ESAIM: M2AN 45 (2011) 335–360
DOI: 10.1051/m2an/2010059

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

NUMERICAL ALGORITHMS FOR BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH 1-D BROWNIAN MOTION: CONVERGENCE AND SIMULATIONS*, **

Shige Peng[1,2] and Mingyu Xu[2,3]

Abstract. In this paper we study different algorithms for backward stochastic differential equations (BSDE in short) basing on random walk framework for 1-dimensional Brownian motion. Implicit and explicit schemes for both BSDE and reflected BSDE are introduced. Then we prove the convergence of different algorithms and present simulation results for different types of BSDEs.

Mathematics Subject Classification. 60H10, 34K28.

Received November 13, 2008. Revised November 26, 2009.
Published online August 24, 2010.

1. Introduction

Non-linear backward stochastic differential equations (BSDEs in short) were firstly studied by Pardoux and Peng [18], who proved the existence and uniqueness of the adapted solution, under smooth square integrability assumptions on the coefficient and the terminal condition, and when the coefficient $g(t,\omega,y,z)$ is Lipschitz in (y,z) uniformly in (t,ω). From then on, the theory of backward stochastic differential equations (BSDE) has been widely and rapidly developed. And many problems in mathematical finance can be treated as BSDEs. The natural connection between BSDE and partial differential equations (PDE) of parabolic and elliptic types is also important applications. It is known that only a limited number of BSDEs can be solved explicitly. To develop numerical methods and numerical algorithms is very helpful, both theoretically and practically.

Keywords and phrases. Backward stochastic differential equations, reflected stochastic differential equations with one barrier, numerical algorithm, numerical simulation.

* *This work is partly supported by the National Basic Research Program of China (973 Program), No. 2007CB814902 and No. 2007CB814906.*

** *Second author is also partly supported by Youth Grant of National Science Foundation (No. 10901154/A0110) and financial support of president fund of AMSS, CAS.*

[1] School of Mathematics and System Science, Shandong University, 250100 Jinan, P.R. China. peng@sdu.edu.cn.

[2] Key Laboratory of Random Complex Structures and Data Science, Academy of Mathematics and Systems Science, CAS (No. 2008DP173182), P.R. China. xumy@amss.ac.cn.

[3] Department of Financial Mathematics and Control science, School of Mathematical Science, Fudan University, 200433 Shanghai, P.R. China.

Article published by EDP Sciences

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https://doi.org/10.1051/m2an/2010059 Published online by Cambridge University Press

The solution of a BSDE is a couple of progressive measurable processes (Y, Z), which satisfies

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s, \tag{1.1}$$

where B is a Brownian motion. Here ξ is terminal condition and g is a generator. From [13,18], we know that when ξ is a square integrable random variable, and g satisfies Lipschitz condition and some integrability condition, BSDE (1.1) admits a unique solution.

The calculation and simulation of BSDEs is essentially different from those of SDEs (see [15]). When g is linear in y and z, we may solve the solution of BSDE by considering its dual equation, which is a forward SDE. However for many nonlinear cases of g, we can not find the solution explicitly. Here we describe a software package that compute our numerical solutions for BSDEs with a convenient user-machine interface[4]. This package computes solutions of BSDEs, reflected BSDEs with one or two barriers as well as BSDEs with constraints. One for significant advantage of this package is that users have a very convenient interface. Any users who know the ABC of BSDE can use this package very easily. The input-output interface was also carefully designed.

This paper is organized as follows. In Section 2, we introduce the discretization of BSDEs, then present implicit and explicit schemes for numerical calculation and consider their convergence. In Section 3, we continue to consider reflected BSDEs with one barrier which is an Itô process, by implicit reflected scheme, explicit reflected scheme, penalized explicit-implicit scheme and penalized explicit scheme, then we prove the convergence of these schemes. In Section 4, we show some numerical simulations for standard BSDE and reflected BSDE. In Section 5, we apply penalized schemes to BSDEs with constraint on z and BSDE with solution y reflecting on a function of z.

We should point out that there have been many recent different algorithms for computing solutions of BSDEs and the related results in numerical analysis, for example [1–8,10,11,14,16,17,23–25]. In contrast to these results, our method uses very simple method.

2. Numerical schemes for standard BSDEs

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $(B_t)_{t\geq 0}$ be a 1-dimensional Brownian motion defined on a fixed interval $[0, T]$. We denote by $\{\mathcal{F}_t\}_{0\leq t\leq T}$ the natural filtration generated by the Brownian motion B, *i.e.*, $\mathcal{F}_t = \sigma\{B_s; 0 \leq s \leq t\}$ augmented with all P-null sets of $\mathcal{F}$. We consider for a fixed $n \in \mathbf{N}$,

$$B_t^n := \sqrt{\delta} \sum_{j=1}^{[t/\delta]} \varepsilon_j^n, \quad \text{for all } \ 0 \leq t \leq T, \ \delta = \frac{T}{n},$$

where $\{\varepsilon_j^n\}_{j=1}^n$ is a $\{1, -1\}$-valued i.i.d. sequence with $P\{\varepsilon_j^n = 1\} = P\{\varepsilon_j^n = -1\} = 0.5$, *i.e.*, a Bernoulli sequence. We set $\mathcal{G}_j^n := \sigma\{\varepsilon_1^n, \ldots, \varepsilon_j^n\}$ and $t_j = \delta j$.

Let $g : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be a Lipschitz function in (y, z) uniformly of t, *i.e.*, g satisfies for a fixed $\mu > 0$

$$|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq \mu(|y_1 - y_2| + |z_1 - z_2|) \ \ \forall t \ \in [0, T], \ \forall (y_1, z_1), (y_2, z_2) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{2.1}$$

And $g(\cdot, 0, 0)$ is square integrable.

[4]The study of simulations of BSDE has been started since 1996 in Shandong University, Mathematical Finance Laboratory directed by Peng Shige. First simulation was done by Zhou Haibin, then following his works Xu Mingyu worked on this software package since her master program (from 2000). This paper is a summary of algorithms for BSDE and reflected BSDE with one barrier that have been used in the package.

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We will approximate a pair of $\mathbb{R}^d$-valued $(\mathcal{F}_t)$-progressively measurable processes (Y, Z) defined on $[0, T]$ such that $E[\sup_{0\le t\le T}|Y_t|^2] + E[\int_0^T |Z_t|^2 \mathrm{d}t] < \infty$, which satisfies

$$-\mathrm{d}Y_t = g(t, Y_t, Z_t)\mathrm{d}t - Z_t \mathrm{d}B_t \tag{2.2}$$

with given terminal condition $Y_T = \xi \in \mathbf{L}^2(\mathcal{F}_T, \mathbb{R}^d)$, where $\mathbf{L}^2(\mathcal{F}_T)$ is the space of $\mathcal{F}_T$ measurable $\mathbb{R}^d$-valued random variable satisfying $E|\xi|^2 < \infty$. It is clear that Y has continuous paths. An existence and uniqueness theorem for equation (2.2) was established in [18], when the generator g satisfies (2.1) and $g(\cdot, 0, 0)$ is a square integrable. In many situations we are also interested in BSDEs of the following form for $d = 1$:

$$-\mathrm{d}Y_t = g(t, Y_t, Z_t)\mathrm{d}t + \mathrm{d}A_t - Z_t \mathrm{d}B_t,\ t \in [0, T], \tag{2.3}$$

where $(A_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t)$-predictable RCLL (right continuous with left limits) process with almost surely bounded variation such that $A_0 = 0$ and $E[\sup_{0\le t\le T}|A_t|^2] < \infty$. By the standard existence and uniqueness theorem for solutions of BSDE, for each given A and $Y_T = \xi \in L^2(\mathcal{F}_T)$, there exists a unique pair (Y, Z) for equation (2.3). Here Y has RCLL paths. We call the triple (Y, Z, A) a g-supersolution (resp. g-subsolution), if A is an increasing process (resp. decreasing process). It is called a g-solution if $A \equiv 0$. It is easy to check that, if both (Y, Z, A) and $(Y, \bar{Z}, \bar{A})$ are g-supersolutions on $[0, T]$, then $(Z, A) \equiv (\bar{Z}, \bar{A})$. Thus we often call Y a g-super(sub)solution (or g-solution when $A \equiv 0$) without specifying the related (Z, A).

2.1. Implicit and explicit schemes for BSDEs

We first give an assumption for discrete terminal condition $\xi^n \in \mathbf{L}^2(\mathcal{F}_T, \mathbb{R}^d)$ and g.

Assumption 2.1. *Consider ξ which is $\mathcal{F}_T$-measurable and ξ^n which is $\mathcal{G}_n^n$-measurable, such that*

$$E[|\xi|^2] + \sup_n E[|\xi^n|^2] < \infty$$

and

$$\lim_{n\to\infty} E[|\xi - \xi^n|^2] = 0.$$

Moreover we assume $\sum_{j=0}^n |g(t_j, 0, 0)|^2\delta$ is uniformly bounded in n.

Example 2.1. Set $\xi = \Phi((B_t)_{0\le t\le T})$, where $\Phi : \mathbf{D}_{[0,T]} :\to \mathbb{R}^d$ and satisfies Lipschitz condition. By Donsker's theorem and Skorokhod representation theorem, there exists a probability space, such that $\sup_{0\le t\le T}|B_t^n - B_t| \to 0$, as $n \to \infty$, in $\mathbf{L}^2(\mathcal{F}_T)$, since ε_j^n is in $\mathbf{L}^{2+\delta}$. So $\xi^n := \Phi((B_t^n)_{0\le t\le T})$, with ξ, satisfies Assumption 2.1.

The numerical solution of (2.2) is obtained by $(Y_t^n, Z_t^n) \equiv (y_j^n, z_j^n)$, $t \in [j\delta, (j+1)\delta)$, $\delta n = T$. $(y_j^n, z_j^n)_{0\le j\le n}$ is the solution of discrete BSDE which starts from $y_n^n = \xi^n$. Our discrete BSDE on the small interval is

$$y_j^n = y_{j+1}^n + g(t_j, y_j^n, z_j^n)\delta - z_j^n \varepsilon_{j+1}^n \sqrt{\delta}. \tag{2.4}$$

Then for given y_{j+1}^n, we want to find $\mathcal{G}_j^n$-measurable (y_j^n, z_j^n). The feasibility of this scheme for small δ is due to the following easy lemma.

Lemma 2.1. *Let y_{j+1}^n be a given $\mathcal{G}_{j+1}^n$-measurable random variable. Then, when $\delta < 1/\mu$, there exists a unique $\mathcal{G}_j^n$-measurable pair (y_j^n, z_j^n) satisfying (2.4).*

Proof. We set $Y_+ = y_{j+1}^n|_{\varepsilon_{j+1}^n=1}$ and $Y_- = y_{j+1}^n|_{\varepsilon_{j+1}^n=-1}$. Both Y_+ and Y_- are $\mathcal{G}_j^n$-measurable. Equation (2.4) is then equivalent to the following algebraic equation:

$$\begin{aligned} y_j^n &= Y_+ + g(t_j, y_j^n, z_j^n)\delta - z_j^n\sqrt{\delta}, \\ y_j^n &= Y_- + g(t_j, y_j^n, z_j^n)\delta + z_j^n\sqrt{\delta}. \end{aligned}$$

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This is equivalent to

$$z_j^n = \frac{1}{2\sqrt{\delta}}(Y_+ - Y_-) = \frac{1}{\sqrt{\delta}} E[y_{j+1}^n \varepsilon_{j+1} | \mathcal{G}_j^n]. \tag{2.5}$$

and

$$y_j^n - g(t_j, y_j^n, z_j^n)\delta = \frac{1}{2}(Y_+ + Y_-) = E[y_{j+1}^n | \mathcal{G}_j^n]. \tag{2.6}$$

Because g is assumed to be Lipschitz, the mapping $\Theta(y) = y - g(t_j, y, z_j^n)\delta$ is strictly monotonic: when $\delta\mu < 1$,

$$\langle \Theta(y) - \Theta(y'), y - y' \rangle \geq (1 - \delta\mu) |y - y'|^2 > 0.$$

So there exists a unique value y_j^n satisfying (2.6). □

This lemma shows a way to solve (2.4), and we named this algorithm as 'implicit scheme'. In many cases, Θ^{-1} cannot be solved explicitly. Thus we introduce the following explicit scheme by using $E[y_{j+1}^n | \mathcal{G}_j^n]$ to approximate y_j^n in g of (2.4). We set $\bar{Y}_T^n = \bar{y}_n^n = \xi^n$ and, starting from $j = n-1$, solve in following reverse order,

$$\bar{y}_j^n = \bar{y}_{j+1}^n + g(t_j, E[\bar{y}_{j+1}^n | \mathcal{G}_j^n], \bar{z}_j^n)\delta - \bar{z}_j^n \varepsilon_{j+1}^n \sqrt{\delta}. \tag{2.7}$$

Then we get,

$$\begin{aligned} \overline{y}_j^n &= E[\bar{y}_{j+1}^n | \mathcal{G}_j^n] + g(t_j, E[\bar{y}_{j+1}^n | \mathcal{G}_j^n], \overline{z}_j^n)\delta, \\ \overline{z}_j^n &= \frac{1}{\sqrt{\delta}} E[\bar{y}_{j+1}^n \varepsilon_{j+1}^n | \mathcal{G}_j^n] = \frac{\bar{y}_{j+1}^n|_{\varepsilon_{j+1}^n=1} - \bar{y}_{j+1}^n|_{\varepsilon_{j+1}^n=-1}}{2\sqrt{\delta}}. \end{aligned}$$

This explicit scheme is useful when g is not linear in y, like $g(t, y, z) = \sin(y)$ and the following example.

Example 2.2. In dynamic risk measure (*cf.* [21]), for a position X in the market, define $\rho(X) := Y_0$, where Y_0 is the solution of BSDE associated with $(g, -X)$. If g is a convex function, then $\rho : \mathbf{L}^2(\mathcal{F}_T) \to \mathbb{R}$ gives a convex dynamic risk measure.

Another advantage of explicit scheme is in the case of $d > 1$. If such case, we consider a multi-dimensional BSDE driven by 1-d Brownian motion. So to solve Θ^{-1} means to solve a d-dimensional equation group with d variables. Even in linear case, it is not easy, since we need to solve a d-dimensional linear equation. In such situation, explicit scheme can reduce many calculations, so we can get the result quicker.

Remark 2.1. To find g-super(sub)solution with an increasing process A as in (2.3), we need to consider the discretization of A, setting $A_0^n = 0$, $A_j^n := \sum_{i=0}^{j-1} E[A_{t_{i+1}} - A_{t_i} | \mathcal{G}_i^n]$. Since A is an increasing process, A_j^n is also increasing. Then instead of (2.4), we get

$$y_j^n = y_{j+1}^n + g(t_j, y_j^n, z_j^n)\delta + (A_{j+1}^n - A_j^n) - z_j^n \varepsilon_{j+1}^n \sqrt{\delta},$$

where $A_{j+1}^n - A_j^n$ is $\mathcal{G}_j^n$-measurable. Then from implicit scheme we get

$$\begin{aligned} y_j^n &= \Theta^{-1}(E[y_{j+1}^n | \mathcal{G}_j^n] + (A_{j+1}^n - A_j^n)), \\ z_j^n &= \frac{1}{\sqrt{\delta}} E[y_{j+1}^n \varepsilon_{j+1}^n | \mathcal{G}_j^n] = \frac{y_{j+1}^n|_{\varepsilon_{j+1}^n=1} - y_{j+1}^n|_{\varepsilon_{j+1}^n=-1}}{2\sqrt{\delta}}. \end{aligned}$$

And from explicit scheme, we get

$$\begin{aligned} \overline{y}_j^n &= E[\bar{y}_{j+1}^n | \mathcal{G}_j^n] + g(t_j, E[\bar{y}_{j+1}^n | \mathcal{G}_j^n], \overline{z}_j^n)\delta + (A_{j+1}^n - A_j^n), \\ \overline{z}_j^n &= \frac{1}{\sqrt{\delta}} E[\bar{y}_{j+1}^n \varepsilon_{j+1}^n | \mathcal{G}_j^n] = \frac{\bar{y}_{j+1}^n|_{\varepsilon_{j+1}^n=1} - \bar{y}_{j+1}^n|_{\varepsilon_{j+1}^n=-1}}{2\sqrt{\delta}}. \end{aligned}$$

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In this paper, we will not make special efforts to study the convergence of discrete g-super(sub)solution. Indeed, if we set

$$\widetilde{y}_j^n = y_j^n + A_j^n, \widetilde{z}_j^n = z_j^n, \quad 0 \le j \le n,$$

then $(\widetilde{y}^n, \widetilde{z}^n)$ is discrete solution of discrete BSDE with coefficient $\widetilde{g}(t, y, z) = g(t, y - A_t, z)$. When $A^n \to A$ in certain sense, then we can get the convergence of (y^n, z^n) by $(\widetilde{y}^n, \widetilde{z}^n)$, which is discrete solution of a classical BSDE.

However in many cases, the increasing process A is not given, it is associated with (Y, Z) in order to keep (Y, Z) to satisfying certain condition, like reflected BSDE and constraint BSDE. We will discuss them later in this paper.

2.2. **Convergence results for numerical schemes for BSDEs**

We set

$$Y_t^n = y_{[t/\delta]}^n, Z_t^n = z_{[t/\delta]}^n, \quad \bar{Y}_t^n = \bar{y}_{[t/\delta]}^n, \overline{Z}_t^n = \overline{z}_{[t/\delta]}^n, \quad 0 \le t \le T,$$

where $(y_j^n, z_j^n)_{0 \le j \le n}$ and $(\overline{y}_j^n, \overline{z}_j^n)_{0 \le j \le n}$ are discrete solutions of (2.4) by implicit and explicit schemes, respectively.

By Donsker's theorem and Skorokhod representation theorem, there exists a probability space, such that $\sup_{0 \le t \le T} |B_t^n - B_t| \to 0$, as $n \to \infty$, in $\mathbf{L}^2(\mathcal{F}_T)$, since ε_k is in $\mathbf{L}^{2+\delta}$. Here $\mathbf{L}^{2+\delta}$ is the space of random variable ϕ satisfying $E[(\phi)^{2+\delta}] < +\infty$. Then we have:

Theorem 2.1. *We suppose that Assumption 2.1 hold and that g is Lipschitz in y and z. Then the discrete solutions $\{(Y^n, Z^n)\}_{n=1}^{\infty}$ under the implicit scheme and $\{(\bar{Y}^n, \bar{Z}^n)\}_{n=1}^{\infty}$ under the explicit scheme converge to the solution (Y, Z) of (2.2) in the following senses: as $n \to \infty$,*

$$E\left[\sup_{0 \le t \le T} |Y_t^n - Y_t|^2 + \int_0^T |Z_s^n - Z_s|^2 \, \mathrm{d}s\right] \to 0, \tag{2.8}$$

and

$$E\left[\sup_{0 \le t \le T} |\bar{Y}_t^n - Y_t|^2 + \int_0^T |\bar{Z}_s^n - Z_s|^2 \, \mathrm{d}s\right] \to 0. \tag{2.9}$$

The convergence (2.8) for this implicit scheme was obtained in 2001 by a profound result of Briand *et al.* [5], which can also be found in [6]. From these results, convergence (2.9) can be derived. Before proving (2.9), we first present following lemmas.

Lemma 2.2. *Let a, b and α be positive constants, $\delta b < 1$ and a sequence $(v_j)_{j=1,\ldots,n}$ of positive numbers such that, for every j*

$$v_j + \alpha \le a + b\delta \sum_{i=1}^{j} v_i.$$

Then

$$\sup_{j \le n} v_j + \alpha \le a \mathrm{e}^{bT}.$$

This is a type of Gronwall lemma for discrete cases. The proof can be found in [17], so we omit it.

Lemma 2.3. *We assume that δ is small enough such that $(1 + 2\mu + 2\mu^2)\delta < 1$. Then*

$$E\left[\sup_j |\overline{y}_j^n|^2 + \sum_{j=0}^{n-1} |\overline{z}_j^n|^2 \delta\right] \le C_{\xi^n, g} \mathrm{e}^{(1+2\mu+2\mu^2)T} \tag{2.10}$$

where $C_{\xi^n, g} = (1 + \delta\mu)E[|\xi^n|^2] + \sum_{j=0}^{n-1} g^2(t_j, 0, 0)\delta$.

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Proof. From explicit scheme

$$\overline{y}_j^n = \overline{y}_{j+1}^n + g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n)\delta - \overline{z}_j^n\sqrt{\delta}\varepsilon_{j+1}^n.$$

We have

$$\begin{aligned} |\bar{y}_j^n|^2 - |\overline{y}_{j+1}^n|^2 &= -|\bar{z}_j^n|^2\delta + 2[\bar{y}_j^n \cdot g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n)]\delta - |g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n)|^2\delta^2 \\ &\quad - 2\bar{y}_j^n\bar{z}_j^n\sqrt{\delta}\varepsilon_{j+1} + 2\bar{z}_j^n g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n)\delta\sqrt{\delta}\varepsilon_{j+1}^n \end{aligned} \tag{2.11}$$

Taking expectation and the sum for $j = i, \ldots, n-1$ yields

$$E|\bar{y}_i^n|^2 \le E|\xi^n|^2 - \sum_{j=i}^{n-1} E|\bar{z}_j^n|^2\delta + 2\delta E\sum_{j=i}^{n-1}\{|\bar{y}_j^n| \cdot (|g(t_j, 0, 0)| + \mu|E[\overline{y}_{j+1}^n|\mathcal{G}_j^n]| + \mu|\bar{z}_j^n|)\}.$$

Since the last term is dominated by

$$\begin{aligned} &\delta E\sum_{j=i}^{n-1}\left\{|\bar{y}_j^n|^2(1+\mu+2\mu^2) + |g(t_j,0,0)|^2 + \mu|E[\overline{y}_{j+1}^n|\mathcal{G}_j^n]|^2 + \frac{1}{2}|\bar{z}_j^n|^2\right\} \\ &\le \delta E\sum_{j=i}^{n-1}\left\{|\bar{y}_j^n|^2(1+2\mu+2\mu^2) + |g(t_j,0,0)|^2 + \frac{1}{2}|\bar{z}_j^n|^2\right\} + \mu\delta E|\xi^n|^2, \end{aligned}$$

we thus have

$$E|\bar{y}_i^n|^2 + \frac{1}{2}\sum_{j=i}^{n-1} E|\bar{z}_j^n|^2\delta \le \sum_{j=i}^{n-1}|g(t_j,0,0)|^2\delta + (1+\mu\delta)E|\xi^n|^2 + \delta(1+2\mu+2\mu^2)\sum_{j=i}^{n-1}E|\bar{y}_j^n|^2$$

Then by Lemma 2.2, we obtain

$$\sup_i E|\bar{y}_i^n|^2 + \frac{1}{2}\sum_{j=0}^{n-1} E|\bar{z}_j^n|^2\delta \le C_{\xi^n,g}\mathrm{e}^{(1+2\mu+2\mu^2)T}.$$

For (2.10), we recall (2.11), and take the sum for $j = i, \ldots, n-1$ and sup over j, then take expectation. Notice that $\sum_{j=0}^i \bar{y}_j^n\bar{z}_j^n\sqrt{\delta}\varepsilon_{j+1}^n$ and $\sum_{j=0}^i g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n)\bar{z}_j^n\delta\sqrt{\delta}\varepsilon_{j+1}^n$ are both martingales with respect to $\mathcal{G}_i^n$, we apply Burkholder-Davis-Gundy inequality for them with similar techniques as before, then get

$$E[\sup_i|\bar{y}_i^n|^2] \le cC_{\xi^n,g^n} + C_\mu\delta\sum_{j=0}^{n-1}E|\bar{y}_j^n|^2 \le cC_{\xi^n,g^n} + C_\mu T\sup_j E|\bar{y}_j^n|^2.$$

With previous results, we obtain (2.10). □

Proof of Theorem 2.1. The convergence of (Y^n, Z^n) to (Y, Z) is proved in [5]. To prove (2.9), the result for $(\overline{Y}^n, \overline{Z}^n)$, it suffices to prove as $n \to \infty$,

$$E\left[\sup_{0\le t\le T}\left|Y_t^n - \bar{Y}_t^n\right|^2 + \int_0^T\left|Z_s^n - \bar{Z}_s^n\right|^2\mathrm{d}s\right] \to 0. \tag{2.12}$$

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From (2.4) and (2.7), we have

$$\begin{aligned}|y_i^n-\overline{y}_i^n|^2 &= \left|y_{i+1}^n-\overline{y}_{i+1}^n\right|^2-|z_i^n-\bar{z}_i^n|^2\delta+2[(y_i^n-\bar{y}_i^n)\cdot(g(t_j,y_i^n,z_i^n)-g(t_j,E[\overline{y}_{i+1}^n|\mathcal{G}_i^n],\overline{z}_i^n))]\delta\\ &\quad-|g(t_j,y_i^n,z_i^n)-g(t_j,E[\overline{y}_{i+1}^n|\mathcal{G}_i^n],\overline{z}_i^n)|^2\delta^2-2(y_i^n-\overline{y}_i^n)(z_i^n-\bar{z}_i^n)\sqrt{\delta}\varepsilon_{j+1}^n\\ &\quad+2(g(t_j,y_i^n,z_i^n)-g(t_j,E[\overline{y}_{i+1}^n|\mathcal{G}_i^n],\overline{z}_i^n))(z_i^n-\bar{z}_i^n)\delta\sqrt{\delta}\varepsilon_{j+1}^n. \qquad (2.13)\end{aligned}$$

Then we take expectation and the sum over i from j to $n-1$. With $\xi^n-\overline{\xi}^n=0$, we get

$$\begin{aligned}E\left|y_j^n-\overline{y}_j^n\right|^2 &\le -E\left[\delta\sum_{i=j}^{n-1}|z_i^n-\overline{z}_i^n|^2\right]+2\sum_{i=j}^{n-1}E[|y_i^n-\overline{y}_i^n|\cdot|g(t_j,y_i^n,z_i^n)-g(t_j,E[\overline{y}_{i+1}^n|\mathcal{G}_i^n],\overline{z}_i^n)|]\delta\\ &\le -\frac{1}{2}E\left[\delta\sum_{i=j}^{n-1}|z_i^n-\overline{z}_i^n|^2\right]+2\mu^2\delta E\left[\sum_{i=j}^{n-1}|y_i^n-\overline{y}_i^n|^2\right]+2\mu\delta E\sum_{i=j}^{n-1}|y_i^n-\overline{y}_i^n|\cdot|y_i^n-E[\overline{y}_{i+1}^n|\mathcal{G}_i^n]|.\end{aligned}$$

Since $\overline{y}_i^n-E[\overline{y}_{i+1}^n|\mathcal{G}_j^n]=g(t_j,E[\overline{y}_{j+1}^n|\mathcal{G}_j^n],\overline{z}_j^n)\delta$, the last term is dominated by

$$\delta\sum_{i=j}^{n-1}(2\mu+1)E|y_i^n-\bar{y}_i^n|^2+\sum_{i=j}^{n-1}\mu^2E|g(t_j,E[\overline{y}_{j+1}^n|\mathcal{G}_j^n],\overline{z}_j^n)|^2\delta^3.$$

But with (2.10), the second term is bounded by $C\delta^2$. We thus have

$$E\left|y_j^n-\overline{y}_j^n\right|^2+\frac{\delta}{2}E\left[\sum_{i=j}^{n-1}|z_i^n-\overline{z}_i^n|^2\right]\le(1+2\mu+2\mu^2)\delta\left[\sum_{i=j}^{n-1}E[|y_i^n-\overline{y}_i^n|^2\right]+C\delta^2.$$

By Lemma 2.2, we get

$$\sup_{j\le n}E\left|y_j^n-\overline{y}_j^n\right|^2\le C\delta^2\mathrm{e}^{(2\mu+2\mu^2+1)T}.$$

Then we reconsider square of the difference between the discrete solutions of implicit scheme and explicit scheme shown in (2.13). This time we first take the sum and $\sup_j$, then take expectation. Using Burkholder-Davis-Gundy inequality and similar techniques, we get

$$E\Big[\sup_j|y_j^n-\bar{y}_j^n|^2\Big] \le C_\mu E\left[\delta\sum_{i=j}^{n-1}|y_i^n-\overline{y}_i^n|^2+\delta\sum_{i=j}^{n-1}|z_i^n-\overline{z}_i^n|^2\right]\le C_\mu T\sup_{j\le n}E\left|y_j^n-\overline{y}_j^n\right|^2,$$

with previous results, (2.9) follows. □

We now prove a more general result which will be useful in proving convergence results for schemes of reflected BSDEs. Consider the following BSDE

$$\begin{aligned}-\mathrm{d}Y_t &= [g_1(t,Y_t,Z_t)+g_2(t,Y_t,Z_t)]\mathrm{d}t-Z_t\mathrm{d}B_t, \qquad (2.14)\\ Y_T &= \xi.\end{aligned}$$

Here g_1 and g_2 are both Lipschitz functions. Then we have the following implicit-explicit scheme to only replace y_j^n by $E[y_{j+1}^n|\mathcal{G}_j^n]$ in g_2,

$$\bar{y}_j^n=\bar{y}_{j+1}^n+g_1(t_j,\overline{y}_j^n,\bar{z}_j^n)\delta+g_2(t_j,[\bar{y}_{j+1}^n|\mathcal{G}_j^n],\bar{z}_j^n)\delta-\bar{z}_j^n\varepsilon_{j+1}^n\sqrt{\delta}, \qquad (2.15)$$

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or, equivalently,

$$\begin{aligned}\overline{y}_j^n &= E[\bar{y}_{j+1}^n|\mathcal{G}_j^n] + g_1(t_j, \overline{y}_j^n, \bar{z}_j^n)\delta + g_2(t_j, E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n)\delta,\\ \overline{z}_j^n &= \frac{1}{\sqrt{\delta}}E[\bar{y}_{j+1}^n\varepsilon_{j+1}^n|\mathcal{G}_j^n] = \frac{\bar{y}_{j+1}^n|_{\varepsilon_{j+1}^n=1} - \bar{y}_{j+1}^n|_{\varepsilon_{j+1}^n=-1}}{2\sqrt{\delta}}.\end{aligned}$$

We also set $\bar{Y}_t^n = \bar{y}_{[t/\delta]}^n$, $\bar{Z}_t^n = \bar{z}_{[t/\delta]}^n$, $0 \le t \le T$. Meanwhile we consider the fully implicit scheme

$$y_j^n = y_{j+1}^n + g_1(t_j, y_j^n, z_j^n)\delta + g_2(t_j, y_j^n, z_j^n)\delta - z_j^n\varepsilon_{j+1}^n\sqrt{\delta},$$

and let $Y_t^n = y_{[t/\delta]}^n$, $Z_t^n = z_{[t/\delta]}^n$, $0 \le t \le 1$.

Proposition 2.1. *Under same assumptions of Theorem 2.1, assume g_1 and g_2 are Lipschitz functions. Let (Y, Z) be the solution of BSDE (2.14). Then as $n \to \infty$,*

$$E\left[\sup_{0\le t\le T}\left|\bar{Y}_t^n - Y_t\right|^2 + \int_0^T \left|\bar{Z}_s^n - Z_s\right|^2 \mathrm{d}s\right] \to 0. \tag{2.16}$$

Moreover there exist a constant C_2 depending on T, μ_1 and μ_2 which are Lipschitz constants of g_1, g_2, such that

$$E\left[\sup_{0\le t\le T}\left|\bar{Y}_t^n - Y_t^n\right|^2 + \int_0^T \left|\bar{Z}_s^n - Z_s^n\right|^2 \mathrm{d}s\right] \le C_2\delta^2.$$

The proof is similar to that of Theorem 2.1 and we omit it.

Remark 2.2. This scheme is very useful. For example, we will use it for penalization BSDE, which will be discusses in Section 4.1.

3. Algorithms for reflected BSDEs with one barrier

In this section, we discuss the algorithms for reflected BSDEs with one continuous lower barrier L. A solution of such equation is a triple (Y, Z, K) on $[0, T]$ satisfying $E[\sup_{0\le t\le T}|Y_t|^2 + \int_0^T |Z_s|^2 \mathrm{d}s + |K_T|^2] < \infty$ and

$$\begin{aligned}Y_t &= \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s + K_T - K_t - \int_t^T Z_s \mathrm{d}B_s, \qquad (3.1)\\ Y_t &\ge L_t,\ \mathrm{d}K_t \ge 0,\ \ 0 \le t \le T,\ \ \text{with } \int_0^T (Y_t - L_t)\mathrm{d}K_t = 0.\end{aligned}$$

In [12], existence and uniqueness of the solution of this equation is proved when g satisfies Lipschitz condition (2.1) and $E[|\xi|^2 + \int_0^T g^2(t, 0, 0)\mathrm{d}t + \sup_{0\le t\le T}(L_t^+)^2] < \infty$. Here we consider the case when L_t is an Itô process, *i.e.* $L_t = L_0 + \int_0^t l_s\mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s$, $0 \le t \le T$ and $\xi = \Phi((B_s)_{0\le s\le T})$ satisfying requires of integrability, for convenience of discretization of processes.

Remark 3.1. We call a progressively measurable process ϕ_t is in space $\mathbf{S}^2(0, T)$, if it satisfies $E[\sup_{0\le t\le T}|\phi_t|^2] < \infty$. If a predictable process ϕ_t is in space $\mathbf{L}^2_{\mathcal{F}}(0, T)$, then it satisfies $E[\int_0^T |\phi_s|^2 \mathrm{d}s] < \infty$. And we define a space of $\mathcal{F}_t$-measurable random variables ξ, which satisfies $E[|\xi|^\beta] < \infty$, as $\mathbf{L}^\beta(\mathcal{F}_t)$, for $\beta \in \mathbf{R}^+$.

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3.1. **Numerical reflected schemes**

Following the same discretization introduced as in Section 2, we will approximate the solution of reflected BSDE. On the small interval $[j\delta,(j+1)\delta]$, the equation (3.1) can be approximated by the discrete equation

$$\begin{aligned} y_j^n &= y_{j+1}^n + g(t_j,y_j^n,z_j^n)\delta + d_j^n - z_j^n\varepsilon_{j+1}^n\sqrt{\delta}, \\ y_j^n &\geq L_j^n, (y_j^n - L_j^n)d_j^n = 0, \end{aligned} \tag{3.2}$$

where $d_j^n = K_{t_{j+1}} - K_{t_j}$, and $L_j^n = L_0 + \delta\sum_{i=0}^{j-1} l_{t_i} + \sum_{i=0}^{j-1}\sigma_{t_i}\varepsilon_{i+1}^n\sqrt{\delta}$. Here (3.2) is called discrete reflected BSDE in [17], with terminal value $\xi^n = \Phi((\sum_{i=0}^{j}\varepsilon_{i+1}^n\sqrt{\delta})_{0\leq j\leq n})$.

Remark 3.2. When $L_t = \Psi(t,B_t)$ with $\Psi\in C^{1,2}([0,T]\times\mathbf{R})$, by Itô formula, we know that $L_t = L_0 + \int_0^t(\frac{\partial}{\partial s} + \frac{1}{2}\frac{\partial^2}{\partial x^2})\psi(s,B_s)\mathrm{d}s + \int_0^t\frac{\partial}{\partial x}\psi(s,B_s)\mathrm{d}B_s$. In fact, our algorithms are available for the case when the barrier L is a functional of Brownian motion, *i.e.* $L_t = \Psi(t,(B_s)_{0\leq s\leq t})$, with its discrete version $L_{[t/\delta]}^n = \Psi(t_{[t/\delta]},(\sum_{k=0}^{i}\varepsilon_{k+1}^n\sqrt{\delta})_{0\leq i\leq[t/\delta]})$. In this section, we focus on Itô process in order to discuss the convergence of discrete solution.

Suppose y_{j+1}^n is known, we now consider to find $\mathcal{G}_j^n$-measurable (y_j^n,z_j^n,d_j^n) to satisfy (3.2). Set $Y_+ = y_{j+1}^n|_{\varepsilon_{j+1}^n=1}$ and $Y_- = y_{j+1}^n|_{\varepsilon_{j+1}^n=-1}$. From (3.2), we get immediately $z_j^n = \frac{1}{\sqrt{\delta}}E[y_{j+1}^n\varepsilon_{j+1}^n|\mathcal{G}_j^n] = \frac{1}{2\sqrt{\delta}}(Y_+ - Y_-)$. Substitute it into the equation, our problem is changed to find (y_j^n,d_j^n) satisfying

$$\begin{aligned} y_j^n &= E[y_{j+1}^n|\mathcal{G}_j^n] + g(t_j,y_j^n,z_j^n)\delta + d_j^n, \\ y_j^n &\geq L_j^n, (y_j^n - L_j^n)d_j^n = 0. \end{aligned} \tag{3.3}$$

Then we introduce two different schemes for this equation.

Implicit reflected scheme. First, we present the implicit reflected scheme which is introduces by Mémin *et al.* in [17]. If we consider the mapping $\Theta(y) := y - (g(t_j,y,z_j^n) - g(t_j,L_j^n,z_j^n))\delta$, then for δ small enough, we have

$$\langle\Theta(y) - \Theta(y'), y - y'\rangle \geq (1-\delta\mu)\left|y-y'\right|^2 > 0,$$

i.e. $\Theta(y)$ is strictly increasing with $\Theta(L_j^n) = L_j^n$, so

$$\Theta^{-1}(y) \geq L_j^n \Longleftrightarrow y \geq L_j^n.$$

It follows

$$\begin{aligned} y_j^n &= \Theta^{-1}(E[y_{j+1}^n|\mathcal{G}_j^n] - g(t_j,L_j^n,z_j^n)\delta + d_j^n), \\ d_j^n &= \left(E[y_{j+1}^n|\mathcal{G}_j^n] + g(t_j,L_j^n,z_j^n)\delta - L_j^n\right)^-. \end{aligned}$$

Notice that $E[y_{j+1}^n|\mathcal{G}_j^n] = \frac{1}{2}(Y_+ + Y_-)$, we get the results.

Explicit reflected scheme. Instead of solving the inverse of the mapping Θ, we replace y_j^n by $E[y_{j+1}^n|\mathcal{G}_j^n]$ on the right side of (3.3) to get an approximal solution. Then it follows

$$\begin{aligned} \overline{y}_j^n &= E[\overline{y}_{j+1}^n|\mathcal{G}_j^n] + g(t_j,E[\overline{y}_{j+1}^n|\mathcal{G}_j^n]),\overline{z}_j^n)\delta + \overline{d}_j^n, \\ \overline{d}_j^n &= \left(E[\overline{y}_{j+1}^n|\mathcal{G}_j^n] + g(t_j,E[\overline{y}_{j+1}^n|\mathcal{G}_j^n],\overline{z}_j^n)\delta - L_j^n\right)^-. \end{aligned} \tag{3.4}$$

Substitute $E[\overline{y}_{j+1}^n|\mathcal{G}_j^n] = \frac{1}{2}(Y_+ + Y_-)$ into it, we get the results.

Remark 3.3. Compared to the implicit reflected scheme, the explicit reflected scheme is much easier to compile programs for simulation or to be a convex function. For example $g(t,y,z) = \sin(y)$.

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3.2. **Numerical penalization schemes**

Another important numerical method is *via* the penalization equations of reflected BSDE. In [12], the authors introduced the penalization method to prove the existence of the solution. For $p \in \mathbf{N}$, the penalization equation with respect to the lower barrier L is

$$Y_t^p = \xi + \int_t^T g(s, Y_s^p, Z_s^p)\mathrm{d}s + p\int_t^T (Y_s^p - L_s)^-\mathrm{d}s - \int_t^T Z_s^p \mathrm{d}B_s, \tag{3.5}$$

thanks to the comparison theorem for BSDE, we have $Y_t^p \le Y_t^{p+1}$, for $p \in \mathbf{N}$. Denote $K_t^p = p\int_0^t (Y_s^p - L_s)^-\mathrm{d}s$. Then we know following results from [12].

Theorem 3.1. *There exists a positive constant c independent on p, such that*

$$E[\sup_{0\le t\le T}|Y_t^p - Y_t|^2 + \int_0^T |Z_t^p - Z_t|^2\mathrm{d}t + \sup_{0\le t\le T}|K_t^p - K_t|^2] \le \frac{c}{\sqrt{p}}.$$

When $p \to \infty$, we know $Y^p \to Y$ in $\mathbf{S}^2(0,T)$, $Z^p \to Z$ in $\mathbf{L}^2_{\mathcal{F}}(0,T)$, $K^p \to K$ in $\mathbf{S}^2(0,T)$.

Numerical penalization scheme. By Theorem 3.1, we know that the solution of reflected BSDE can be approximated by the solution of penalization equations (3.5), for some large p. Then on the small time $[j\delta, (j+1)\delta]$, we consider the following discrete penalized BSDE

$$y_j^{p,n} = y_{j+1}^{p,n} + g(t_j, y_j^{p,n}, z_j^{p,n})\delta + p(y_j^{p,n} - L_j^n)^-\delta - z_j^{p,n}\sqrt{\delta}\varepsilon_{j+1}^n.$$

The scheme is the repetition of the following procedure running from $j = n$ backwardly to $i = 1$. If we have already known $(y_{j+1}^{p,n}, z_{j+1}^{p,n})$, then to solve $(y_j^{p,n}, z_j^{n,p})$ from above equation, we first get $z_j^{p,n} = \frac{1}{\sqrt{\delta}}E[y_{j+1}^{p,n}\varepsilon_{j+1}^n|\mathcal{G}_j^n] = \frac{1}{2\sqrt{\delta}}(Y_+^p - Y_-^p)$, where $Y_+^p = y_{j+1}^{p,n}|_{\varepsilon_{j+1}^n=1}, Y_-^p = y_{j+1}^{p,n}|_{\varepsilon_{j+1}^n=-1}$.

Then $y_j^{p,n}$ satisfies following equation

$$y_j^{p,n} = E[y_{j+1}^{p,n}|\mathcal{G}_j^n] + g(t_j, y_j^{p,n}, z_j^{p,n})\delta + p(y_j^{p,n} - L_j^n)^-\delta. \tag{3.6}$$

There are two ways to find suitable $y_j^{p,n}$. One is implicit penalization scheme, *i.e.* to solve the equation:

$$y_j^{p,n} = (\Theta^p)^{-1}(E[y_{j+1}^{p,n}|\mathcal{G}_j^n]) = (\Theta^p)^{-1}\left(\frac{1}{2}(Y_+^p + Y_-^p)\right).$$

Here Θ^p is a mapping, $\Theta^p(y) = y - (g(t_j, y, z_j^{p,n}) + p(y - L_j^n)^-)\delta$. Let $d_j^{p,n} = p(y_j^{p,n} - L_j^n)^-\delta$.

The other is implicit-explicit scheme, we only replace $y_j^{p,n}$ of g in (3.6) by $E[y_{j+1}^{p,n}|\mathcal{G}_j^n]$. Then we get, penalization explicit-implicit scheme, *i.e.*

$$\overline{y}_j^{p,n} = E[\overline{y}_{j+1}^{p,n}|\mathcal{G}_j^n] + g(t_j, E[\overline{y}_{j+1}^{p,n}|\mathcal{G}_j^n], \overline{z}_j^{p,n})\delta + \frac{p\delta}{1+p\delta}(E[\overline{y}_{j+1}^{p,n}|\mathcal{G}_j^n] + g(t_j, E[\overline{y}_{j+1}^{p,n}|\mathcal{G}_j^n], \overline{z}_j^{p,n})\delta - L_j^n)^-.$$

With $E[\overline{y}_{j+1}^{p,n}|\mathcal{G}_j^n] = \frac{1}{2}(\overline{y}_{j+1}^{p,n}|_{\varepsilon_{j+1}^n=1} + \overline{y}_{j+1}^{p,n}|_{\varepsilon_{j+1}^n=-1})$, results follow easily. And we set $\overline{d}_j^{p,n} = p(\overline{y}_j^{p,n} - L_j^n)^-\delta$.

3.3. **Convergence results of penalization schemes**

We first study the penalization scheme of reflected BSDE with one lower barrier. For penalization implicit scheme, define $Y_t^{p,n} = y_{[t/\delta]}^{p,n}$, $Z_t^{p,n} = z_{[t/\delta]}^{p,n}$ and $K_t^{p,n} = \sum_{m=0}^{[t/\delta]} d_m^{p,n}$. By Donsker's theorem and Skorokhod representation theorem, there exists a probability space, such that $\sup_{0\le t\le T}|B_t^n - B_t| \to 0$, as $n \to \infty$, in $\mathbf{L}^2(\mathcal{F}_T)$, since ε_k is in $\mathbf{L}^{2+\delta}$. For convergence of scheme, we have:

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Proposition 3.1. *Under Assumption 2.1 and g satisfying Lipschitz condition. The sequence $(Y_t^{p,n}, Z_t^{p,n})$ converges to (Y_t, Z_t) in the following sense*

$$\lim_{p\to\infty}\lim_{n\to\infty} E\left[\sup_{0\le t\le T}|Y_t^{p,n}-Y_t|^2+\int_0^T|Z_s^{p,n}-Z_s|^2\,\mathrm{d}s\right]=0, \tag{3.7}$$

and for $0\le t\le T$, $K_t^{p,n}\to K_t$ in $\mathbf{L}^2(\mathcal{F}_t)$, as $n\to\infty$, $p\to\infty$.

Proof. Since

$$\begin{aligned}E\left[\sup_{0\le t\le T}|Y_t^{p,n}-Y_t|^2+\int_0^T|Z_s^{p,n}-Z_s|^2\,\mathrm{d}s\right] &\le 2E\left[\sup_{0\le t\le T}|Y_t^{p,n}-Y_t^p|^2+\int_0^T|Z_s^{p,n}-Z_s^p|^2\,\mathrm{d}s\right]\\ &\quad+2E\left[\sup_{0\le t\le T}|Y_t^p-Y_t|^2+\int_0^T|Z_s^p-Z_s|^2\,\mathrm{d}s\right],\end{aligned}$$

by the convergence results of numerical solutions for BSDE and penalization method for reflected BSDE, Theorem 3.1, we know (3.7) hold. For the increasing processes, we have

$$E[(K_t^{p,n}-K_t)^2]\le 2E[(K_t^{p,n}-K_t^p)^2]+2E[(K_t^p-K_t)^2].$$

While for fixed p,

$$\begin{aligned}K_t^{p,n} &= Y_0^{p,n}-Y_t^{p,n}-\int_0^t g(s,Y_s^{p,n},Z_s^{p,n})\mathrm{d}s+\int_0^t Z_s^{p,n}\mathrm{d}B_s^n,\\ K_t^p &= Y_0^p-Y_t^p-\int_0^t g(s,Y_s^p,Z_s^p)\mathrm{d}s+\int_0^t Z_s^p\mathrm{d}B_s,\end{aligned}$$

from Corollary 14 in [6], we know that $\int_0^\cdot Z_s^{p,n}\mathrm{d}B_s^n$ converges to $\int_0^\cdot Z_s^p\mathrm{d}B_s$ in $\mathbf{S}^2(0,T)$, as $n\to\infty$, then with Lipschitz condition of g and (3.7), we get $E[(K_t^{p,n}-K_t^p)^2]\to 0$, as $n\to\infty$. With convergence result of penalization methods, the result follows. □

Then we consider the penalization explicit-implicit scheme, note $\overline{Y}_t^{p,n}=\overline{y}_{[t/\delta]}^{p,n}$, $\overline{Z}_t^{p,n}=\overline{z}_{[t/\delta]}^{p,n}$ and $\overline{K}_t^{p,n}=\sum_{m=0}^{[t/\delta]}\overline{d}_m^{p,n}$, it follows that

Proposition 3.2. *Under same assumptions of Proposition 3.1, $(\overline{Y}_t^{p,n},\overline{Z}_t^{p,n})$ converges to (Y_t,Z_t) in the following sense*

$$\lim_{p\to\infty}\lim_{n\to\infty} E\left[\sup_{0\le t\le T}\left|\overline{Y}_t^{p,n}-Y_t\right|^2+\int_0^T\left|\overline{Z}_s^{p,n}-Z_s\right|^2\mathrm{d}s\right]=0,$$

with $\overline{K}_t^{p,n}\to K_t$ in $\mathbf{L}^2(\mathcal{F}_t)$, for $0\le t\le T$, as $n\to\infty$, $p\to\infty$.

Proof. The convergence of $(\overline{Y}_t^{p,n},\overline{Z}_t^{p,n})$ is a direct result of Proposition 2.1 and (3.7). We consider the increasing process, notice that for $0\le t\le T$,

$$\overline{K}_t^{p,n}=\overline{Y}_0^{p,n}-\overline{Y}_t^{p,n}-\int_0^t g(s,\overline{Y}_s^{p,n},\overline{Z}_s^{p,n})\mathrm{d}s+\int_0^t\overline{Z}_s^{p,n}\mathrm{d}B_s^n,$$

with $K_t^{p,n}=Y_0^{p,n}-Y_t^{p,n}-\int_0^t g(s,Y_s^{p,n},Z_s^{p,n})\mathrm{d}s+\int_0^t Z_s^{p,n}\mathrm{d}B_s^n$, thanks to Lipschitz condition of g and the convergence of $(\overline{Y}^{p,n},\overline{Z}^{p,n})$, we get $E[(K_t^{p,n}-\overline{K}_t^{p,n})^2]\to 0$, as $n\to\infty$, $p\to\infty$. With convergence results of penalization method, results follow. □

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3.4. **Convergence results of reflected schemes**

Now we study the convergence of reflected schemes. First for the implicit reflected scheme, denote $Y_t^n = y_{[t/\delta]}^n$, $Z_t^n = z_{[t/\delta]}^n$, $K_t^n = \sum_{j=0}^{[t/\delta]} d_j^n$, for $0 \le t \le T$, from the results in [17], we know:

Theorem 3.2 (Thm. 3.2 in [17]). *Under Assumption 2.1 and (2.1) for g, as $n \to +\infty$,*

$$E[\sup_{0\le t\le T} |Y_t^n - Y_t|^2] + E\int_0^T |Z_t^n - Z_t|^2 \mathrm{d}t \to 0.$$

For the increasing process, we have

Proposition 3.3. *For $t \in [0,T]$, $E[(K_t - K_t^n)^2] \to 0$, as $n \to \infty$.*

Proof. For $t \in [0,T]$, we have

$$E[(K_t - K_t^n)^2] \le 3E[(K_t - K_t^p)^2] + 3E[(K_t^p - K_t^{p,n})^2] + 3E[(K_t^{p,n} - K_t^n)^2]$$

where K^p is from penalization equation (3.1), and $K_t^{p,n}$ is discrete solution of (3.6), with $K_t^{p,n} = \sum_{j=0}^{[t/\delta]} d_j^{p,n}$. Similar as Lemma 2.5 in [17], we have $E[\sup_t |Y_t^{p,n} - Y_t^n|^2] + E\int_0^T |Z_t^{p,n} - Z_t^n|^2 \mathrm{d}t \le \frac{C^R_{\xi^n,g,L}}{\sqrt{p}}$, where $C^R_{\xi^n,g,L}$ only depends on ξ^n, g, L and μ. Since

$$\begin{aligned} K_t^{p,n} &= Y_0^{p,n} - Y_t^{p,n} - \int_0^t g(s, Y_s^{p,n}, Z_s^{p,n})\mathrm{d}s + \int_0^t Z_s^{p,n}\mathrm{d}B_s^n, \\ K_t^n &= Y_0^n - Y_t^n - \int_0^t g(s, Y_s^n, Z_s^n)\mathrm{d}s + \int_0^t Z_s^n \mathrm{d}B_s^n, \end{aligned}$$

with Lipschitz condition of g, we deduce that $E[(K_t^{p,n} - K_t^n)^2] \le \frac{C^R_{\xi^n,g,L}}{\sqrt{p}}$. It follows

$$E[(K_t - K_t^n)^2] \le (C^R_{\xi^n,g,L} + C^R_{\xi,g,L})\frac{1}{\sqrt{p}} + 3E[(K_t^p - K_t^{p,n})^2].$$

Since $K_t^{p,n} \to K_t^p$ in $\mathbf{L}^2(\mathcal{F}_t)$ as $n \to \infty$, for fixed p, we can choose n large enough to get right side very small. Then result of K^n follows. □

Then we consider the convergence of the reflected explicit scheme. We set

$$\bar{Y}_t^n = \bar{y}_{[t/\delta]}^n, \quad \overline{Z}_t^n = \overline{z}_{[t/\delta]}^n, \quad \overline{K}_t^n = \sum_{j=0}^{[t/\delta]} \overline{d}_j^n \quad 0 \le t \le T.$$

First as Lemma 2.3, we have similar estimation of $\overline{y}_j$ of reflected BSDE, given by (3.4).

Lemma 3.1. *We assume that δ is small enough such that $(2 + 2\mu + 6\mu^2)\delta < 1$. Then*

$$E\left[\sup_j |\overline{y}_j^n|^2 + \sum_{j=0}^{n-1} |\overline{z}_j^n|^2 \delta\right] + E\left[\left(\sum_{j=0}^{n-1} \overline{d}_j^n\right)^2\right] \le C^R_{\xi^n,g,L}$$

where $C^R_{\xi^n,g,L}$ only depends on μ, $E[|\xi^n|^2]$, $\sum_{j=0}^{n-1} g^2(t_j,0,0)\delta$ and $E[\sup_j((L_j^n)^+)^2]$.

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Proof. Recall that for $j = 0, 1, \ldots, n-1$, $(\overline{y}_j^n, \bar{z}_j^n)$ satisfies

$$\begin{aligned} \overline{y}_j^n &= \overline{y}_{j+1}^n + g(t_j, (E[\overline{y}_{j+1}^n|\mathcal{G}_j^n]), \overline{z}_j^n)\delta + \overline{d}_j^n - \bar{z}_j^n \varepsilon_{j+1}^n \sqrt{\delta}, \\ \overline{y}_j^n &\geq L_j^n, (\overline{y}_j^n - L_j^n)\overline{d}_j^n = 0. \end{aligned} \tag{3.8}$$

Apply similar techniques of Lemma 2.3 to (3.8), we have

$$E|\bar{y}_j^n|^2 = E|\overline{y}_{j+1}^n|^2 - E|\bar{z}_j^n|^2\delta + 2E[\bar{y}_{j+1}^n \cdot g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n)]\delta + 2E[\bar{y}_j^n \cdot \overline{d}_j^n] + E|g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n)|^2\delta^2 - E[(\overline{d}_j^n)^2].$$

In view of $(\overline{y}_j^n - L_j^n)\overline{d}_j^n = 0$ and $\overline{d}_j^n \geq 0$, it follows

$$\begin{aligned} E|\bar{y}_j^n|^2 + E|\bar{z}_j^n|^2\delta &\leq E|\overline{y}_{j+1}^n|^2 + 2E[\bar{y}_{j+1}^n \cdot g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n)]\delta + 2E[(L_j^n)^+ \cdot \overline{d}_j^n] \\ &\quad + E[|g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n)|^2\delta^2] \\ &\leq E|\overline{y}_{j+1}^n|^2 + (\delta + 3\delta^2)E[|g(t_j, 0, 0)|^2] + \left(\frac{1}{4}\delta + 3\mu^2\delta^2\right)E[(\overline{z}_j^n)^2] \\ &\quad + \delta(1 + 2\mu + 4\mu^2 + 3\mu^2\delta)E|\overline{y}_{j+1}^n|^2 + 2E[(L_j^n)^+ \cdot \overline{d}_j^n]. \end{aligned}$$

Notice that $3\mu^2\delta < \frac{1}{2}$, since $6\mu^2\delta < 1$. Taking the sum for $j = i, \ldots, n-1$, it yields

$$\begin{aligned} E|\bar{y}_i^n|^2 + \frac{1}{4}\sum_{j=i}^{n-1} E|\bar{z}_j^n|^2\delta \leq E|\xi^n|^2 + (\delta + 3\delta^2)E\sum_{j=i}^{n-1}[|g(t_j, 0, 0)|^2] + \delta\left(\frac{3}{2} + 2\mu + 4\mu^2\right)E\sum_{j=i}^{n-1}|\overline{y}_{j+1}^n|^2 \\ + \alpha E[\sup_j((L_j^n)^+)^2] + \frac{1}{\alpha}E\left[\left(\sum_{j=i}^{n-1}\overline{d}_j^n\right)^2\right], \end{aligned}$$

where α is a constant to be decided later. Since $\overline{d}_j^n = \overline{y}_j^n - \overline{y}_{j+1}^n - g(t_j, (E[\overline{y}_{j+1}^n|\mathcal{G}_j^n]), \overline{z}_j^n)\delta + \bar{z}_j^n\varepsilon_{j+1}^n\sqrt{\delta}$, we get

$$\sum_{j=i}^{n-1}\overline{d}_j^n = \overline{y}_i^n - \xi^n - \sum_{j=i}^{n-1} g(t_j, (E[\overline{y}_{j+1}^n|\mathcal{G}_j^n]), \overline{z}_j^n)\delta + \sum_{j=i}^{n-1}\bar{z}_j^n\varepsilon_{j+1}^n\sqrt{\delta},$$

taking square and expectation on both sides, it follows

$$E\left[\left(\sum_{j=i}^{n-1}\overline{d}_j^n\right)^2\right] \leq 4E|\bar{y}_i^n|^2 + 4E|\xi^n|^2 + 12\delta T E\sum_{j=i}^{n-1}[|g(t_j, 0, 0)|^2] + 12\mu^2\delta\sum_{j=i}^{n-1}E|\overline{y}_{j+1}^n|^2 + 4\delta(3\mu^2+1)\sum_{j=i}^{n-1}|\bar{z}_j^n|^2. \tag{3.9}$$

Set $\alpha = 32$, notice that $\delta(3\mu^2 + 1) < \frac{1}{2}$, then $\frac{\delta(3\mu^2+1)}{8} < \frac{1}{16}$, we get

$$\frac{7}{8}E|\bar{y}_i^n|^2 \leq \frac{11}{8}E|\xi^n|^2 + \left(\frac{9}{8}\delta + 3\delta^2\right)E\sum_{j=i}^{n-1}[|g(t_j, 0, 0)|^2] + 32E[\sup_j((L_j^n)^+)^2] + \delta\left(\frac{3}{2} + 2\mu + \frac{35}{8}\mu^2\right)E\sum_{j=i}^{n-1}|\overline{y}_{j+1}^n|^2.$$

Then apply Lemma 2.2, in view of assumption that implies $\delta(\frac{3}{2} + 2\mu + \frac{35}{8}\mu^2) < 1$, we obtain

$$\sup_j E\left[\left|\overline{y}_j^n\right|^2\right] \leq C^R_{\xi^n, g, L}.$$

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It follows from the estimations of $\overline{z}_j^n$ and $\overline{d}_j^n$ that

$$E\left[\sum_{j=0}^{n-1}\left|\overline{z}_j^n\right|^2\delta+\left(\sum_{j=0}^{n-1}\overline{d}_j^n\right)^2\right]\leq C^R_{\xi^n,g,L}.$$

As Lemma 2.3, using Burkholder-Davis-Gundy inequality and similar techniques, we get the results. □

Then we have following convergence result for explicit reflected scheme.

Theorem 3.3. *Under the same assumptions of Theorem 3.2, the discrete solutions* $\{(\bar{Y}^n,\bar{Z}^n)\}_{n=1}^{\infty}$ *of the explicit reflected scheme converges to the solution* (Y,Z) *of* (3.1) *in the following senses: as* $n\to\infty$

$$E\left[\sup_{0\leq t\leq T}\left|\bar{Y}_t^n-Y_t\right|^2+\int_0^T\left|\bar{Z}_s^n-Z_s\right|^2\mathrm{d}s\right]\to 0. \tag{3.10}$$

Moreover $E[\sup_{0\leq t\leq T}(K_t-\overline{K}_t^n)^2]\to 0$.

Proof. Thanks to convergence results of Theorem 3.2, it suffices to prove

$$E\left[\sup_{0\leq t\leq T}\left|Y_t^n-\bar{Y}_t^n\right|^2+\int_0^T\left|Z_s^n-\bar{Z}_s^n\right|^2\mathrm{d}s\right]\to 0. \tag{3.11}$$

Recall the implicit reflected scheme and explicit reflected scheme: for $0\leq j\leq n-1$,

$$\begin{aligned}y_j^n &= y_{j+1}^n+g(t_j,y_j^n,z_j^n)\delta+d_j^n-z_j^n\varepsilon_{j+1}^n\sqrt{\delta},\\ \overline{y}_j^n &= \overline{y}_{j+1}^n+g(t_j,E[\overline{y}_{j+1}^n|\mathcal{G}_j^n]),z_j^n)\delta+\overline{d}_j^n-\overline{z}_j^n\varepsilon_{j+1}^n\sqrt{\delta}.\end{aligned}$$

Consider the difference, we have

$$\begin{aligned}E\left|y_j^n-\overline{y}_j^n\right|^2 &= E\left|y_{j+1}^n-\overline{y}_{j+1}^n\right|^2-\delta E\left|z_j^n-\overline{z}_j^n\right|^2+2\delta E[(y_j^n-\overline{y}_j^n)(g(t_j,y_j^n,z_j^n)-g(t_j,E[\overline{y}_{j+1}^n|\mathcal{G}_j^n],\overline{z}_j^n))]\\ &\quad+2E[(y_j^n-\overline{y}_j^n)(d_j^n-\overline{d}_j^n)]-\delta^2E[(g(t_j,y_j^n,z_j^n)-g(t_j,E[\overline{y}_{j+1}^n|\mathcal{G}_j^n],\overline{z}_j^n))^2]\\ &\quad-2\delta E[(d_j^n-\overline{d}_j^n)(g(t_j,y_j^n,z_j^n)-g(t_j,E[\overline{y}_{j+1}^n|\mathcal{G}_j^n],\overline{z}_j^n))]-E\left|d_j^n-\overline{d}_j^n\right|^2\\ &\leq E\left|y_{j+1}^n-\overline{y}_{j+1}^n\right|^2-\delta E\left|z_j^n-\overline{z}_j^n\right|^2+2\delta E[(y_j^n-\overline{y}_j^n)(g(t_j,y_j^n,z_j^n)-g(t_j,E[\overline{y}_{j+1}^n|\mathcal{G}_j^n],\overline{z}_j^n))],\end{aligned}$$

in view of $-a^2-2ab-b^2=-(a+b)^2\leq 0$ and

$$(y_j^n-\overline{y}_j^n)(d_j^n-\overline{d}_j^n)=(y_j^n-L_j^n)d_j^n+(\overline{y}_j^n-L_j^n)(\overline{d}_j^n)-(\overline{y}_j^n-L_j^n)d_j^n-(\overline{y}_j^n-L_j^n)(d_j^n)\leq 0.$$

We take sum over j from i to $n-1$, with $\xi^n-\overline{\xi}^n=0$, then get

$$E\left|y_i^n-\overline{y}_i^n\right|^2+\delta\sum_{j=i}^{n-1}E\left|z_j^n-\overline{z}_j^n\right|^2\leq 2\delta\sum_{j=i}^{n-1}E[(y_j^n-\overline{y}_j^n)(g(t_j,y_j^n,z_j^n)-g(t_j,E[\overline{y}_{j+1}^n|\mathcal{G}_j^n],\overline{z}_j^n))].$$

Now we are in the same situation as in the proof of Theorem 2.1. By similar methods, with Lemma 3.1, and

$$\begin{aligned}2\mu\delta E[|y_j^n-\overline{y}_j^n|\cdot|y_j^n-E[\overline{y}_{j+1}^n|\mathcal{G}_j^n]|] &= 2\mu\delta E\left[|y_j^n-\overline{y}_j^n|\cdot\left|y_j^n-\overline{y}_j^n+g(t_j,E[\overline{y}_{j+1}^n|\mathcal{G}_j^n]),z_j^n)\delta+\overline{d}_j^n\right|\right]\\ &\leq(2\mu+1)\delta E[|y_j^n-\overline{y}_j^n|^2]+2\mu^2\delta E[\delta^2\left|g(t_j,E[\overline{y}_{j+1}^n|\mathcal{G}_j^n]),z_j^n)\right|^2+(\overline{d}_j^n)^2],\end{aligned}$$

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we obtain

$$E\left|y_i^n-\overline{y}_i^n\right|^2+\frac{\delta}{2}\sum_{j=i}^{n-1}E\left|z_j^n-\overline{z}_j^n\right|^2\leq(2\mu^2+2\mu+1)\delta\sum_{j=i}^{n-1}E\left|y_j^n-\overline{y}_j^n\right|^2+\delta C^R_{\xi^n,g,L}, \tag{3.12}$$

where $C^R_{\xi^n,g,L}$ is a constant only depends on ξ^n, $g(\cdot,0,0)$, μ and L. By Lemma 2.2, we get

$$\sup_{j\leq n}E\left|y_j^n-\overline{y}_j^n\right|^2\leq C\delta \mathrm{e}^{(2\mu+2\mu^2+1)T}.$$

From (3.12), it follows (3.11), which implies $\lim_{n\to\infty}\delta\sum_{j=i}^{n-1}E\left|z_j^n-\overline{z}_j^n\right|^2=0$. Then (3.10) follows by using Burkholder-Davis-Gundy inequality, similar techniques and estimations results from Lemma 3.1. In fact, we get

$$E[\sup_{j\leq n}\left|y_j^n-\overline{y}_j^n\right|^2]\leq C_\mu\delta\sum_{j=i}^{n-1}E\left|y_j^n-\overline{y}_j^n\right|^2+\delta\sum_{j=i}^{n-1}E\left|z_j^n-\overline{z}_j^n\right|^2.$$

For the convergence of $\overline{K}^n$, for $0\leq t\leq T$, since

$$\begin{aligned}K_t^n &= Y_0^n-Y_t^n-\sum_{j=0}^{[t/\delta]}g(t_j,y_j^n,z_j^n)\delta+\sum_{j=0}^{[t/\delta]}z_j^n\varepsilon_{j+1}^n\sqrt{\delta},\\ \overline{K}_t^n &= \overline{Y}_0^n-\overline{Y}_t^n-\sum_{j=0}^{[t/\delta]}g(t_j,E[\overline{y}_{j+1}^n|\mathcal{G}_j^n],\overline{z}_j^n)\delta+\sum_{j=0}^{[t/\delta]}\overline{z}_j^n\varepsilon_{j+1}^n\sqrt{\delta},\end{aligned}$$

with Lipschitz condition of g and BDG inequality, we get

$$\begin{aligned}E[\sup_{0\leq t\leq T}(K_t^n-\overline{K}_t^n)^2] &\leq 4\left|Y_0^n-\overline{Y}_0^n\right|^2+4E\left[\sup_{0\leq t\leq T}\left|Y_t^n-\overline{Y}_t^n\right|^2\right]+8\delta\mu^2\sum_{j=0}^{[t/\delta]}\left|y_j^n-E[\overline{y}_{j+1}^n|\mathcal{G}_j^n]\right|^2\\ &\quad+4\delta(2\mu^2+c_2)\sum_{j=0}^{[t/\delta]}\left|z_j^n-\overline{z}_j^n\right|^2\\ &= 4\left|Y_0^n-\overline{Y}_0^n\right|^2+4E\left[\sup_{0\leq t\leq T}\left|Y_t^n-\overline{Y}_t^n\right|^2\right]+4\delta(2\mu^2+c_2)\sum_{j=0}^{[t/\delta]}\left|z_j^n-\overline{z}_j^n\right|^2\\ &\quad+24\delta\mu^2\sum_{j=0}^{[t/\delta]}\left|y_j^n-\overline{y}_j^n\right|^2+\delta C^R_{\xi^n,g,L}.\end{aligned}$$

From (3.11) and convergence of K_t^n to K_t in $\mathbf{L}^2(\mathcal{F}_t)$, we obtain the convergence of $\overline{K}_t^n$ to K_t. □

4. Simulation results

4.1. Simulation results for standard BSDE

We consider the terminal condition $Y_T=\xi$ which is a function of B_T: $Y_T=\xi=\Phi(B_T)$. In this case we set $y_n^n=\xi^n=\Phi(B_{n\delta}^n)$. It can be checked that our explicit schemes (2.7) (as well as the implicit scheme) will automatically derive

$$y_j^n:=u(j,B_{j\delta}^n)=u\left(j,\sqrt{\delta}\sum_{i=1}^{j}\varepsilon_i^n\right),\ z_j^n=v(j,B_{j\delta}^n)=v\left(j,\sqrt{\delta}\sum_{i=1}^{j}\varepsilon_i^n\right).$$

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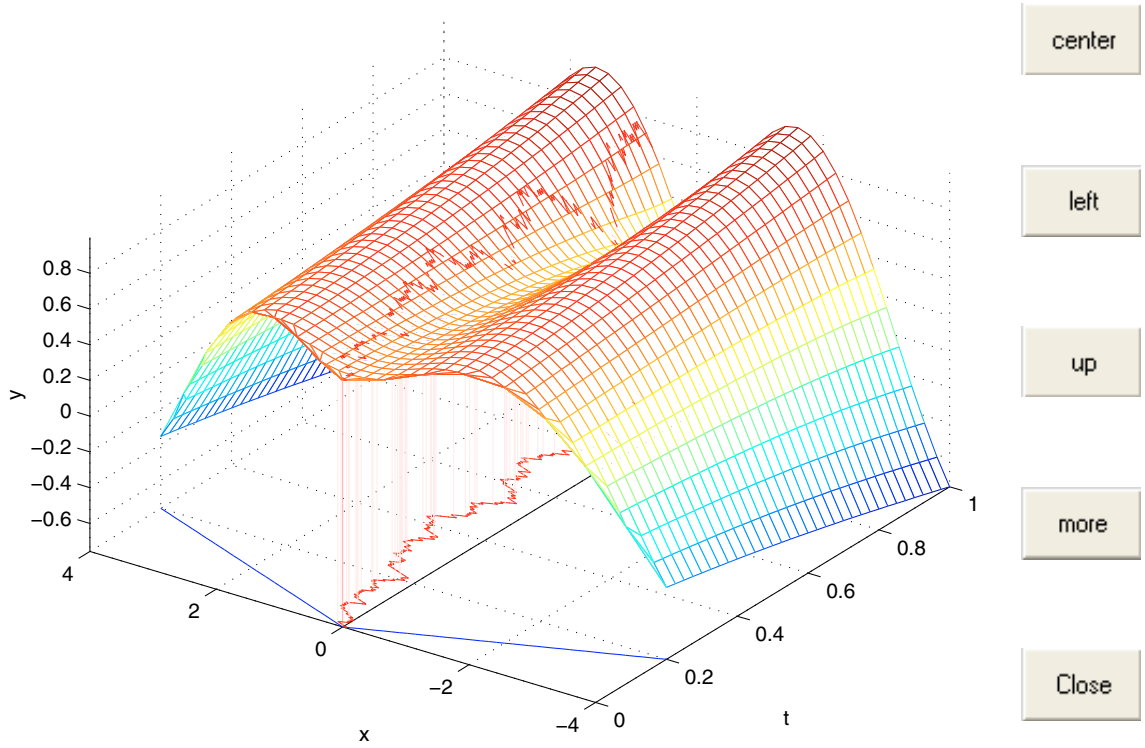


FIGURE 1. The solution surface with one trajectory.

Since $B^n_{j\delta}$ takes on $j+1$ different values, the whole solution $\{y^n_j, z^n_j\}_{0\le j\le n-1}$ is a 2-vector with $\frac{n\times(n+1)}{2}$ values. For convenience, we set $T=1$ in our simulation part.

Applying the above numerical schemes, we have developed a Matlab toolbox for calculating and simulating solutions of BSDEs. This toolbox starts with a Matlab figure window with input area for generator $g=g(t,y,z)$ and terminal function $\xi=\Phi(x)$, where x stands for B_T. Here g and Φ can be any functions accepted by Matlab. These toolboxes can be downloaded from http://159.226.47.50:8080/iam/xumingyu/English.jsp, by clicking 'Preprint' on the left side, or http://www.sciencenet.cn/u/xvmingyu.

Here we consider the case: $g(t,y,z)=-5\,|y+z|$, $\xi=\Phi(B_1)=\sin(|B_1|)$. After inputting these parameters of a BSDE, the numerical calculation for the BSDE are launched after clicking the button "calculate". When the toolbox indicates "the calculation is complete", clicking any other button in button-area will produce different types of simulations.

Clicking the button "B.M. and solution y" it will generate the dynamic simulation of (t,B_t,Y_t), shown in Figure 1. Here a trajectory of Y_t runs on a colored 2-dimensional surface represented $u=u(t,x)$, where x stands for the coordinate of Brownian motion B_t.

Clicking "solution (y, z)" will generate another Matlab figure, displayed in Figure 2. This figure shows the 2-dimensional dynamic trajectories of (t,B_t,Y_t) and (t,B_t,Z_t) and, simultaneously, 2-dimensional trajectories of (t,Y_t) and (t,Z_t). And there are two groups of trajectories on the figure

We now compare some numerical solutions calculated by these algorithms: implicit scheme, explicit scheme and Monte-Carlo method in some particular situations.

Case I. If g is a linear function (y,z): $g(s,y,z)=by+cz+r$. The solution Y_0 of the BSDE is

$$Y_0=\exp\left(\left(b-\frac{1}{2}c^2\right)T\right)E[\xi\exp(cB_T)]+\frac{r}{b}[\exp(bT)-1].$$

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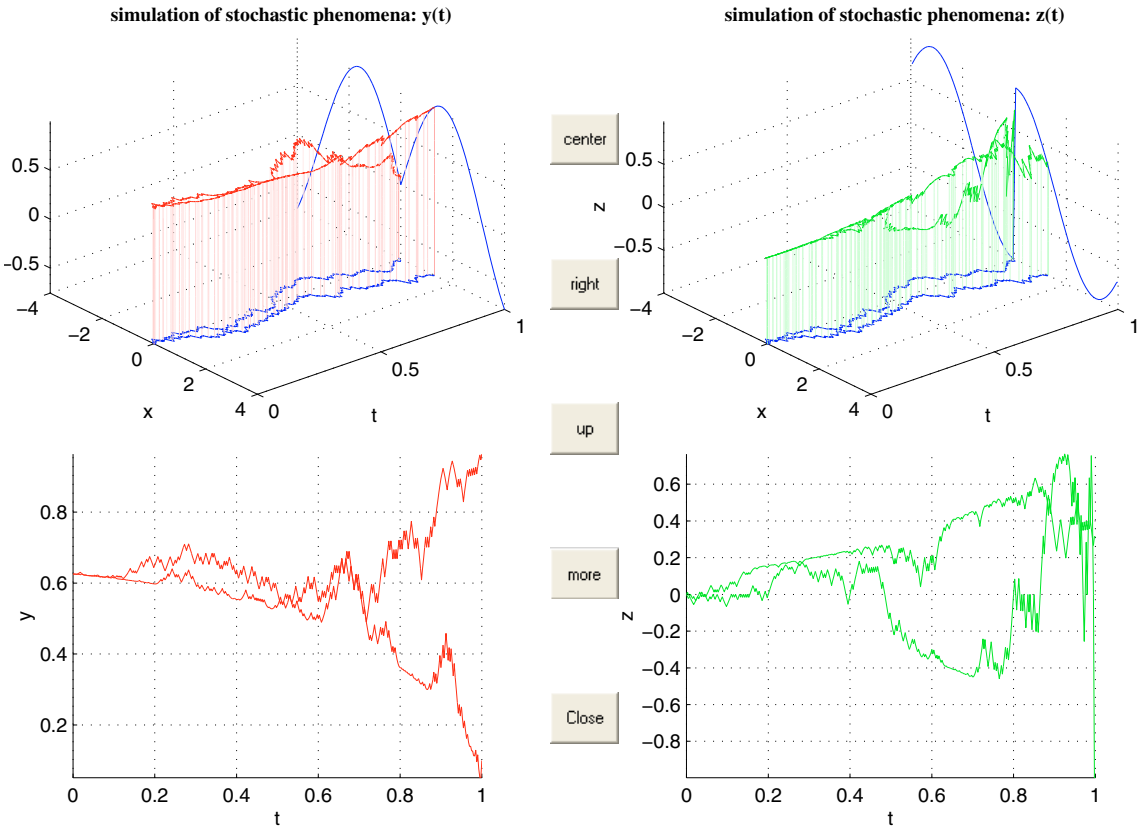


FIGURE 2. The trajectories of the solution.

Example 4.1. Set $b = c = r = 1$, $\xi = \sin(|B_T|)$. The numerical results obtained with the implicit and explicit schemes are shown in the following table:

n	100	500	1000	2000	5000
Y_0^n	3.5106	3.4916	3.4879	3.4866	3.4859
$\overline{Y}_0^n$	3.4171	3.4716	3.4785	3.4819	3.4840

The exact solution is expressed by $Y_0 = \exp(\frac{1}{2})E[\sin(|B_1|)\exp(B_1)] + \exp(1) - 1$. We apply the Monte-Carlo method, with 10 000 000 samples, to calculate Y_0. The result is $Y_0 = 3.4850$.

Example 4.2. Set $b = c = 1$, $r = 0$, $\xi = |B_T|$. The numerical results obtained with the implicit and explicit schemes are:

n	100	500	1000	2000	5000
Y_0	3.1806	3.1731	3.1722	3.1719	3.1714
$\overline{Y}_0$	3.0818	3.1531	3.1621	3.1667	3.1694

Applying Monte-Carlo method with 10 000 000 samples to the exact solution $Y_0 = \exp(\frac{1}{2})E[|B_1|\exp(B_1)]$, we get $Y_0 = 3.1710$.

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Case II. If $g = \frac{1}{2}z^2$, then we have the exact solution $Y_0 = \ln(E[\exp(\xi)])$. Since g does not depend on y, implicit schemes and explicit scheme give same results.

Example 4.3. For $\xi = \sin(|B_1|)$, applying the implicit scheme, we obtain:

n	100	400	800	1000	2000
Y_0^n	0.6249	0.6253	0.6254	0.6254	0.6255

By Monte-Carlo method with 10 000 000 samples to the exact expression $Y_0 = \ln(E[\exp(\sin|B_1|)])$, we get $Y_0 = 0.6255$.

4.2. **Simulations of reflected BSDEs with one lower barrier**

Consider the amount of total calculation for most general case, we only treat a very simple situation: $\xi = \phi(B_1)$, $L_t = \psi(t, B(t))$, where ϕ and ψ are real regular functions defined on $\mathbb{R}$ and $[0,1]\times\mathbb{R}$ respectively. As for BSDE, we have also developed a Matlab toolbox for calculating and simulating solutions of reflected BSDEs, which can be downloaded at the same webpages.

Here we consider following case: $g(t,y,z) = -|y+z|$, $\xi = \Phi(B_1) = 2\sin(B_1)$, $L_t = \Psi(t,B_t) = \sin(B_t + \frac{\pi}{2}) - 2$ and $n = 400$.

After inputting the parameters, we run the calculation program using reflected explicit scheme, then get all possible results of y. We may notice that at $t = 1$, $\xi \geq L_1$ does not always hold. But the numerical scheme still works as well. In fact, in such case the increasing process K as well as y has a jump of size $(L_1 - \xi)^+$ at $t = 1$, which pushes the solution y_{t-}, *i.e.* y_{n-1} in our case, to stay above the barrier L. Then both K and y act as the terminal condition is $(\xi - L_1)^+ + L_1$, which is always bigger than L_1.

Now we will see some properties of the trajectory of solution y in Figure 3. We see two surfaces in the upper portion of Figure 3. The lower surface shows the barrier L in 3-dimensional, as well the upper one is for the solution y. Then we use programs to generate two trajectories of the discrete Brownian motion $(B_j^{n,i})_{0\leq j\leq n}$, for $i = 1, 2$, which are drawn on the horizontal plane. The corresponding two paths of $y_j^{n,i}$ $(i = 1, 2)$ with respect to two Brownian samples, are showed on the solution surface, and we use the fine vertical line to give correspondence between two group of trajectories of y and B. The remainder of the figure shows respectively the trajectories of the reflecting force $K_j^{n,i} = \sum_{k=0}^{j} d_k^{n,i}$ $(i = 1, 2)$ corresponding to the value of $y_j^{n,i}$ $(i = 1, 2)$, and $y_j^{n,i} - L_j^{n,i}$ $(i = 1, 2)$.

In the upper portion we can see that there is an area where two surfaces (the solution surface and the barrier surface) stick together. The force $K_j^{n,i}$ will push y_j^n upward, only when the trajectory of solution y_j^n goes into this area. Indeed, if there is not a barrier, $y_j^{n,i}$ intends becoming smaller than the reflecting barrier $L_j^{n,i}$, so to keep $y_j^{n,i}$ being no less than $L_j^{n,i}$, the action of forces $K_j^{n,i}$ are necessary. Comparing these two trajectories, we can see that one trajectory, noted as $K_j^{n,1}$ pushes upwards the corresponding trajectory of solution $y_j^{n,1}$, while the other one noted as $K_j^{n,2}$, keeps to be zero, the trajectory $y_j^{n,2}$ with respect to $K_j^{n,2}$ does not go into the sticking area.

Comparing the two lower sub-figures of Figure 2, we can see that $K_j^{n,i}$ only increases when $y_j^{n,i} - L_j^{n,i}$ takes the value 0; but the converse is not always true, when $y_j^{n,i} - L_j^{n,i} = 0$, $K_j^{n,i}$ does not necessary increase.

About this point, we can also see in Figure 4. This figure shows two groups of 3-dimensional dynamic trajectories $(t_j, B_j^{n,i}, Y_j^{n,i})_{(i=1,2)}$ and $(t_j, B_j^{n,i}, Z_j^{n,i})_{(i=1,2)}$ and, simultaneously, two groups of 2-dimensional trajectories of $(t_j, Y_j^{n,i})_{(i=1,2)}$ and $(t_j, Z_j^{n,i})_{(i=1,2)}$. For remainder sub-figures, the above-right one is for the trajectories $K_j^{n,i}$ $(i = 1, 2)$, and while the below-left one is for $y_j^{n,i} - L_j^{n,i}$ $(i = 1, 2)$, then comparing these two sub-figures, as in Figure 3, we can see clearly the relation between $K_j^{n,i}$ $(i = 1, 2)$ and $y_j^{n,i} - L_j^{n,i}$ $(i = 1, 2)$. Moreover one trajectory, noted as $K^{n,1}$ as well as $y^{n,1}$jumps at $t = 1$, since its terminal value is less than

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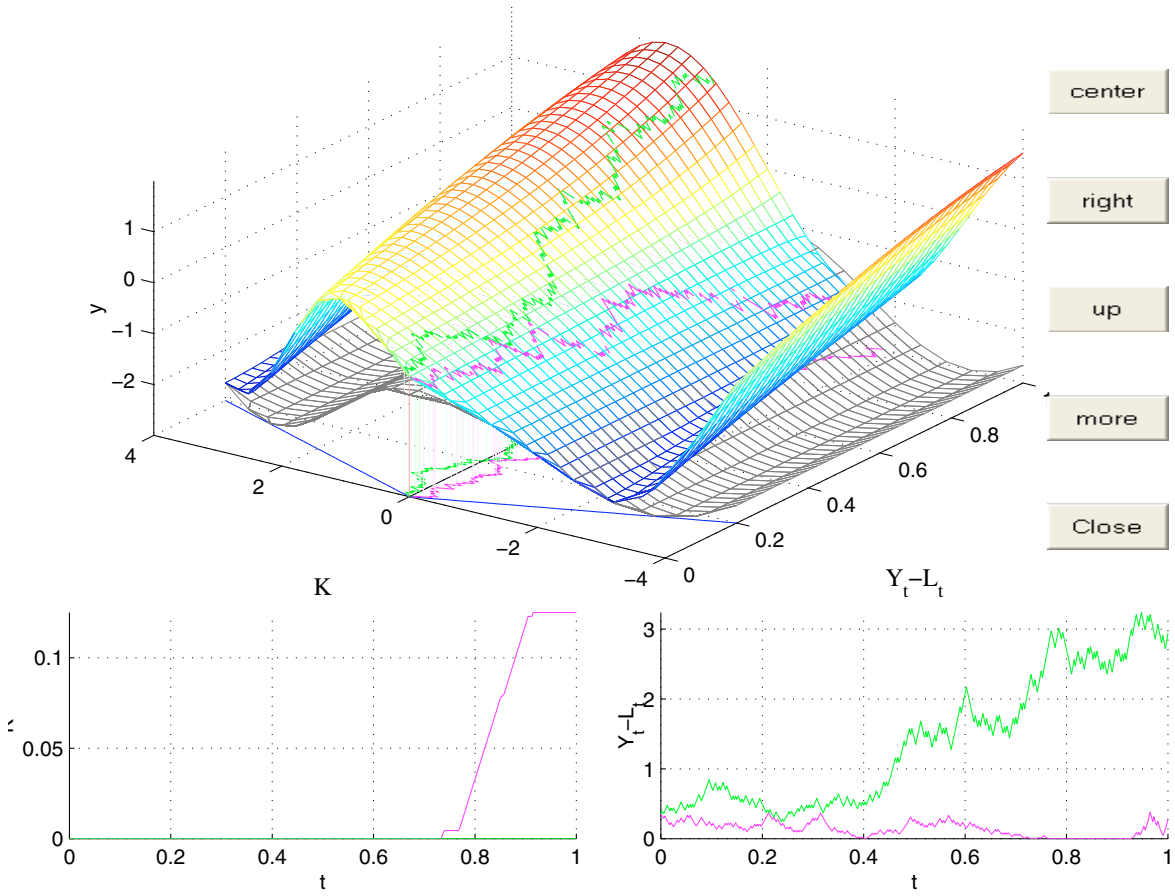


FIGURE 3. The solution on surface.

the barrier. Figure 4 shows two trajectories of our simulation in 3-dimension, for $(t_j, B_j^{n,i}, y_j^{n,i})$, $(t_j, B_j^{n,i}, z_j^{n,i})$ as well as in 2-dimension for $(t_j, y_j^{n,i})$, $(t_j, z_j^{n,i})$, $(t_j, K_j^{n,i})$, $(t_j, y_j^{n,i} - L_j)$, for $i = 1, 2$.

Now we list out some numerical results for the reflected scheme and explicit implicit penalization scheme,where we can see that as the penalized parameter p tends to infinity, $y_0^{p,n}$ converge to y_0^n. Consider the same parameters as above: $f(y,z) = -|y+z|$, $\xi = \Phi(B_1) = 2\sin(B_1)$, $L_t = \Psi(t, B_t) = \sin(B_t + \frac{\pi}{2}) - 2$. Then as the following tablet showing:

$n = 400$, reflected explicit scheme: $y_0^n = -0.6430$,

penalization scheme:

p	20	200	2000	2×10^4
$y_0^{p,n}$	-0.6553	-0.6444	-0.6431	-0.6430

$n = 1000$, reflected explicit scheme: $y_0^n = -0.6425$,

penalization scheme:

p	20	200	2000	2×10^4
$y_0^{p,n}$	-0.6550	-0.6441	-0.6427	-0.6425

$n = 2000$, reflected explicit scheme: $y_0^n = -0.6424$,

penalization scheme:

p	20	200	2000	2×10^4
$y_0^{p,n}$	-0.6549	-0.6439	-0.6426	-0.6424

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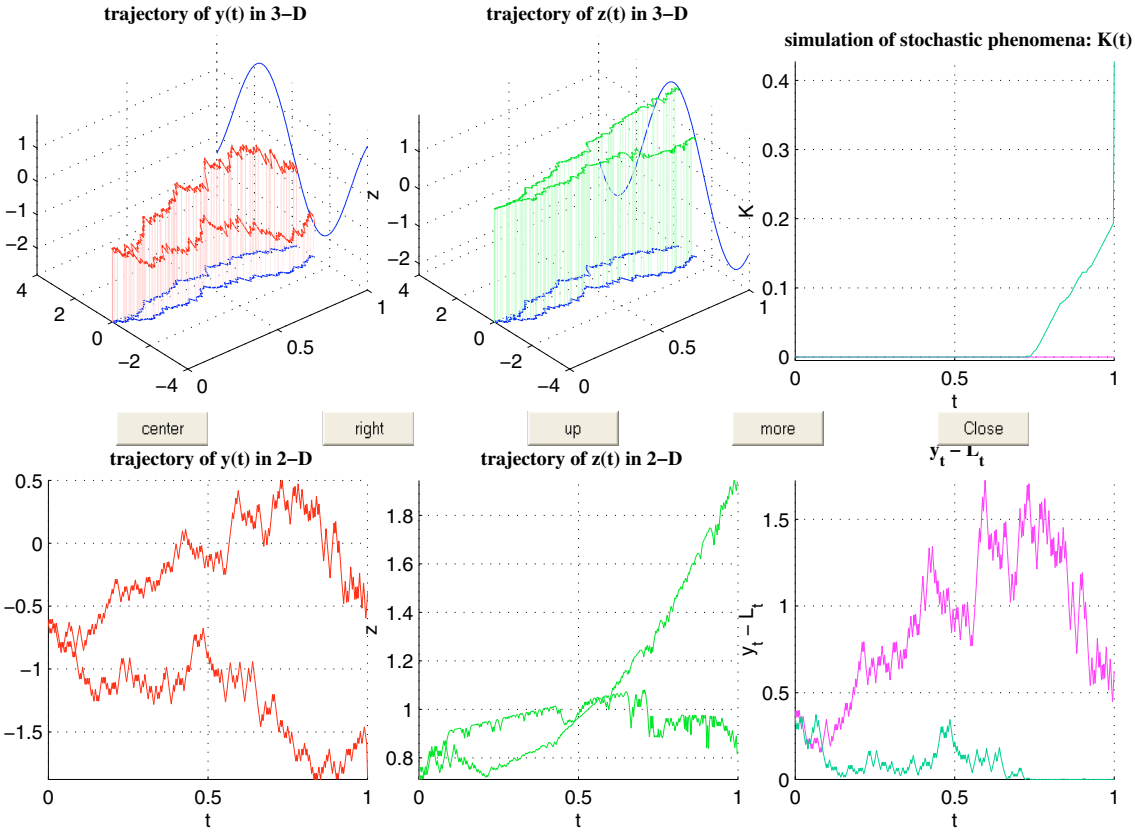


FIGURE 4. Simulation of y, z, K.

Here the penalization parameter can be choose as large as we want, even if we have a comparing small discretization of time interval.

Remark 4.1. For BSDE with two reflecting barriers, we introduced also reflected implicit and explicit scheme as well as penalization schemes. The proofs of convergence and simulations results can be found in [22].

5. Γ-CONSTRAINED BSDES

In this section, we consider the 1-d smallest g-supersolution with constraint $(Y_t, Z_t) \in \Gamma_t$ of the following form:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s + A_T - A_t - \int_t^T Z_s \mathrm{d}B_s, \quad \mathrm{d}A_t \geq 0. \tag{5.1}$$

with $d_\Gamma(Y_t, Z_t) = 0$, *a.e.*, *a.s.*, where Γ is a nonempty closed subset of $\mathbb{R} \times \mathbb{R}$ and d_Γ is the distance function of Γ, *i.e.*, $d_\Gamma(y, z) = \inf_{(y',z')\in\Gamma}\{|y - y'| + |z - z'|\}$. It is clear that Γ is a Lipschitz function

$$|d_\Gamma(y, z) - d_\Gamma(y', z')| \leq |y - y'| + |z - z'|.$$

Such a g-supersolution (Y, Z, A) is called a Γ-constrained g-supersolution.

As before we assume that g satisfies Lipschitz condition (2.1) and that Assumption 2.1 holds for ξ. From [9,19], we have the existence of the smallest solution for (5.1):

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Theorem 5.1. *If there exists at least one* Γ*-constrained* g*-supersolution of* (5.1)*, then the equation admits a smallest* Γ*-constrained* g*-supersolution* (Y, Z, A)*. Moreover,* (Y, Z) *is the limit of the following sequence of penalization solutions:*

$$Y_t^p = \xi + \int_t^T g(s, Y_s^p, Z_s^p)\mathrm{d}s + p\int_t^T \mathrm{d}_\Gamma(Y_s^p, Z_s^p)\mathrm{d}s - \int_t^T Z_s^p \mathrm{d}B_s, \tag{5.2}$$

in the sense of

$$\lim_{p\to\infty} E\int_0^T [|Y_t - Y_t^p|^2 + |Z_t - Z_t^p|^\beta]\mathrm{d}t = 0,\ 1 \le \beta < 2.$$

This smallest Γ-constrained g-supersolution is called g_Γ-solution. In fact such equation can be considered as a BSDE with singular coefficient $g_\Gamma := g(t, y, z) + \infty \cdot d_\Gamma(y, z)$. It easy to check that when $\xi^+ \in L^\infty(\mathcal{F}_T)$, and there exists a large enough constant C_0 such that for $y \ge C_0$

$$g(t, y, 0) \le C_0 + \mu|y|, \quad \text{and} \quad (y, 0) \in \Gamma,$$

then there exists a Γ-constrained g-supersolution of (5.1) (see Peng and Xu [20]). Then by Theorem 5.1, a g_Γ-solution exists. In this section we will work under these assumptions. We now derive a numerical scheme applying convergence results in Theorem 5.1.

5.1. Constraint on Z

First we consider the case when constraint is only on process Z and invariant in t, *i.e.* Γ is a close subset in $\mathbf{R}$. And we require $Z \in \Gamma$, *i.e.* $d_\Gamma(Z_t) = 0$, *a.e. a.s.* By the same discretization as for BSDEs introduced in Section 2, for each positive number p we have the following penalization discrete equation on small interval $[j\delta, (j+1)\delta]$

$$y_j^{p,n} = y_{j+1}^{p,n} + g(t_j, y_j^{p,n}, z_j^{p,n})\delta + pd_\Gamma(z_j^{p,n})\delta - z_j^{p,n}\sqrt{\delta}\varepsilon_{j+1}, \tag{5.3}$$

with discrete terminal condition: $y_n^n := \xi^n$.

Now we need to find a way to find $\mathcal{G}_j^n$-measurable $(y_j^{p,n}, z_j^{n,p})$ to satisfy (5.3) with $(y_{j+1}^{p,n}, z_{j+1}^{p,n})$. It is easy to get $z_j^{p,n} = \frac{1}{\sqrt{\delta}}E[y_{j+1}^{p,n}\varepsilon_{j+1}^n|\mathcal{F}_j^n] = \frac{1}{2\sqrt{\delta}}(y_{j+1}^{p,n}|_{\varepsilon_j^n=1} - y_{j+1}^{p,n}|_{\varepsilon_j^n=-1})$. Substitute it into (5.3), it follows a equation of $y_j^{p,n}$ as

$$y_j^{p,n} = E[y_{j+1}^{p,n}|\mathcal{G}_j^n] + g(t_j, y_j^{p,n}, z_j^{p,n})\delta + pd_\Gamma(z_j^{p,n})\delta.$$

So apply the implicit scheme for BSDE in Section 2, we get

$$y_j^{p,n} = \Theta^{-1}(E[y_{j+1}^{p,n}|\mathcal{G}_j^n] + pd_\Gamma(z_j^{p,n})\delta),$$

where $\Theta(y) = y - g(t_j, y, z_j^{p,n})\delta$. While the explicit method gives

$$\overline{y}_j^{p,n} = E[\overline{y}_{j+1}^{p,n}|\mathcal{G}_j^n] + g(t_j, E[\overline{y}_{j+1}^{p,n}|\mathcal{G}_j^n], \overline{z}_j^{p,n})\delta + pd_\Gamma(\overline{z}_j^{p,n})\delta.$$

The interesting point here is that the penalization of $z^{p,n}$ with respect to z is not directly on $z^{p,n}$, it act on $y^{n,p}$ to influence $z^{p,n}$.

We have:

Theorem 5.2 (convergence theorem)**.** *Define*

$$Y_t^{p,n} = y_{[t/\delta]}^{p,n}, Z_t^{p,n} = z_{[t/\delta]}^{p,n}, \overline{Y}_t^{p,n} = \overline{y}_{[t/\delta]}^{p,n}, \overline{Z}_t^{p,n} = \overline{z}_{[t/\delta]}^{p,n}.$$

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Here $y_j^{p,n}$ and $\overline{y}_j^{p,n}$, $0 \le j \le n$, can come from either implicit scheme or explicit scheme respectively. Under Assumption 2.1, and g satisfying Lipschitz condition. Then

$$\lim_{p\to\infty}\lim_{n\to\infty} E\left[\int_0^T |Y_s^{p,n} - Y_s|^2\,\mathrm{d}s + \int_0^T |Z_s^{p,n} - Z_s|^\beta\,\mathrm{d}s\right] = 0, \quad 1 \le \beta < 2.$$

Proof. By Theorem 5.1, for any $\varepsilon > 0$, there exists $p_0 > 0$ such that for each $p > p_0$,

$$E\left[\int_0^T |Y_s^{p} - Y_s|^2\,\mathrm{d}s^2 + \int_0^T |Z_s^{p} - Z_s|^\beta\,\mathrm{d}s\right] \le \varepsilon.$$

Moreover, by Theorem 2.1, for implicit scheme, we have as $n \to \infty$

$$E\left[\sup_{0\le t\le T} |Y_t^{p_0,n} - Y_t^{p_0}|^2 + \int_0^T |Z_s^{p_0,n} - Z_s^{p_0}|^2\,\mathrm{d}s\right] \to 0.$$

For explicit scheme, the result follows from

$$\sup_{0\le t\le T} E\left[\left|\overline{Y}_t^{p_0,n} - Y_t^{p_0}\right|^2 + \int_0^T \left|\overline{Z}_s^{p_0,n} - Z_s^{p_0}\right|^2\,\mathrm{d}s\right] \to 0. \qquad \square$$

To illustrate calculation and simulation in our software package, we consider the case $\Gamma = [a, b]$ with $a \le 0 \le b$. Then $d_\Gamma(z) = (z-a)^- + (z-b)^+$. The default setting is $g(t, y, z) = -2|y + z| - 1$, $\xi = |B_1|$, with $a = -0.5$, $b = 0.8$, $p = 20$ and $n = 400$. The surface $u = u^{p,n}(t, x)$ and $v = v^{n,p}(t, x)$ are given with dynamic simulation $Y_t^{p,n} = u^{p,n}(t, B_t^n)$ and $Z_t^{p,n} = v^{p,n}(t, B_t^n)$ as shown as the upper part of Figure 5. The lower part of the figure displays the simultaneous action of the process $A_t^{p,n} = p\sum_{j\le[t/\delta]} d_\Gamma(z_j^{p,n})$. The effect of increases in the $A_t^{p,n}$ when $Z^{p,n}$ is less than -0.5 and larger than 0.8 are clearly shown in Figure 6. But it seems that the solution is still too sensitive to the choice of p and n. If $p\sqrt{\delta} > 1$, then the numerical solution will explode.

5.2. **BSDE reflected on process *Z***

Now we consider a very interesting special case, when the constraint is $d_\Gamma(y, z) = (y - \phi(z))^-$, for a given function ϕ, in other words, y_t is reflected on $\phi(z_t)$ *i.e.* $y_t \ge \phi(z_t)$. After the same discretization of the time interval, we have following discrete penalization equation for some p large enough, on the small interval $[j\delta, (j+1)\delta]$, $0 \le j \le n-1$

$$y_j^{p,n} = y_{j+1}^{p,n} + g(t_j, y_j^{p,n}, z_j^{p,n})\delta + p(\phi(z_j^{p.n}) - y_j^{p,n})^+\delta - z_j^{p,n}\delta\varepsilon_{j+1}^n.$$

Similarly, $z_j^{p,n} = \frac{1}{\sqrt{\delta}} E[y_{j+1}^{p,n}\varepsilon_{j+1}^n|\mathcal{F}_j^n] = \frac{1}{2\sqrt{\delta}}(y_{j+1}^{p,n}|_{\varepsilon_j^n=1} - y_{j+1}^{p,n}|_{\varepsilon_j^n=-1})$. Then $y_j^{p,n}$ satisfies

$$y_j^{p,n} = E[y_{j+1}^{p,n}|\mathcal{G}_j^n] + g(t_j, y_j^{p,n}, z_j^{p,n})\delta + p(\phi(z_j^{p.n}) - y_j^{p,n})^+\delta.$$

Set $\Theta(y) = y - (g(t_j, y, z_j^{p,n})\delta + p(\phi(z_j^{p,n}) - y)^+\delta)$, with $E[y_{j+1}^{p,n}|\mathcal{G}_j^n] = \frac{1}{2}(y_{j+1}^{p,n}|_{\varepsilon_j^n=1} + y_{j+1}^{p,n}|_{\varepsilon_j^n=-1})$, then our implicit scheme is given by solving following equation

$$y_j^{p,n} = \Theta^{-1}(E[y_{j+1}^{p,n}|\mathcal{G}_j^n]).$$

Meanwhile, we have also explicit-implicit scheme, which is

$$\overline{y}_j^{p,n} = E[\overline{y}_{j+1}^{p,n}|\mathcal{G}_i^n] + g(t_j, E[\overline{y}_{j+1}^{p,n}|\mathcal{G}_j^n], \overline{z}_j^{p,n})\delta + \frac{p\delta}{1+p\delta}(E[\overline{y}_{j+1}^{p,n}|\mathcal{G}_j^n] + g(t_j, E[\overline{y}_{j+1}^{p,n}|\mathcal{G}_j^n], \overline{z}_j^{p,n})\delta - \phi(\overline{z}_j^{p,n}))^-.$$

As in the previous section, we have convergence results of these two schemes.

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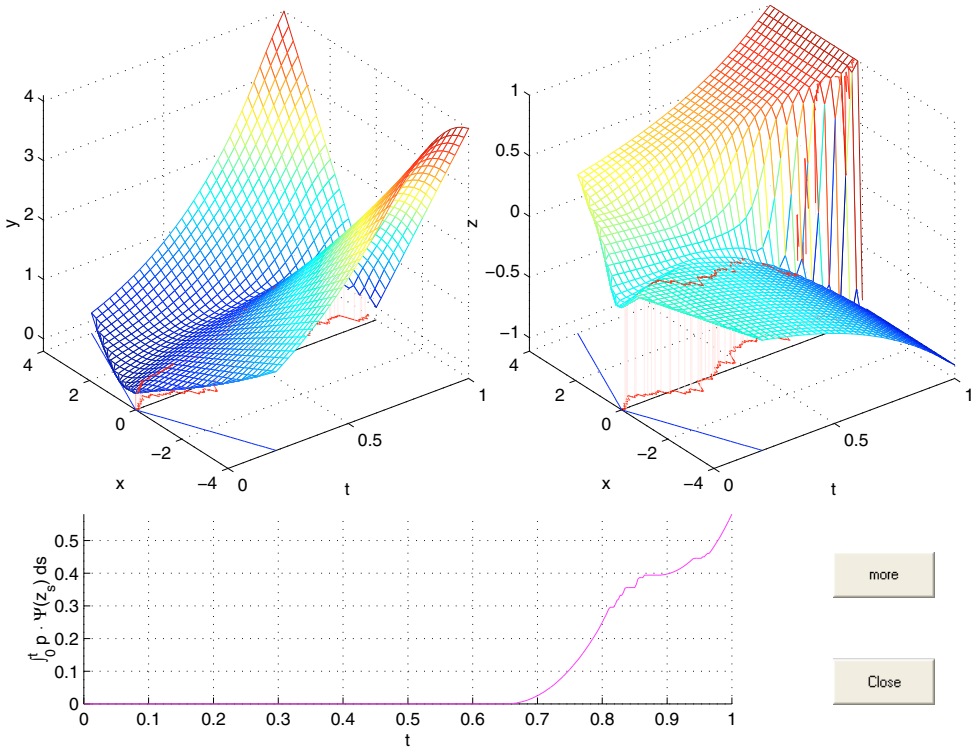


FIGURE 5. The solution surface of BSDE (5.3).

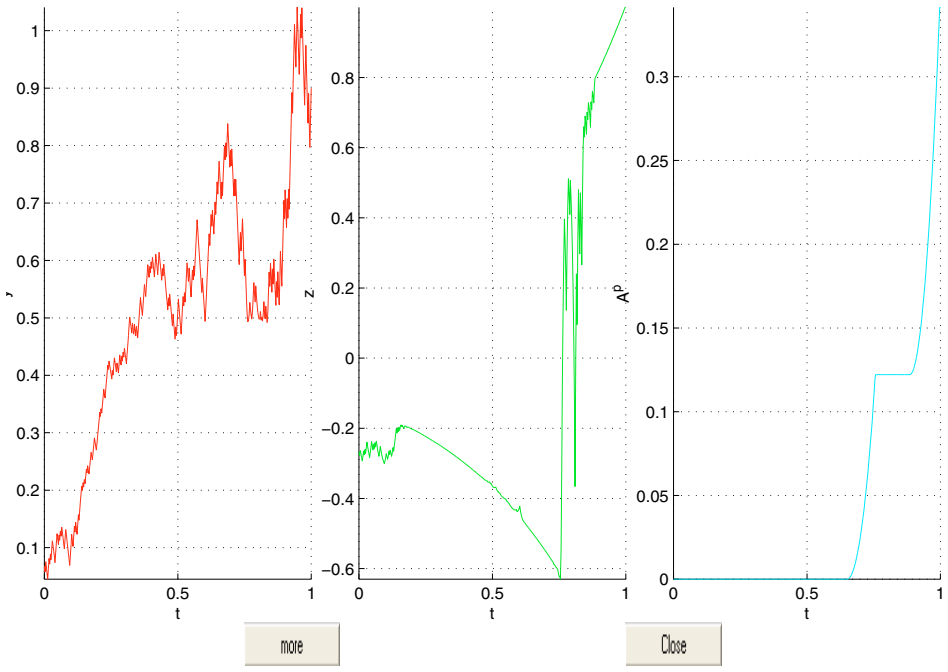


FIGURE 6. A trajectory of solutions of (5.3).

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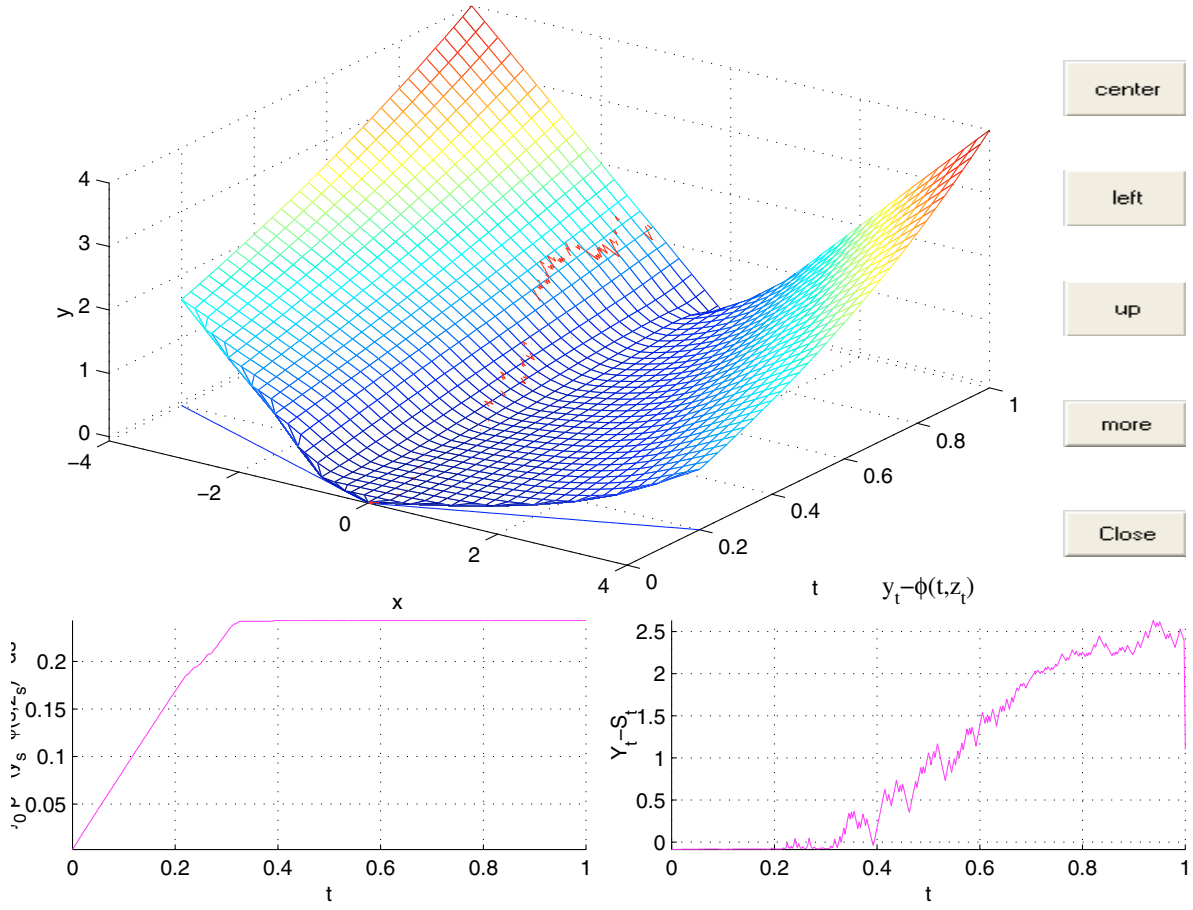


FIGURE 7. The solution surface of penalization BSDE with $d_\Gamma(y,z) = (y - \phi(z))^-$.

Theorem 5.3. *Define* $Y_t^{p,n} = y_{[t/\delta]}^{p,n}$, $Z_t^{p,n} = z_{[t/\delta]}^{p,n}$ *and* $\overline{Y}_t^{p,n} = \overline{y}_{[t/\delta]}^{p,n}$, $\overline{Z}_t^{p,n} = \overline{z}_{[t/\delta]}^{p,n}$. *Then we have, for* $1 \le \beta < 2$,

$$\lim_{p\to\infty}\lim_{n\to\infty} E\left[g\int_0^T |Y_s^{p,n} - Y_s|^2\,\mathrm{d}s + \int_0^T |Z_s^{p,n} - Z_s|^\beta\,\mathrm{d}s\right] = 0,$$

$$\lim_{p\to\infty}\lim_{n\to\infty} E\left[\int_0^T \left|\overline{Y}_s^{p,n} - Y_s\right|^2\,\mathrm{d}s + \int_0^T \left|\overline{Z}_s^{p,n} - Z_s\right|^\beta\,\mathrm{d}s\right] = 0.$$

Proof. The results follow from Theorem 5.1 and Proposition 2.1, so we omit the proof. □

Now we do simulations by explicit-implicit scheme. We consider the case $g = -2|y+z| - 1$, $\xi = |B_1|$, $\phi(z) = 1.25 \times z$, with penalization parameter $p = 10$, and discretization number $n = 400$. In Figure 7, we see the surface of solution $Y^{p,n}$ with a trajectory of $Y^{p,n}$ on the surface in upper portion, while in two lower subfigures there presents the trajectory $A^{p,n} = p\sum_{j\le[t/\delta]}(y_j^{p,n} - \phi(z_j^{p,n}))^-\delta$ and $y_j^{p,n} - \phi(z_j^{p,n})$ corresponding to the one on the surface. We can see that when $y_j^{p,n} - \phi(z_j^{p,n})$ is positive, the penalization term will not work to the process $y_j^{p,n}$. About this point we can see more clear in Figure 8, which presents trajectories of $y_j^{p,n}$, $z_j^{p,n}$, $A^{p,n}$ and $y_j^{p,n} - \phi(z_j^{p,n})$ in 3 or 2-dimensional subfigures.

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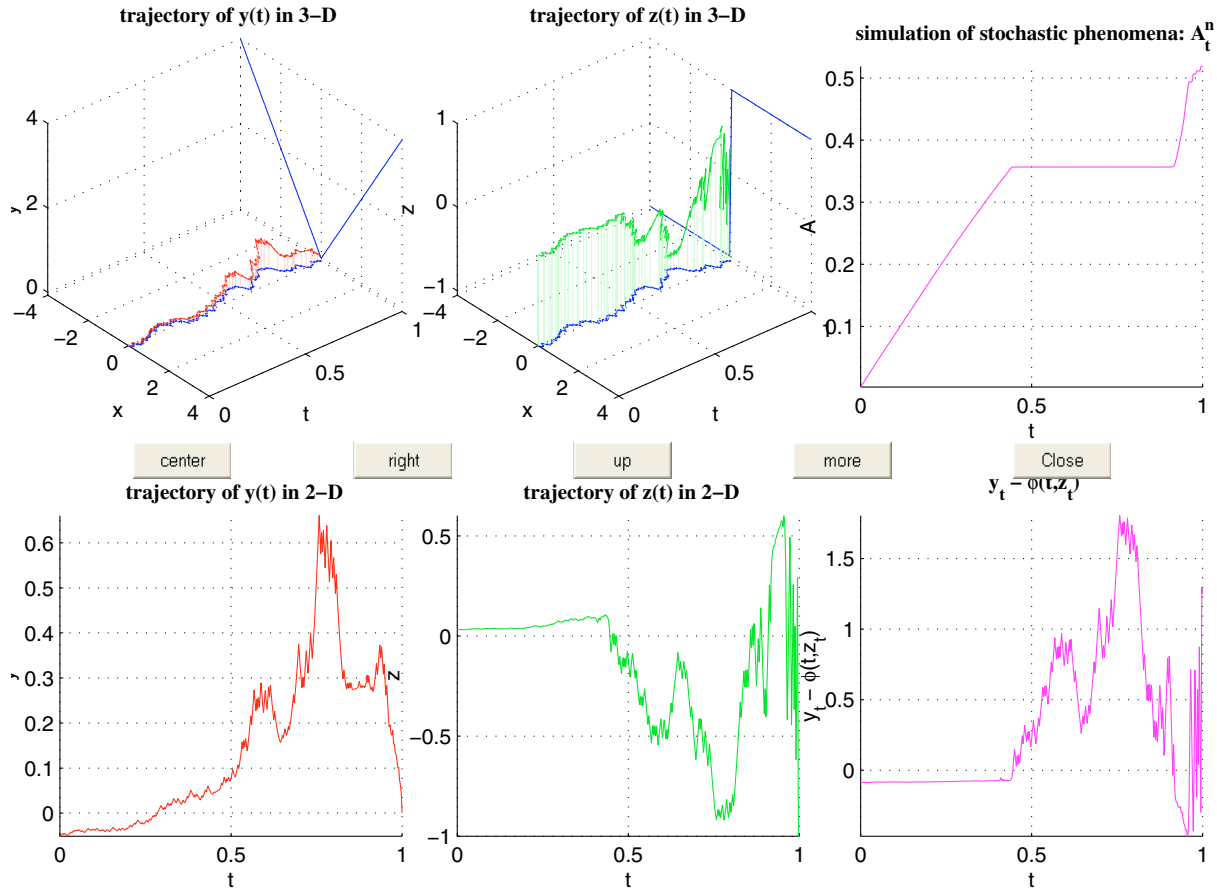


FIGURE 8. A trajectory of solutions of penalization BSDE with $d_\Gamma(y,z) = (y - \phi(z))^-$.

Acknowledgements. We appreciate the anonymous referees for valuable remarks and suggestions.

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