Math. Proc. Camb. Phil. Soc. (2002), **132**, 435 © 2002 *Cambridge Philosophical Society*
DOI: 10.1017/S030500410100562X *Printed in the United Kingdom*

Euler characteristics for orbifold generalized triangle groups

By ALUN G. T. WILLIAMS

*Department of Mathematics, Heriot Watt University, Edinburgh EH*14 4*AS.*
e-mail: gerald@ma.hw.ac.uk

(*Received* 28 *June* 2000)

1. *Introduction*

A *generalized triangle group* is a group with a presentation

$$GT(l, m, n; w) = \langle a, b \,|\, a^l = b^m = w^n = 1 \rangle,$$

where $l, m, n \geqslant 2$, and w is a cyclically reduced word in the free product on $\{a, b\}$. In the setting of one-relator products of cyclic groups, Fine *et al.* [**3**] calculated their Euler characteristics for $n \geqslant 3$ under certain conditions on the defining word w.

Property E. w is conjugate in the free product on a, b to a word uv for some elements u, v of orders p, q ($\geqslant 2$) respectively with

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{n} > 1.$$

Theorem 1·1 ([**3**]). *Let* $G = GT(l, m, n; w)$ *and suppose one of the following holds*:
(1) $n \geqslant 4$ *and* G *does not have Property E.*
(2) $n = 3$ *and the word* w *does not involve an element of order* 2.
Then

$$\chi(G) = 1/l + 1/m + 1/n - 1.$$

In [**6**] Jones and Reid investigate connections between generalized triangle groups and 3-dimensional orbifolds. They show that any group $GT(l, m, n; W(r, s, a, b))$ where $r < s$ are positive coprime integers and

$$W(r, s, a, b) = \prod_{i=0}^{s-1} a^{(-1)^{\lfloor (2i+1)r/s \rfloor}} b^{(-1)^{\lfloor (2i+2)r/s \rfloor}}$$

arises as the fundamental group of a 3-dimensional orbifold Q whose singular set in S^3 is a graph (with 3 edges and 2 vertices) formed by adding an extra edge to a 2-bridge knot or link L and labelling the edges l, m, n to denote cone angles of $2\pi/l$, $2\pi/m$, $2\pi/n$ respectively. Moreover they show that the fundamental group of every such orbifold is of the form $GT(l, m, n; W(r, s, a, b))$ for some pair (r, s). We write $Q = Q_{r,s}(l, m, n)$. Using Thurston's Orbifold Conjecture [**5**, **9**], Jones and Reid [**6**] go on to prove that $Q_{r,s}(l, m, n)$ has a geometric structure unless L is a link of two unknotted, unlinked components, and that in most cases Q is hyperbolic. Thus $G = GT(l, m, n; W(r, s, a, b))$ admits a faithful, discrete representation in $PSL(2, \mathbb{C})$, that is, a faithful representation as a *Kleinian group*.

Here we calculate the Euler characteristic of every group $G = GT(l, m, n; W(r, s, a, b))$ arising as the fundamental group of a hyperbolic orbifold $Q =$

https://doi.org/10.1017/S030500410100562X Published online by Cambridge University Press

$Q_{r,s}(l,m,n)$. A corollary of this result provides necessary and sufficient conditions for G to admit a faithful, discrete, finite co-volume representation in $PSL(2,\mathbb{C})$. This improves upon theorem 3·2 of [**4**] where necessary conditions were provided for the groups $GT(l,m,n;w)$ to admit such representations when either $n \geqslant 4$, or $n = 3$ and w involves no element of order 2. We also prove a result which provides conditions for G to admit a faithful, discrete, co-compact representation in $PSL(2,\mathbb{C})$.

2. *The main theorem*

Suppose $Q = Q_{r,s}(l,m,n)$ is a hyperbolic orbifold with fundamental group $G = \pi_1^{\mathrm{orb}}(Q) = GT(l,m,n;W(r,s,a,b))$. If s is odd then L is a knot, and thus both vertices of the graph representing the singular set of $Q = Q_{r,s}(l,m,n)$ are labelled (l,m,n). If s is even then L is a link, and one vertex is labelled (l,l,n) and the other is labelled (m,m,n). Our main result proves that the Euler characteristic of G depends only on the vertex labelling of Q.

Define

$$c_1(G) = \begin{cases} 1/l + 1/m + 1/n - 1 & \text{if } s \text{ is odd,} \\ 2/l + 1/n - 1 & \text{if } s \text{ is even,} \end{cases}$$

$$c_2(G) = \begin{cases} 1/l + 1/m + 1/n - 1 & \text{if } s \text{ is odd,} \\ 2/m + 1/n - 1 & \text{if } s \text{ is even.} \end{cases}$$

THEOREM 2·1. *Suppose $Q = Q_{r,s}(l,m,n)$ is a hyperbolic orbifold with $G = \pi_1^{\mathrm{orb}}(Q) = GT(l,m,n;W(r,s,a,b))$ and let*

$$\chi_1 = \min\{0, c_1(G)\},$$
$$\chi_2 = \min\{0, c_2(G)\}.$$

Then the Euler characteristic

$$\chi(G) = \frac{\chi_1 + \chi_2}{2}.$$

Proof. Since Q is hyperbolic, G admits a faithful discrete representation ρ to $PSL(2,\mathbb{C})$. By Selberg's lemma [**8**], there exists a torsion-free subgroup K of $\rho(G)$ of finite index p (say) so $\chi(G) = \chi(K)/p$. By example 2·4·8 of [**1**, p. 40] the manifold $M = H^3/K$ is aspherical and thus $\chi(K) = \chi(M)$. This manifold is a p-fold branched covering of Q; we denote the covering map by $\pi\colon M \to Q$.

Let $Q_0 = Q\backslash(B(v_1)\dot{\cup}B(v_2))$ be the orbifold with boundary $\partial(B(v_1))\dot{\cup}\partial(B(v_2))$ formed by removing neighbourhoods $B(v_1), B(v_2)$ of the vertices v_1, v_2 of the singular set of Q. Then $M_0 = \pi^{-1}(Q_0)$ is a manifold with boundary $V_1\dot{\cup}V_2$ where $V_i = \pi^{-1}(\partial B(v_i))$. Now M_0 is a deformation retract of M, so $\chi(M_0) = \chi(M)$.

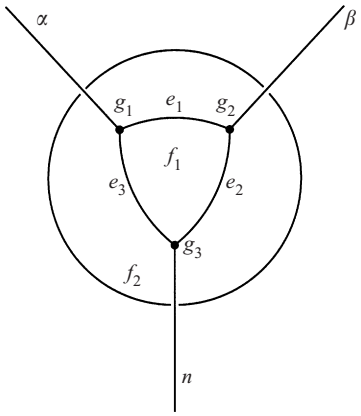
Form the manifold M_1 by capping off the 2-spheres in V_i if the vertex v_i has spherical labelling. Then $\partial M_1 = U_1\dot{\cup}U_2$ where $U_i = \varnothing$ if the labelling of v_i is spherical and $U_i = V_i$ otherwise. Now

$$\chi(M_0) = \chi(M_1) = \chi(\partial M_1)/2 = (\chi(U_1) + \chi(U_2))/2,$$

so all that remains is to show that $\chi(U_i) = p \cdot \chi_i$ for $i = 1, 2$. If the labelling of v_i is spherical then $\chi(U_i) = 0 = \chi_i$. Thus we may assume the labelling is euclidean or hyperbolic and we are left with an exercise in branched coverings.

https://doi.org/10.1017/S030500410100562X Published online by Cambridge University Press

The edges incident to the vertex v_i are labelled (α, β, n) where $\alpha = l$, $\beta = m$ if s is odd, and if s is even then $\alpha = \beta = l$, when $i = 1$ and $\alpha = \beta = m$, when $i = 2$. The boundary $\partial B(v_i)$ has the following triangulation:



This lifts to a triangulation of U_i: the two 2-cells f_1, f_2 lift to $2p$ 2-cells in V_i, the three 1-cells e_1, e_2, e_3 each lift to p 1-cells in V_i. Suppose that there are N_1 0-cells in U_i covering the 0-cell g_1. Taking account of the branching index, adjacent to each 0-cell in U_i covering g_1 there are α copies of e_1. Since there is a total of p 1-cells in V_i which cover e_1 we have

$$\alpha \cdot N_1 = p.$$

Similarly if N_2, N_3 are the number of 0-cells in U_i which cover g_2, g_3 then

$$\beta \cdot N_2 = p,$$
$$n \cdot N_3 = p.$$

Hence the total number of 0-cells in U_i is given by

$$N = N_1 + N_2 + N_3 = p \cdot (1/\alpha + 1/\beta + 1/n).$$

Thus

$$\begin{aligned} \chi(U_i) &= p \cdot (1/\alpha + 1/\beta + 1/n) - 3p + 2p \\ &= p \cdot (1/\alpha + 1/\beta + 1/n - 1) \\ &= p \cdot \chi_i \end{aligned}$$

as required.

By theorem 8 of [**7**] a non-elementary Kleinian group Γ is of finite co-volume if and only if $\chi(\Gamma) \geqslant 0$, so we can deduce

Corollary 2·2. *Suppose $G = GT(l, m, n; W(r, s, a, b))$ is the fundamental group of a hyperbolic orbifold $Q_{r,s}(l, m, n)$. Then G admits a faithful discrete representation in $PSL(2, \mathbb{C})$ of finite co-volume if and only if*

$$c_1(G) \geqslant 0 \quad \textit{and} \quad c_2(G) \geqslant 0.$$

A countable group G is said to be *SQ-universal* if for every countable group C, there exists a quotient group of G in which C can be embedded. In connection with

https://doi.org/10.1017/S030500410100562X Published online by Cambridge University Press

a conjecture of Rosenberger ([**2**]) we also note the following, which is immediate from theorem 5 of [**7**].

Corollary 2·3. *Suppose $G = GT(l, m, n; W(r, s, a, b))$ is the fundamental group of a hyperbolic orbifold $Q_{r,s}(l, m, n)$. If $c_1(G) < 0$ or $c_2(G) < 0$ then G is SQ-universal.*

3. *Co-compact representations*

If $c_1(G), c_2(G) > 0$ then Q is a compact orbifold, and thus this is a sufficient condition for G to admit a faithful discrete co-compact representation in $PSL(2, \mathbb{C})$. Again we can show that this is also a necessary condition.

Theorem 3·1. *Suppose a group Γ admits faithful discrete representations ρ_1, ρ_2 into $PSL(2, \mathbb{C})$, then $\rho_1(\Gamma), \rho_2(\Gamma)$ are either both co-compact or both non-co-compact.*

Proof. Suppose (for contradiction) that $\rho_1(\Gamma)$ is a co-compact and $\rho_2(\Gamma)$ is non-co-compact. By Selberg's lemma [**8**], there exist torsion-free subgroups H_1, H_2 of finite index in $\rho_1(\Gamma), \rho_2(\Gamma)$ respectively. These are discrete, and H_1 is co-compact while H_2 is non-co-compact. Let K be a maximal co-compact subgroup of $PSL(2, \mathbb{C})$, and let $d = \dim(PSL(2, \mathbb{C})) - \dim K$. Then by example 8·9·4 of [**1**, p. 217] the cohomological dimensions

$$\operatorname{cd}(H_1) = d,$$
$$\operatorname{cd}(H_2) < d.$$

But the virtual cohomological dimension $\operatorname{vcd}(G) = \operatorname{cd}(H_1) = \operatorname{cd}(H_2)$, a contradiction.

Corollary 3·2. *Suppose $G = GT(l, m, n; W(r, s, a, b))$ is the fundamental group of a hyperbolic orbifold $Q_{r,s}(l, m, n)$. Then G admits a faithful discrete co-compact representation in $PSL(2, \mathbb{C})$ if and only if*

$$c_1(G) > 0 \quad \text{and} \quad c_2(G) > 0.$$

Acknowledgements. I am grateful to my Ph.D. supervisor Professor James Howie for his help with this work, and to Dr Colin Maclachlan for comments which led to a more precise proof of the main theorem.

References

[**1**] K. S. Brown. *Cohomology of groups* (Springer-Verlag, 1982).
[**2**] B. Fine, F. Levin and G. Rosenberger. Free subgroups and decompositions of one-relator products of cyclics. I. The Tits alternative. *Arch. Math. (Basel)* **50** (1988), 97–109.
[**3**] B. Fine, G. Rosenberger and M. Stille. Euler characteristic for one-relator products of cyclics. *Comm. Algebra* **21** (1993), 4353–4359.
[**4**] M. Hagelberg, C. Maclachlan and G. Rosenberger. On discrete generalised triangle groups. *Proc. Edinburgh Math. Soc.* (2), **38** (1995), 397–412.
[**5**] C. D. Hodgson. Geometric structures on 3-dimensional orbifolds: notes on Thurston's proof. Preprint.
[**6**] K. N. Jones and A. W. Reid. Minimal index torsion-free subgroups of Kleinian groups. *Math. Ann.* **310** (1998), 235–250.
[**7**] J. G. Ratcliffe. Euler characteristics of 3-manifold groups and discrete subgroups of $SL(2, \mathbb{C})$. In *Proceedings of the Northwestern Conference on Cohomology of Groups* (Evanston, IL, 1985), vol. 44, pp. 303–314, 1987.
[**8**] A. Selberg. On discontinuous groups in higher-dimensional symmetric spaces. In *Contributions to function theory* (Internat. Colloq. Function Theory, Bombay, 1960), (Tata Institute of Fundamental Research, Bombay, 1960), pp. 147–164.
[**9**] W. P. Thurston. Three manifolds with symmetry. Preprint.

https://doi.org/10.1017/S030500410100562X Published online by Cambridge University Press