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109.47 Invariant lines in the Euclidean plane

Introduction

The action of a real 2×2 matrix A on the Euclidean plane $\mathbb{R}^2$, together with elementary ideas about the eigenvalues and eigenvectors of A, and points and lines in $\mathbb{R}^2$ that are invariant under A, all appear on some of the current AS/A Further Mathematics specifications. Here we expand on the discussion given in [1] by giving an elementary description of all such invariant points and lines.

For any matrix (or vector) M, we denote the transpose of M by M^t. Points in $\mathbb{R}^2$ will be denoted by column vectors $\mathbf{x}$, $\mathbf{y}$, …, where, for example, $\mathbf{x} = (x_1, x_2)^t$, and $\mathbf{0}$ is the zero vector. Throughout, we say that a set E is *invariant* if, and only if, $A(E) = E$ (although in many other contexts 'invariance' means $A(E) \subseteq E$).

Invariant (i.e. fixed) points are the solutions of the equation $A\mathbf{x} = \mathbf{x}$. Obviously, $\mathbf{0}$ is invariant for every A. A non-zero point $\mathbf{x}$ is invariant if, and only if, it is an eigenvector of A with associated eigenvalue equal to 1. By definition, an eigenvector is non-zero, and it is convenient to refer to the set of all real multiples of a given eigenvector as an *eigenline*. Then *the set of invariant points is the union of all eigenlines whose associated eigenvalue* is 1.

The eigenvalues and eigenvectors of A^t

The key idea is to use the transpose A^t of the matrix A, and it is well known that the eigenvalues of A coincide with those of A^t: indeed, both are solutions of the equation

$$t^2 - (a + d)t + (ad - bc) = 0, \qquad \text{where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \qquad (1)$$

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We shall now compare the eigenvectors of A with those of A^t: see Figure 1.

Theorem 1: Suppose that A has real eigenvalues λ and μ. Then

$$A\mathbf{v} = \lambda\mathbf{v} \text{ if, and only if, } A^t\mathbf{v}^* = \mu\mathbf{v}^*;$$

$$A\mathbf{v} = \mu\mathbf{v} \text{ if, and only if, } A^t\mathbf{v}^* = \lambda\mathbf{v}^*,$$

where $\mathbf{v} = (v_1, v_2)^t$ and $\mathbf{v}^* = (v_2, -v_1)^t$. Further, $\mathbf{v}^* \perp \mathbf{v}$ since $\mathbf{v}\cdot\mathbf{v}^* = 0$.

Proof: We see from (1) that $\lambda + \mu = a + d$. Suppose now that $A\mathbf{v} = \lambda\mathbf{v}$: then

$$av_1 + bv_2 = \lambda v_1;$$

$$cv_1 + dv_2 = \lambda v_2.$$

If we now replace λ by $a + d - \mu$, we find that $A^t\mathbf{v}^* = \mu\mathbf{v}^*$. The rest of the proof follows in a similar way.

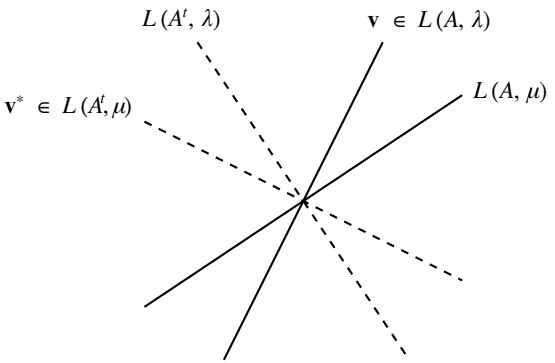


FIGURE 1: Eigenlines for A and A^t

Invariant lines when A is non-singular

In this section we shall assume that A is non-singular. Then A is a bijection of $\mathbb{R}^2$ onto itself, and, for each line L, $A(L) \subseteq L$ implies $A(L) = L$. We begin with some examples to show the different ways in which invariant lines can occur:

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ A_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ A_4 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix},\ A_5 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If $A = A_1$ then every line through $\mathbf{0}$ is invariant, and there are no other invariant lines. If $A = A_2$, then A acts as the reflection across the horizontal axis, so this axis, together with each vertical line, is invariant. If $A = A_3$, then A acts as a shear, and the horizontal lines form the complete set of invariant lines. If $A = A_4$, then A acts as a shear followed by an expansion, and the horizontal axis is the only invariant line. Finally, if $A = A_5$, then A

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acts as a rotation by $\frac{1}{2}\pi$ so there are no invariant lines. The reader may find it helpful to return to these examples after reading the following discussion.

It is important to distinguish between those invariant lines that pass through $\mathbf{0}$ and those that do not. Clearly, a line that passes through $\mathbf{0}$ is invariant if, and only if, it is an eigenline. We shall now show that a line L that does not pass through $\mathbf{0}$ is invariant if, and only if, its normal $\mathbf{n}$ satisfies $A^t\mathbf{n} = \mathbf{n}$ (that is, $\mathbf{n}$ is an eigenvector of A^t with eigenvalue 1). Better still, we have the next two results.

Theorem 2: Let L be the line given by $\mathbf{n}^t\mathbf{x} = r$, where $\mathbf{n} \neq \mathbf{0}$. If $A^t\mathbf{n} = \mathbf{n}$ then each line parallel to L (including L itself) is invariant.

Proof: Suppose that $A^t\mathbf{n} = \mathbf{n}$, and that $\mathbf{x} \in L$. Then $\mathbf{n}^t(A\mathbf{x}) = (\mathbf{n}^tA)\mathbf{x} = \mathbf{n}^t\mathbf{x} = r$ so that L is invariant. Since each line parallel to L has equation $\mathbf{n}^t\mathbf{x} = s$ for some s, these lines are also invariant.

Theorem 3: Let L be the line given by $\mathbf{n}^t\mathbf{x} = r$, where $\mathbf{n} \neq \mathbf{0}$. If L is invariant and $\mathbf{0} \notin L$ then $A^t\mathbf{n} = \mathbf{n}$, and A has an eigenvalue 1.

Proof: First, suppose that L is invariant, and that $\mathbf{x} \in L$. Then $(\mathbf{n}^tA)\mathbf{x} = \mathbf{n}^t(A\mathbf{x}) = r$. Thus L is given by both of the equations $\mathbf{n}^t\mathbf{x} = r$ and $\mathbf{m}^t\mathbf{x} = r$, where $\mathbf{m}^t = \mathbf{n}^tA$. Since $\mathbf{0} \notin L$ we find that $r \neq 0$, and this implies that $\mathbf{m} = \mathbf{n}$: thus $A^t\mathbf{n} = \mathbf{n}$.

In conclusion, consider the case in which A has two real eigenvalues 1 and μ, where $\mu \neq 1$. Let $L(\mu)$ be the eigenline of vectors $\mathbf{x}$ with $A\mathbf{x} = \mu\mathbf{x}$. Theorem 1 shows that $L(\mu)$ has normal $\mathbf{n}$, where $A^t\mathbf{n} = \mathbf{n}$ so, by Theorem 2, every line parallel to $L(\mu)$ is invariant. By Theorem 3 the only other invariant line is the eigenline of fixed points of A, and this crosses every line parallel to $L(\mu)$.

Finally, suppose that A has the repeated eigenvalue 1, but only one eigenline $L(1)$. By Theorem 1, $L(1)$ has normal $\mathbf{n}$, where $A^t\mathbf{n} = \mathbf{n}$, so, by Theorem 2, every line parallel to $L(1)$ is invariant. Further, by Theorem 3, no other line is invariant.

Invariant lines when A is singular

Suppose that A is a singular matrix. Then $A(\mathbb{R}^2)$ is either $\{\mathbf{0}\}$ or a line L, and if $A(\mathbb{R}^2) = L$ then $A(L)$ is either $\{\mathbf{0}\}$ or a line L'. Therefore we must consider the following three cases: (i) $A(\mathbb{R}^2) = \{\mathbf{0}\}$; (ii) $A(\mathbb{R}^2) = L$ and $A(L) = \{\mathbf{0}\}$, and (iii) $A(\mathbb{R}^2) = L$ and $A(L) = L'$. Obviously, (i) occurs if, and only if, A is the zero matrix, and there are no invariant lines. Next, (ii) occurs if, and only if, A^2, but not A, is the zero matrix, and again there are no invariant lines. Suppose now that (iii) holds. Then $L' = A(L) \subseteq A(\mathbb{R}^2) = L$. As L and L' are lines, this shows that $L = L'$ so that $A(L) = L$, and L is an invariant line. Clearly, in this case L is the only invariant line.

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109.48 A generalisation and proof of the Steiner–Lehmus Theorem

The Steiner–Lehmus theorem stands as one of the cornerstones of classical Euclidean geometry. It articulates that when the bisectors of two internal angles within a triangle are equal, the triangle becomes isosceles. This theorem has been extensively studied, leading to various extensions and adaptations found in references [1, 2] for broader applications, alongside alternative formulations available in [3, 4, 5, 6]. Malesevic presented a compelling direct proof of this theorem in [7]. Building upon this rich foundation, our Note introduces a novel, straightforward generalisation of the theorem, accompanied by a new proof.

Theorem 1 (Generalisation of the Steiner–Lehmus Theorem):

Let ABC be a triangle. Points E and F lie on sides CA and AB, respectively, such that

$$\angle ABE - \angle ACF = \frac{\angle ABC - \angle ACB}{2}. \tag{1}$$

If $BE = CF$, then triangle ABC is isosceles with $AB = AC$.

When BE and CF are the bisectors of angles $\angle ABC$ and $\angle ACB$ respectively, the equality (1) is satisfied. At this point, the theorem simplifies to the classical Steiner–Lehmus Theorem.

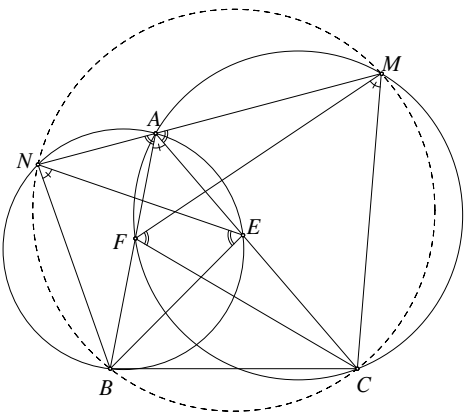


FIGURE 1: A direct proof of the generalised Steiner–Lehmus Theorem

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