 to construct the conjugacy. Now it is essential that S is a non-compact Markov shift, since for a compact Markov shift S and a presentation T of S it always holds that $H_+(T) = H_-(T) = h_{\mathrm{G}}(S)$.

Since we want to apply the Main Spindle Lemma we need the graph presentation we start with not to have too many short loops at a vertex. The following lemma shows that we can recode a given graph presentation to a new one which suits our purposes.

LEMMA 2.10. *Let S be any Markov shift. Let $\kappa \in \mathbb{N}$. Then there is a graph presentation of S, say G, such that for each edge e of G it holds that*

$$\#\{w \mid ewe \text{ is a path in } G, 0 \leq |w| \leq \kappa, w \text{ does not see } e\} \leq 1.$$

Proof. Begin with any graph presentation G' of S. Any higher block system of G' is again a graph presentation for S. Take a higher block system G such that the edges in G are blocks of G' of the form $x_{-n} \ldots x_n$, where $n > \kappa$. Let e be an edge in G, say e corresponds to the path $x_{-n} \ldots x_n$ in G'. So, if we have ewe in G with $0 \leq |w| \leq \kappa$, then this means $x_{-n} \ldots x_{n-|w|-1} = x_{-n+|w|+1} \ldots x_n$, thus for $e' = x_0$ and $w' = x_1 \ldots x_{|w|}$ we get that the block $x_{-n} \ldots x_n$ presenting e is the subblock $p[-n, n]$ of the periodic point $p \in S$ with $S^{|w|+1}p = p$ and $p[0, |w|] = e'w'$. Since $w_i = p[-n+i, i+n]$, $0 \leq i \leq |w|$, we get thus, if w does not see e then the only possibility is that $|w| + 1 =$ the least period of p. □

From now on and for the rest of the paper let S be a locally compact non-compact transitive Markov shift with $h_{\mathrm{G}}(S) = \log \lambda < \infty$. Let $\alpha, \beta \in [\lambda, \infty]$ and $p(\alpha)$, $p(\beta) \in \mathbb{N} \cup \{\infty\}$ such that $p(\alpha), p(\beta) > \lambda$ and $p(\alpha) \geq \alpha$ and $p(\beta) \geq \beta$.

To prove the Strong Degree Theorem it suffices to construct a conjugacy $\Phi : S \to \Phi(S)$ such that:

- every symbol of $\Phi(S)$ is synchronizing;
- $\deg_+(\Phi(S)) = p(\alpha)$ and $H_+(\Phi(S)) = \log \alpha$;
- $\deg_-(\Phi(S)) = p(\beta)$ and $H_-(\Phi(S)) = \log \beta$;

since then we can define a graph G such that the vertices of G are the symbols of $\Phi(S)$ and there is an edge named (e, f) from vertex e to vertex f if and only if $ef \in \mathcal{B}_2(\Phi(S))$. Then S_{G} is conjugate to $\Phi(S)$ with the same degrees and Salama entropies and is thus the desired graph presentation of S.

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General preparations for constructing the conjugacy. Let p be the least integer greater than λ. Since S is not compact, S is not trivial and thus $\lambda > 1$. Thus, $p(\alpha) \geq p \geq 2$ and $p(\beta) \geq p \geq 2$. Let λ_n be a sequence with

(1) $\lim_{n\to\infty} \log \lambda_n = \log \lambda = h_{\mathrm{G}}(S)$ and $\lambda < \lambda_{n+1} < \lambda_n < p$ for all n.

Fix integers $c_i, d_i \geq 3$ such that for all $i \geq 1$ it holds that

(2) $(\lambda_i)^{d_i+2c_i} < p^{d_i} < (\lambda_{i-1})^{d_i+2c_i}$.

Choose an increasing sequence $M(\lambda_i, c_i, p, d_i)$ from the Main Spindle Lemma 2.9 such that for all $i \geq 1$ it holds that

(3) $n \geq M(\lambda_i, c_i, p, d_i) + 2i + 4 \Rightarrow \lambda^n < (\lambda_i)^{n-2i-4}$.

Then fix an integer $\kappa \geq M(\lambda_1, c_1, p, d_1) + 5$. Then by Lemma 2.10 we may assume that S is in graph presentation such that for each edge e of S it holds that

(4) $\#\{w \mid ewe \text{ is a path in } S, 0 \leq |w| \leq \kappa, w \text{ does not see } e\} \leq 1$.

By renaming the edges of S we may assume that the edge set of S is $\mathbb{N}$.

Choosing trees for coding marked rays. Now fix integers

(5) $c, d \geq 3$ so that $p^d < \lambda^{d+2c}$ and let $r := d + 2c$, $R = p^d$.

For each $e \in \mathbb{N}$ fix a copy $T^{c,p,d,e}$ of $T^{c,p,d}$ and for each edge f in $T^{c,p,d}$ denote the corresponding edge in the copy $T^{c,p,d,e}$ by (f, e). Thus, in particular, the unique edge corresponding to e^* is (e^*, e). Since $p \geq 2$, we can choose, by the Tree Lemma 2.7, a sequence

(6) $1 < n_1 < n_2 < \cdots$ with $n_k = 0 \bmod 3r$ for each k

and for each edge e of S a subgraph T_e of $T^{c,p,d,e}$ and bijections $F_e : \{x \in S \mid x_0 = e\} \to \{t \in S_{T_e} \mid t_0 = (e^*, e)\}$ such that for any $x, x' \in S$ with $x_0 = x'_0$ and $k \geq x_0$ it holds that

$$x[0,k] = x'[0,k] \iff (F_{x_0}x)[0, n_k - 1] = (F_{x_0}x')[0, n_k - 1] \tag{7a}$$

$$x[-k,0] = x'[-k,0] \iff (F_{x_0}x)[-n_k + 1, 0] = (F_{x_0}x')[-n_k + 1, 0]. \tag{7b}$$

Note that the edges of the tree T_e are of the form (f, e), where f is an edge in $T^{c,p,d}$.

Choosing the marker set. We shall define a weight function on the edges. For each edge f let

$$w_-(f) = \min\{n \geq 1 \mid x_0 = 1, x_n = f \text{ for some } x \in S\}$$
$$w_+(f) = \min\{n \geq 1 \mid x_0 = f, x_n = 1 \text{ for some } x \in S\}$$
$$w(f) = w_+(f) + w_-(f).$$

By (1), (6) and the Strong Marker Lemma 2.6 we can choose a strictly increasing function $L : \mathbb{N} \to \mathbb{N}$ satisfying:

(8) $L(i) \geq \kappa + w(i)$ and $m > L(i)$ implies $m - 7\log_p m - 6n_i > M(\lambda_i, c_i, p, d_i)$ for all i;

(9) $\#\{x[0,m] \mid x \in S, x_0 = e, x_m = f\} < (\lambda_i)^{m-7\log_p m - 6n_k}$ for all $m > L(i)$, $i \geq k := \max(e, f)$.

Define the marker set $M := \{x \in S \mid x_0 \leq x_i \text{ for all } -L(x_0) \leq i \leq L(x_0)\}$. Recall that we call a marked interval $[0, m]$ of x long if and only if $m > \max(L(x_0), L(x_m))$.

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Choosing spindles for coding short marked intervals. For each edge e let

$$H(e) := \{x[0,m] \mid x \in S, x_0 = x_m = e, m > \kappa + 1 \text{ and } [0,m] \text{ is a short marked interval for } x\}$$

and let $H_m(e) := \{w \in H(e) \mid |w| = m\}$. Note that $H_m(e) \neq \emptyset$ implies $m > \kappa + 1 \geq M(\lambda_1, c_1, p, d_1) + 6$. If $H_m(e) \neq \emptyset$ and $m \geq M(\lambda_i, c_i, p, d_i) + 2i + 4$ then $\#H_m(e) \leq \#\{x[0, m-1] \mid x \in S,\ x_0 = x_{m-1} = e\} \leq \lambda^m \leq (\lambda_1)^{m-2i-4}$ by (3). Thus we can apply the Main Spindle Lemma 2.9 with $M(\lambda_i, c_i, p, d_i)$, and we can choose a spindle $G_i(e)$ of degree less than or equal to p such that for each $M(\lambda_i, c_i, p, d_i) + 2i + 4 \leq m < M(\lambda_{i+1}, c_{i+1}, p, d_{i+1}) + 2(i+1) + 4$ there are $\#H_m(e)$ paths of length $m - 2i - 4$ and such that $\#\mathcal{P}_{+,n}(G_i(e)) < p^{(n \cdot d_i)/(d_i + 2c_i)}$ and $\#\mathcal{P}_{-,n}(G_i(e)) < p^{(n \cdot d_i)/(d_i + 2c_i)}$ for all n.

Now we glue the spindles $G_i(e)$ together to obtain a spindle $G(e)$ as follows. Since $H_m(e) \neq \emptyset$ implies $m - 1 \leq L(e)$ there is a maximal i_0 such that there is m with $\mathrm{H}_m(e) \neq \emptyset$ and $M(\lambda_{i_0}, c_{i_0}, p, d_{i_0}) + 2i_0 + 4 \leq m < M(\lambda_{i_0+1}, c_{i_0+1}, p, d_{i_0+1}) + 2(i_0+1) + 4$. Let $H_+ := E^{i_0}$ be the outgoing 1-tree of height i_0 and let $H_- = \mathcal{R}(H_+)$ be the incoming 1-tree of height i_0. For each $1 \leq i \leq i_0$ such that there is some m with $H_m(e) \neq \emptyset$ and $M(\lambda_i, c_i, p, d_i) + 2i + 4 \leq m < M(\lambda_{i+1}, c_{i+1}, p, d_{i+1}) + 2(i+1) + 4$ we constructed a spindle $G_i(e)$ and we now identify the root of $G_i(e)$ with the unique vertex at height i in H_+ and we identify the end of $G_i(e)$ with the unique vertex at height i in H_-. We call the obtained graph $G(e)$.

Since $\deg(H_+) = \deg(H_-) = 1$ and since in the spindle $G_i(e)$ the root has outdegree 1 and the end has indegree 1, $G(e)$ is a spindle of degree less than or equal to p. Since in H_+ the path from the root to vertex i has length i and in H_- the path from vertex i to the end also has length i, a maximal path $w \in \mathcal{MP}(G(e))$ with a subpath $w' \in \mathcal{MP}(G_i(e))$ satisfies $|w| = |w'| + 2i$. Thus for each m we obtain that in $G(e)$ there are $\#H_m(e)$ maximal paths of length $m-4$. Furthermore, if $n \geq M(\lambda_i, c_i, p, d_i) + 2i + 4$ a path $w \in \mathcal{P}_{+,n}(G(e))$ can be uniquely written as $w'w''$, where w' is a path of length j in H_+ starting from the root and w'' begins with a path in $\mathrm{G}_j(e)$ starting from the root ($|w'| = 0$ is possible), where $L(e) \geq j \geq i$. Thus by the choice of the spindles $G_j(e)$, $j \geq i$, and by (2) we obtain that $n \geq M(\lambda_i, c_i, p, d_i) + 2i + 4$ implies $\#\mathcal{P}_{+,n}(G(e)) \leq \sum_{j=i}^{L(e)} (\lambda_{j-1})^n$. Thus (1) implies $\#\mathcal{P}_{+,n}(G(e)) \leq L(e)(\lambda_{i-1})^n$.

If $n < M(\lambda_i, c_i, p, d_i) + 6$ then a path $w \in \mathcal{P}_{+,n}(G(e))$ can be uniquely written as $w'w''$, where w' is a path of length j in H_+ starting from the root and w'' begins with a path in $\mathrm{G}_j(e)$ starting from the root, where $j < n$. Thus using (1) we obtain $\#\mathcal{P}_{+,n}(G(e)) \leq \sum_{j=1}^{n} (\lambda_{j-1})^n \leq n(\lambda_0)^n$. Similarly for $\#\mathcal{P}_{-,n}(G(e))$.

Thus for each e with $H(e) \neq \emptyset$ we can fix:

(10) a spindle $G(e)$ of degree less than or equal to p having $\#H(e)$ maximal paths;

(10b) $n \geq M(\lambda_i, c_i, p, d_i) + 2i + 4$ implies $\#\mathcal{P}_{+,n}(G(e)) \leq L(e)(\lambda_{i-1})^n$ and $\#\mathcal{P}_{-,n}(G(e)) \leq L(e)(\lambda_{i-1})^n$;

(10c) $n < M(\lambda_i, c_i, p, d_i) + 2i + 4$ implies $\#\mathcal{P}_{+,n}(G(e)) \leq n(\lambda_0)^n$ and $\#\mathcal{P}_{-,n}(G(e)) \leq n(\lambda_0)^n$;

(11) a bijection $g_e : H(e) \to \mathcal{MP}(G(e))$ such that $|g_e(w)| = |w| - 4$ for all $w \in H(e)$.

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We use the symbols (f, e, e), $1 \leq f \leq \#edge(G(e))$, to denote the edges of the spindle $G(e)$.

Remark. If $\lambda < \alpha$ and $\lambda < \beta$ then one can also choose all $\lambda_i < \max(\alpha, \beta)$ and then one can simply choose the spindle $G(e)$ from the Main Spindle Lemma 2.9 with $M(\lambda_1, c_1, p, d_1)$ such that (10) and (11) are satisfied. The extra conditions (10b) and (10c) are only to ensure $H_+(\Phi(S)) = \log\alpha$ and $H_-(\Phi(S)) = \log\beta$ in case $\lambda = \alpha$ or $\lambda = \beta$.

Choosing spindles for obtaining the Salama entropies. Since S is locally compact, but not compact, $\sup\{w(f) \mid f \geq 1\} = \infty$ and thus we can choose inductively a sequence of edges f_n such that $w(f_1) > L(1) + 1$ and $w(f_{n+1}) > w(f_n) + 1$ for all $n \geq 1$. Then for each n choose a point $x^n \in S$ with

$$x^n_0 = 1, \quad x^n_{w_-(f_n)} = f_n, x^n_{w(f_n)} = 1 \quad \text{and} \quad x^n[-w(f_n), 0] = x^n[0, w(f_n)].$$

Since $x^n_0 = 1$ for all n by passing to a subsequence we may assume that the x^n converge. Let $x^* \in S$ be the limit point.

LEMMA 2.11. *Let $s \in S$ with $s[0, w(f_n)] = x^n[0, w(f_n)]$. Let $m = w(f_n)$. Then $[0, m]$ is a marked interval for s.*

Proof. Since $s_0 = s_m = 1$, s is marked at time 0 and at time m. Let $0 < i \leq w_-(f_n)$. Then $w_-(s_i) = i$ and $s_i \neq 1$. Thus by (8), $i - L(s_i) \leq i - w_-(s_i) = 0$. Since $1 < s_i$, s is thus not marked at time i. A symmetric argument using w_+ instead of w_- shows that s is not marked at time i for $w_-(f_n) + 1 \leq i < m$. Since $m > L(1) = L(s_0) = L(s_m)$, $[0, m]$ is a long marked interval. □

Let $P' = \{x^n[0, w(f_n)] \mid n \geq 1\}$ and choose a family of pairwise disjoint infinite subsets $P_+(h)$ and $P_-(h)$, $h \geq 1$, of P'. By Lemma 2.11 if $x \in S$ with $x[0, m] \in P'$ then $[0, m]$ is a long marked interval for x. We shall now choose for all large enough $h \in \mathbb{N}$ a spindle $Z_+(h)$ and an injection $g_{+,h} : \mathcal{M}P(Z_+(h)) \to P_+(h)$ as follows. Recall that we have already chosen $c, d \geq 1$ such that $p^d < \lambda^{d+2c}$ and that $r = d + 2c$. Recall also that $\mathcal{P}_{+,m}(G)$ denotes the set of paths of length m in the spindle G starting from the root and $\mathcal{P}_{-,m}(G)$ is the set of paths of length m ending at the end and that $\mathcal{M}P(G)$ denotes the set of maximal paths.

If $\alpha = \infty$ then let $H_\alpha = p$ and if $\alpha < \infty$ then let $H_\alpha \geq \max(\alpha, p)$ such that:

- $p(\alpha) < \infty \Rightarrow p(\alpha) < \alpha^{h+1}$ for all $h \geq H_\alpha$;
- $p(\alpha) = \infty \Rightarrow h < \alpha^{h+1}$ for all $h \geq H_\alpha$.

Then for $h \geq H_\alpha$ define

(12) $p_h(\alpha) = p(\alpha)$ if $p(\alpha) < \infty$ and $p_h(\alpha) = h$ if $p(\alpha) = \infty$.

If $\alpha = p(\alpha)$ then fix a sequence $k_h \in \mathbb{N}$, $h \geq H_\alpha$, such that

$$\lim_{h\to\infty} \frac{k_h}{k_h + 1 + 3rh + r} = 1.$$

If $\alpha < p(\alpha)$ then $\alpha < \infty$ and for each $h \geq H_\alpha$ let $k_h \in \mathbb{N}$ be maximal such that

(13) $p_h(\alpha)^{k_h} < \alpha^{k_h+1+3rh+r}$.

Note that in both cases $\alpha = p(\alpha)$ and $\alpha < p(\alpha)$, the sequence k_h satisfies

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(14)
$$\lim_{h\to\infty}\frac{1}{k_h+1+3rh+r}\log p_h(\alpha)^{k_h}=\log\alpha.$$

Let $h\geq H_\alpha$ and $k:=k_h$. Let H be the subtree of height k of the outgoing $p_h(\alpha)$ tree and $G_+=EH$, so G_+ is a tree of height $k+1$ having $p_h(\alpha)^k$ ends. Choose N so large that $p^{dN}\geq p_h(\alpha)^k$ and let G_- be a subtree of height Nr of $T_-=$ the reversed tree of $T_+(c,p,d)^\infty$ having $p_h(\alpha)^k$ roots. Fix a bijection $f:end(G_+)\to root(G_-)$. Let Q be a subset of $P_+(h)$ of cardinality $p_h(\alpha)^k$ such that each path $p\in Q$ has length $>m:=k+1+Nr+2(3rh+r)+2$. Fix a bijection $q:end(G_+)\to Q$. For each $v\in end(G_+)$ identify the root of $E^{|q(v)|-m}$ with v and identify the end of $E^{|q(v)|-m}$ with $f(v)\in root(G_-)$. The obtained graph is a spindle $Z_+(h)$ with the properties:

(15) $\deg_+(Z_+(h))=p_h(\alpha)$ and $\deg_-(Z_+(h))\leq p$;
- $\#\mathcal{P}_{+,k_h+1}(Z_+(h))=\#\mathcal{MP}(Z_+(h))=p_h(\alpha)^{k_h}$;
- $\#\mathcal{P}_{-,n}(Z_+(h))\leq\lambda^{n+r}$ for all n;
- there is an injection $g_{+,h}:\mathcal{MP}(Z_+(h))\to P_+(h)$ such that for all $w\in\mathcal{MP}(Z_+(h))$ it holds that $|g_{+,h}(w)|=|w|+2(3rh+r)+2$.

Now we perform the analogue for the subsets $P_-(h)$. If $\beta=\infty$ then let $H_\beta=p$ and if $\beta<\infty$ then let $H_\beta\geq\max(\beta,p)$ such that:
- $p(\beta)<\infty\Rightarrow p(\beta)<\beta^{h+1}$ for all $h\geq H_\beta$;
- $p(\beta)=\infty\Rightarrow h<\beta^{h+1}$ for all $h\geq H_\beta$.

Then for $h\geq H_\beta$ define

(16) $p_h(\beta)=p(\beta)$ if $p(\beta)<\infty$ and $p_h(\beta)=h$ if $p(\beta)=\infty$.

If $\beta=p(\beta)$ then fix a sequence $m_h\in\mathbb{N}$, $h\geq H_\beta$, such that
$$\lim_{h\to\infty}\frac{m_h}{m_h+1+3rh+2r}=1.$$
If $\beta<p(\beta)$ then $\beta<\infty$ and for each $h\geq H_\beta$ let $m_h\in\mathbb{N}$ be maximal such that

(17) $p_h(\beta)^{m_h}<\beta^{m_h+1+3rh+2h}$.

Note that in both cases, $\beta=p(\beta)$ and $\beta<p(\beta)$, the sequence k_h satisfies

(18)
$$\lim_{h\to\infty}\frac{1}{m_h+1+3rh+2r}\log p_h(\beta)^{m_h}=\log\beta.$$

Then for $h\geq H_\beta$ we fix spindles $Z_-(h)$ and maps $g_{-,h}$ with:

(19) $\deg_-(Z_-(h))=p_h(\beta)$ and $\deg_+(Z_-(h))\leq p$;
- $\#\mathcal{P}_{-,m_h+1}(Z_-(h))=\#\mathcal{MP}(Z_-(h))=p_h(\beta)^{m_h}$;
- $\#\mathcal{P}_{+,n}(Z_-(h))\leq\lambda^{n+r}$ for all n;
- $g_{-,h}:\mathcal{MP}(Z_-(h))\to P_-(h)$ injective and for all $w\in\mathcal{MP}(Z_-(h))$ it holds that $|g_{-,h}(w)|=|w|+2(3rh+2r)+2$.

We denote the edges of the spindle $Z_+(h)$ by $(f,+,h)$, $1\leq f\leq\#edge(Z_+(h))$ and the edges of the spindle $Z_-(h)$ by $(f,-,h)$, $1\leq f\leq\#edge(Z_-(h))$. Let
$$P:=\bigcup_{h\geq H_\alpha}g_{+,h}(\mathcal{MP}(Z_+(h)))\cup\bigcup_{h\geq H_\beta}g_{-,h}(\mathcal{MP}(Z_-(h)))$$
be the special path set of S.

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Choosing spindles for coding long marked intervals. Recall that we have already chosen the marker set M, the increasing sequence n_k, $k \in \mathbb{N}$, and the special path set P. Let a, b be S-blocks of the same length $q+1$, $q \geq 1$. Let e be the first symbol of a and let f be the last symbol of b and let $k = \max(e, f)$. Let

$$\begin{aligned} H(a,b) := \{x[0,m] \mid x \in S \text{ with } x[0,m] = awb \text{ for some } w \text{ and} \\ [0,m] \text{ is a long marked interval for } x \text{ and} \\ awb \notin P \text{ and} \\ m \text{ satisfies } n_q \leq n_k + \log_p m < n_{q+1}\}. \end{aligned}$$

Note that $H(a,b) \neq \emptyset$ implies $k = \max(e, f) \leq q$ since $n_k < n_{q+1}$ and n_i is a strictly increasing sequence by (6).

If $w \in H(a,b)$ and $|w| = m+1$ then $m > L(k)$. Since L is strictly increasing there is a largest $i = i(a,b)$ such that $|w| - 1 > L(i)$ for all $w \in H(a,b)$. Then

$$\begin{aligned} \#\{w \in H(a,b) \mid |w| = m+1\} &\leq \#\{x[0,m] \mid x \in S, x_0 = e, x_m = f\} \\ &\leq (\lambda_i)^{m - 7\log_p m - 6n_k} \quad \text{by (9), since } m > L(i), i \geq k \\ &\leq (\lambda_i)^{m-1-6n_q} \end{aligned}$$

since $n_q \leq n_k + \log_p m$ and $\log_p m \geq 1$ by (8). Then

$$\begin{aligned} (m+1) - 2 - 6n_q &\geq m - 7\log_p m - 6n_k \\ &\geq m - 7\log_p m - 6n_i \quad \text{since } i \geq k \text{ and by (6)} \\ &\geq M(\lambda_i, c_i, p, d_i) \quad \text{by (8), since } m > L(i). \end{aligned}$$

Thus, by the Main Spindle Lemma 2.9 there is a spindle $W(a,b)$ of degree p such that for all $h \geq 1$ it holds that

(20) $\#\mathcal{P}_{+,h}(W(a,b)) < p^{hd_i/(d_i+2c_i)}$ and $\#\mathcal{P}_{-,h}(W(a,b)) < p^{hd_i/(d_i+2c_i)}$ and there is

(21) a bijection $\omega_{a,b} : H(a,b) \to \mathcal{M}P(W(a,b))$ with $|\omega_{a,b}(w)| + 6n_q + 2 = |w|$.

We denote the edges of the spindle $W(a,b)$ by $(f, *, a, b)$, $1 \leq f \leq \#edge(W(a,b))$.

Remark. As for the spindles $G(e)$, if $\lambda < \alpha$ and $\lambda < \beta$ then one can also choose all $\lambda_i < \max(\alpha, \beta)$ and then one can simply choose the spindles $W(a,b)$ from the Main Spindle Lemma 2.9 with $M(\lambda_1, c_1, p, d_1)$ such that (21) is satisfied. The extra condition (20) is only to ensure $H_+(\Phi(S)) = \log\alpha$ and $H_-(\Phi(S)) = \log\beta$ in case $\lambda = \alpha$ or $\lambda = \beta$.

Choosing intermediate trees for gluing spindles to trees. Let a be an S-block of length $q+1$ starting with a symbol $e \leq q$ such that there is an S-block b with $H(a,b) \neq \emptyset$. Any such S-block b has length $q+1$ and ends with a symbol $f \leq q$; see the remark after the definition of $H(a,b)$.

Let $f \leq q$ and consider the map $F_f : \{x \in S \mid x_0 = f\} \to \{t \in S_{T_f} \mid t_0 = (e^*, f)\}$. Since T_f is a subtree of a copy of $T^{c,p,d}$ the continuity property (7b) of F_f implies thus that $\#\{x[-q,0] \mid x \in S, x_0 = f\} \leq \#\{\text{vertices at height } n_q - 1 \text{ in } T_-^{c,p,d}\} = \#\{\text{vertices at height } n_q \text{ in } T_+^{c,p,d}\}$. Since, by (6), $p^{n_q/r} \geq p^q \geq q$, a subtree of height $2n_q$ of T_+ has at least $\#\{x[-q,0] \mid x \in S, x_0 \leq q\}$ ends. Thus we can fix:

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(22) a subtree $B_+(a)$ of height $2n_q$ of a copy of $T_+^{c,p,d}$;

- a bijection $\beta_{+,a} : \{b \mid H(a,b) \neq \emptyset\} \to \mathcal{M}P(B_+(a))$.

Now let b be an S-block of length $q+1$ such that there is an S-block a with $H(a,b) \neq \emptyset$. By a symmetric argument we can fix:

(23) a subtree $B_-(b)$ of height $2n_q$ of $T_-^{c,p,d}E$;

- a bijection $\beta_{-,b} : \{a \mid H(a,b) \neq \emptyset\} \to \mathcal{M}P(B_-(b))$.

We denote the edges of the tree $B_+(a)$ by $(f, *, +, a)$, $1 \leq f \leq \#edge(B_+(a))$ and the edges of $B_-(b)$ by $(f, *, -, b)$, $1 \leq f \leq \#edge(B_-(b))$.

The map. Before defining the map we give an informal description of the resulting graph. We build a graph which has as its basic part a forest of trees T_e, each of which is infinitely high, has infinitely deep rootage, and the distinguished edge (e^*, e) is at ground level. Edges in T_e leading to (e^*, e) belong to the rootage, and edges reached from (e^*, e) belong to the greenery of the tree. The trees are connected in the following way. Each tree has in its greenery (possibly infinitely) many 'flowers', which are the finite trees of the form $B_+(a)$, and in its rootage (possibly infinitely) many 'fibres', which are the finite trees of the form $B_-(b)$. Starting from an end of a flower, say an end of the finite tree $B_+(a)$, the spindle $W(a,b)$ leads to a root vertex of the fibre $B_-(b)$ which is located in the rootage of the tree T_f, say. The end of $B_+(a)$ determines b and thus the whole spindle $W(a,b)$ and the root vertex of $B_-(b)$ to which the spindle $W(a,b)$ leads to. The higher the flower $B_+(a)$ is located in T_e the longer is each path in the spindle $W(a,b)$ and the deeper is the fibre $B_-(b)$ located in T_f. One exceptional case is the tree T_1. It has additionally (always) infinitely many connections via a spindle $Z_+(h)$ or $Z_-(h)$ from a top vertex in T_1 to a rootage vertex in T_1. The height of these vertices where the spindles $Z_+(h)$ and $Z_-(h)$ are connected to is a linear function in h. The purpose of those spindles is to get the desired Salama entropies. Finally, each tree T_e may have a finite spindle $G(e)$ leading from the terminal vertex of (e^*, e) to the initial vertex of (e^*, e). These special spindles code the blocks $x[i, j]$ with $x_i = x_j = e$ whenever $[i, j]$ is a short marked interval. A path from (e^*, e) in the tree T_e to (f^*, f) in the tree T_f through the flower $B_+(a)$, the spindle $W(a,b)$ and the fibre $B_-(b)$ codes blocks $x[i, j]$ of that length which begin with the block a and end with the block b whenever $[i, j]$ is a long marked interval (here $|a|$ and $|b|$ are determined by $j - i$ and $x_i = e, x_j = f$). The path in the spindle $W(a,b)$ alone is enough to recover the whole block $x[i, j]$. The task of the flower $B_+(a)$ and the fibre $B_-(b)$ is to keep the desired degrees, the special placement of those, is to make the coding and its inverse continuous. The rays $x[i, \infty)$ with $x_i = e$ are coded in the paths in T_e starting in (e^*, e) and running the whole time through the greenery of T_e whenever $[i, \infty)$ is a marked ray, while the rays $x(-\infty, j]$ with $x_j = e$ are coded into paths leading to (e^*, e) and running the whole time through the rootage of T_e whenever $(-\infty, j]$ is a marked ray. The derived graph for S is the smallest connected subgraph of this forest of connected trees which contains all spindles $W(a,b)$, all spindles $Z_+(h)$, $Z_-(h)$ and all spindles $G(e)$.

We now give an exact definition of the map.

Let $x \in S$. We describe $\Phi(x)$. Let $i \in \mathbb{Z}$ and $m \geq 0$.

If $[i, i+m]$ is a long marked interval for x and $x[i, i+m] \notin P$ then let $e := x_i$, $f := x_{i+m}$ and $k := \max(e, f)$. Since the sequence n_q is strictly increasing there is

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a unique q such that $n_q \le n_k + \log_p m < n_{q+1}$. Then $x[i, i+m] \in H(a, b)$, where $a := x[i, i+q]$, $b := x[i+m-q, i+m]$. We define $(\Phi x)_i = (e, *)$, $(\Phi x)_{i+m} = (f, *)$ and $(\Phi x)[i+1, i+m-1] = ABCB'A'$, where

$$A := (F_e S^i x)[0, n_q - 1], \quad B := \beta_{+,a}(b), \quad C := \omega_{a,b}(x[i, i+m]),$$
$$B' := \beta_{-,b}(a), \quad A' := (F_f S^{i+m} x)[-n_q + 1, 0].$$

Since $|ABCB'A'| = n_q + 2n_q + (m+1-6n_q-2) + 2n_q + n_q = m-1$, then $(\Phi x)[i, i+m]$ is well defined.

If $[i, i+m]$ is a long marked interval for x and $x[i, i+m] \in P$ then $x_i = x_{i+m} = 1$ and either there is a unique $h \ge H_\alpha$ with $x[i, i+m] \in P_+(h)$ or there is a unique $h \ge H_\beta$ with $x[i, i+m] \in P_-(h)$. Recall that $x^* \in S$ was a chosen point while defining the special path set with $x_0^* = 1$. Let $(\Phi x)_i = (\Phi x)_{i+m} = (1, *)$ and $(\Phi x)[i+1, i+m-1] = ABA'$, where

$$\begin{aligned} x[i, i+m] \in P_+(h) \Rightarrow A &= (F_1 x^*)[0, 3rh + r - 1] \\ B &= (g_{+,h})^{-1}(x[i, i+m]) \\ A' &= (F_1 x^*)[-3rh - r + 1, 0], \\ x[i, i+m] \in P_-(h) \Rightarrow A &= (F_1 x^*)[0, 3rh + 2r - 1] \\ B &= (g_{-,h})^{-1}(x[i, i+m]) \\ A' &= (F_1 x^*)[-3rh - 2r + 1, 0]. \end{aligned}$$

Note here that $|ABA'| = m - 1$, by the definition of $g_{+,h}$ respectively $g_{-,h}$.

If $[i, i+m]$ is a short marked interval for x then let $e = x_i$. Then $x_{i+m} = e$, also and $1 \le m \le L(e)$. We put $(\Phi x)_i = (\Phi x)_{i+m} = (e, *)$. If $m \ge 2$ then if $x[i, i+m] \notin H(e)$ then $m \le \kappa + 1$ and $x[i, i+m]$ is the only path with this property by (4), then let $\Phi(s)_{i+j} = (e, *, j)$, for $1 \le j \le m-1$. If $m \ge 2$ and $x[i, i+m] \in H(e)$ then $m \ge \kappa + 2$ and we can let $(\Phi x)_{i+1} = (\Phi x)_{i+m-1} := (e^*, e)$ and $(\Phi x)[i+2, i+m-2] = g_e(x[i, i+m])$.

If $[i, \infty)$ is a marked ray for x then $(\Phi x)[i, \infty) := (x_i, *)(F_{x_i} S^i x)[0, \infty)$. Note that $(F_{x_i} S^i x)[0, \infty)$ is a path in T_{x_i} starting with the special edge (x_i^*, x_i).

If $(-\infty, i]$ is a marked ray for x then $(\Phi x)(-\infty, i] = (F_{x_i} S^i x)(-\infty, 0](x_i, *)$.

This defines Φx, since x is marked at least once. Since $(\Phi x)_i = (x_i, *)$ if $S^i x \in M$, Φ is well defined. The map is shift commuting by definition.

2.5. *Verification of the asserted properties.* In this section Φ always denotes the map defined on S that we have just defined.

2.5.1. *Continuity.* We show that $\Phi : S \to \Phi(S)$ is continuous. Let $x \in S$. By the paragraph after Definition 2.1, there is some i such that x is marked at time i. Choose such i with smallest $|i|$.

If $i = 0$ then, since M is open, there is an open neighborhood of x such that $y \in U$ implies $y \in M$ and $y_0 = x_0$ and thus $(\Phi y)_0 = (\Phi x)_0 = (x_0, *)$.

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Now assume $i < 0$ (the case $i > 0$ is symmetric). Then either $[i, \infty)$ is a marked ray for x or there is $m > -i$ such that $[i, i+m]$ is a marked interval for x. If $[i, i+m]$ is a long marked interval for x and $x[i, i+m] \notin P$ then $(\Phi x)[i+1, i+n_q] = (F_{x_i} S^i x)[0, n_q - 1]$, where n_q is determined by $x[i, i+m]$ and increases if m increases. Thus for m large enough $n_q \geq k := \max(|i|, x_i)$. If $[i, i+m]$ is a marked interval for x and $x[i, i+m] \in P \cap (P_+(h) \cup P_-(h))$ then $(\Phi x)[i+1, i+3rh+1] = (F_{x_i} S^i x)[0, 3rh]$. By the definition of P, h is increasing when m increases. Thus for m large enough we obtain $3rh + 1 \geq k$.

Thus there is some m' such that if $[i, i+m]$ is a marked interval for x with $m \geq m'$ or $[i, \infty)$ is a marked ray for x then $(\Phi x)[i+1, i+k] = (F_{x_i} S^i x)[0, k-1]$. In this case, since M is closed open, there is a neighborhood U of x such that $y \in U$ implies $y[i, i+k] = x[i, i+k]$ and y is not marked at j for $i < j \leq i + m'$. Thus $(\Phi y)[i+1, i+k] = (F_{y_i} S^i y)[0, k-1]$ and since $y[i, i+k] = x[i, i+k]$ thus $(\Phi y)[i+1, i+k] = (\Phi x)[i+1, i+k]$ by (7a) and since $n_k \geq k$ by (6). Since $k = \max(|i|, x_i)$, thus $(\Phi x)_0 = (\Phi y)_0$.

If there is some $m < m'$ such that $[i, i+m]$ is a marked interval for x then, since M is closed and open, there is a neighborhood U of x such that $y \in U$ implies $y[i, i+m] = x[i, i+m]$ and $[i, i+m]$ is a marked interval for y. Thus $(\Phi x)[i, i+m] = (\Phi y)[i, i+m]$, and since $i + m > 0$ thus $(\Phi y)_0 = (\Phi x)_0$.

2.5.2. *Conjugacy.* We show that $\Phi : S \to \Phi(S)$ is a conjugacy. Let $x \in S$. We show that there is $N \geq 0$ such that if $y \in S$ with $(\Phi y)[-N, N] = (\Phi x)[-N, N]$ then $y_0 = x_0$. Then, since Φ is continuous, it follows that Φ is a conjugacy. Note that $(\Phi x)_i = (x_i, *)$ if and only if x is marked at time i. Since x is marked at least once we can choose $i \in \mathbb{Z}$ with least $|i|$ and $(\Phi x)_i = (x_i, *)$.

If $i = 0$, then we can choose $N = 0$.

Assume $i < 0$ (the case $i > 0$ is symmetric). If there is no $j > i$ such that $(\Phi x)_j = (x_j, *)$ then $[i, \infty)$ is a marked ray for x. Then let $k = \max(|i|, x_i)$ and $N = i + n_k + L(k)$. Then for $y \in S$ with $(\Phi y)[-N, N] = (\Phi x)[-N, N]$ we have $(\Phi y)[i+1, i+n_k] = (\Phi x)[i+1, i+n_k] = (F_{x_i} S^i x)[0, n_k - 1]$ and this is a block with symbols of the form (f, x_i). Thus we know that either $[i, \infty)$ is a marked ray for y also or $[i, i+m]$ is a long marked interval for y for some m. Thus $(\Phi y)[i+1, i+n_k] = (F_{y_i} S^i y)[0, n_k - 1]$ and thus $y[i, i+k] = x[i, i+k]$ by the continuity property (7a) of F_{y_i} and $x_i = y_i$ which follows from $(x_i, *) = (\Phi x)_i = (\Phi y)_i$.

If there is $m > -i$ such that $[i, i+m]$ is a marked interval for x, then we choose $N = m$. Let $y \in S$ with $(\Phi y)[-N, N] = (\Phi x)[-N, N]$. Since $-N < i < i+m < N$, $[i, i+m]$ is a marked interval for y also. The block $(\Phi x)[i, i+m]$ contains a maximal subblock w with symbols either of the form $(f, *, a, b)$ or of the form $(f, +, h)$ or of the form $(f, -, h)$ or $(f, *, j)$ or (f, e, e). In the first case we know w is a maximal path in the spindle $W(a, b)$, $[i, i+m]$ is a long marked interval for y and $y[i, i+m] \notin P$ and furthermore $y[i, i+m] = (\omega_{a,b})^{-1}(w)$. Similarly, in the second case $y[i, i+m] = g_{+,h}(w)$. In the third case $y[i, i+m] = g_{-,h}(w)$, in the fourth $y[i, i+m]$ is the unique path in S starting with edge x_i and ending with edge x_i and having length less than or equal to $\kappa + 1$. In the last case $y[i, i+m] = (g_{x_i})^{-1}(w)$.

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2.5.3. *Degree.* Let z be a symbol of $\Phi(S)$. In this section we determine the number of follower symbols $\mathcal{F}(z) := \{(\Phi x)_1 \mid x \in S \text{ with } (\Phi x)_0 = z\}$ as well as the number of predecessor symbols $\mathcal{P}(z) := \{(\Phi x)_{-1} \mid x \in S \text{ with } (\Phi x)_0 = z\}$. For that let $x \in S$ and $\Phi x = y$ with $y_0 = z$.

If z is a marker symbol, say $z = (e, *)$ for some edge e of S, then we shall show that $\#\mathcal{F}(z) \leq p$ and $\#\mathcal{P}(z) \leq p$ also. If $z = (e, *)$ then x is marked at time 0 with $x_0 = e$. If $[0, \infty)$ is a marked ray for x or there is some m such that $[0, m]$ is a long marked interval for x then $y_1 = (F_e x)_0$ or $y_1 = (F_e x^*)_0$ and thus $y_1 = (e^*, e)$. If $[0, m]$ is a short marked interval, then if $m = 1$, $y_1 = (e, *)$, and if $1 < m \leq \kappa + 1$ then $y_1 = (e, *, 1)$ and if $m > \kappa + 1$ then $y_1 = (e^*, e)$. By (4) at most one of the cases $m = 1$ or $1 < m \leq \kappa + 1$ occurs, we thus obtain that $\#\mathcal{F}(z) \leq 2 \leq p$. The argument for $\#\mathcal{P}(z) \leq 2 \leq p$ is symmetric.

If $z = (f, e)$ is a tree symbol of some T_e then we shall show that $\#\mathcal{F}(z) \leq p$ and $\#\mathcal{P}(z) = 1$ if f is in $T_+^{c,p,d}$ and $\#\mathcal{F}(z) = 1$ and $\#\mathcal{P}(z) \leq p$ if f is in $T_-^{c,p,d}$. Assume that f is in $T_+^{c,p,d}$ (the other case is symmetric). Thus f is an edge in $T_+^{c,p,d}$ of height h, say. Thus z has in T_e a unique predecessor and at most p followers. The number of followers of z in T_e is 1 if $h = 0 \bmod r$, since $r = d + 2c$, $c \geq 3$ by (5) and $T_+^{c,p,d} = T(c, p, d)^\infty$ and T_e is a copy of a subtree of $T^{c,p,d}$.

If $h = 1$ then $f = e^*$ and x is marked at time -1 with $x_{-1} = e$ and thus $y_{-1} = (e, *)$ and $\#\mathcal{P}(z) = 1$. If $[-1, \infty)$ is a marked ray for x or $[-1, m]$ is a long marked interval for x then y_1 is a follower of z in T_e, since $n_1 \geq 3$ by (6) and $r \geq 9$ by (5). If $[-1, m]$ is a short marked interval for some m then $y_0 = (e^*, e)$ implies $m > \kappa + 1$ and y_1 is the edge in $G(e)$ which starts at the root. Thus $\#\mathcal{F}(z) \leq 2 \leq p$.

If $h > 1$ then we know that x is marked at time $-h$ with $x_{-h} = e$ and $(\Phi x)[-h + 1, 0]$ is the unique path in T_e starting with (e^*, e) and ending with z. Thus $\#\mathcal{P}(z) = 1$. Furthermore, we know that $[-h, \infty)$ is a marked ray or there is some $m > 0$ such that $[-h, m]$ is a long marked interval for x. Assume that y_1 is not a follower of z in T_e. Then $h = 0 \bmod r$. If $h = 0 \bmod 3r$ then $h = n_q$ for some q and y_1 is the unique edge in $B_+(a)$ starting from the root, where $a = (F_e^{-1} t)[0, q]$ for any $t \in T_e$ with $t[0, h - 1] = (\Phi x)[-h + 1, 0]$ by (7a). If $h = 3rh' + r$ for some $h' \geq 1$ then y_1 is the unique edge in $Z_{+,h'}$ starting from the root. If $h = 3rh' + 2r$ for some $h' \geq 1$ then y_1 is the unique edge in $Z_{-,h'}$ starting from the root. Thus if y_1 is not a follower of z in T_e then y_1 is uniquely determined by z, thus altogether we obtain $\#\mathcal{F}(z) \leq p$.

If $z = (f, *, +, a)$ from the tree $B_+(a)$, for some S-block a then we shall show that $\#\mathcal{F}(z) \leq p$ and $\#\mathcal{P}(z) = 1$.

First, $z = (f, *, +, a)$ implies that there is $i < 0 < j$ such that $[i, j]$ is a long marked interval for x and $x[i, j] \notin P$. Furthermore, there are $i < i' \leq 0 \leq j' < j$ such that $(\Phi x)[i', j']$ is a maximal path in $B_+(a)$. If z is at height h in $B_+(a)$ with $1 < h < 2n_q$, where $q + 1 = |a|$, then we are done, since by (22) $B_+(a)$ is a subtree of $T_+^{c,p,d}$ of height $2n_q$. If $h = 1$ then $\#\mathcal{F}(z) \leq p$, by (22), and y_{-1} is the last edge of the path $(F_{x'_0} x')[0, n_q - 1]$ for any $x' \in S$ with $x'[0, q] = a$ by (7a). Thus $\#\mathcal{P}(z) = 1$. If $h = 2n_q$ then $\#\mathcal{P}(z) = 1$, by (22) and y_1 is the unique edge in the spindle $W(a, b)$ starting from the root, where $b = \beta_{+,a}^{-1}(B)$ and B is the unique maximal path in $B_+(a)$ ending with z. Thus $\#\mathcal{F}(z) = 1$ in this case.

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If $z = (f, *, -, a)$ from the tree $B_-(b)$, for some S-block b then $\#\mathcal{F}(z) = 1$ and $\#\mathcal{P}(z) \leq p$ by an argument symmetric to the previous one.

If z is a spindle symbol (f, e, e) from the spindle $G(e)$ for some edge e in S then we show $\#\mathcal{F}(z), \#\mathcal{P}(z) \leq p$.

Now, $z = (f, e, e)$ implies that there are $i + 2 \leq 0 \leq j - 2$ such that $[i, j]$ is a short marked interval for x with $j - i > \kappa + 1$ and $(\Phi x)[i + 2, j - 2]$ is a maximal path in $G(e)$. If $i + 2 = 0$ then $y_{-1} = (e^*, e)$ and if $j - 2 = 0$ then $y_1 = (e^*, e)$. The statement now follows from the fact that $G(e)$ is a spindle of degree p by (10).

If z is a spindle symbol $(f, *, a, b)$ from the spindle $W(a, b)$ where a, b are S-blocks of the same length, $q + 1$ say, then we show $\#\mathcal{F}(z), \#\mathcal{P}(z) \leq p$.

From z we know there are $i < i' \leq 0 \leq j' < j$ such that $[i, j]$ is a long marked interval for x with $x[i, j] \notin P$ and $(\Phi x)[i', j']$ is a maximal path in the spindle $W(a, b)$. If $i' = 0$ then y_{-1} is the last edge in the path $\beta_{+,a}(b)$ in the spindle $B_+(a)$ and if $j' = 0$ then y_1 is the first edge in the path $\beta_{-,b}(a)$ in the spindle $B_-(b)$. The statement now follows from the fact that $W(a, b)$ is a spindle of degree less than or equal to p.

If $z = (e, *, i)$ for some edge e of S and some $i \geq 1$ then $\#\mathcal{F}(z) = \mathcal{P}(z) = 1$, since then by (4) there is a unique $i < m \leq \kappa + 1$ such that $[-i, -i + m]$ is a short marked interval for x with $x_{-i} = e$. (Also, $x[-i, -i + m]$ is uniquely determined.)

Note that we have shown that all symbols that we considered so far have degree less than or equal to p. Now we consider the symbols from $Z_+(h)$ and $Z_-(h)$ and we shall see that they have indegree $p_h(\beta)$ and outdegree $p_h(\alpha)$.

Let $z = (f, +, h)$ be a symbol from the spindle $Z_+(h)$. Then there are $i < i' \leq 0 \leq j' < j$ such that $[i, j]$ is a long marked interval for x with $x[i, j] \in P_+(h)$ and $(\Phi x)[i', j']$ is a maximal path in the spindle $Z_+(h)$. If $i' = 0$ then $y_{-1} = (F_1 x^*)_{3rh+r-1}$ and if $j' = 0$ then $y_1 = (F_1 x^*)_{-3rh-r+1}$. Thus by (15), $\#\mathcal{P}(z) \leq p \leq p_h(\beta)$ and $\#\mathcal{F}(z) \leq p_h(\alpha)$.

Now let $z = (f, +, h)$ be an edge from $Z_+(h)$ with exactly $p_h(\alpha)$ followers z' in $Z_+(h)$; such z exists by (15). For each z' let $w^{z'}$ be a maximal path in $Z_+(h)$ which has zz' as a subblock. Let $u^{z'} := g_{+,h}(w^{z'}) \in P_+(h)$, say $u^{z'}$ has length $m + 1$. Since $u^{z'}$ begins and ends with the symbol 1, there is $x^{z'} \in S$ with $S^m x^{z'} = x^{z'}$ and $x^{z'}[0, m] = u^{z'}$. By Lemma 2.11, $[0, m]$ is a long marked interval for $x^{z'}$ and, of course, $x^{z'}[0, m] \in P_+(h)$. Thus $(\Phi x^{z'})[3rh + r + 1, m - 3rh - r - 1] = w^{z'}$. In particular, there is some i with $(\Phi x^{z'})[i, i + 1] = zz'$. This shows that $\#\mathcal{F}(z) \geq p_h(\alpha)$. Thus we have shown that for each $h \geq H_\alpha$ there is a symbol z of $\Phi(S)$ with $\#\mathcal{F}(z) = p_h(\alpha)$.

The argument for $z = (f, -, h)$ a symbol in the spindle $Z_-(h)$ is symmetric. Thus altogether, if $p(\alpha) = \infty$ then $p_h(\alpha) = h$ and thus $\deg_+(\Phi(S)) = \infty = p(\alpha)$. If $p(\alpha) < \infty$, then $p_h(\alpha) = p(\alpha)$ and the arguments above show $\deg_+(\Phi(S)) = p(\alpha)$. Similarly, $\deg_-(\Phi(S)) = p(\beta)$.

2.5.4. *Synchronizing.* From §2.5.2 we know that $\Phi(S) \in \mathcal{P}res(S)$. Now we shall show that every symbol z of $\Phi(S)$ is synchronizing. Let $s, t \in S$ with $(\Phi s)_0 = (\Phi t)_0 = z$. Let y be defined by $y(-\infty, 0] = \Phi(s)(-\infty, 0]$ and $y[0, \infty) = \Phi(t)[0, \infty)$. We have to show $y \in \Phi(S)$.

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We first treat the case that z is not a spindle symbol. We shall use the information that the symbol z carries to find a coordinate $i \in \mathbb{Z}$ such that:

(a) s and t are both marked at time i with $s_i = t_i$;

(b) $(\Phi s)(-\infty, i] = y(-\infty, i]$ and $(\Phi t)[i, \infty) = y[i, \infty)$.

Then we let $x \in S$ with $x(-\infty, i] = s(-\infty, i]$ and $x[i, \infty) = t[i, \infty)$ and by Lemma 2.4 we have that $M(x) = (M(s) \cap (-\infty, i]) \cup (M(t) \cap [i, \infty))$ and thus $\Phi x = y$ and thus $y \in \Phi(S)$. We call such a coordinate i a *magic coordinate* for s and t.

We consider the different cases for z to show how to find a magic coordinate.

If $z = (e, *)$ for an edge e of S, then 0 is a magic coordinate.

If z is a tree symbol of some T_e which can be reached from the edge (e^*, e), or z is an edge of some $B_+(a)$, or $z = (e', *, i)$, then as shown in §2.5.3, there is a unique $h < 0$, a unique edge e of S and a unique block w such that for any $u \in S$ with $\Phi(u)_0 = z$ then $u_h = e$, u is marked at time h, u is not marked at time j, $h < j \leq 0$ and $\Phi(u)[h + 1, 0] = w$. Thus, h is a magic coordinate for s and t.

Similarly, if z is a tree symbol from some T_e which leads to the edge (e^*, e), or z is an edge of some $B_-(b)$ then there is $h > 0$ which is a magic coordinate for s and t.

To show that the spindle symbols are synchronizing we use a different argument.

First let z be an edge of a spindle $W(a, b)$, where a and b are S-blocks of the same length, say $q + 1$. Let $u = 3n_q + 1$. Then there is some h_s, $m_s \geq 1$ such that $\Phi(s)[-h_s, m_s]$ is a maximal path is $W(a, b)$. Then $[-h_s - u, m_s + u]$ is a long marked interval for s and $s[-h_s - u, m_s + u] = \omega_{a,b}^{-1}((\Phi s)[-h_s, m_s])$. Similarly, h_t, $m_t \geq 1$ with $(\Phi t)[-h_t, m_t]$ is a maximal path is $W(a, b)$. Since $(\Phi s)_0 = y_0 = z = (\Phi t)_0$, then $y[-h_s, m_t]$ is a maximal path in $W(a, b)$. By (21), $w = \omega_{a,b}^{-1}(y[-h_s, m_t]) \in H(a, b)$ and $|w| = |y[-h_s, m_t]| + 6n_q + 2 = h_s + 1 + m_t + 2u$. Thus x defined by $x(-\infty, -h_s - u] = s(-\infty, -h_s - u]$, $x[-h_s - u, m_t + u] = w$ and $x[m_t + u, \infty) = t[m_t + u, \infty)$ is a point in S.

We want to show now that $[-h_s - u, m_t + u]$ is a long marked interval for x. Let e be the initial symbol of the block a and let f be the last symbol of the block b and $k = \max(e, f)$. Since $w \in H(a, b)$, w begins with a and ends with b and there is a point $x' \in S$ with $x'[-h_s - u, m_t + u] = w$ such that $[-h_s - u, m_t + u]$ is a long marked interval for x'. Thus by Lemma 2.2, x is not marked at i for $-h_s - u < i < m_t + u$. Now consider the point $x'' \in S$ which has $x''(-\infty, -h_s - u] = s(-\infty, -h_s - u]$ and $x''[-h_s - u, \infty) = x'[-h_s - u, \infty)$. Since s and x' are both marked at time $-h_s - u$, Lemma 2.3 implies that x'' is marked at time $-h_s - u$. Since $L(e) < |w|$, we obtain that $x(-\infty, -h_s - u + L(e)] = x''(-\infty, -h_s - u + L(e)]$ and thus x is marked at $-h_s - u$ also. Similarly, using t shows that x is marked at time $m_t + u$. Thus $[-h_s - u, m_t + u]$ is a marked interval for x and since $L(e) < |w|$ it is a long marked interval.

Since $x_{-h_s-u} = s_{-h_s-u}$ and x and s are both marked at time $-h_s - u$, Lemma 2.4 implies that $M(x) \cap (-\infty, -h_s - u] = M(s) \cap (-\infty, -h_s - u]$ and thus $(\Phi x)(-\infty, -h_s - u] = (\Phi s)(-\infty, -h_s - u]$. Similarly, $(\Phi x)[m_t + u, \infty) = (\Phi t)[m_t + u, \infty)$. Since $w \in H(a, b)$, $w = x[-h_s - u, m_t + u] \notin P$ and $|w|$ satisfies $n_q \leq n_k + \log_p |w| < n_{q+1}$. Thus $(\Phi x)[-h_s - u + 1, m_t + u - 1] = (F_e S^{m'} x)[0, n_q - 1]\beta_{+,a}(b) w' \beta_{-,b}(a)(F_f S^m x)[-n_q + 1, 0]$, where $m' = -h_s + u$, $m = m' + |w|$. By (7a),

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$(F_e S^{m'} x)[0, n_q - 1]$ is determined by $x[-h_s - u, -h_s - u + q] = a$ and by (7b), $(F_f S^m x)[-n_q + 1, 0]$ is determined by $x[-q + m_t + u, m_t + u] = b$ and we thus obtain that $(\Phi x)[-h_s - u + 1, m_t + u - 1] = y[-h_s - u + 1, m_t + u - 1]$. Thus by the definition of y, $\Phi x = y$.

If z is a symbol of the spindle $Z_+(h)$ then one can give the analogous argument as for the spindle $W(a, b)$ by replacing u by $u = 3rh + r + 1$ and using (15) (instead of (21)) to define $w = g_{+,h}(y[-h_s, m_t]) \in P_+(h)$.

If z is a symbol of the spindle $Z_-(h)$ then one replaces u by $u = 3rh + 2r + 1$ and uses (19) to define $w = g_{-,h}(y[-h_s, m_t]) \in P_-(h)$.

If z is a symbol of a spindle $G(e)$ for some edge e of S, then we know there is some $\kappa + 1 < m_s \le L(e)$ and some $1 \le h_s \le m_s - 3$ such that $[-h_s - 1, -h_s - 1 + m_s]$ is a short marked interval with $s_{-h_s - 1} = e$. Similarly, there is $\kappa + 1 < m_t \le L(e)$ and some $1 \le h_t \le m_t - 3$ such that $[-h_t - 1, -h_t - 1 + m_t]$ is a short marked interval for t and $t_{-h_t - 1 + m_t} = e$. Thus, $(\Phi s)[-h_s + 1, 0]$ is a path in $G(e)$ starting at the root and ending with z and $\Phi(t)[0, -h_t + m_t - 3]$ is a path in $G(e)$ starting with z and ending at the end of $G(e)$. Thus $y[-h_s + 1, -h_t + m_t - 3]$ is a maximal path in $G(e)$. By (11), $w := g_e^{-1}(y[-h_s + 1, -h_t + m_t - 3]) \in H(e)$ with $|w| = h_s + 1 - h_t + m_t$ and x defined by $x(-\infty, -h_s - 1] = s(-\infty, -h_s - 1]$, $x[-h_s - 1, -h_t - 1 + m_t] = w$ and $x[-h_t - 1 + m_t, \infty) = t[-h_t - 1 + m_t, \infty)$ is a point in S. Since $x_i > e$ for all $-h_s \le i < -h_t - 3 + m_t$, and s is marked at $-h_s - 1$ and t is marked at $-h_t - 1 + m_t$, we obtain that $[-h_s - 1, -h_t - 1 + m_t]$ is a marked interval for x. It is a short marked interval, since $w \in H(e)$. Thus $\Phi(x)[-h_s, -h_t - 2 + m_t] = (e^*, e) g_e(w)(e^*, e) = y[-h_s, -h_t - 2 + m_t]$. Since $x_{-h_s - 1} = s_{-h_s - 1}$ and x, s are marked at time $-h_s - 1$, by Lemma 2.4, $M(x) \cap (-\infty, -h_s - 1] = M(s) \cap (-\infty, -h_s - 1]$ and thus $\Phi(x)(-\infty, -h_s - 1] = \Phi(s)(-\infty, -h_s - 1] = y(-\infty, -h_s - 1]$. Similarly, using x and t, $\Phi(x)[-h_t - 1 + m_t, \infty) = \Phi(t)[-h_t - 1 + m_t, \infty) = y[-h_t - 1 + m_t, \infty)$. Thus $\Phi(x) = y$.

Thus, all symbols in $\Phi(S)$ are synchronizing and there is a graph presentation T of S with $\deg_+(T) = \deg_+(\Phi(S)) = p(\alpha)$, $\deg_-(T) = \deg_-(\Phi(S)) = p(\beta)$ and $H_+(T) = H_+(\Phi(S))$, $H_-(T) = H_-(\Phi(S))$. It remains to calculate the Salama entropies of $\Phi(S)$.

2.5.5. *Salama entropies.* We show that $H_+(\Phi(S)) = \log \alpha$; the argument for $H_-(\Phi(S)) = \log \beta$ is symmetric. Note that for every $x \in S$ with $x_0 = 1$ it holds that x is marked at time 0, thus $(1, *)$ is a symbol in $\Phi(S)$. Let

$$D(n) := \{(\Phi x)[0, n] \mid x \in S \text{ with } (\Phi x)_0 = (1, *)\}.$$

Thus $H_+(\Phi(S)) = \limsup_{n \to \infty} (1/n) \log \# D(n)$. For each $h \ge H_\alpha$, if $x \in S$ such that $[0, m]$ is a long marked interval with $x[0, m] \in P \cap P_+(h)$ then $(\Phi x)[0, 3rh + r] = (1, *)(F_1 x^*)[0, 3rh + r - 1]$ and $(\Phi x)[3rh + r + 1, \infty)$ begins with a maximal path in $Z_+(h)$. Every maximal path in $Z_+(h)$ occurs in this way; see argument for $\deg_+(\Phi(S)) \ge p_h(\alpha)$. Thus $\# D(3rh + r + i) \ge \# \mathcal{P}_{+,i}(Z_+(h))$ for all i. Thus by (15), $\# D(3rh + r + k_h + 1) \ge$

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$p_h(\alpha)^{k_h}$ and

$$\begin{aligned} H_+(\Phi(S)) &= \limsup_{n\to\infty} \frac{1}{n} \log \#D(n) \\ &\geq \limsup_{h} \frac{1}{3rh + r + k_h + 1} \log \#D(3rh + r + k_h + 1) \\ &\geq \lim_{h} \frac{1}{3rh + r + k_h + 1} \log p_h(\alpha)^{k_h} = \log \alpha \quad \text{by (14).} \end{aligned}$$

Thus $H_+(\Phi(S)) \geq \log \alpha$.

We show that $H_+(\Phi(S)) \leq \log \alpha$. We may assume that $\alpha < \infty$. Let $\mathcal{M}$ denote the marker symbols in $\Phi(S)$, that is $\mathcal{M} = \{(\Phi x)_0 \mid x \in M\} = \{(e, *) \mid M \cap [e]_0 \neq \emptyset\}$. We partition $D(n)$ according to the last occurrence of a marker symbol, i.e. we consider for $0 \leq k \leq n$ the sets $D(n, k) := \{w = w_0 \dots w_n \in D(n) \mid w_k \in \mathcal{M}, w_i \notin \mathcal{M} \text{ for } i > k\}$. Note that $w \in D(n, 1)$ implies $w_1 = (1, *)$ and thus $\#D(n, 1) = \#D(n - 1, 0)$. Now let $k \geq 2$ and consider $x \in S$, $y = \Phi x$ with $y[0, n] \in D(n, k)$. If there is $0 \leq i < i + m \leq k$ such that $[i, i + m]$ is a long marked interval for x then $L(x_i)$, $L(x_{i+m}) < m \leq k$ and thus by the strict monotonicity of L we obtain $x_i < k$ and $x_{i+m} < k$. Thus $x_k < k$ and $L(x_k) < k \leq n$. If there is no $0 \leq i < i + r \leq k$ such that $[i, j]$ is a long marked interval for x then $x_k = 1 < k$ also. Thus we can write

$$D(n, k) = \bigcup_{e=1}^{k-1} D(n, k, e), \quad \text{where } D(n, k, e) = \{w \in D(n, k) \mid w_k = (e, *)\}.$$

Now let $U(n, e) := \{(\Phi x)[1, n] \mid x \in S, (\Phi x)_0 = (e, *), (\Phi x)_i \notin \mathcal{M} \text{ for } i > 0\}$. Then we have $\#D(n, k, e) \leq \#D(k, k, e)\#U(n - k, e)$ and thus for $n \geq L(1)$ we have

$$\begin{aligned} \#D(n) &= \sum_{k=0}^{n} \#D(n, k) \\ &= \#D(n, 0) + \#D(n, 1) + \sum_{k=2}^{n} \sum_{\{1 \leq e < k \mid L(e) < k\}} \#D(n, k, e) \\ &\leq \#U(n, 1) + \#U(n - 1, 1) + \sum_{k=2}^{n} \sum_{\{1 \leq e < k \mid L(e) < k\}} \#D(k, k, e)\#U(n - k, e). \end{aligned}$$

Thus it suffices to estimate $\#D(k, k, e)$ and $\#U(n, e)$. Let $\varepsilon > 0$. Since $\lambda \leq \alpha < \infty$ and by (1) we can fix I such that $\max(\lambda_I, \alpha) < \alpha + \varepsilon$.

First we estimate $\#D(k, k, e)$ for $L(e) < k$. If $w = (\Phi x)[0, k] \in D(k, k, e)$ then $x[0, k]$ is uniquely determined by w and $x_0 = 1$ and $x_k = e$. Thus $\#D(k, k, e) \leq \#\{x[0, k] \mid x \in S, x_0 = 1, x_k = e\}$. Thus there is a constant Q_1 such that $\#D(k, k, e) < Q_1 \lambda^k$ for all k and all $e \leq I$. Now consider $e > I$. Then $\#D(k, k, e) < (\lambda_e)^k$ by (9), since $k > L(e)$ and $e = \max(1, e)$. Since $\lambda_e < \lambda_I$ by (1) we thus obtain

$$\#D(k, k, e) < Q_1(\lambda_I)^k < Q_1(\alpha + \varepsilon)^k \quad \text{for } L(e) < k.$$

Now we estimate $\#U(n, e)$ for all e and n. If $w = (\Phi x)[1, n] \in U(n, e)$ then $x_0 = e$ and x is marked at time 0, but not at times $1 \leq i \leq n$. Thus $[0, \infty)$ is a

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marked ray for x and w is a path in T_e of length n starting with the symbol (e^*, e), or there is some $m > n$ such that $[0, m]$ is a marked interval for x. Thus we can write $U(n, e) = U_1(n, e) \cup U_2(n, e) \cup U_3(n, e)$ where

$$U_1(n, e) := \{w \in U(n, e) \mid w \text{ is a path of length } n \text{ in } T_e \text{ starting with } (e^*, e)\},$$
$$U_2(n, e) := \{(\Phi x)[1, n] \in U(n, e) \mid [0, m] \text{ is a short marked interval for } x, m > n\},$$
$$U_3(n, e) := \{(\Phi x)[1, n] \in U(n, e) \mid [0, m] \text{ is a long marked interval for } x, m > n\}.$$

Since T_e is a subgraph of $T^{c,p,d}$ we can thus write $n = ar + b$, where $0 \le b < r$ and $r = d + 2c$, and obtain by (5) that

$$\#U_1(n, e) \le p^{d\cdot(a+1)} = p^d p^{da} < p^d \lambda^{ar} \le p^d \lambda^n.$$

If $(\Phi x)[1, n] \in U_2(n, e)$ and $m > n$ such that $x[0, m]$ is a short marked interval for x then $n < m \le L(e)$ and $x_0 = x_m = e$ and thus by (10b) and (10c) with $i = I + 1$,

$$\#U_2(n, e) \le L(e)(\lambda_I)^n \quad \text{if } n \ge M(\lambda_{I+1}, c_{I+1}, p, d_{I+1}) + 2(I + 1) + 4,$$
$$\#U_2(n, e) \le n(\lambda_0)^n \quad \text{if } n < M(\lambda_{I+1}, c_{I+1}, p, d_{I+1}) + 2(I + 1) + 4.$$

Now we write $U_3(n, e) = V(n, e) \cup V'(n, e)$, where

$$V(n, e) = \{(\Phi x)[1, n] \in U_3(n, e) \mid [0, m] \text{ is a marked interval for } x, x[0, m] \notin P\},$$
$$V'(n, e) = \{(\Phi x)[1, n] \in U_3(n, e) \mid [0, m] \text{ is a marked interval for } x, x[0, m] \in P\}.$$

We first estimate $V(n, e)$. Let $w = (\Phi x)[1, n] \in V(n, e)$ and $m > n$ such that $[0, m]$ is a marked interval for x. Then w is the prefix of length n of the block $ABCB'A'$, where

$$A := (F_e x)[0, n_q - 1] \quad \text{is a path in } T_e \text{ starting with } (e^*, e),$$
$$B := \beta_{+,a}(b) \quad \text{is a path in } B_+(a),$$
$$C := \omega_{a,b}(x[i, i + m]) \quad \text{is a path in } W(a, b),$$
$$B' := \beta_{-,b}(a) \quad \text{is a path in } B_-(b),$$
$$A' := (F_f S^m x)[-n_q + 1, 0] \quad \text{is a path in } T_f \text{ ending with } (f^*, f).$$

where $e = x_0$, $f = x_m$, $k = \max(e, f)$ and q is the integer with $n_q \le n_k + \log_p m < n_{q+1}$, and $a = x[0, q]$, $b = [m-q, m]$ by the definition of Φ. The symbol w_n determines whether $n \le |A|$ or $|A| < n \le |AB|$ or $|AB| < n \le |ABC|$ or $[ABC| < n$ without knowing the whole block $ABCB'A'$, since the blocks A, B, C, B', A' use disjoint symbol sets. So we obtain

$$\#V_1(n, e) := \#\{w \in V(n, e) \mid w \quad \text{is a path in } T_e \text{ starting with } (e^*, e)\} < p^d \lambda^n.$$

Let $V_2(n, e) := \{w \in V(n, e) \mid w_n \text{ is a symbol of some } B_+(a)\}$. If $w \in V_2(n, e)$ then there is a unique q such that $n_q < n$ and $A := w_1 \dots w_{n_q}$ is a path in T_e and $w_{n_q+1} \dots w_n$ is a path in $B_+(a)$ starting from the root, where $a = (F_e^{-1} t)[0, q]$ for any $t \in S_{T_e}$ with $t[0, n_q - 1] = A$, by (7a) and $q \ge k \ge e$. Fixing A and using (5) we thus get $\#\{w \in V_2(n, e) \mid w_1 \dots w_{|A|} = A\} \le p^d \lambda^{n-|A|}$, since $B_+(a)$ is a subtree of height $2|A|$ of $T(c, p, d)^\infty$ by (22). Thus we obtain

$$\#V_2(n, e) \le \sum_{\{q \mid n_q < n\}} \#V_1(n_q, e) p^d \lambda^{n-n_q} \le n p^{2d} \lambda^n.$$

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Now let $V_3(n,e) := \{w \in V(n,e) \mid w_n \text{ is a symbol of some spindle } W(a,b)\}$. If $w \in V_3(n,e)$ then there is a unique q such that $3n_q < n$, $A := w_1 \dots w_{n_q}$ is a path in T_e, $B = w_{n_q+1} \dots w_{3n_q}$ is a maximal path in $B_+(a)$ and $w_{3n_q+1} \dots w_n$ is a path in $W(a,b)$ starting from the root where $a = (F_e^{-1}t)[0,q]$ for any $t \in S_{T_e}$ with $t[0,n_q-1] = A$ and $b = \beta_{+,a}^{-1}(B)$.

Now let $x \in S$ such that $w = (\Phi x)[1,n]$ and let $m > n$ such that $[0,m]$ is a long marked interval for x. Note that the block AB already determines q, a, b and $k = k(AB) = \max(x_0, x_m) \geq x_0 = e$. Note further that the blocks a, b determine AB by $A = (F_e s)[0, n_q - 1]$ for any $s \in S$ with $s[0,q] = a$, using (7a), and $B = \beta_{+,a}(b)$. So AB determines a, b and a, b determines AB. Recall that $i = i(a,b)$ is the largest integer such that $|u| - 1 > L(i)$ for all $u \in H(a,b)$, thus we get by (20) that

$$\begin{aligned}\#\{w \in V_3(n,e) \mid w_1 \dots w_{|AB|} = AB\} &\leq \#\mathcal{P}_{+,n-|AB|}(W(a,b)) \\ &< p^{(n-|AB|)d_i/(d_i+2c_i)} \\ &\leq (\lambda_{i-1})^{n-|AB|} = (\lambda_{i(a,b)-1})^{n-|AB|} \quad \text{by (2).}\end{aligned}$$

We now want to show that there is some N such that $i(a,b) > I$ for all AB occurring as proper prefixes of some $w \in V_3(n,e)$ for all $n \geq N$.

Fix q_0 such that $n_{q_0} - n_I > \log_p L(I+1)$. Let N be so large that $\log_p N < n_{q+1}$ implies $q \geq q_0$. Let $n \geq N$ and let AB occur as proper prefixes of some $w \in V_3(n,e)$.

If $k(AB) > I$ then $i(a,b) \geq k(AB) > I$.

If $k(AB) \leq I$ then AB determines a, b, q, k. By the definition of $H(a,b)$, $n_q \leq n_k + \log_p m < n_{q+1}$ for any m such that there is $u \in H(a,b)$ with $|u| = m+1$. Since $n < m$ we thus obtain $\log_p n < n_{q+1}$. Thus since $n \geq N$, $q \geq q_0$ by the choice of N. Thus (6) implies $\log_p m \geq n_q - n_k \geq n_{q_0} - n_I > \log_p L(I+1)$. Thus $m > L(I+1)$ for all m such that there is $u \in H(a,b)$ with $|u| = m+1$. Thus $i(a,b) > I$.

Thus we obtain for $n > N$ or $e \geq I$ and by (1) that

$$\begin{aligned}\#V_3(n,e) &\leq \sum_{\{q \mid 3n_q < n\}} \sum_{\{AB, |AB| = 3n_q\}} \#\{w \in V_3(n,e) \mid w_1 \dots w_{3n_q} = AB\} \\ &\leq \sum_{\{q \mid 3n_q < n\}} \sum_{\{AB, |AB| = 3n_q\}} (\lambda_{i(a,b)-1})^{n-3n_q} \\ &\leq \sum_{\{q \mid 3n_q < n\}} \sum_{\{AB, |AB| = 3n_q\}} (\lambda_I)^{n-3n_q} \leq \sum_{\{q \mid 3n_q < n\}} \#V_2(3n_q, e)(\lambda_I)^{n-3n_q} \\ &\leq \sum_{\{q \mid 3n_q < n\}} 3n_q p^{2d} \lambda^{3n_q} (\lambda_I)^{n-3n_q} \leq n^2 p^{2d} (\lambda_I)^n.\end{aligned}$$

For $Q_2 := \max\{\#V_3(n,e) \mid n \leq N, e < I\} + 1$ we have

$$\#V_3(n,e) \leq Q_2 n^2 p^{2d} (\lambda_I)^n \quad \text{for all } n \text{ and all } e.$$

Let $V_4(n,e) := \{w \in V(n,e) \mid w \text{ contains a symbol from some } B_-(b)\}$. Let $w = (\Phi x)[1,n] \in V_4(n,e)$. Let m be such that $[0,m]$ is a marked interval for x. Then there is a smallest j such that w_{j+1} is a symbol from some $B_-(b)$. Let $ABC = w_1 \dots w_j$. Then AB

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determines a, b and then $x[0,m] = \omega_{a,b}^{-1}(C)$. Thus ABC determines $ABCB'A'$. Therefore

$$\begin{aligned}\#V_4(n,e) &= \sum_{k=1}^{n-1} \sum_{\{ABC, |ABC|=k\}} \#\{w \in U_4(n,e) \mid w_1 \dots w_k = ABC\} \\ &\le \sum_{k=1}^{n-1} \#V_3(k,e) \le Q_2 n^3 p^{2d} (\lambda_I)^n \quad \text{for all } n \text{ and all } e.\end{aligned}$$

By adding up the inequalities and since $\lambda < \lambda_I < \alpha + \varepsilon$ we obtain for a suitable Q_3 that

$$\#V(n,e) \le \sum_{k=1}^{4} \#V_k(n,e) \le 4 Q_2 n^3 p^{2d} (\lambda_I)^n \le Q_3 (\alpha+\varepsilon)^n.$$

Now we estimate $\#V'(n,e)$. Let $V_1'(n,e) = \{w \in V'(n,e) \mid w$ contains a symbol from some $Z_+(h)\}$. Let $w = (\Phi x)[1,n] \in V'(n,e)$. Then there is some $m > n$ such that $x[0,m]$ is a long marked interval with $x[0,m] \in P$. Thus $(\Phi x)[1, m-1]$ is of the form ABA' and $|A| < n \le |ABA'|$ and there is a unique h such that $|A| = 3rh + r$ and B is a maximal path in $Z_+(h)$. Note that h uniquely determines A and A'. Thus by (15),

$$\begin{aligned}\#\{w \in V_1'(n,e) \mid w_1 \dots w_{3rh+r} = A\} &\le \#\mathcal{P}_{+,n-3rh-r}(Z_+(h)) \\ &\le p_h(\alpha)^{\min(k_h, n-3rh-r-1)}.\end{aligned}$$

Recall that $\alpha < \infty$. If $\alpha = p(\alpha)$ then $p_h(\alpha) = p(\alpha)$ also. If $\alpha < p(\alpha)$ then by (13), $p_h(\alpha)^{\min(k_h, n-3rh-r-1)} \le \alpha^{\min(k_h, n-3rh-r-1)+3rh+r+1} \le \alpha^n$. Thus in both cases $\#\{w \in V_1'(n,e) \mid w_1 \dots w_{3rh+r} = A\} \le \alpha^n$. Thus

$$\begin{aligned}\#V_1'(n,e) &= \sum_{\{h \ge H_\alpha \mid 3rh+r \le n\}} \#\{w \in V_1'(n,e) \mid w_1 \dots w_{3rh+r} = (F_1 x^*)[0, 3rh+r-1]\} \\ &\le \sum_{\{h \ge 1 \mid 3rh+r \le n\}} \alpha^n \le n\alpha^n.\end{aligned}$$

Let $V_2'(n,e) = \{w \in U(n,e) \mid w$ contains a symbol from some $Z_-(h)\}$. Let $(\Phi x)[1,n] \in V_2'(n,e)$. Then there is some $m > n$ such that $x[0,m]$ is a long marked interval with $x[0,m] \in P$. Thus $(\Phi x)[1, m-1]$ is of the form ABA' and $|A| < n \le |ABA'|$. Say $(\Phi x)[1, m-1]$ contains a symbol from $Z_-(h)$. Then $A = (F_1 x^*)[0, 3rh+2r-1]$ and $A' = (F_1 x^*)[-3rh-2r+1, 0]$ are uniquely determined by h and we have $3rh + 2r < n$. B is a maximal path in the spindle $Z_-(h)$ and $(\Phi x)[3rh+2r, n]$ is a path of length $n - 3rh - 2r$ in $Z_-(h)$ starting from the root. The number of these paths in $Z_-(h)$ is at most $\lambda^{n-3rh-2r+d+2c}$ by (19). Adding up these for the different values of h we obtain $\#V_2'(n,e) \le n\lambda^n$ and thus for a suitable Q_4 we have

$$\begin{aligned}\#V'(n,e) &\le \#U_1(n,e) + \#V_1'(n,e) + \#V_2'(n,e) \\ &\le p^d \lambda^n + n\alpha^n + n\lambda^n \le 3p^d n\alpha^n \le Q_4(\alpha+\varepsilon)^n.\end{aligned}$$

Let $M_I := M(\lambda_{I+1}, c_{I+1}, p, d_{I+1}) + 2(I+1) + 4$. Since $U(n,e) = U_1(n,e) \cup U_2(n,e) \cup U_3(n,e)$ and $U_3(n,e) = V(n,e) \cup V'(n,e)$ we thus obtain for $n \ge M_I$ that

$$\begin{aligned}\#U(n,e) &\le p^d \lambda^n + L(e)(\lambda_I)^n + Q_3(\alpha+\varepsilon)^n + Q_4(\alpha+\varepsilon)^n \\ &\le (L(e) + Q_5)(\alpha+\varepsilon)^n\end{aligned}$$

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for a suitable Q_5. For $n < M_I$ we obtain that

$$\begin{aligned}\#U(n,e) &\le p^d\lambda^n + n(\lambda_0)^n + Q_3(\alpha+\varepsilon)^n + Q_4(\alpha+\varepsilon)^n \\ &\le n(\lambda_0)^n + Q_5(\alpha+\varepsilon)^n \le Q_6\end{aligned}$$

for a suitable constant Q_6. Recall that $\#D(k,k,e) < Q_1(\alpha+\varepsilon)^k$ for $L(e) < k$ and that

$$\#D(n) \le \#U(n,1) + \#U(n-1,1) + \sum_{k=2}^{n} \sum_{\{1\le e<k \mid L(e)<k\}} \#D(k,k,e)\#U(n-k,e).$$

For $1 \le e < k \le n$ with $n-k < M_I$ we have

$$\#D(k,k,e)\#U(n-k,e) \le Q_1(\alpha+\varepsilon)^k Q_6 \le Q_1 Q_6(\alpha+\varepsilon)^n.$$

For $1 \le e < k \le n$ with $n-k \ge M_I$ we have

$$\begin{aligned}\#D(k,k,e)\#U(n-k,e) &\le Q_1(\alpha+\varepsilon)^k(L(e)+Q_5)(\alpha+\varepsilon)^{n-k} \\ &= Q_1(L(e)+Q_5)(\alpha+\varepsilon)^n.\end{aligned}$$

Thus for $n > M_I$ we obtain

$$\begin{aligned}\#D(n) &\le 2(L(1)+Q_5)(\alpha+\varepsilon)^n + \sum_{k=2}^{n-M_I} \sum_{\{1\le e<k \mid L(e)<k\}} Q_1(L(e)+Q_5)(\alpha+\varepsilon)^n \\ &\quad + \sum_{k=n-M_I+1}^{n} \sum_{\{1\le e<k \mid L(e)<k\}} Q_1 Q_6(\alpha+\varepsilon)^n \\ &\le 3n^3 Q_7(\alpha+\varepsilon)^n\end{aligned}$$

for a suitable constant Q_7. Thus we have shown that for every $\varepsilon > 0$

$$\limsup_{n\to\infty} \frac{1}{n} \log \#D(n) \le \log(\alpha+\varepsilon).$$

This finishes the proof of $h_+(\Phi(S)) = \log\alpha$. The argument for $\mathrm{H}_-(\Phi(S)) = \log\beta$ is symmetric.

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