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ON A CLASS OF MULTIVARIATE COUNTING PROCESSES

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Abstract

In this paper we define and study a new class of multivariate counting processes, named 'multivariate generalized Pólya process'. Initially, we define and study the bivariate generalized Pólya process and briefly discuss its reliability application. In order to derive the main properties of the process, we suggest some key properties and an important characterization of the process. Due to these properties and the characterization, the main properties of the bivariate generalized Pólya process are obtained efficiently. The marginal processes of the multivariate generalized Pólya process are shown to be the univariate generalized Pólya processes studied in Cha (2014). Given the history of a marginal process, the conditional property of the other process is also discussed. The bivariate generalized Pólya process is extended to the multivariate case. We define a new dependence concept for multivariate point processes and, based on it, we analyze the dependence structure of the multivariate generalized Pólya process.

Keywords: Multivariate generalized Pólya process; marginal process; complete stochastic intensity function; dependence structure; conditional counting process

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1. Introduction

A point process (or counting process) is a type of random process for which any generic realization consists of a set of isolated points either in time or geographical space, or in even more general spaces. Generally, in order to describe the event (point) occurrences over time, a point process relies on the sequential arrival points over $(0, \infty)$. On the other hand, a counting process describes the random occurrences of events in terms of the counts of events observed over time intervals. In the following, for convenience, we will use the terms 'point process' and 'counting process' interchangeably.

Until now, various univariate point processes have been proposed and the properties of these have been intensively studied. The univariate point processes which are most frequently applied in practice include the Markov process, homogeneous and nonhomogeneous Poisson processes, and the renewal process. In particular, homogeneous and nonhomogeneous Poisson processes are the most commonly used point processes in practice due to their nice properties, which generally yield explicit results in most applications even when the stochastic model under study is complex; see, for example, Cha and Finkelstein (2009), Cha and Lee (2011), and Liang and Wang (2012).

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Stochastically dependent multivariate series of events arise in many contexts. For instance, queues are a well-known situation where bivariate point processes arise as the input and output processes (Daley (1968)). In reliability applications, the consecutive failures in two parts of a system are frequently positively dependent (see also Section 2). In a finance context, the bankruptcy of a financial company in one group may also affect those in other groups (Allen and Gale (2000)). In econometrics, multivariate point processes are frequently used to model multivariate market events; see Bowsher (2007). In insurance, two types of claims can be modeled by a bivariate point process (Partrat (1994)). For more examples, see Cox and Lewis (1972). Although the general theoretical framework was constructed in Cox and Lewis (1972), practically available multivariate point processes have not yet been developed to the same extent as in the univariate case and, accordingly, there exists a large gap between the desired practical applications and available useful models. In this regard, the aim of this paper is to develop a new class of multivariate point processes which possess nice properties, allowing mathematical tractability in various applications.

The structure of the paper is as follows. In Section 2 we start with the bivariate case and define a new class of bivariate counting processes, named 'bivariate generalized Pólya process', based on a stochastic intensity approach. Then we study some important properties of this process. The joint distribution of the number of events will be derived based on the obtained properties. As a possible application of the bivariate generalized Pólya process, a specific example for modeling the 'dependent failure and repair process' will be suggested. In Section 3, in order to derive the main properties of the process, we suggest an important characterization of the process. Due to this characterization, the main properties of the bivariate generalized Pólya process are obtained efficiently. We define a new dependence concept for bivariate point processes and, based on it, we analyze the dependence structure of the bivariate generalized Pólya process. The marginal and conditional processes of the bivariate generalized Pólya process are also discussed. In Section 4 the bivariate generalized Pólya process is extended to the multivariate case. We generalize the dependence concept defined for the bivariate point processes to the multivariate case and, based on it, we analyze the dependence structure of the multivariate generalized Pólya process. Finally, in Section 5, some concluding remarks will be given.

2. Definition and basic properties: bivariate case

In this section we define the bivariate generalized Pólya process and discuss basic properties of it. Let $\{\boldsymbol{N}(t),\ t \geq 0\}$, where $\boldsymbol{N}(t) = (N_1(t), N_2(t))$, be a bivariate process of our interest and define the corresponding 'pooled' point process $\{M(t),\ t \geq 0\}$, where $M(t) = N_1(t) + N_2(t)$. We can then define the marginal point processes $\{N_i(t),\ t \geq 0\}$ and, for convenience, they will be called type i point process, $i = 1, 2$, respectively. Furthermore, the events from type i point process $\{N_i(t),\ t \geq 0\}$ will also be called type i events. In this paper we will consider *regular* (also known as *orderly*) multivariate point processes. For a univariate point process $\{N(t),\ t \geq 0\}$, it is called 'regular' (or 'orderly') if

$$\mathbb{P}(N(t + \Delta t) - N(t) > 1) = o(\Delta t) \quad \text{for all } t \geq 0.$$

Regularity is intuitively the nonoccurrence of multiple events in a small interval. Note that the 'regularity' in the multivariate process should be more precisely defined (see also Cox and Lewis (1972)). There are two types of regularity that occur in multivariate point processes: marginal regularity and regularity. For a multivariate point process, we say the process is *marginally regular* if its marginal processes, considered as univariate point processes, are all

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regular. The multivariate process is said to be *regular* if the 'pooled' process is regular. This type of regularity, of course, implies the *marginal regularity*. Throughout this paper, we will assume that the multivariate process $\{\boldsymbol{N}(t),\ t \geq 0\}$ of our interest is a regular process.

Let $\mathcal{H}_{\mathrm{P}t-} \equiv \{M(u),\ 0 \leq u < t\}$ be the history (internal filtration) of the pooled process in $[0, t)$, i.e. the set of all point events in $[0, t)$. Observe that $\mathcal{H}_{\mathrm{P}t-}$ can equivalently be defined in terms of $M(t-)$ and the sequential arrival points of the events $0 \leq T_1 \leq T_2 \leq \cdots \leq T_{M(t-)} < t$ in $[0, t)$, where $M(t-)$ is the total number of events in $[0, t)$ and T_i is the time from 0 until the arrival of the ith event in $[0, t)$ of the pooled process $\{M(t),\ t \geq 0\}$. Similarly, define the marginal histories of the marginal processes $\mathcal{H}_{it-} \equiv \{N_i(u),\ 0 \leq u < t\}$, $i = 1, 2$. Then, $\mathcal{H}_{it-} \equiv \{N_i(u),\ 0 \leq u < t\}$ can also be completely defined in terms of $N_i(t-)$ and the sequential arrival points of the events $0 \leq T_{i1} \leq T_{i2} \leq \cdots \leq T_{iN_i(t-)} < t$ in $[0, t)$, $i = 1, 2$, where $N_i(t-)$ is the total number of events of type i point process in $[0, t)$, $i = 1, 2$.

A convenient way of mathematical description of the univariate point processes is using the concept of the stochastic intensity (or intensity process); see Aven and Jensen (1999), (2000) and Finkelstein and Cha (2013). Although multivariate point processes can be defined in different ways, the most convenient general specification can also be achieved through the stochastic intensity approach. A 'marginally regular bivariate process' can be specified by

$$\begin{aligned}
\lambda_{1t} &\equiv \lim_{\Delta t \to 0} \frac{\mathbb{P}(N_1(t, t+\Delta t) \geq 1 \mid \mathcal{H}_{1t-}; \mathcal{H}_{2t-})}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\mathbb{P}(N_1(t, t+\Delta t) = 1 \mid \mathcal{H}_{1t-}; \mathcal{H}_{2t-})}{\Delta t}, \qquad (1) \\
\lambda_{2t} &\equiv \lim_{\Delta t \to 0} \frac{\mathbb{P}(N_2(t, t+\Delta t) \geq 1 \mid \mathcal{H}_{1t-}; \mathcal{H}_{2t-})}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\mathbb{P}(N_2(t, t+\Delta t) = 1 \mid \mathcal{H}_{1t-}; \mathcal{H}_{2t-})}{\Delta t}, \qquad (2) \\
\lambda_{12t} &\equiv \lim_{\Delta t \to 0} \frac{\mathbb{P}(N_1(t, t+\Delta t) N_2(t, t+\Delta t) \geq 1 \mid \mathcal{H}_{1t-}; \mathcal{H}_{2t-})}{\Delta t}, \qquad (3)
\end{aligned}$$

where $N_i(t_1, t_2)$, $t_1 < t_2$, represents the number of events in $[t_1, t_2)$, $i = 1, 2$, respectively (see Cox and Lewis (1972)). The functions in (1)–(3) are called the *complete intensity functions*. For a regular process, $\lambda_{12t} = 0$, and it is sufficient to specify just λ_{1t} and λ_{2t} in (1) and (2) in order to define a regular process. Now we define the bivariate generalized Pólya process based on the stochastic intensity functions.

Definition 1. (*Bivariate generalized Pólya process.*) A bivariate counting process $\{\boldsymbol{N}(t),\ t \geq 0\}$ is called the bivariate generalized Pólya process (BVGPP) with the set of parameters $(\lambda_1(t), \lambda_2(t), \alpha, \beta)$, $\lambda_i(t) \geq 0$ for all $t \geq 0$, $i = 1, 2$, $\alpha \geq 0$, $\beta > 0$, if

(i) $N_1(0) = 0$, $N_2(0) = 0$;

(ii) $\lambda_{1t} = (\alpha(N_1(t-) + N_2(t-)) + \beta)\lambda_1(t)$;

(iii) $\lambda_{2t} = (\alpha(N_1(t-) + N_2(t-)) + \beta)\lambda_2(t)$.

The bivariate generalized Pólya process defined in Definition 1 will be denoted by

$$\mathrm{BVGPP}(\lambda_1(t), \lambda_2(t), \alpha, \beta).$$

The conditions (ii) and (iii) in Definition 1 specify the dependence structure of the process in a fully intuitive way. That is, the occurrences of any type of events in the previous interval

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increase the occurrence probabilities of both types of events in the next interval. This type of dependency in a bivariate point process (or in a multivariate point process) can be frequently observed in practice. For instance, in reliability applications, the consecutive failures of two parts of a system are frequently positively dependent (see Example 1). In the following example, we will consider an example of the application of the BVGPP in a reliability context.

Example 1. (*Dependent failure and repair process.*) As mentioned in Cha (2014), 'one counting process corresponds to one repair type and vice versa' in a reliability application. Thus, based on the BVGPP, a new type of dependent failure and repair process can be modeled. Suppose that a system is composed of two parts (part 1 and part 2) having respective failure rates $r_i(t) = \beta\lambda_i(t)$, $i = 1, 2$. We now consider the following three different types of failure and repair process. In the following discussions, the repairs of the failed parts are performed whenever any part of the system fails. Furthermore, we assume that the repairs are instantaneous and the repair times are negligible.

Type I repair process. Under this model, each failed part is instantly 'minimally repaired', which implies that the reliability performance of the repaired part is the same as that just before the failure. Furthermore, we assume that this failure does not affect the reliability performance of the whole system. This failure and repair process can be specified as

(i) $N_1(0) = 0, N_2(0) = 0$;

(ii) $\lambda_{1t} = \beta\lambda_1(t)$;

(iii) $\lambda_{2t} = \beta\lambda_2(t)$.

Type II repair process. This failure and repair process is defined by

(i) $N_1(0) = 0, N_2(0) = 0$;

(ii) $\lambda_{1t} = (\alpha(N_1(t-)) + \beta)\lambda_1(t)$;

(iii) $\lambda_{2t} = (\alpha(N_2(t-)) + \beta)\lambda_2(t)$.

Type III repair process. This failure and repair process is defined by Definition 1.

Clearly, a type I repair process defines 'independent minimal repair processes (IMRP)'. Of course, under this model, the joint failure processes can be modeled by two independent nonhomogeneous Poisson processes (NHPP) with intensities $r_i(t)$, $i = 1, 2$. Under a type III repair process, the reliability performances of both parts after a repair of any part are worse than before the failure (thus resulting in increased susceptibilities to failures). This type of repair process can be applied when a failure of any part negatively affects both parts. For instance, in an electric system, a failure of a component may affect the whole system by causing a sudden electric surge and then increasing the chances for sudden excessive electric current spikes in the system afterwards. In this sense, we have defined a type of 'dependent worse than minimal repair processes (DWMRP)' in dependent point process setting. On the other hand, a type II repair process defines 'independent worse than minimal repair processes (IWMRP)'.

Clearly, our main interest in this paper is the type III repair process and we suppose that the system defined above undergoes this type of repair process. First of all, one would be interested in the number of repairs of each part until a specified time t. Thus, in this case, we need to consider the joint distribution $\mathbb{P}(N_1(t) = n_1, N_2(t) = n_2)$. In some cases, the number of repairs for each part can be limited (for example, due to the limitation of the number of components for repairs), say, l_1 and l_2, respectively, for parts 1 and 2. Then, in this case, the

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probability that the system will work properly until time t under the repair process is given by $\mathbb{P}(N_1(t) \le l_1, N_2(t) \le l_2)$.

In general, the reliability performance of the system deteriorates with time and it eventually results in a low efficiency of the item and, at the same time, a high operational cost. Thus, until now, various repair and replacement policies for repairable systems have been studied and thoroughly discussed in the literature. Suppose that the systems in operation are repaired according to a type III repair process on each failure. Under the model, the reliability performances of 'both parts' deteriorates as there are more failures of 'any part' of the system due to the dependency between the two parts. For example, even though most of failures have occurred in part 1, due to the dependency, the susceptibilities to failures of both parts have increased. Thus, it could be reasonable to consider a replacement policy based on $N_1(t) + N_2(t)$. Accordingly, we can consider a maintenance policy where the system is replaced by a new one whenever $N_1(t) + N_2(t)$ reaches n, $n > 1$. Other variants of a replacement policy could also be developed under a type III repair process.

Example 2. (*Dependent bankruptcies in finance.*) In a finance context, the bankruptcy of a financial company in one group may also affect those of other companies in different group(s). In this case, a couple of events (bankruptcies) in a particular region spread to the rest of the financial industry, causing waves of consecutive defaults and bankruptcies in a global financial market. This is called 'financial contagion' (Allen and Gale (2000)). Let $N_i(t)$ be the total number of bankruptcies in group i, $i = 1, 2$, in the interval $(0, t]$. By employing the BVGPP in Definition 1 for modeling $\{(N_1(t), N_2(t)),\ t \ge 0\}$, one could model the financial phenomenon where the occurrences of bankruptcies in any group increase the susceptibilities to occurrences of bankruptcies in all groups in the next interval.

In the following, we will derive the properties of the BVGPP and, for this, we need to understand the univariate generalized Pólya process and the definition of the restarting process.

Definition 2. (*Generalized Pólya process, Cha (2014).*) A univariate counting process $\{N(t),\ t \ge 0\}$ is called the generalized Pólya process (GPP) with the set of parameters $(\gamma(t), \alpha, \beta)$, $\alpha \ge 0$, $\beta > 0$, if

(i) $N(0) = 0$;

(ii) $\lambda_t \equiv \lim_{\Delta t \to 0} \mathbb{P}(N(t, t + \Delta t) \ge 1 \mid \mathcal{H}_{t-})/\Delta t = (\alpha N(t-) + \beta)\gamma(t)$, where $\mathcal{H}_{t-}$ is the history of the process in $[0, t)$.

In the following, we denote the generalized Pólya process defined in Definition 2 by the GPP$(\lambda(t), \alpha, \beta)$. In Cha (2014), the following restarting property was defined and usefully used for deriving the properties of the GPP. The concept of the restarting property can now be directly extended for the multivariate processes without modification.

Definition 3. (*Restarting property.*) Let $t > 0$ be an 'arbitrary' time point. If the conditional future stochastic process from t, given the history until time t, follows the same type of stochastic process with possibly a different set of process parameters, then the process is said to possess the *restarting property*. A stochastic process with the restarting property is called the *restarting process*.

Several examples for univariate restarting processes are given in Cha (2014). As mentioned in Cha (2014), the GPP possesses the restarting property: let $\{N(t),\ t \ge 0\}$ be GPP$(\lambda(t), \alpha, \beta)$

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and the corresponding sequential arrival points of the events be denoted by $0 \le S_1 \le S_2 \le \cdots \le S_{N(t-)} < t$ in $[0, t)$. For an arbitrary time $u > 0$, given the history up to u, $\{N(u-) = n, S_1 = s_1, S_2 = s_2, \ldots, S_n = s_n\}$, the conditional future process $\{N_u(t),\ t \ge 0\}$, where $N_u(t) \equiv N(u+t) - N(u)$, is also the GPP with the set of parameters $(\lambda(u+t), \alpha, \beta + n\alpha)$, $t \ge 0$.

The following proposition provides some important insights into the structural relationship between the pooled process and the marginal processes in addition to some basic properties of the BVGPP. The understanding of this structural relationship will be crucial for our further analysis of the BVGPP. For our discussions we need to define the following concept of 'thinning' of the process.

Definition 4. ($p(t)$*-thinning.*) Let $\{N(t),\ t \ge 0\}$ be a univariate point process and denote by $\{N_{p(\cdot)}(t),\ t \ge 0\}$ the point process obtained by retaining (in the same location) every point of the process with probability $p(t)$ and deleting it with probability $q(t) = 1 - p(t)$, independently of everything else. Denote by $\{N_{q(\cdot)}(t),\ t \ge 0\}$ the point process constructed by the deleted points. Then the process $\{N_{p(\cdot)}(t),\ t \ge 0\}$ and $\{N_{q(\cdot)}(t),\ t \ge 0\}$ are $p(t)$-thinning of $\{N(t),\ t \ge 0\}$.

For convenience, we will define the following notation: $N_{ui}(t) \equiv N_i(u+t) - N_i(u)$, $\Lambda_i(t) \equiv \int_0^t \lambda_i(u)\,\mathrm{d}u$, $i = 1, 2$, $\lambda(t) \equiv \lambda_1(t) + \lambda_2(t)$, $\Lambda(t) = \int_0^t \lambda(u)\,\mathrm{d}u = \Lambda_1(t) + \Lambda_2(t)$, and $p_i(t) \equiv \lambda_i(t)/\lambda(t)$, $i = 1, 2$.

Proposition 1. *Let* $\{\boldsymbol{N}(t),\ t \ge 0\}$ *be* $\mathrm{BVGPP}(\lambda_1(t), \lambda_2(t), \alpha, \beta)$. *Then*

(i) $\{M(t),\ t \ge 0\}$ *is* $\mathrm{GPP}(\lambda(t), \alpha, \beta)$.

(ii) *The process* $\{\boldsymbol{N}(t),\ t \ge 0\}$ *is constructed by* $p_1(t)$*-thinning of* $\{M(t),\ t \ge 0\}$*:*

$$\{(M_{p_1(\cdot)}(t), M_{p_2(\cdot)}(t)),\ t \ge 0\}.$$

(iii) *Given* $(\mathcal{H}_{1u-}, \mathcal{H}_{2u-})$, $\{\boldsymbol{N}_u(t),\ t \ge 0\}$ *is*

$$\mathrm{BVGPP}(\lambda_1(u+t), \lambda_2(u+t), \alpha, \beta + \alpha(n_1 + n_2)),$$

where $\boldsymbol{N}_u(t) = (N_{u1}(t), N_{u2}(t))$ *and* n_i *is the realization of* $N_i(t-)$, $i = 1, 2$, *respectively.*

(iv) *For any fixed* $u \ge 0$, $\{\boldsymbol{N}_u(t),\ t \ge 0\}$ *is 'unconditionally'*

$$\mathrm{BVGPP}(\psi_1(t,u), \psi_2(t,u), \alpha, \beta),$$

where

$$\psi_i(t,u) \equiv \frac{\lambda_i(u+t)\exp\{\alpha\Lambda(u+t)\}}{1 + \exp\{\alpha\Lambda(u+t)\} - \exp\{\alpha\Lambda(u)\}}, \qquad i = 1, 2.$$

Proof. Propositions 1(i) and 1(ii) immediately follow from Definitions 1 and 2. For Proposition 1(iii), given $(\mathcal{H}_{1t-}, \mathcal{H}_{2t-})$, $\{M_u(t),\ t \ge 0\}$, where $M_u(t) \equiv M(u+t) - M(u)$, is $\mathrm{GPP}(\lambda(u+t), \alpha, \beta + \alpha(n_1 + n_2))$ due to the restarting property of the GPP. Then, by applying $p_1(u+t)$-thinning to $\{M_u(t),\ t \ge 0\}$, $\{\boldsymbol{N}_u(t),\ t \ge 0\}$ can be constructed (Proposition 1(ii)). For Proposition 1(iv), note that $\{M_u(t),\ t \ge 0\}$ is 'unconditionally' $\mathrm{GPP}(\psi(t,u), \alpha, \beta)$ (see Cha (2014, Theorem 2)), where

$$\psi(t,u) \equiv \frac{\lambda(u+t)\exp\{\alpha\Lambda(u+t)\}}{1 + \exp\{\alpha\Lambda(u+t)\} - \exp\{\alpha\Lambda(u))\}}.$$

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Then, by applying $p_1(u+t)$-thinning to $\{M_u(t),\ t \geq 0\}$, we have, for $i = 1, 2$,

$$\begin{aligned}\lambda_{it}^u &\equiv \lim_{\Delta t \to 0} \frac{\mathbb{P}(N_{ui}(t, t+\Delta t) = 1 \mid \mathcal{H}_{1[u,u+t)}; \mathcal{H}_{2[u,u+t)})}{\Delta t} \\ &= (\alpha[(N_1((u+t)-) - N_1(u-)) + (N_2((u+t)-) - N_2(u-))] + \beta)\psi(t,u)p_i(u+t),\end{aligned}$$

where $N_{ui}(t_1, t_2)$, $t_1 < t_2$, represents the number of events in $[u+t_1, u+t_2)$ of the type i process and $\mathcal{H}_{i[u,u+t)}$ is the history of type i process in $[u, u+t)$, $i = 1, 2$. □

Remark 1. Proposition 1(iii) implies that the BVGPP is a restarting process (conditionally on the history of the process). Furthermore, Proposition 1(iv) says that the BVGPP 'unconditionally' restarts at any time point.

The restarting properties in Propositions 1(iii) and 1(iv) will be usefully used in obtaining the joint distributions of numbers of events. The following theorem provides these joint distributions.

Theorem 1. *Let $t > 0$ and $0 \equiv u_0 < u_1 < u_2 < \cdots < u_m$.*

(i) *It holds that*

$$\begin{aligned}&\mathbb{P}(N_i(t) = n_i,\ i = 1, 2) \\ &\quad = \frac{\Gamma(\beta/\alpha + n_1 + n_2)}{\Gamma(\beta/\alpha)n_1!n_2!}\left(\alpha\int_0^t \lambda_1(x)\exp\{-\alpha[\Lambda(t) - \Lambda(x)]\}\,\mathrm{d}x\right)^{n_1} \\ &\qquad \times \left(\alpha\int_0^t \lambda_2(x)\exp\{-\alpha[\Lambda(t) - \Lambda(x)]\}\,\mathrm{d}x\right)^{n_2} (\exp\{-\alpha\Lambda(t)\})^{\beta/\alpha}.\end{aligned}$$

(ii) *It holds that*

$$\begin{aligned}&\mathbb{P}(N_i(u_2) - N_i(u_1) = n_i,\ i = 1, 2) \\ &\quad = \frac{\Gamma(\beta/\alpha + n_1 + n_2)}{\Gamma(\beta/\alpha)n_1!n_2!}\left(\frac{\alpha\int_{u_1}^{u_2} \lambda_1(x)\exp\{\alpha\Lambda(x)\}\,\mathrm{d}x}{1 + \exp\{\alpha\Lambda(u_2)\} - \exp\{\alpha\Lambda(u_1)\}}\right)^{n_1} \\ &\qquad \times \left(\frac{\alpha\int_{u_1}^{u_2} \lambda_2(x)\exp\{\alpha\Lambda(x)\}\,\mathrm{d}x}{1 + \exp\{\alpha\Lambda(u_2)\} - \exp\{\alpha\Lambda(u_1)\}}\right)^{n_2} \\ &\qquad \times \left(\frac{1}{1 + \exp\{\alpha\Lambda(u_2)\} - \exp\{\alpha\Lambda(u_1)\}}\right)^{\beta/\alpha}.\end{aligned}$$

(iii) *It holds that*

$$\begin{aligned}&\mathbb{P}(N_i(u_j) - N_i(u_{j-1}) = n_{ij},\ i = 1, 2,\ j = 1, 2, \ldots, m) \\ &\quad = \prod_{k=1}^m \Bigg[\frac{\Gamma(\beta/\alpha + \sum_{j=1}^k(\sum_{i=1}^2 n_{ij}))}{\Gamma(\beta/\alpha + \sum_{j=1}^{k-1}(\sum_{i=1}^2 n_{ij}))n_{1k}!n_{2k}!} \\ &\qquad \times \left(\alpha\int_{u_{k-1}}^{u_k} \lambda_1(x)\exp\{-\alpha[\Lambda(u_k) - \Lambda(x)]\}\,\mathrm{d}x\right)^{n_{1k}}\end{aligned}$$

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$$\times \left(\alpha \int_{u_{k-1}}^{u_k} \lambda_2(x) \exp\{-\alpha[\Lambda(u_k) - \Lambda(x)]\}\, \mathrm{d}x\right)^{n_{2k}}$$
$$\times \left(\exp\{-\alpha[\Lambda(u_k) - \Lambda(u_{k-1})]\}\right)^{\beta/\alpha + \sum_{j=1}^{k-1}(\sum_{i=1}^{2} n_{ij})}\Bigg],$$

where $\sum_{j=1}^{k-1}(\cdot) \equiv 0$ *when* $k = 1$.

Proof. (i) Observe that

$$\mathbb{P}(N_i(t) = n_i,\ i = 1, 2)$$
$$= \mathbb{P}(N_i(t) = n_i,\ i = 1, 2 \mid M(t) = n_1 + n_2)\mathbb{P}(M(t) = n_1 + n_2).$$

As $\{M(t),\ t \geq 0\}$ is GPP$(\lambda(t), \alpha, \beta)$, from Cha (2014, Theorem 1), we have

$$\mathbb{P}(M(t) = n_1 + n_2) = \frac{\Gamma(\beta/\alpha + n_1 + n_2)}{\Gamma(\beta/\alpha)(n_1 + n_2)!}(1 - \exp\{-\alpha\Lambda(t)\})^{n_1+n_2}(\exp\{-\alpha\Lambda(t)\})^{\beta/\alpha}.$$

From Cha (2014, Theorem 3), given $M(t) = n_1 + n_2$, the sequential arrival times $0 < T_1 < T_2 < \cdots < T_{n_1+n_2} \leq t$ have the same distribution as the order statistics corresponding to n independent random variables identically distributed according to the probability density function (PDF)

$$\left(\frac{\alpha\lambda(x)\exp\{\alpha\Lambda(x)\}}{\exp\{\alpha\Lambda(t)\} - 1}\right), \qquad 0 < x \leq t.$$

Thus, if an event has occurred in $(0, t]$, the probability that it will be a type 1 event is given by

$$\phi(t) \equiv \left(\frac{\alpha \int_0^t \lambda_1(x)\exp\{\alpha\Lambda(x)\}\, \mathrm{d}x}{\exp\{\alpha\Lambda(t)\} - 1}\right).$$

Hence,

$$\mathbb{P}(N_i(t) = n_i,\ i = 1, 2 \mid M(t) = n_1 + n_2) = \binom{n_1 + n_2}{n_1}(\phi(t))^{n_1}(1 - \phi(t))^{n_2}.$$

Therefore,

$$\mathbb{P}(N_i(t) = n_i,\ i = 1, 2)$$
$$= \frac{\Gamma(\beta/\alpha + n_1 + n_2)}{\Gamma(\beta/\alpha)n_1!\,n_2!}\left(\alpha \int_0^t \lambda_1(x)\exp\{-\alpha[\Lambda(t) - \Lambda(x)]\}\, \mathrm{d}x\right)^{n_1}$$
$$\times \left(\alpha \int_0^t \lambda_2(x)\exp\{-\alpha[\Lambda(t) - \Lambda(x)]\}\, \mathrm{d}x\right)^{n_2}(\exp\{-\alpha\Lambda(t)\})^{\beta/\alpha}.$$

(ii) By applying the restarting property of Proposition 1(iv), Theorem 1(i), and using

$$\int_0^t \psi_1(w, u_1) + \psi_2(w, u_1)\, \mathrm{d}w = \frac{1}{\alpha}\ln(1 + \exp\{\alpha\Lambda(u_1 + t)\} - \exp\{\alpha\Lambda(u_1)\}),$$

we can immediately obtain the desired result.

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(iii) Observe that

$$\begin{aligned}\mathbb{P}(N_i(u_j) - N_i(u_{j-1}) = n_{ij},\ i = 1,2,\ j = 1,2) \\ = \mathbb{P}(N_i(u_2) - N_i(u_1) = n_{i2},\ i = 1,2 \mid N_i(u_1) - N_i(u_0) = n_{i1},\ i = 1,2) \\ \times \mathbb{P}(N_i(u_1) - N_i(u_0) = n_{i1},\ i = 1,2),\end{aligned}$$

where $u_0 \equiv 0$. By Proposition 1(iii), given $N_i(u_1) - N_i(u_0) = n_{i1},\ i = 1,2$, $\{\boldsymbol{N}_{u_1}(t),\ t \geq 0\}$ is BVGPP$(\lambda_1(u_1 + t), \lambda_2(u_1 + t), \alpha, \beta + \alpha(n_{11} + n_{21}))$. Then, by applying Theorem 1(i), we have

$$\begin{aligned}&\mathbb{P}(N_i(u_2) - N_i(u_1) = n_{i2},\ i = 1,2 \mid N_i(u_1) - N_i(u_0) = n_{i1},\ i = 1,2) \\ &= \frac{\Gamma(\beta/\alpha + n_{11} + n_{21} + n_{12} + n_{22})}{\Gamma(\beta/\alpha + n_{11} + n_{21}) n_{12}!\, n_{22}!} \left(\alpha \int_{u_1}^{u_2} \lambda_1(x) \exp\{-\alpha[\Lambda(u_2) - \Lambda(x)]\}\, \mathrm{d}x\right)^{n_{12}} \\ &\quad \times \left(\alpha \int_{u_1}^{u_2} \lambda_2(x) \exp\{-\alpha[\Lambda(u_2) - \Lambda(x)]\}\, \mathrm{d}x\right)^{n_{22}} \\ &\quad \times (\exp\{-\alpha[\Lambda(u_2) - \Lambda(u_1)]\})^{\beta/\alpha + n_{11} + n_{21}}.\end{aligned}$$

Thus,

$$\begin{aligned}&\mathbb{P}(N_i(u_j) - N_i(u_{j-1}) = n_{ij},\ i = 1,2,\ j = 1,2) \\ &= \prod_{k=1}^{2} \Bigg[\frac{\Gamma(\beta/\alpha + \sum_{j=1}^{k} (\sum_{i=1}^{2} n_{ij}))}{\Gamma(\beta/\alpha + \sum_{j=1}^{k-1} (\sum_{i=1}^{2} n_{ij})) n_{1k}!\, n_{2k}!} \\ &\qquad \times \left(\alpha \int_{u_{k-1}}^{u_k} \lambda_1(x) \exp\{-\alpha[\Lambda(u_k) - \Lambda(x)]\}\, \mathrm{d}x\right)^{n_{1k}} \\ &\qquad \times \left(\alpha \int_{u_{k-1}}^{u_k} \lambda_2(x) \exp\{-\alpha[\Lambda(u_k) - \Lambda(x)]\}\, \mathrm{d}x\right)^{n_{2k}} \\ &\qquad \times (\exp\{-\alpha[\Lambda(u_k) - \Lambda(u_{k-1})]\})^{\beta/\alpha + \sum_{j=1}^{k-1} (\sum_{i=1}^{2} n_{ij})} \Bigg],\end{aligned}$$

where $\sum_{j=1}^{k-1}(\cdot) \equiv 0$ when $k = 1$. By applying similar procedure recursively, we have the desired result. □

Remark 2. (i) Note that the joint distribution of $N_1(t)$ and $N_2(t)$ is a negative multinomial distribution. The marginal distribution of $N_i(t)$ is given by a negative binomial distribution:

$$\begin{aligned}\mathbb{P}(N_i(t) = n_i) = \frac{\Gamma(\beta/\alpha + n_i)}{\Gamma(\beta/\alpha) n_i!} \left(\frac{\alpha \int_0^t \lambda_i(x) \exp\{-\alpha[\Lambda(t) - \Lambda(x)]\}\, \mathrm{d}x}{\alpha \int_0^t \lambda_i(x) \exp\{-\alpha[\Lambda(t) - \Lambda(x)]\}\, \mathrm{d}x + \exp\{-\alpha\Lambda(t)\}}\right)^{n_i} \\ \times \left(\frac{\exp\{-\alpha\Lambda(t)\}}{\alpha \int_0^t \lambda_i(x) \exp\{-\alpha[\Lambda(t) - \Lambda(x)]\}\, \mathrm{d}x + \exp\{-\alpha\Lambda(t)\}}\right)^{\beta/\alpha}\end{aligned}$$

for $i = 1, 2$.

(ii) Using Theorem 1(iii), the joint distribution of number of events in different time intervals, for example, $\mathbb{P}(N_1(t) = n_1, N_2(s) = n_2)$, $t > s$, can also be obtained. However, such distributions would be more conveniently obtained by using the characterization of the BVGPP suggested in the next section.

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3. Characterization and further properties

In this section we will define a new dependence concept for dependent point processes and analyze the dependence structure of the BVGPP. Furthermore, the marginal and conditional processes of the BVGPP will be discussed. For all of these, we will characterize the BVGPP in Definition 1 in a 'novel way' using special parametrization: the mixture of bivariate Poisson process with independent univariate Poisson components which 'share all the elementary parameter functions $\lambda_1(t)$ and $\lambda_2(t)$'. This new characterization would substantially simplify the derivation of further properties of the BVGPP. For our discussions, we denote by MBVPP($\nu_1(t)$, $\nu_2(t)$, $f_Z(z)$) the mixture of bivariate Poisson process with independent univariate (nonhomogeneous) Poisson components with respective intensities $z\nu_1(t)$ and $z\nu_2(t)$ (given $Z = z$) and the corresponding mixing distribution of Z (PDF), $f_Z(z)$. The following theorem characterizes the BVGPP based on the MBVPP.

Theorem 2. *Let*

$$\nu_i(t) \equiv \lambda_i(t) \exp\{\alpha(\Lambda_1(t) + \Lambda_2(t))\}, \quad i = 1, 2, \qquad f_Z(z) \equiv \frac{b^a z^{a-1}}{\Gamma(a)} \exp\{-bz\},$$

where $a \equiv \beta/\alpha$, $b \equiv 1/\alpha$. *Then* BVGPP($\lambda_1(t)$, $\lambda_2(t)$, α, β) *and* MBVPP($\nu_1(t)$, $\nu_2(t)$, $f_Z(z)$) *share the same stochastic properties.*

Proof. The corresponding bivariate process of MBVPP($\nu_1(t)$, $\nu_2(t)$, $f_Z(z)$) will be denoted by $\{(N_1^*(t), N_2^*(t)),\ t \geq 0\}$, but we will use the same notation $\mathcal{H}_{it-}, i = 1, 2$ for the corresponding histories and the sequential arrival points of the events $0 \leq T_{i1} \leq T_{i2} \leq \cdots \leq T_{iN_i^*(t-)} < t$ in $[0, t)$, $i = 1, 2$. It is sufficient to show that the complete stochastic intensity functions λ_{1t}^*, λ_{2t}^*, and λ_{12t}^* of MBVPP($\nu_1(t)$, $\nu_2(t)$, $f_Z(z)$) are the same as those of BVGPP($\lambda_1(t)$, $\lambda_2(t)$, α, β). Clearly, λ_{12t}^* of MBVPP($\nu_1(t)$, $\nu_2(t)$, $f_Z(z)$) is given by $\lambda_{12t}^* = 0$. Now we will derive λ_{1t}^* of MBVPP($\nu_1(t)$, $\nu_2(t)$, $f_Z(z)$), where

$$f_Z(z) \equiv \frac{1}{\Gamma(a)} b^a z^{a-1} \exp\{-bz\}, \qquad a > 0,\ b > 0.$$

Observe that

$$\begin{aligned} \lambda_{1t}^* &= \lim_{\Delta t \to 0} \frac{\mathbb{P}(N_1^*(t, t+\Delta t) = 1 \mid \mathcal{H}_{1t-}; \mathcal{H}_{2t-})}{\Delta t} \\ &= \mathbb{E}_{(Z \mid \mathcal{H}_{1t-}; \mathcal{H}_{2t-})} \left[\lim_{\Delta t \to 0} \frac{\mathbb{P}(N_1^*(t, t+\Delta t) = 1 \mid \mathcal{H}_{1t-}; \mathcal{H}_{2t-}, Z)}{\Delta t} \right], \end{aligned} \tag{4}$$

where '$\mathbb{E}_{(Z \mid \mathcal{H}_{1t-}; \mathcal{H}_{2t-})}[\cdot]$' stands for the expectation with respect to the conditional distribution of $(Z \mid \mathcal{H}_{1t-}; \mathcal{H}_{2t-})$. Observe that

$$\lim_{\Delta t \to 0} \frac{\mathbb{P}(N_1^*(t, t+\Delta t) = 1 \mid \mathcal{H}_{1t-}; \mathcal{H}_{2t-}, Z = z)}{\Delta t} = z\nu_1(t),$$

as, given $Z = z$, the two NHPP are independent and a NHPP possesses independent increments property. Now we will derive the conditional distribution of $(Z \mid \mathcal{H}_{1t-}; \mathcal{H}_{2t-})$. Observe that the conditional distribution of

$$(T_{i1}, T_{i2}, \ldots, T_{iN_i^*(t-)}, N_i^*(t-), i = 1, 2 \mid Z = z)$$

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is given by

$$\prod_{i=1,2}\left[z^{n_i}\left(\prod_{j=1}^{n_i}\nu_i(t_{ij})\right)\exp\left\{-z\int_0^t \nu_i(x)\,\mathrm{d}x\right\}\right],$$
$$0 < t_{i1} < t_{i2} < \cdots < t_{in_i} \le t,\ n_i = 0, 1, 2, \ldots.$$

Thus, the conditional distribution of

$$(Z \mid T_{i1} = t_{i1}, T_{i2} = t_{i2}, \ldots, T_{iN_i^*(t-)} = t_{in}, N_i^*(t-) = n_i, i = 1, 2)$$

is given by

$$\frac{\prod_{i=1,2}[z^{n_i}(\prod_{j=1}^{n_i}\nu_i(t_{ij}))\exp\{-z\int_0^t \nu_i(x)\,\mathrm{d}x\}]f_Z(z)}{\int_0^\infty \prod_{i=1,2}[v^{n_i}(\prod_{j=1}^{n_i}\nu_i(t_{ij}))\exp\{-v\int_0^t \nu_i(x)\,\mathrm{d}x\}]f_Z(v)\,\mathrm{d}v}$$
$$= \frac{z^{n_1+n_2}\exp\{-z\int_0^t \nu_1(x)+\nu_2(x)\,\mathrm{d}x\}f_Z(z)}{\int_0^\infty v^{n_1+n_2}\exp\{-v\int_0^t \nu_1(x)+\nu_2(x)\,\mathrm{d}x\}f_Z(v)\,\mathrm{d}v}.$$

Thus, the conditional distribution $(Z \mid \mathcal{H}_{1t-}; \mathcal{H}_{2t-})$ depends on the past history only through $N_1^*(t-) + N_2^*(t-)$:

$$(Z \mid \mathcal{H}_{1t-}; \mathcal{H}_{2t-}) \stackrel{\mathrm{D}}{=} (Z \mid N_1^*(t-) + N_2^*(t-)),$$

where '$\stackrel{\mathrm{D}}{=}$' stands for 'equality in distribution'. Then, by taking expectation of $Z\nu_1(t)$ with respect to the conditional distribution of $(Z \mid T_{i1} = t_{i1}, T_{i2} = t_{i2}, \ldots, T_{iN_i^*(t-)} = t_{in}, N_i^*(t-) = n_i, i = 1, 2)$, we have

$$\frac{\int_0^\infty z^{n_1+n_2+1}\exp\{-z\int_0^t \nu_1(x)+\nu_2(x)\,\mathrm{d}x\}f_Z(z)\,\mathrm{d}z}{\int_0^\infty v^{n_1+n_2}\exp\{-v\int_0^t \nu_1(x)+\nu_2(x)\,\mathrm{d}x\}f_Z(v)\,\mathrm{d}v}\nu_1(t)$$
$$= (n_1+n_2+a)\frac{1}{b+\int_0^t \nu_1(x)+\nu_2(x)\,\mathrm{d}x}\nu_1(t).$$

Therefore, from (4), we have

$$\lambda_{1t}^* = ((N_1^*(t-) + N_2^*(t-)) + a)\frac{1}{b+\int_0^t \nu_1(x)+\nu_2(x)\,\mathrm{d}x}\nu_1(t).$$

Now reparameterizing $\nu_i(t) \equiv \lambda_i(t)\exp\{\alpha\int_0^t \lambda_1(x)+\lambda_2(x)\,\mathrm{d}x\}$, $i = 1, 2$, $a \equiv \alpha/\beta$, and $b \equiv \alpha^{-1}$, we have

$$\lambda_{1t}^* = (\alpha(N_1^*(t-) + N_2^*(t-)) + \beta)\lambda_1(t).$$

By applying similar arguments, we can also obtain

$$\lambda_{2t}^* = (\alpha(N_1^*(t-) + N_2^*(t-)) + \beta)\lambda_2(t). \qquad \square$$

In the following discussions, for convenience, we denote by $\{(N_1^*(t), N_2^*(t)),\ t \ge 0\}$ the MBVPP$(\nu_1(t), \nu_2(t), f_Z(z))$ with $\nu_i(t) \equiv \lambda_i(t)\exp\{\alpha\int_0^t \lambda_1(x)+\lambda_2(x)\,\mathrm{d}x\}$, $i = 1, 2$, $f_Z(z) = \Gamma(\beta/\alpha)^{-1}\alpha^{-\alpha/\beta}z^{\alpha/\beta-1}\exp\{-\alpha^{-1}z\}$. But, as in the proof of Theorem 2, we will use the same

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notation $\mathcal{H}_{it-}$, $i = 1, 2$, for the corresponding histories and the sequential arrival points of the events $0 \le T_{i1} \le T_{i2} \le \cdots \le T_{iN_i^*(t-)} < t$ in $[0, t)$, $i = 1, 2$.

From Definition 1, intuitively, the BVGPP should possess positive dependence between the occurrences of events in two involved processes. We will now define the dependence of the point processes more precisely and then analyze the dependence structure of the BVGPP. For this, we start our discussions with a dependence concept for two random variables. Two random variables U_1 and U_2 are *positive quadrant dependent* (PQD) if

$$\mathbb{P}(U_1 > u_1, U_2 > u_2) \ge \mathbb{P}(U_1 > u_1)\mathbb{P}(U_2 > u_2) \quad \text{for all } u_1,\ u_2 \tag{5}$$

holds (Lehmann (1966)). Intuitively, (5) implies that U_1, U_2 are more likely simultaneously to have large values, compared with two independent random variables with the same univariate marginal distributions. It can be shown that if U_1 and U_2 are PQD then $\operatorname{cov}(U_1, U_2) \ge 0$ (Lehmann (1966)). Thus, the PQD is a stronger type of dependency than the positive covariance. Now we extend this concept in a more general way to define a concept for the positive dependence of bivariate point processes.

Definition 5. (*PQD bivariate process.*) A bivariate point process

$$\{(U_1(t), U_2(t)),\ t \ge 0\}$$

is *PQD bivariate process* (PQDBP) if

$$\begin{aligned}&\mathbb{P}(U_1(t_2) - U_1(t_1) > n_1, U_2(s_2) - U_2(s_1) > n_2)\\ &\quad \ge \mathbb{P}(U_1(t_2) - U_1(t_1) > n_1)\mathbb{P}(U_2(s_2) - U_2(s_1) > n_2) \quad \text{for all } t_2 > t_1,\ s_2 > s_1,\ n_1,\ n_2.\end{aligned}$$

We will now analyze the dependence structure of the BVGPP. For this purpose, we need the following preliminary lemma, whose proof can be found in Joe (1997) and Cuadras (2002).

Lemma 1. *Let X be a random variable and $g(x)$, $h(x)$ be real valued functions.*

(i) *If both $g(x)$ and $h(x)$ are increasing; or if both $g(x)$ and $h(x)$ are decreasing, then $\mathbb{E}[g(X)h(X)] \ge \mathbb{E}[g(X)]\mathbb{E}[h(X)]$.*

(ii) *If $g(x)$ is increasing and $h(x)$ is decreasing; or if $g(x)$ is decreasing and $h(x)$ is increasing, then $\mathbb{E}[g(X)h(X)] \le \mathbb{E}[g(X)]\mathbb{E}[h(X)]$.*

The following result states the dependence structure of the BVGPP.

Theorem 3. *The BVGPP is a PQDBP:*

$$\begin{aligned}&\mathbb{P}(N_1(t_2) - N_1(t_1) > n_1, N_2(s_2) - N_2(s_1) > n_2)\\ &\quad \ge \mathbb{P}(N_1(t_2) - N_1(t_1) > n_1)\mathbb{P}(N_2(s_2) - N_2(s_1) > n_2) \quad \textit{for all } t_2 > t_1,\ s_2 > s_1,\ n_1,\ n_2.\end{aligned}$$

Proof. Due to Theorem 2, it suffices to show that

$$\begin{aligned}&\mathbb{P}(N_1^*(t_2) - N_1^*(t_1) > n_1, N_2^*(s_2) - N_2^*(s_1) > n_2)\\ &\quad \ge \mathbb{P}(N_1^*(t_2) - N_1^*(t_1) > n_1)\mathbb{P}(N_2^*(s_2) - N_2^*(s_1) > n_2).\end{aligned}$$

Note that

$$\begin{aligned}&\mathbb{P}(N_1^*(t_2) - N_1^*(t_1) > n_1, N_2^*(s_2) - N_2^*(s_1) > n_2)\\ &\quad = \mathbb{E}[\mathbb{P}(N_1^*(t_2) - N_1^*(t_1) > n_1, N_2^*(s_2) - N_2^*(s_1) > n_2 \mid Z)].\end{aligned} \tag{6}$$

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Clearly,

$$\begin{aligned}\mathbb{P}(N_1^*(t_2) - N_1^*(t_1) > n_1, N_2^*(s_2) - N_2^*(s_1) > n_2 \mid Z = z) \\ = \sum_{n=n_1+1}^{\infty} \frac{(z\int_{t_1}^{t_2} \nu_1(x)\,\mathrm{d}x)^n}{n!} \exp\left\{-z\int_{t_1}^{t_2} \nu_1(x)\,\mathrm{d}x\right\} \\ \times \sum_{n=n_2+1}^{\infty} \frac{(z\int_{s_1}^{s_2} \nu_2(x)\,\mathrm{d}x)^n}{n!} \exp\left\{-z\int_{s_1}^{s_2} \nu_2(x)\,\mathrm{d}x\right\}.\end{aligned}$$

For $z_1 < z_2$,

$$\begin{aligned}&\left(\frac{(z_1\int_{t_1}^{t_2} \nu_1(x)\,\mathrm{d}x)^n}{n!} \exp\left\{-z_1\int_{t_1}^{t_2} \nu_1(x)\,\mathrm{d}x\right\}\right) \\ &\times \left(\frac{(z_2\int_{t_1}^{t_2} \nu_1(x)\,\mathrm{d}x)^n}{n!} \exp\left\{-z_2\int_{t_1}^{t_2} \nu_1(x)\,\mathrm{d}x\right\}\right)^{-1} \qquad (7)\end{aligned}$$

is decreasing in n. This implies that the Poisson distribution in the denominator of (7) is stochastically larger than that in the numerator of (7) in the sense of 'likelihood ratio order'; see Shaked and Shanthikumar (2007). This in turn implies that the Poisson distribution in the denominator of (7) is also stochastically larger than that in the numerator of (7) in the sense of 'usual stochastic order' and the survival function of the Poisson distribution in the denominator should be greater than that in the numerator; see also Shaked and Shanthikumar (2007). This implies that

$$g(z) \equiv \sum_{n=n_1+1}^{\infty} \frac{(z\int_{t_1}^{t_2} \nu_1(x)\,\mathrm{d}x)^n}{n!} \exp\left\{-z\int_{t_1}^{t_2} \nu_1(x)\,\mathrm{d}x\right\}$$

is increasing in z. By the same arguments,

$$h(z) \equiv \sum_{n=n_2+1}^{\infty} \frac{(z\int_{s_1}^{s_2} \nu_2(x)\,\mathrm{d}x)^n}{n!} \exp\left\{-z\int_{s_1}^{s_2} \nu_2(x)\,\mathrm{d}x\right\}$$

is increasing in z. Then, by (6) and Lemma 1,

$$\begin{aligned}&\mathbb{P}(N_1^*(t_2) - N_1^*(t_1) > n_1, N_2^*(s_2) - N_2^*(s_1) > n_2) \\ &\quad = \mathbb{E}[g(Z)h(Z)] \\ &\quad \geq \mathbb{E}[g(Z)]\mathbb{E}[h(Z)] \\ &\quad = \mathbb{P}(N_1^*(t_2) - N_1^*(t_1) > n_1)\mathbb{P}(N_2^*(s_2) - N_2^*(s_1) > n_2).\end{aligned}$$

This completes the proof. □

Now, we will derive the conditional joint distribution of the arrival times of the two types of processes in the interval $(0, t]$, given $N_1(t)$ and $N_2(t)$ in the BVGPP:

$$(T_{i1}, T_{i2}, \ldots, T_{in_i}, i = 1, 2 \mid N_i(t) = n_i, i = 1, 2).$$

In Theorem 3, we observed that the BVGPP possesses quite a strong type of dependence between the two involved processes. However, somewhat surprisingly, the following result shows that the joint arrival times distribution of type 1 events and that of type 2 events are independent conditionally on $N_i(t) = n_i, i = 1, 2$.

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Theorem 4. *The conditional joint arrival times distribution of*

$$(T_{i1}, T_{i2}, \ldots, T_{in_i}, i = 1, 2 \mid N_i(t) = n_i, i = 1, 2)$$

is

$$f_{(T_{i1},T_{i2},\ldots,T_{in_i},i=1,2 \mid N_i(t),i=1,2)}(t_{11}, \ldots, t_{1n_1}, t_{21}, \ldots, t_{2n_2} \mid n_1, n_2)$$
$$= \prod_{i=1}^{2}\left[n_i! \prod_{j=1}^{n_i}\left(\frac{\lambda_i(t_{ij}) \exp\{\alpha \int_0^{t_{ij}} \lambda_1(x) + \lambda_2(x)\,\mathrm{d}x\}}{\int_0^t \lambda_i(v) \exp\{\alpha \int_0^v \lambda_1(x) + \lambda_2(x)\,\mathrm{d}x\}\,\mathrm{d}v}\right)\right],$$

$0 < t_{11} < \cdots < t_{1n_1} \le t; 0 < t_{21} < \cdots < t_{2n_2} \le t.$

Proof. Due to Theorem 2, the conditional joint arrival times distribution in

$$\mathrm{BVGPP}(\lambda_1(t), \lambda_2(t), \alpha, \beta)$$

is the same as that in $\mathrm{MBVPP}(\nu_1(t), \nu_2(t), f_Z(z))$, where $\nu_i(t) \equiv \lambda_i(t) \exp\{\alpha \int_0^t \lambda_1(x) + \lambda_2(x)\,\mathrm{d}x\}, i = 1, 2, f_Z(z) = \Gamma(\beta/\alpha)^{-1}\alpha^{-\alpha/\beta} z^{\alpha/\beta-1} \exp\{-\alpha^{-1}z\}$.

Observe that the conditional joint arrival times distribution (PDF) in the MBVPP can be expressed as

$$f_{(T_{i1},T_{i2},\ldots,T_{in_i},i=1,2 \mid N_i^*(t),i=1,2)}(t_{11}, \ldots, t_{1n_1}, t_{21}, \ldots, t_{2n_2} \mid n_1, n_2)$$
$$= \mathbb{E}_{(Z \mid N_i^*(t),i=1,2)}[f_{(T_{i1},T_{i2},\ldots,T_{in_i},i=1,2 \mid N_i^*(t),i=1,2,Z)}$$
$$(t_{11}, \ldots, t_{1n_1}, t_{21}, \ldots, t_{2n_2} \mid n_1, n_2, z)],$$

where $f_{(T_{i1},T_{i2},\ldots,T_{in_i},i=1,2 \mid N_i^*(t),i=1,2,Z)}(t_{11}, \ldots, t_{1n_1}, t_{21}, \ldots, t_{2n_2} \mid n_1, n_2, z)$ is the conditional joint distribution of

$$(T_{i1}, T_{i2}, \ldots, T_{in_i}, i = 1, 2 \mid N_i(t) = n_i, i = 1, 2, Z = z).$$

If the process $\{N(t),\ t \ge 0\}$ is the NHPP with the intensity function $\lambda(t)$, then it is well known that the conditional arrival time distribution of $S_1, S_2, \ldots, S_{N(t)}$ in $(0, t]$, given that $N(t) = n$, is given by (Ross, (1996))

$$n! \prod_{i=1}^{n}\left(\frac{\lambda(s_i)}{\Lambda(t)}\right), \qquad 0 \le s_1 \le s_2 \le \cdots \le s_n \le t.$$

Thus,

$$f_{(T_{i1},T_{i2},\ldots,T_{in_i},i=1,2 \mid N_i^*(t),i=1,2,Z)}(t_{11}, \ldots, t_{1n_1}, t_{21}, \ldots, t_{2n_2} \mid n_1, n_2, z)$$
$$= \prod_{i=1}^{2}\left[n_i! \prod_{j=1}^{n_i}\left(\frac{z\nu_i(t_{ij})}{\int_0^t z\nu_i(v)\,\mathrm{d}v}\right)\right]$$
$$= \prod_{i=1}^{2}\left[n_i! \prod_{j=1}^{n_i}\left(\frac{\lambda_i(t_{ij}) \exp\{\alpha \int_0^{t_{ij}} \lambda_1(x) + \lambda_2(x)\,\mathrm{d}x\}}{\int_0^t \lambda_i(v) \exp\{\alpha \int_0^v \lambda_1(x) + \lambda_2(x)\,\mathrm{d}x\}\,\mathrm{d}v}\right)\right],$$

which is independent of z. Therefore, we now have the desired result. □

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Remark 3. From Theorem 4,

$$\begin{aligned}&\text{Joint PDF of } (T_{i1}, T_{i2}, \ldots, T_{in_i}, i = 1, 2 \mid N_i(t) = n_i, i = 1, 2)\\ &= \prod_{i=1}^{2} \text{Joint PDF of } (T_{i1}, T_{i2}, \ldots, T_{in_i} \mid N_i(t) = n_i).\end{aligned}$$

Therefore, as mentioned before, the joint arrival times distribution of type 1 events and that of type 2 events are independent conditionally on $N_i(t) = n_i$, $i = 1, 2$.

Now, we will characterize the marginal and conditional processes. As mentioned in Remark 2, the marginal distributions of $N_i(t)$ are given by negative binomial distributions and this implies that there is fairly high potential for the marginal process $\{N_i(t),\ t \geq 0\}$ to be GPP. The following theorem provides the answer to this conjecture. In some cases, we are interested in the conditional process of $\{N_1(t),\ t \geq 0\}$ given the process history of the type 2 process $\mathcal{H}_{2s-}$ for some $s > 0$. We also discuss this issue.

Theorem 5. *For* BVGPP$(\lambda_1(t), \lambda_2(t), \alpha, \beta)$,

(i) *the marginal process* $\{N_i(t),\ t \geq 0\}$ *is* GPP$(\gamma_i(t), \alpha, \beta)$, *where*

$$\gamma_i(t) = \frac{\lambda_i(t) \exp\{\alpha \int_0^t \lambda_1(x) + \lambda_2(x)\, \mathrm{d}x\}}{\alpha \int_0^t \lambda_i(v) \exp\{\alpha \int_0^v \lambda_1(x) + \lambda_2(x)\, \mathrm{d}x\}\, \mathrm{d}v + 1}, \qquad i = 1, 2;$$

(ii) *given* $\mathcal{H}_{2s-}$ *for some* $s > 0$, *the conditional process of* $\{N_1(t),\ t \geq 0\}$ *is*

$$\mathrm{GPP}(\psi(t), \alpha, \alpha n_2 + \beta),$$

where

$$\psi(t) \equiv \frac{\lambda_1(t) \exp\{\alpha \int_0^t \lambda_1(x) + \lambda_2(x)\, \mathrm{d}x\}}{\alpha(\int_0^t \lambda_1(v) \exp\{\alpha \int_0^v \lambda_1(x) + \lambda_2(x)\, \mathrm{d}x\}\, \mathrm{d}v + \int_0^s \lambda_2(v) \exp\{\alpha \int_0^v \lambda_1(x) + \lambda_2(x)\, \mathrm{d}x\}\, \mathrm{d}v) + 1},$$

and n_2 *is the realization of* $N_2(s-)$.

Proof. Due to Theorem 2, it is sufficient to specify the stochastic intensity of the marginal process $\{N_i^*(t),\ t \geq 0\}$, $i = 1, 2$, of the MBVPP$(\nu_1(t), \nu_2(t), f_Z(z))$, where

$$\nu_i(t) \equiv \lambda_i(t) \exp\left\{\alpha \int_0^t \lambda_1(x) + \lambda_2(x)\, \mathrm{d}x\right\}, \qquad i = 1, 2,$$

$$f_Z(z) = \Gamma(\beta/\alpha)^{-1} \alpha^{-\alpha/\beta} z^{\alpha/\beta - 1} \exp\{-\alpha^{-1} z\} :$$

$$\begin{aligned}\lambda_{it}^{0*} &\equiv \lim_{\Delta t \to 0} \frac{\mathbb{P}(N_i^*(t, t + \Delta t) = 1 \mid \mathcal{H}_{it-})}{\Delta t}\\ &= \mathbb{E}_{(Z \mid \mathcal{H}_{it-})}\left[\lim_{\Delta t \to 0} \frac{\mathbb{P}(N_i^*(t, t + \Delta t) = 1 \mid \mathcal{H}_{it-}, Z)}{\Delta t}\right], \qquad i = 1, 2.\end{aligned}$$

It is clear that

$$\lim_{\Delta t \to 0} \frac{\mathbb{P}(N_i^*(t, t + \Delta t) = 1 \mid \mathcal{H}_{it-}, Z = z)}{\Delta t} = z\nu_i(t), \qquad i = 1, 2.$$

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For $i = 1$, the conditional distribution of $(Z \mid \mathcal{H}_{1t-})$ is given by

$$\frac{[z^{n_1}(\prod_{j=1}^{n_1} \nu_1(t_{1j})) \exp\{-z \int_0^t \nu_1(x)\,\mathrm{d}x\}] f_Z(z)}{\int_0^\infty [v^{n_1}(\prod_{j=1}^{n_1} \nu_1(t_{1j})) \exp\{-v \int_0^t \nu_1(x)\,\mathrm{d}x\}] f_Z(v)\,\mathrm{d}v} = \frac{z^{n_1} \exp\{-z \int_0^t \nu_1(x)\,\mathrm{d}x\} f_Z(z)}{\int_0^\infty v^{n_1} \exp\{-v \int_0^t \nu_1(x)\,\mathrm{d}x\} f_Z(v)\,\mathrm{d}v}.$$

Thus, by taking the expectation of $Z\nu_1(t)$ with respect to the conditional distribution of $(Z \mid \mathcal{H}_{1t-})$, we have

$$(\alpha n_1 + \beta) \frac{\nu_1(t)}{\alpha \int_0^t \nu_1(v)\,\mathrm{d}v + 1}.$$

Therefore,

$$\lambda_{1t}^{0*} = (\alpha N_1^*(t-) + \beta) \frac{\lambda_1(t) \exp\{\alpha \int_0^t \lambda_1(x) + \lambda_2(x)\,\mathrm{d}x\}}{\alpha \int_0^t \lambda_1(v) \exp\{\alpha \int_0^v \lambda_1(x) + \lambda_2(x)\,\mathrm{d}x\}\,\mathrm{d}v + 1}.$$

The marginal stochastic intensity for the type 2 process λ_{2t}^{0*} can also be similarly obtained.

In order to characterize the conditional process of $\{N_1(t),\ t \geq 0\}$ given the process history of the type 2 process $\mathcal{H}_{2s-}$, we have to specify the following conditional stochastic intensity function:

$$\lim_{\Delta t \to 0} \frac{\mathbb{P}(N_1^*(t, t+\Delta t) = 1 \mid \mathcal{H}_{1t-}; \mathcal{H}_{2s-})}{\Delta t} = \mathbb{E}_{(Z \mid \mathcal{H}_{1t-}; \mathcal{H}_{2s-})} \left[\lim_{\Delta t \to 0} \frac{\mathbb{P}(N_1^*(t, t+\Delta t) = 1 \mid \mathcal{H}_{1t-}; \mathcal{H}_{2s-}, Z)}{\Delta t} \right].$$

Then, following the procedure described above, it is given by

$$(\alpha N_1^*(t-) + \alpha n_2 + \beta) \times \frac{\lambda_1(t) \exp\{\alpha \int_0^t \lambda_1(x) + \lambda_2(x)\,\mathrm{d}x\}}{\alpha(\int_0^t \lambda_1(v) \exp\{\alpha \int_0^v \lambda_1(x) + \lambda_2(x)\,\mathrm{d}x\}\,\mathrm{d}v + \int_0^s \lambda_2(v) \exp\{\alpha \int_0^v \lambda_1(x) + \lambda_2(x)\,\mathrm{d}x\}\,\mathrm{d}v) + 1},$$

which implies that the corresponding conditional process is GPP. □

4. Generalization to multivariate processes

In this section we extend the discussions in the previous sections to the multivariate case. Let $\{\boldsymbol{N}(t),\ t \geq 0\}$, where $\boldsymbol{N}(t) = (N_1(t), N_2(t), \ldots, N_m(t))$, be a multivariate process of our interest and define the corresponding 'pooled' point process $\{M(t),\ t \geq 0\}$, where $M(t) = N_1(t) + N_2(t) + \cdots + N_m(t)$. We can then define the marginal point processes $\{N_i(t),\ t \geq 0\}$, $i = 1, 2, \ldots, m$, and the corresponding marginal histories of the marginal processes: $\mathcal{H}_{it-}$, $i = 1, 2, \ldots, m$. As mentioned before, we are considering a 'regular' process and, for a specification of the multivariate process, it is sufficient to define the following set of stochastic intensity functions:

$$\lambda_{it} \equiv \lim_{\Delta t \to 0} \frac{\mathbb{P}(N_i(t, t+\Delta t) = 1 \mid \mathcal{H}_{1t-}; \mathcal{H}_{2t-}; \ldots; \mathcal{H}_{mt-})}{\Delta t}, \qquad i = 1, 2, \ldots, m.$$

Now we formally define the multivariate generalized Pólya process.

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Definition 6. (*Multivariate generalized Pólya process.*) A multivariate counting process $\{\boldsymbol{N}(t), t \geq 0\}$ is called the multivariate generalized Pólya process (MVGPP) with the set of parameters $(\lambda_1(t), \lambda_2(t), \ldots, \lambda_m(t), \alpha, \beta)$, $\alpha \geq 0$, $\beta > 0$, if

(i) $N_i(0) = 0$, for $i = 1, 2, \ldots, m$;

(ii) $\lambda_{it} = (\alpha(\sum_{i=1}^m N_i(t-)) + \beta)\lambda_i(t)$, $i = 1, 2, \ldots, m$.

For the MVGPP, we can obtain similar but more general properties without difficulty. For the statement of these basic properties, we define $\lambda(t) = \sum_{i=1}^m \lambda_i(t)$, $\Lambda(t) = \int_0^t \lambda(v)\,\mathrm{d}v$, $p_i(t) = \lambda_i(t)/\lambda(t)$, $i = 1, 2, \ldots, m$. Furthermore, denote by

$$\mathrm{MMVPP}(\nu_1(t), \nu_2(t), \ldots, \nu_m(t), f_Z(z))$$

the mixture of multivariate NHPP (dimension m) with respective intensities $z\nu_i(t)$, $i = 1, 2, \ldots, m$ (given $Z = z$) and the corresponding mixing distribution of Z (PDF), $f_Z(z)$. The main properties are briefly stated as a proposition.

Proposition 2. *Let* $\{\boldsymbol{N}(t),\ t \geq 0\}$ *be* $\mathrm{MVGPP}(\lambda_1(t), \lambda_2(t), \ldots, \lambda_m(t), \alpha, \beta)$. *Then*

(i) $\{M(t),\ t \geq 0\}$ *is* $\mathrm{GPP}(\lambda(t), \alpha, \beta)$.

(ii) *The process* $\{\boldsymbol{N}(t),\ t \geq 0\}$ *is constructed by thinning of* $\{M(t),\ t \geq 0\}$ *with thinning probabilities* $p_i(t)$, $i = 1, 2, \ldots, m$: $\{(N_{p_1(t)}, N_{p_2(t)}, \ldots, N_{p_m(t)}),\ t \geq 0\}$.

(iii) *The process* $\{\boldsymbol{N}_u(t),\ t \geq 0\}$ *is both conditionally and unconditionally MVGPP.*

(iv) *The processes*

$$\mathrm{MVGPP}(\lambda_1(t), \lambda_2(t), \ldots, \lambda_m(t), \alpha, \beta) \quad \textit{and} \quad \mathrm{MMVPP}(\nu_1(t), \nu_2(t), \ldots, \nu_m(t), f_Z(z))$$

share the same stochastic properties, where $\nu_i(t) \equiv \lambda_i(t)\exp\{\alpha\Lambda(t)\}$, $i = 1, 2, \ldots, m$, *and* $f_Z(z) = \Gamma(\beta/\alpha)^{-1}\alpha^{-\alpha/\beta}z^{\alpha/\beta-1}\exp\{-\alpha^{-1}z\}$.

Based on the above basic properties, the results on the joint distributions of the numbers of events, the conditional joint arrival times distribution, and the marginal processes, etc., can be obtained following similar procedures described in Sections 2 and 3. For instance, the joint distributions of the numbers of events in the interval $(0, t]$ is given by the following negative multinomial distribution:

$$\begin{aligned}
&\mathbb{P}(N_i(t) = n_i,\ i = 1, 2, \ldots, m) \\
&\quad = \frac{\Gamma(\beta/\alpha + \sum_{i=1}^m n_i)}{\Gamma(\beta/\alpha)\prod_{i=1}^m n_i!}\left[\prod_{i=1}^m\left(\alpha\int_0^t \lambda_i(x)\exp\{-\alpha[\Lambda(t) - \Lambda(x)]\}\,\mathrm{d}x\right)^{n_i}\right] \\
&\qquad \times (\exp\{-\alpha\Lambda(t)\})^{\beta/\alpha}.
\end{aligned}$$

The conditional joint arrival times distribution of

$$(T_{i1}, T_{i2}, \ldots, T_{in_i}, i = 1, 2, \ldots, m \mid N_i(t-) = n_i, i = 1, 2, \ldots, m)$$

is

$$\begin{aligned}
&f_{(T_{i1}, T_{i2}, \ldots, T_{in_i}, i=1,2,\ldots,m \mid N_i(t-), i=1,2,\ldots,m)}(t_{i1}, \ldots, t_{in_i}, i = 1, 2, \ldots, m \mid n_1, n_2, \ldots, n_m) \\
&\quad = \prod_{i=1}^m\left[n_i!\prod_{j=1}^{n_i}\left(\frac{\lambda_i(t_{ij})\exp\{\alpha\int_0^{t_{ij}}\sum_{i=1}^m\lambda_i(x)\,\mathrm{d}x\}}{\int_0^t\lambda_i(v)\exp\{\alpha\int_0^v\sum_{i=1}^m\lambda_i(x)\,\mathrm{d}x\}\,\mathrm{d}v}\right)\right],
\end{aligned}$$

$0 < t_{i1} < \cdots < t_{in_i} \leq t$, $i = 1, 2, \ldots, m$.

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For the dependence concept of the MVGPP we need a new notion. The random variables $U_1, U_2, \ldots, U_m$ are *positively upper orthant dependent* (PUOD) if the inequality

$$\mathbb{P}(U_i > u_i, i = 1, 2, \ldots, m) \geq \prod_{i=1}^{m} \mathbb{P}(U_i > u_i) \quad \text{for all } u_i, i = 1, 2, \ldots, m \tag{8}$$

holds (see, for example, Joe (1997)). Intuitively, the inequality (8) implies that $U_1, U_2, \ldots, U_m$ are more likely simultaneously to have large values, compared with a vector of independent random variables with the same univariate marginal distributions. Also, it can be seen that if $U_1, U_2, \ldots, U_m$ are PUOD, then, for arbitrary $1 \leq i_1 < i_2 < \cdots < i_r \leq m$,

$$\mathbb{P}(U_{i_j} > u_{i_j}, j = 1, 2, \ldots, r) \geq \prod_{j=1}^{r} \mathbb{P}(U_{i_j} > u_{i_j}) \quad \text{for all } u_{i_j}, j = 1, 2, \ldots, r.$$

Accordingly, PUOD is a concept of multivariate positive dependency which generalizes the bivariate PQD. Now we define a similar concept for multivariate point processes.

Definition 7. (*Positive upper orthant dependent multivariate process.*) A multivariate point process $\{(U_1(t), \ldots, U_m(t)),\ t \geq 0\}$ is *positive upper orthant dependent multivariate process* (PUODMP) if

$$\mathbb{P}(U_i(t_{i2}) - U_i(t_{i1}) > n_i, i = 1, 2, \ldots, m) \geq \prod_{i=1}^{m} \mathbb{P}(U_i(t_{i2}) - U_i(t_{i1}) > n_i)$$

for all $t_{i2} > t_{i1}$ and $n_i, i = 1, 2, \ldots, m$.

We will now analyze the dependence structure of the MVGPP.

Theorem 6. *The MVGPP is a positive upper orthant dependent multivariate process:*

$$\mathbb{P}(N_i(t_{i2}) - N_i(t_{i1}) > n_i, i = 1, 2, \ldots, m) \geq \prod_{i=1}^{m} \mathbb{P}(N_i(t_{i2}) - N_i(t_{i1}) > n_i)$$

for all $t_{i2} > t_{i1}$ and $n_i, i = 1, 2, \ldots, m$.

Proof. Due to Proposition 2(iv), it suffices to show that

$$\mathbb{P}(N_i^*(t_{i2}) - N_i^*(t_{i1}) > n_i, i = 1, 2, \ldots, m) \geq \prod_{i=1}^{m} \mathbb{P}(N_i^*(t_{i2}) - N_i(t_{i1}) > n_i),$$

where $N_i^*(t)$ is the counting random variable in the corresponding MMVPP. Note that

$$\begin{aligned}&\mathbb{P}(N_i^*(t_{i2}) - N_i^*(t_{i1}) > n_i, i = 1, 2, \ldots, m)\\ &\quad = \mathbb{E}[\mathbb{P}(N_i^*(t_{i2}) - N_i^*(t_{i1}) > n_i, i = 1, 2, \ldots, m \mid Z)]\end{aligned}$$

and

$$\begin{aligned}&\mathbb{P}(N_i^*(t_{i2}) - N_i^*(t_{i1}) > n_i, i = 1, 2, \ldots, m \mid Z = z)\\ &\quad = \prod_{i=1}^{m}\left[\sum_{n=n_i+1}^{\infty} \frac{(z\int_{t_{i1}}^{t_{i2}} \nu_i(x)\,\mathrm{d}x)^n}{n!} \exp\left\{-z\int_{t_{i1}}^{t_{i2}} \nu_i(x)\,\mathrm{d}x\right\}\right]\\ &\quad = \prod_{i=1}^{m} g_i(z),\end{aligned}$$

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where

$$g_i(z) \equiv \sum_{n=n_i+1}^{\infty} \frac{(z\int_{t_{i1}}^{t_{i2}} \nu_i(x)\,\mathrm{d}x)^n}{n!} \exp\left\{-z\int_{t_{i1}}^{t_{i2}} \nu_i(x)\,\mathrm{d}x\right\}, \qquad i = 1, 2, \ldots, m.$$

Note that $g_i(z)$ is increasing in z (the proof of Theorem 3) and thus, by applying Lemma 1 recursively, we have

$$\begin{aligned}
&\mathbb{P}(N_i^*(t_{i2}) - N_i^*(t_{i1}) > n_i, i = 1, 2, \ldots, m) \\
&\quad = \mathbb{E}\left[\prod_{i=1}^{m} g_i(Z)\right] \geq \cdots \geq \mathbb{E}[g_1(Z)g_2(Z)] \prod_{i=3}^{m} \mathbb{E}[g_i(Z)] \\
&\quad \geq \prod_{i=1}^{m} \mathbb{E}[g_i(Z)] \\
&\quad = \prod_{i=1}^{m} \mathbb{P}(N_i(t_{i2}) - N_i(t_{i1}) > n_i).
\end{aligned}$$

This completes the proof. □

5. Concluding remarks

The multivariate point processes are practically very useful tools for modeling the random occurrences of multivariate series of events arising over time intervals. However, until now, very few practically available multivariate point processes have been developed and, accordingly, there has been a great discrepancy between desired practical applications and available useful models. In this regard, our aim of this paper was to develop a new class of multivariate point processes which possess nice properties, allowing mathematical tractability in various applications. In this paper, we have defined the MVGPP and studied its various stochastic properties. It has been shown that the pooled process and all the marginal and conditional processes are the univariate GPP studied in Cha (2014). Furthermore, the MVGPP possesses the multivariate version of 'restarting property' and this property has simplified the analysis of the MVGPP greatly. The MVGPP has been characterized in terms of the mixture of multivariate Poisson processes and, due to this characterization, further properties of the MVGPP have been obtained efficiently. One of the most important aspects of the MVGPP is that it allows mathematical tractability almost to the same extent as in the univariate Poisson process.

As illustrated in the introductory section, the multivariate counting processes can be usefully applied to many areas such as reliability, queueing analysis, insurance risk analysis, finance, econometrics, and so forth. Especially in the reliability area, numerous new repair types have been suggested, generalizing the 'minimal repair'. However, until now, there have been few works which deal with bivariate dependent failure and repair processes. In this paper, it has been shown that the dependent failure and repair process can be modeled via the MVGPP and it allows a clear interpretation on the effect of the dependent failure and repair. This will be a crucial contribution of this work especially to the area of reliability.

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