Canad. J. Math. Vol. 73 (4), 2021 pp. 1153–1170
http://dx.doi.org/10.4153/S0008414X2000036X
© Canadian Mathematical Society 2020

A certain structure of Artin groups and the isomorphism conjecture

S.K. Roushon

Abstract. We observe an inductive structure in a large class of Artin groups of finite real, complex and affine types and exploit this information to deduce the Farrell–Jones isomorphism conjecture for these groups.

1 Introduction

We prove the isomorphism conjecture of Farrell and Jones ([11, 17]) in the K-, L- and A-theories, for a large class of Artin groups of finite real, complex and affine types. The classical braid group is an example of a finite type real Artin group (type A_n), and we considered this group in [12] and [22] in the pseudoisotopy case of the conjecture.

The classical braid group case as in [12] and [22], used the crucial idea that the geometry of a certain class of 3-manifolds is involved in the building of the group. In the situation of the Artin groups considered in this paper, we find a similar structure. We exploit this information to prove the conjecture. More precisely, some of the Artin groups are realized as a subgroup of the orbifold fundamental group of the configuration space of unordered n-tuples of distinct points, on some 2-dimensional orbifolds. In the classical braid group case, the complex plane played the role of this 2-dimensional orbifold.

The isomorphism conjecture is an important conjecture in Geometry and Topology. Much work has been done in this area in recent times (*e.g.*, [2, 3, 20, 21, 22]). The motivations behind the isomorphism conjecture are the Borel and the Novikov conjectures, which claim that any two closed aspherical homotopy equivalent manifolds are homeomorphic, and the homotopy invariance of higher signatures of manifolds, respectively. Furthermore, the isomorphism conjecture for a group provides a better understanding of the K- and L-theories of the group. Also it is well known that the isomorphism conjecture for a group implies vanishing of the lower K-theory ($Wh(-), \widetilde{K}_0(\mathbb{Z}[-])$ and $K_{-i}(\mathbb{Z}[-])$ for $i \geq 1$) of any torsion free subgroup of the group. In fact, it is still an open conjecture that this should be the case for all torsion free groups. A consequence of the isomorphism conjecture is another conjecture by Hsiang, which says that $K_{-i}(\mathbb{Z}[-]) = 0$ for all $i \geq 2$ and for all groups ([15]).

Received by the editors January 3, 2019; revised May 5, 2020.
Published online on Cambridge Core May 21, 2020.
AMS subject classification: **19B99, 19G24, 20F36, 57R67**, 57N37.
Keywords: Artin group, isomorphism conjecture, Whitehead group, reduced projective class group, surgery obstruction group, Waldhausen A-theory.

https://doi.org/10.4153/S0008414X2000036X Published online by Cambridge University Press

There are two classical exact sequences in K- and L-theories, involving the following two assembly maps, where R is a regular ring in A_K, and is a ring with involution in A_L. This summarizes the history of the Borel and the Novikov conjectures.

$$A_K : H_*(B\Gamma, \mathbb{K}(R)) \to K_*(R[\Gamma]).$$
$$A_L : H_*(B\Gamma, \mathbb{L}(R)) \to L_*(R[\Gamma]).$$

In the case of torsion free groups, the isomorphism conjecture says that the above two maps are isomorphisms. See [[11], §1.6.1] or [[17], Remark 20.12.3].

Following Farrell–Jones and Farrell–Hsiang, in all recent works on the isomorphism conjecture, geometric input on the group plays a significant role. Here also we follow a similar path to prove the conjecture, but we need to use an inductive structure in the groups, where the building blocks carry certain (non-positively curved) geometry. The existence of any global geometric structure on Artin groups is not yet known. A problem is stated in [6], asking if these groups should be $CAT(0)$, and for $CAT(0)$ groups the isomorphism conjecture is known ([3], [27]).

We prove the following theorem.

Theorem 1.1 *Let Γ be an Artin group of type A_n, $B_n(=C_n)$, D_n, F_4, G_2, $I_2(p)$, $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$ or $G(de, e, r)$ $(d, r \geq 2)$. Then the isomorphism conjecture in K-, L- and A-theories with coefficients and finite wreath product is true for any subgroup of Γ. That is, the isomorphism conjecture with coefficients is true for $H \wr G$, for any subgroup H of Γ, and for any finite group G.*

Proof In Section 3 we introduce a class $\mathcal{C}$ (Definition 3.1) of groups defined inductively, using suitable group extensions, from fundamental groups of manifold of dimension ≤ 3. Then in Theorems 3.1 and 3.2 we show that the groups considered in Theorem 1.1 belong to $\mathcal{C}$. Finally, in Theorem 3.3 we observe that the groups in $\mathcal{C}$ satisfy the Farrell–Jones isomorphism conjecture with coefficients and finite wreath product. ∎

Since the groups considered in the theorem are torsion free, a consequence of the theorem is that the two assembly maps above are isomorphisms for the subgroups of any of these groups.

Let Γ be a finite type pure Artin group appearing in Theorem 1.1. Then as an application, in Theorem 3.4, we extend our earlier computation of the surgery groups of the classical pure Artin braid groups ([22]) to Γ.

Below we mention a well-known consequence of the isomorphism of the K-theory assembly map. See [[11], §1.6.1].

Corollary 1.1 The Whitehead group $Wh(-)$, the reduced projective class group $\widetilde{K}_0(\mathbb{Z}[-])$ and the negative K-groups $K_{-i}(\mathbb{Z}[-])$, for $i \geq 1$, vanish for any subgroup of the groups considered in Theorem 1.1.

The paper is organized as follows. In Section 2 we recall some background on Artin groups and in Section 3 we state our main results. We describe orbifold braid groups, some work from [1] and also prove a crucial proposition in Section 4, which are the key inputs in this work. Section 5 contains the proofs of Theorems 3.1 and 3.2. The

https://doi.org/10.4153/S0008414X2000036X Published online by Cambridge University Press

proof of Theorem 3.4, on the computation of the surgery groups of pure Artin groups, is given in Section 6.

2 Artin Groups

The Artin groups are an important class of groups, and appear in different areas of mathematics.

We are interested in those Artin groups which appear as extensions of Coxeter groups by the fundamental group of hyperplane arrangement complements in $\mathbb{C}^n$, for some n. We need the topology and geometry of this complement to work on the Artin groups.

The Coxeter groups are a generalization of reflection groups, and are yet another useful class of groups ([8]). Several Coxeter groups appear as Weyl groups of simple Lie algebras. In fact, all the Weyl groups are Coxeter groups. There are two different types of reflection groups we consider in this article; spherical and affine. The spherical type is generated by reflections on hyperplanes passing through the origin in a Euclidean space, and the affine type is generated by reflections on hyperplanes without the requirement that the hyperplanes pass through the origin.

Next we give the presentation of the Coxeter groups in terms of generators and relations. This way one has a better understanding of the corresponding Artin groups.

Let $K = \{s_1, s_2, \dots, s_k\}$ be a finite set, and $m : K \times K \to \{1, 2, \cdots \dots, \infty\}$ be a function with the property that $m(s,s) = 1$, and $m(s',s) = m(s,s') \geq 2$ for $s \neq s'$. The *Coxeter group* associated to the pair (K,m), is by definition the following group.

$$\mathcal{W}_{(K,m)} = \langle K \mid (ss')^{m(s,s')} = 1,\ s,s' \in S \text{ and } m(s,s') < \infty \rangle.$$

(K,m) is called a *Coxeter system*.

Associated to a Coxeter system there is a graph called *Coxeter diagram*. This graph has vertex set K, two vertices s and s' are connected by an edge if $m(s,s') \geq 3$. If $m(s,s') = 3$, then the edge is not labeled; otherwise it gets the label $m(s.s')$. From this diagram, the presentation of the Coxeter group can be reproduced.

A complete classification of finite, irreducible Coxeter groups is known (see [8]). Here irreducible means the corresponding Coxeter diagram is connected. That is, the Coxeter group becomes the direct product of the Coxeter groups corresponding to the connected components of the Coxeter diagram. In this article, without any loss, by a Coxeter group we will always mean an irreducible Coxeter group, since the Artin group (see below for definition) corresponding to a reducible Coxeter group is the direct product of the Artin groups of its irreducible factors (see Condition (3) in Definition 3.1). We reproduce the list of all finite Coxeter groups in Table 1. These are exactly the finite (spherical) reflection groups. For a general reference on this subject we refer the reader to [16].

Also there are infinite Coxeter groups which are affine reflection groups. See Table 2, which shows a list of only those which we need for this paper. For a complete list see [16]. In the tables the associated Coxeter diagrams are also shown.

The symmetric groups S_n on n letters and the finite dihedral group are examples of Coxeter groups. These are the Coxeter group of type A_n and $I_2(p)$

https://doi.org/10.4153/S0008414X2000036X Published online by Cambridge University Press

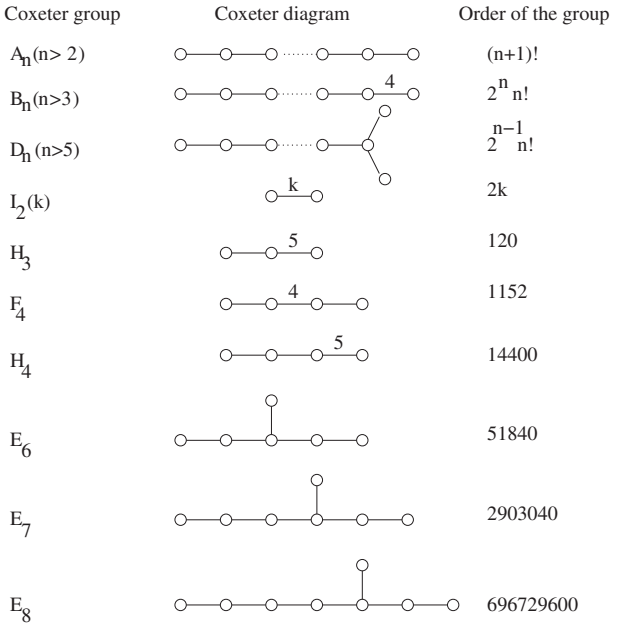


Table 1: Table of finite Coxeter groups.

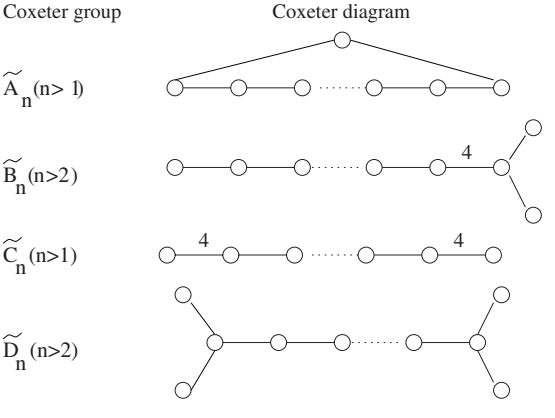


Table 2: Table of some affine Coxeter groups.

respectively in the table. The *Artin group* associated to the Coxeter group $\mathcal{W}_{(K,m)}$ is, by definition,

$$\mathcal{A}_{(K,m)} = \langle K \mid ss's\cdots = s'ss'\cdots;\ s, s' \in K \rangle.$$

Here the number of times the factors in $ss's\cdots$ appear is $m(s,s')$; for example, if $m(s,s') = 3$, then the relation is $ss's = s'ss'$. $\mathcal{A}_{(K,m)}$ is called the Artin group of type $\mathcal{W}_{(K,m)}$. There is an obvious surjective homomorphism $\mathcal{A}_{(K,m)} \to \mathcal{W}_{(K,m)}$. The kernel

https://doi.org/10.4153/S0008414X2000036X Published online by Cambridge University Press

$\mathcal{PA}_{(K,m)}$ (say) of this homomorphism is called the associated *pure Artin group*. In the case of type A_n, the (pure) Artin group is also known as the *classical (pure) braid group*. We will justify the word 'braid' in Section 4. When a Coxeter group is a finite or an affine reflection group, the associated Artin group is called of *finite* or *affine type*, respectively.

Now given a finite Coxeter group (equivalently, a finite reflection group) $\mathcal{W}_{(K,m)}$, and its standard faithful representation in $GL(n,\mathbb{R})$, for some n, consider the hyperplane arrangement in $\mathbb{R}^n$, where each hyperplane is fixed pointwise by an involution of the group. Next complexify $\mathbb{R}^n$ to $\mathbb{C}^n$ and consider the corresponding complexified hyperplanes. We call these complex hyperplanes in this arrangement, *reflecting hyperplanes* associated to the reflection group $\mathcal{W}_{(K,m)}$. Let $PA_{(K,m)}$ be the complement of this hyperplane arrangement in $\mathbb{C}^n$. The fundamental group of $PA_{(K,m)}$ is identified with the pure Artin group $\mathcal{PA}_{(K,m)}$, associated to the reflection group ([4]). The group $\mathcal{W}_{(K,m)}$ acts freely on this complement. The quotient space has fundamental group isomorphic to the Artin group, associated to the reflection group ([4]). That is, we get an exact sequence of the following type.

$$1 \longrightarrow \mathcal{PA}_{(K,m)} \longrightarrow \mathcal{A}_{(K,m)} \longrightarrow \mathcal{W}_{(K,m)} \longrightarrow 1.$$

We call the space $(PA_{(K,m)})$ $PA_{(K,m)}/\mathcal{W}_{(K,m)}$ the *(pure) Artin space* of the Artin group of type $\mathcal{W}_{(K,m)}$.

In this article, this geometric interpretation of the Artin groups is relevant for us.

Next our aim is to discuss some generalization of this interpretation, replacing $\mathbb{C}$ by some 2-dimensional orbifold, in the classical braid group case. These generalized groups are related to some of the Artin groups. This connection is exploited here to help extend our earlier result in [12] and [22], to the Artin groups of types A_n, $B_n(=C_n)$, D_n, $\widetilde{A}_n$, $\widetilde{B}_n$ and $\widetilde{C}_n$. Then, using a different method, we treat the cases F_4, G_2 and $I_2(p)$. However, we will see the core idea behind the proofs is the same; some kind of inductive structure, with 3-manifold groups as building blocks.

Remark 2.1 Recall that a complex reflection group is a subgroup of $GL(n,\mathbb{C})$ generated by complex reflections. A *complex reflection* is an element of $GL(n,\mathbb{C})$ which fixes a hyperplane in $\mathbb{C}^n$ pointwise. There is a classification of finite complex reflection groups ([25]) in the following two classes.

1. $G(de,e,r)$, where d,e,r are positive integers.
2. 34 exceptional groups denoted by $G_4,\ldots,G_{37}$.

There are associated Artin groups of the finite complex reflection groups.

Definition 2.1 For a connected topological space X, we define the *pure braid space* of X of n strings by the following.

$$PB_n(X) := X^n - \{(x_1,x_2,\ldots,x_n) \in X^n \mid x_i = x_j \text{ for some } i \neq j\}.$$

For $n=1$ we define $PB_1(X)=X$. For $n \geq 2$, the symmetric group S_n acts freely on the pure braid space, by permuting coordinates. The quotient space $PB_n(X)/S_n$

https://doi.org/10.4153/S0008414X2000036X Published online by Cambridge University Press

is denoted by $B_n(X)$, and is called the *braid space* of X. The fundamental group of $(PB_n(X))$ $B_n(X)$ is called the (*pure*) *braid group* of n strings of the topological space X.

We will need the following well-known Fadell–Neuwirth fibration theorem.

Theorem 2.1 [10] *Let M be a connected manifold of dimension ≥ 2. Then the projection $M^n \to M^{n-1}$ to the first $n-1$ coordinates induces a locally trivial fibration $PB_n(M) \to PB_{n-1}(M)$, with fiber homeomorphic to $M - \{(n-1)$ points$\}$.*

3 Statements of Results

In this section we state the results we referred to in the Introduction.

In [12] and [22], we proved the fibered isomorphism conjecture for $H \wr F$, where H is any subgroup of the classical braid group and F is a finite group.

In this article we consider a general statement of the conjecture with coefficients, and we denote the *isomorphism conjecture with coefficients and finite wreath product in the K-, L- and A-theories* by $FICwF$. The finite wreath product version of the conjecture was introduced in [20], and its general properties were proved in [23]. The importance of this version of the conjecture was first observed in [12] and [20]. We do not need the exact statement of the isomorphism conjecture, but need some already known results. Therefore, we do not state the conjecture and refer the reader to [23] or [17] for more on this subject.

Now we are in a position to state the results we prove in this paper.

We need to define the following class of groups, which contains the groups defined in [[21], Definition 1.2.1].

Definition 3.1 Let $\mathcal{C}$ denote the smallest class of groups satisfying the following conditions.

1. The fundamental group of any connected manifold of dimension ≤ 3 belongs to $\mathcal{C}$.
2. If H is a subgroup of a group G, then $G \in \mathcal{C}$ implies $H \in \mathcal{C}$. This reverse implication is also true if H is of finite index in G.
3. If $G_1, G_2 \in \mathcal{C}$ then $G_1 \times G_2 \in \mathcal{C}$.
4. If $\{G_i\}_{i \in I}$ is a directed system of groups and $G_i \in \mathcal{C}$ for each $i \in I$, then the $colim_{i \in I} G_i \in \mathcal{C}$.
5. Let $1 \to K \to G \to H \to 1$ be a short exact sequence of groups with $p : G \to H$ being the last surjective homomorphism. If K, H and $p^{-1}(C)$, for any infinite cyclic subgroup C of H, belong to $\mathcal{C}$ then G also belongs to $\mathcal{C}$.

In the following two theorems, we prove that the Artin groups considered in this paper belong to $\mathcal{C}$.

Theorem 3.1 *Let S be an aspherical connected 2-manifold. Then the (pure) braid group of S belongs to $\mathcal{C}$. Consequently, the Artin groups of types A_n, $B_n(= C_n)$, $\widetilde{A}_n$ and $\widetilde{C}_n$ belong to $\mathcal{C}$.*

Theorem 3.2 *The Artin groups of types D_n, F_4, G_2, $I_2(p)$, $\widetilde{B}_n$ and $G(de, e, r)$, $d, r \geq 2$ belong to $\mathcal{C}$.*

https://doi.org/10.4153/S0008414X2000036X Published online by Cambridge University Press

$\mathcal{PA}$ = **pure Artin group of type**	N
A_{n-1}	$\frac{n(n-1)}{2}$
$B_n (= C_n)$	n^2
D_n	$n(n-1)$
F_4	24
$I_2(p)$	p
G_2	6

Table 3: Number of reflecting hyperplanes.

Theorem 1.1 is then a consequence of Theorems 3.1, 3.2 and the following theorem.

Theorem 3.3 *The FICwF is true for any group in* $\mathcal{C}$.

Proof This is a special case of [[17], Theorem 20.8.6]. ∎

Remark 3.1 It will be interesting to know if all the Artin groups belong to $\mathcal{C}$.

Finally, we state a corollary regarding an explicit computation of the surgery groups $L_*(\mathbb{Z}[-])$ of the pure Artin groups of finite type. This comes out of the isomorphisms of the two assembly maps A_K and A_L, for the finite type Artin groups, stated in Theorem 1.1. The calculation was done in [22] for the classical pure braid group case. Together with the isomorphisms of A_K and A_L, the proof basically used the homotopy type of the suspension of the pure braid space and the knowledge of the surgery groups of the trivial group.

Theorem 3.4 *The surgery groups of the finite type pure Artin groups take the following form.*

$$L_i(\mathcal{PA}) = \begin{cases} \mathbb{Z} & \text{if } i = 4k, \\ \mathbb{Z}^N & \text{if } i = 4k+1, \\ \mathbb{Z}_2 & \text{if } i = 4k+2, \\ \mathbb{Z}_2^N & \text{if } i = 4k+3. \end{cases}$$

Here N is the number of reflecting hyperplanes associated to the finite Coxeter group, as given in the following table.

Remark 3.2 Recall that there are surgery groups for different kinds of surgery problems, and they appear in the literature with the notations $L_i^*(-)$, where $* = h, s, \langle -\infty \rangle$ or $\langle j \rangle$ for $j \leq 0$. But all of them are naturally isomorphic for a group G, if $Wh(G)$, $\widetilde{K}_0(\mathbb{Z}[G])$, and $K_{-i}(\mathbb{Z}[G])$, for $i \geq 1$, vanish. This can be checked by the

https://doi.org/10.4153/S0008414X2000036X Published online by Cambridge University Press

Rothenberg exact sequence ([[24], 4.13]).

$$\cdots \to L_i^{\langle j+1\rangle}(R) \to L_i^{\langle j\rangle}(R) \to \hat{H}^i(\mathbb{Z}/2; \widetilde{K}_j(R)) \to L_{i-1}^{\langle j+1\rangle}(R) \to L_{i-1}^{\langle j\rangle}(R) \to \cdots.$$

Where $R = \mathbb{Z}[G]$, $j \leq 1$, $Wh(G) = \widetilde{K}_1(R)$, $L_i^{\langle 1\rangle} = L_i^h$, $L_i^{\langle 2\rangle} = L_i^s$ and $L_i^{\langle -\infty\rangle}$ is the limit of $L_i^{\langle j\rangle}$. Therefore, because of Corollary 1.1, we use the simplified notation $L_i(-)$ in the above corollary.

Remark 3.3 The same calculation holds for the other pure Artin groups corresponding to the finite type Coxeter groups and finite complex reflection groups (see Remark 2.1) also, provided we know the $FICwF$ for these groups. We further need the fact that the Artin spaces are aspherical ([9]), which implies that the Artin groups are torsion free. The last fact can also be proved group theoretically ([13]). The Artin groups corresponding to the finite complex reflection groups $G(de, e, r)$ are torsion free, since $\mathcal{A}_{G(de,e,r)}$ is a subgroup of $\mathcal{A}_{B_r}$, for $d, r \geq 2$ ([[7], Proposition 4.1]).

4 Artin Groups and Orbifold Braid Groups

In this section we give a short introduction to orbifold braid groups and describe a connection with Artin groups.

Let $\mathbb{C}(k, l; q_1, q_2, \cdots \ldots, q_l)$ denote the complex plane with k punctures at the points $p_1, p_2, \cdots \ldots, p_k \in \mathbb{C}$, l distinguished points (also called *cone* points) $x_1, x_2, \cdots, x_l \in \mathbb{C} - \{p_1, p_2, \cdots, p_k\}$ and an integer $q_i > 1$ attached to x_i for $i = 1, 2, \cdots, l$. q_i is called the *order* of the cone point x_i.

One can define (pure) braid group of n strings of $\mathbb{C}(k, l; q_1, q_2, \cdots, q_l)$. There are two different ways one can do this, which give the same result. The first way is topological and the second way is pictorial as in the classical braid group case. In the former, one realizes $\mathbb{C}(k, l; q_1, q_2, \cdots, q_l)$ as a 2-dimensional orbifold and then considers $PB_n(\mathbb{C}(k, l; q_1, q_2, \cdots, q_l))$. Note that, $PB_n(\mathbb{C}(k, l; q_1, q_2, \cdots, q_l))$ is a high-dimensional orbifold and one considers its orbifold fundamental group and follow Definition 2.1. We call this orbifold fundamental group the *(pure) orbifold braid group* of n strings of $\mathbb{C}(k, l; q_1, q_2, \cdots, q_l)$. For basics on orbifolds see [26]. The latter pictorial definition is relevant for us. We describe it now from [1].

Recall that any element of the classical braid group $\pi_1(B_n(\mathbb{C}))$ can be represented by a braid as in the following picture. Juxtaposing one braid over another gives the group operation.

Let S be the complex plane with only one cone point. The underlying topological space of S is nothing but the complex plane. Therefore, $B_n(S)$ is an orbifold when we consider the orbifold structure of S; otherwise it is the classical braid space.

Therefore, although the fundamental group of the underlying topological space of $B_n(S)$ has the classical pictorial braid representation as above, the orbifold fundamental group of $B_n(S)$ needs different treatment. We point out here a similar pictorial braid representation of a typical element of the orbifold fundamental group of $B_n(S)$

https://doi.org/10.4153/S0008414X2000036X Published online by Cambridge University Press

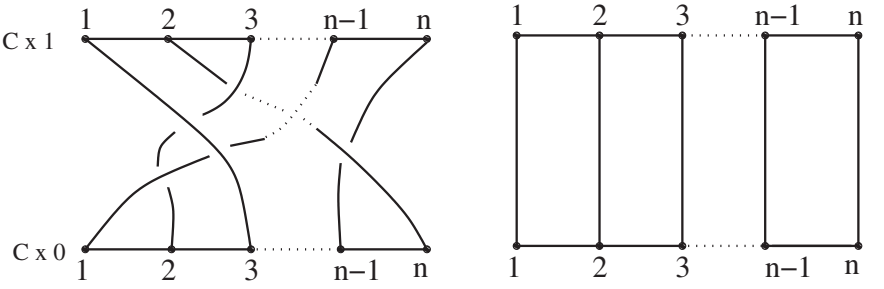


Figure 1: A typical braid and the trivial (identity) braid.

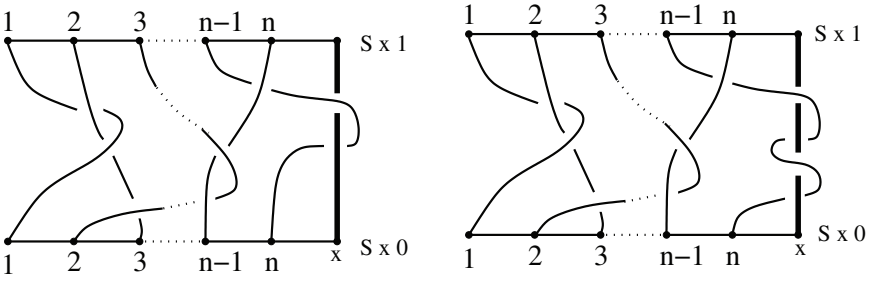


Figure 2: A typical orbifold braid.

from [1]. The following picture shows the case for one cone point x. The thick line represents $x \times I$.

The group operation is again given by juxtaposing one braid onto another, and the identity element is also obvious.

Note that, in Figure 2, both the braids represent the same element in $\pi_1(B_n(S))$, but different in $\pi_1^{orb}(B_n(S))$, depending on the order of the cone point.

In this orbifold situation the movement of the braids is restricted, because of the presence of the cone point. Therefore, one has to define new relations among braids, respecting the singular set of the orbifold. We produce one situation to see how this is done. The second picture in Figure 3 represents part of a typical element of $\pi_1^{orb}(B_n(S))$.

Now if a string in the braid wraps the thick line $x \times I$, q times, then it is equal to the third picture. This is because, if a loop circles q times around the cone point x, then the loop gives the trivial element in the orbifold fundamental group of S. Therefore, both braids represents the same element in the orbifold fundamental group of $B_n(S)$. Furthermore, if the string wraps the thick line, but less than q number of times, then it is not equal to the unwrapped braid. For more details see [1].

When there is a puncture p in the complex plane then the braids will have to satisfy a similar property, but in this case, if any of the string wraps $p \times I$ once then the braid will have infinite order. See the first picture in Figure 3.

https://doi.org/10.4153/S0008414X2000036X Published online by Cambridge University Press

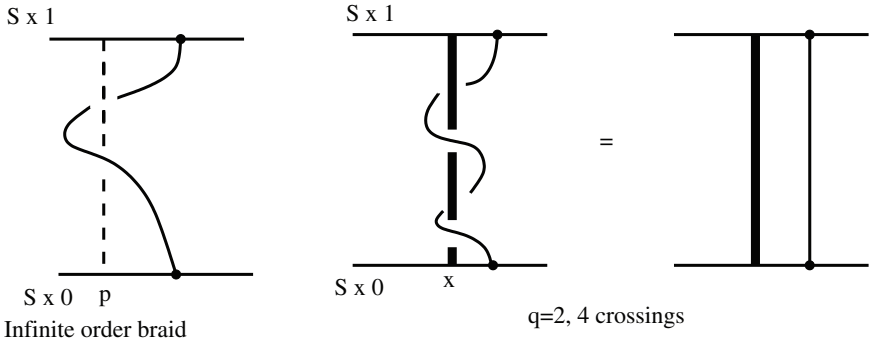


Figure 3: Braid movement around a puncture and a cone point.

A = **Artin group of type**	**Orbifold** S	**Quotient group**	n
A_{n-1}	$\mathbb{C}$	$< 1 >$	$n > 1$
$B_n (= C_n)$	$\mathbb{C}(1,0)$	$< 1 >$	$n > 1$
D_n	$\mathbb{C}(0,1;2)$	$\mathbb{Z}/2$	$n > 1$
$\widetilde{A}_{n-1}$	$\mathbb{C}(1,0)$	$\mathbb{Z}$	$n > 2$
$\widetilde{B}_n$	$\mathbb{C}(1,1;2)$	$\mathbb{Z}/2$	$n > 2$
$\widetilde{C}_n$	$\mathbb{C}(2,0)$	$< 1 >$	$n > 1$
$\widetilde{D}_n$	$\mathbb{C}(0,2;2,2)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$n > 2$

Table 4: Artin groups embedded in orbifold braid groups.

There is a useful connection between the orbifold braid group of the orbifold $\mathbb{C}(k,l;q_1,q_2,\cdots,q_l)$ and some of the Artin groups. More precisely, to associate braid type representation to elements of an Artin group.

For a given Artin group one has to choose a suitable 2-dimensional orbifold, as in Table 4, in Theorem 4.1. In Remark 2.1, we recalled a classification of finite complex reflection groups. In the case of finite complex reflection groups of type $G(de,e,r)$, for $d, r \geq 2$, one has to consider the braid group of the punctured plane (annulus). See the case B_n in Table 4 in Theorem 4.1, and use [[7], Proposition 4.1].

Theorem 4.1 ([1]) *Let A be an Artin group, and S be an orbifold as described in the following table. Then A is a (normal) subgroup of the orbifold braid group* $\pi_1^{orb}(B_n(S))$. *The third column gives the quotient group* $\pi_1^{orb}(B_n(S))/A$.

https://doi.org/10.4153/S0008414X2000036X Published online by Cambridge University Press

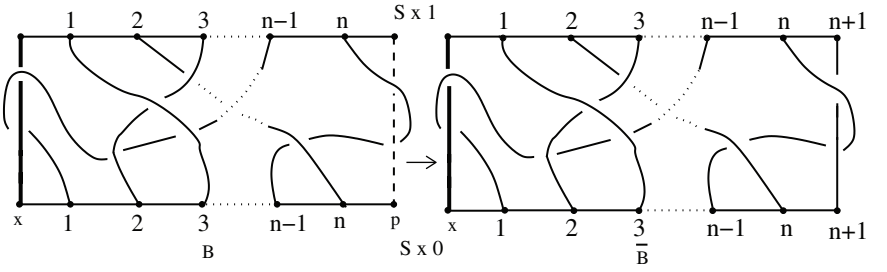


Figure 4: Mapping a braid after filling a puncture with a string.

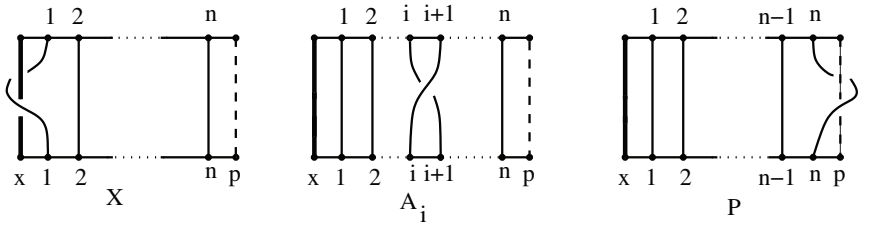


Figure 5: Generators of $\pi_1^{orb}(B_n(\mathbb{C}(1,1;q)))$.

Proof See [1] for the proof. ∎

The following proposition is crucial for this paper, which helps to deduce some more inclusions as in the previous theorem.

Proposition 4.1 There is an injective homomorphism from the orbifold braid group $\pi_1^{orb}(B_n(\mathbb{C}(1,1;q)))$ to $\pi_1^{orb}(B_{n+1}(\mathbb{C}(0,1;q)))$. That is, the orbifold braid group of n strings of $\mathbb{C}(1,1;q)$ can be embedded into the orbifold braid group of $n+1$ strings of $\mathbb{C}(0,1;q)$.

Proof We give a pictorial proof (following [1]) and also define the explicit map.

Let $B \in \pi_1^{orb}(B_n(\mathbb{C}(1,1;q)))$ as in the following figure. Here x is the cone point and p is the puncture. We send B to $\overline{B}$ where the line $p \times I$ is sent to a new string ($n+1$ as in the figure) and hence $\overline{B} \in \pi_1^{orb}(B_{n+1}(\mathbb{C}(0,1;q)))$.

It is easy to see that the map $B \mapsto \overline{B}$ is an injective homomorphism. Here recall that the composition is a juxtaposition of braids. In the first picture of Figure 4, $S = \mathbb{C}(1,1;q)$ and it is $\mathbb{C}(0,1;q)$ in the second one.

The group $\pi_1^{orb}(B_n(\mathbb{C}(1,1;q)))$ has the following generators:

$$X, A_1, \cdots, A_{n-1}, P.$$

The relations are

$$X^q = 1, XA_1XA_1 = A_1XA_1X, A_iA_{i+1}A_i = A_{i+1}A_iA_{i+1},$$

$$A_iA_j = A_jA_i, \text{ for } |i-j| > 1 \text{ and } PA_{n-1}PA_{n-1} = A_{n-1}PA_{n-1}P.$$

https://doi.org/10.4153/S0008414X2000036X Published online by Cambridge University Press

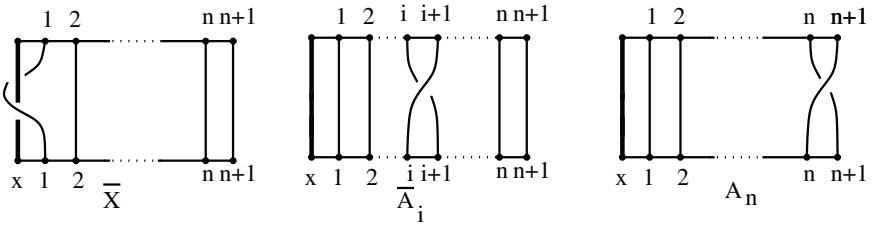


Figure 6: Generators of $\pi_1^{orb}(B_{n+1}(\mathbb{C}(0,1;q)))$.

And the group $\pi_1^{orb}(B_{n+1}(\mathbb{C}(0,1;q)))$ has the following generators.

$$\overline{X}, \overline{A}_1, \cdots, \overline{A}_{n-1}, \overline{A}_n.$$

The relations are

$$\overline{X}^q = 1,\ \overline{X}\,\overline{A}_1\overline{X}\,\overline{A}_1 = \overline{A}_1\overline{X}\,\overline{A}_1\overline{X},\ \overline{A}_i\overline{A}_{i+1}\overline{A}_i = \overline{A}_{i+1}\overline{A}_i\overline{A}_{i+1}$$

$$\text{and } \overline{A}_i\overline{A}_j = \overline{A}_j\overline{A}_i,\ \text{for } |i-j| > 1.$$

See [1] for more on this matter.

It is easy to see that the map defined above is obtained by the following map on the generators level.

$$X \mapsto \overline{X}, A_i \mapsto \overline{A}_i \text{ for } i = 1, 2, \ldots, n-1 \text{ and } P \mapsto \overline{A}_n^2.$$

This completes the proof of the proposition. ∎

5 Proofs of Theorems 3.1 and 3.2

In this section we give the proofs of Theorems 3.1 and 3.2. Throughout this section we refer to the conditions in Definition 3.1 as Condition (n). We need the following lemma for some induction argument.

Lemma 5.1 *Let $f : M \to N$ be a locally trivial fibration of connected manifolds with fiber a 2-manifold, and assume that $\pi_2(N) = 1$. If $\pi_1(N) \in \mathcal{C}$, then $\pi_1(M) \in \mathcal{C}$.*

Proof Let F be the 2-manifold fiber of the fibration over some base point. Then we have the following short exact sequence of groups coming from the long exact sequence of homotopy groups applied to the fibration f.

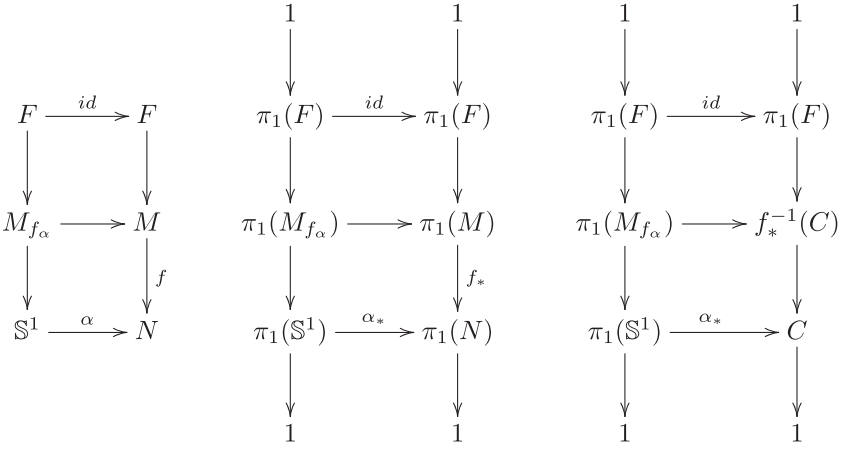
$$1 \longrightarrow \pi_1(F) \longrightarrow \pi_1(M) \xrightarrow{f_*} \pi_1(N) \longrightarrow 1.$$

By Conditions (1) and (2), $\pi_1(M) \in \mathcal{C}$ if $\pi_1(N)$ is finite. Therefore, we can assume that $\pi_1(N)$ is infinite. Let $C < \pi_1(N)$ be an infinite cyclic subgroup generated by $[\alpha] \in \pi_1(N)$. Let f_α be the monodromy homeomorphism of F, corresponding to α. Then we get the following first diagram, as the mapping torus M_{f_α} of f_α is identified with the pullback $\{(s,m) \in \mathbb{S}^1 \times M \mid \alpha(s) = f(m)\}$ of the fibration f, under $\alpha : \mathbb{S}^1 \to N$. The

https://doi.org/10.4153/S0008414X2000036X Published online by Cambridge University Press

second diagram is obtained by applying the long exact homotopy sequence and its naturality on the two fibrations f and $M_{f_\alpha} \to \mathbb{S}^1$.

Note that $\pi_1(M_{f_\alpha})$ goes into $f_*^{-1}(C)$, which gives the third diagram.



From the third commutative diagram, and applying the Five lemma we get that $f_*^{-1}(C)$ is isomorphic to the fundamental group of M_{f_α}, which is a 3-manifold. Hence, $f_*^{-1}(C)$ belongs to $\mathcal{C}$ by Condition (2). Using Condition (5) we complete the proof of the lemma. ∎

Proof of Theorem 3.1 Recall that S is an aspherical, connected 2-manifold, and we have to show that the braid group of S belong to $\mathcal{C}$.

First, note that the second half of the theorem follows from Theorem 4.1 and Condition (2), once we prove that the braid group of S belongs to $\mathcal{C}$.

By Condition (2), it is enough to prove that the pure braid group of S belongs to $\mathcal{C}$.

So, let $\Gamma_n = \pi_1(PB_n(S))$ be the pure braid group of S. The proof is by induction on n.

Case $n = 1$. In this case $\Gamma_1 \simeq \pi_1(S)$, and hence the proof is completed using Condition (1).

Case $n \geq 2$. Assume that we have proved the theorem for Γ_{n-1}. Consider the following projection to the first $n-1$ coordinates; $PB_n(S) \to PB_{n-1}(S)$. This is a locally trivial fibration with fiber $S - \{(n-1) \text{ points}\}$ (Theorem 2.1). By the induction hypothesis, $\pi_1(PB_{n-1}(S)) \in \mathcal{C}$. Therefore, by Lemma 5.1, $\pi_1(PB_n(S)) \in \mathcal{C}$.

This completes the proof of the Theorem. ∎

Remark 5.1 The same proof also applies to show that the fundamental group of any fiber-type hyperplane arrangement (see [[19], Definition 5.11]) complement in $\mathbb{C}^n$ belongs to $\mathcal{C}$.

Proof of Theorem 3.2 Except for the $\widetilde{B}_n$ case, by Condition (2), it is enough to prove that the pure Artin groups of the other types belong to $\mathcal{C}$. We will prove the $\widetilde{B}_n$ case at the end.

First, we deduce the proof of the theorem in the F_4 and D_n cases.

https://doi.org/10.4153/S0008414X2000036X Published online by Cambridge University Press

The idea is to see the pure Artin spaces of types F_4 and D_n as the total space of a locally trivial fibration over another aspherical manifold, whose fundamental group belong to $\mathcal{C}$, and the fiber is a 2-manifold. Then Lemma 5.1 will be applicable.

For the rest of the section we refer to the discussion in the proof of [[5], Proposition 2].

We begin with the following lemma. ■

Lemma 5.2 *Let $Z_n = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq 0$ for all $i, z_i \neq z_j$ for $i \neq j\}$. Then Z_n is aspherical and $\pi_1(Z_n)$ belongs to $\mathcal{C}$ for all n.*

Proof The proof that Z_n is aspherical follows easily by an induction argument, on taking successive projections and using the long exact sequence of homotopy groups of a fibration. (See Theorem 2.1 and note that $Z_n = PB_n(\mathbb{C}^*)$).

The proof that $\pi_1(Z_n)$ belongs to $\mathcal{C}$, follows from the first half of Theorem 3.1. ■

F_4 case.

The pure Artin group of type F_4 is isomorphic to the fundamental group of the following hyperplane arrangement complement in $\mathbb{C}^4$.

$$PA_{F_4} = \{(y_1, y_2, y_3, y_4) \in \mathbb{C}^4 \mid y_i \neq 0 \text{ for all } i,\ y_i \pm y_j \neq 0 \text{ for } i \neq j,$$
$$y_1 \pm y_2 \pm y_3 \pm y_4 \neq 0\}.$$

The signs above appear in arbitrary combinations. Next consider the map $PA_{F_4} \to Z_3$ defined by the following formula.

$$z_i = y_1 y_2 y_3 y_4 (y_4^2 - y_i^2),\ i = 1, 2, 3.$$

This map is a locally trivial fibration, with fiber a 2-manifold. Hence, by Lemmas 5.1 and 5.2, $\pi_1(PA_{F_4})$ belongs to $\mathcal{C}$.

D_n-case.

Again, the reference for this discussion is [[5], Proposition 2].

This proof goes along the same lines as the proof of the F_4 case. First, note that the pure Artin space of type D_n has the following form.

$$PA_{D_n} = \{(y_1, \ldots, y_n) \in \mathbb{C}^n \mid y_i \pm y_j \neq 0 \text{ for } i \neq j\}.$$

Next the map $PA_{D_n} \to Z_{n-1}$ defined by $z_i = y_n^2 - y_i^2$, $i = 1, 2, \cdots, n-1$, is a locally trivial fibration with fiber a 2-manifold. The proof now follows from Lemmas 5.1 and 5.2.

$I_2(p)$-case.

First, note that this is a rank 2 case. Therefore, the pure Artin group of type $I_2(p)$ belongs to $\mathcal{C}$, by the following more general statement.

Lemma 5.3 *The fundamental group of any hyperplane arrangement complement in $\mathbb{C}^2$ belongs to $\mathcal{C}$.*

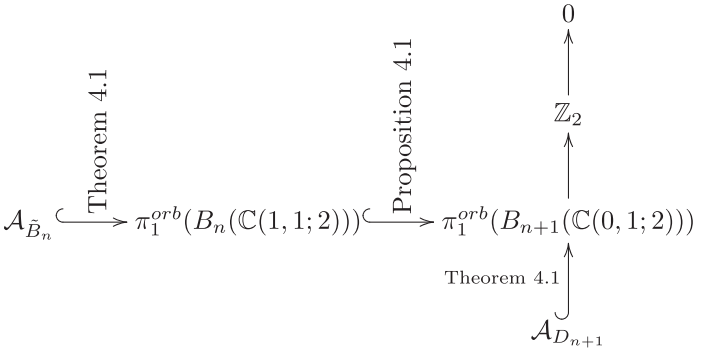
Proof Let $\mathcal{H}$ be the union of the hyperplanes in $\mathbb{C}^2$. Consider the unit sphere $\mathbb{S}^3 \subset \mathbb{C}^2$. Then $\mathbb{C}^2 - \mathcal{H}$ deformation retracts to $\mathbb{S}^3 - (\mathbb{S}^3 \cap \mathcal{H})$, which is a 3-manifold. Hence, by Condition 1, $\pi_1(\mathbb{C}^2 - \mathcal{H}) \in \mathcal{C}$. ■

https://doi.org/10.4153/S0008414X2000036X Published online by Cambridge University Press

G_2-case.
Since $G_2 = I_2(6)$, the proof in this case is completed using the previous case.

$G(de, e, r)$-case.
For the proof of the theorem in the $G(de, e, r)$-case, we refer to Remark 2.1. In ([[7], Proposition 4.1]) it was shown that, for $d, r \geq 2$, $\mathcal{A}_{G(de,e,r)}$ is a subgroup of $\mathcal{A}_{B_r}$. (In [7], these are called the braid groups associated to the reflection groups). Therefore, using Condition (2) and the B_n case of Theorem 3.1, $\mathcal{A}_{G(de,e,r)} \in \mathcal{C}$.

$\widetilde{B}_n$-case.
By Theorem 4.1, the Artin group of type $\widetilde{B}_n$ is isomorphic to a subgroup of $\pi_1^{orb}(B_n(\mathbb{C}(1,1;2)))$, and by Proposition 4.1, $\pi_1^{orb}(B_n(\mathbb{C}(1,1;2)))$ is isomorphic to a subgroup of $\pi_1^{orb}(B_{n+1}(\mathbb{C}(0,1;2)))$. On the other hand, again by Theorem 4.1, $\pi_1^{orb}(B_{n+1}(\mathbb{C}(0,1;2)))$ contains the Artin group of type D_{n+1} as a subgroup of index 2. Therefore, using Condition (2) and the above D_n-case, we complete the proof. The flow of the argument is given in the following diagram.



6 Computation of the Surgery Groups

This section is devoted to some applications related to computation of the surgery groups of some of the discrete groups considered in this paper.

Since the finite type pure Artin groups are torsion free, Corollary 1.1 is applicable to the finite type pure Artin groups considered in Theorem 1.1. A parallel to this K-theoretic vanishing result is the computation of the surgery groups.

For finite groups, the computation of the surgery groups is well established ([14]). The infinite groups case needs different techniques and is difficult, even when we have the isomorphism of the L-theory assembly map. For a survey on known results and techniques, on computation of surgery groups for infinite groups, see [23].

In the case of classical pure braid group, we carried out the computation in [22]. Here we extend it to the pure Artin groups of the finite type Artin groups considered in Theorem 1.1.

The main idea behind the computation is the following lemma on the homotopy type of the first suspension of a hyperplane arrangement complement. This lemma was stated and proved in [22].

https://doi.org/10.4153/S0008414X2000036X Published online by Cambridge University Press

Lemma 6.1 ([22], Lemma 4.1) *The first suspension* $\Sigma(\mathbb{C}^n - \cup_{j=1}^N A_j)$ *of the complement of a hyperplane arrangement* $\mathcal{A} = \{A_1, A_2, \ldots, A_N\}$ *in* $\mathbb{C}^n$*, is homotopically equivalent to the wedge of spheres* $\vee_{j=1}^N S_j$*, where* S_j *is homeomorphic to the 2-sphere* $\mathbb{S}^2$ *for* $j = 1, 2, \ldots, N$.

We need the following result to prove Theorem 3.4.

Proposition 6.1 Let $\mathcal{A} = \{A_1, A_2, \ldots, A_N\}$ be an aspherical hyperplane arrangement in $\mathbb{C}^n$. Assume that the two assembly maps A_K and A_L are isomorphisms, for the group $\Gamma = \pi_1(\mathbb{C}^n - \cup_{j=1}^N A_j)$. Then the surgery groups of Γ are given by the following.

$$L_i(\Gamma) = \begin{cases} \mathbb{Z} & \text{if } i \equiv 0 \bmod 4, \\ \mathbb{Z}^N & \text{if } i \equiv 1 \bmod 4, \\ \mathbb{Z}_2 & \text{if } i \equiv 2 \bmod 4, \\ \mathbb{Z}_2^N & \text{if } i \equiv 3 \bmod 4. \end{cases}$$

Proof The proposition was stated in [[22], Theorem 2.2] for fiber-type hyperplane arrangement complements. But the proof of this more general statement is the same. First, by Lemma 6.1, the first suspension of $\mathbb{C}^n - \cup_{j=1}^N A_j = X$ (say) is homotopically equivalent to the wedge of N many 2-spheres. Hence, for any generalized homology theory h_*, applied over X, we get the following equation.

$$h_i(X) = h_i(*) \oplus h_{i-2}(*)^N.$$

Where $*$ denotes a single point space. The proof is completed using the known computation of the surgery groups of the trivial group, and since A_L is an isomorphism. ∎

Proof of Theorem 3.4 Note that the pure Artin groups of the finite type Artin groups are torsion free, since they are fundamental groups of finite dimensional aspherical hyperplane arrangement complements in some complex space. See Remark 3.3. Hence, by Theorem 1.1 the assembly map A_L is an isomorphism for these hyperplane arrangement complements.

Therefore, by Proposition 6.1 together with the calculation in Table 5 of the number of hyperplanes associated to the finite reflection groups, we complete the proof of the corollary. We refer the reader to [5], for the equations of the hyperplanes given in the table. Or see [[16], pages 41–43] for a complete root structure. ∎

We end with the following remark and a problem.

Remark 6.1 The asphericity of certain hyperplane arrangement complements, including the configuration space of aspherical 2-manifolds, was needed in this work, to conclude that they have torsion free fundamental groups, and also for some induction arguments to work (see Lemma 5.1). This result is well known for configuration spaces of aspherical 2-manifolds (see Theorem 2.1 and [5]). It is known that the finite type real Artin groups are fundamental groups of aspherical manifolds ([9]). In the affine type Artin group case, this is proved recently in [19].

For the configuration space of aspherical 2-dimensional good orbifold, we state the following Problem.

https://doi.org/10.4153/S0008414X2000036X Published online by Cambridge University Press

$\mathcal{PA}$ = **pure Artin group type**	**Reflecting hyperplanes in** $\mathbb{C}^n = \{(z_1, z_2, \ldots, z_n) \mid z_i \in \mathbb{C}\}$	**Number of reflecting hyperplanes**
A_{n-1}	$z_i = z_j$ for $i \neq j$.	$\frac{n(n-1)}{2}$
$B_n (= C_n)$	$z_i = 0$ for all i; $z_i = z_j$, $z_i = -z_j$ for $i \neq j$.	n^2
D_n	$z_i = z_j$, $z_i = -z_j$ for $i \neq j$.	$n(n-1)$
F_4	$n = 4$; $z_i = 0$ for all i; $z_i = z_j$, $z_i = -z_j$ for $i \neq j$; $z_1 \pm z_2 \pm z_3 \pm z_4 = 0$.	24
$I_2(p)$	$n = 2$; roots are in one-to-one correspondence with the lines of symmetries of a regular p-gon in $\mathbb{R}^2$. See [[16], page 4]. Hence p hyperplanes.	p
$G_2 = I_2(6)$		6

Table 5: Reflecting hyperplanes of some Artin groups.

Problem. Show that the orbifold braid group of any 2-dimensional good orbifold, whose universal cover is contractible, belongs to $\mathcal{C}$ or to the class of groups defined in [[17], Theorem 20.8.6]. In particular, this will prove that the FICwF is true for the affine Artin group of type $\widetilde{D}_n$. See Theorem 4.1. Note that, the pure braid space of such a 2-dimensional good orbifold is again a good orbifold. However, it is not yet known if the universal cover of this pure braid space is contractible. An interesting simple case to prove this will be $\mathbb{C}(0, 2; 2, 2)$.

Acknowledgment The author is grateful to the referees for their valuable comments and suggestions, which made some significant improvements in the paper.

References

[1] D. Allcock, *Braid pictures of Artin groups*. Trans. Amer. Math. Soc. **354**(2002), no. 9, 3455–3474.
[2] A. Bartels and M. Bestvina, *The Farrell-Jones Conjecture for mapping class groups*. Invent. Math. **215**(2019), no. 2, 651–712.
[3] A. Bartels and W. Lück, *The Borel conjecture for hyperbolic and CAT(0) groups*. Ann. Math. **175**(2012), no. 2, 631–689.
[4] E. Brieskorn, *Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe*. Invent. Math. **12**(1971), 57–61.
[5] E. Brieskorn, *Sur les groupes de tresses [d'après V.I. Arnol'd]*. In: Séminaire Bourbaki 24ème année (1971/1972), Exp. No. 401, Lecture Notes in Math., 317, Springer, Berlin, 1973, pp. 21–44.
[6] R. Charney, *Problems related to Artin groups*. 2006. http://people.brandeis.edu/~charney/papers/Artin_probs.pdf
[7] R. Corran, E. Lee, and S. Lee, *Braid groups of imprimitive complex reflection groups*. J. Algebra **427**(2015), 387–425.
[8] H. S. M. Coxeter, *The complete enumeration of finite groups of the form* $R_i^2 = \left(R_i R_j\right)^{k_{ij}} = 1$. J. Lond. Math. Soc. **10**(1935), 21–25.

https://doi.org/10.4153/S0008414X2000036X Published online by Cambridge University Press

[9] P. Deligne, *Les immeubles des groupes de tresses généralisés*. Invent. Math. **17**(1972), 273–302.
[10] E. Fadell and L. Neuwirth, *Configuration spaces*. Math. Scand. **10**(1962), 111–118.
[11] F. T. Farrell and L. E. Jones, *Isomorphism conjectures in algebraic K-theory*. J. Amer. Math. Soc. **6**(1993), 249–297.
[12] F. T. Farrell and S. K. Roushon, *The Whitehead groups of braid groups vanish.* In. Math. Res. Notices **10**(2000), 515–526.
[13] F. A. Garside, *The braid group and other groups*. Quart. J. Math. Oxford **20**(1969), 235–254.
[14] I. Hambleton and L. R. Taylor, *A guide to the calculation of surgery obstruction groups for finite groups.* Surveys on surgery theory, 1, Princeton University Press, Princeton, NJ, 2000, pp. 225–274.
[15] W.-C. Hsiang, *Geometric applications of algebraic K-theory*. Proc. Int. Cong. Math. **1**(1983), 99–118.
[16] J. E. Humphreys, *Reflection groups and Coxeter groups*. Cambridge Studies in Advanced Mathematics, 29, Cambridge University Press, Cambridge, UK, 1990.
[17] W. Lück, *Assembly maps.* In: H. R. Miller (ed.), Handbook of homotopy theory, CRC Press/Chapman & Hall, Assembly Maps, Boca Raton, FL, 2019, pp. 853–892.
[18] P. Orlik and H. Terao, *Arrangements of hyperplanes*. Springer-Verlag, Berlin, 1992.
[19] G. Paolini and M. Salvetti, *Proof of the K* $(\pi, 1)$*conjecture for affine Artin groups*. Preprint, 2019. arXiv:1907.11795.
[20] S. K. Roushon, *The Farrell-Jones isomorphism conjecture for* 3*-manifold groups*. J. K-Theory **1**(2008), 49–82.
[21] S. K. Roushon, *The isomorphism conjecture for* 3*-manifold groups and K-theory of virtually poly-surface groups*. J. K-Theory **1**(2008), 83–93.
[22] S. K. Roushon, *Surgery groups of the fundamental groups of hyperplane arrangement complements*. Arch. Math. (Basel) **96**(2011) no. 5, 491–500.
[23] S. K. Roushon, *The isomorphism conjecture for groups with generalized free product structure*. Handbook of Group Actions, II, Higher Education Press and International Press, Beijing, 2014, pp. 77–119.
[24] J. Shaneson, *Wall's surgery obstruction groups for* $G \times \mathbb{Z}$. Ann. Math. **90**(1969), no. 2, 296–334.
[25] G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*. Canad. J. Math. **25**(1954), 274–304.
[26] W. P. Thurston, *Three-dimensional geometry and topology*. Mathematical Sciences Research Institute Notes, Berkeley, CA, 1991.
[27] C. Wegner, *The K-theoretic Farrell-Jones conjecture for* $CAT(0)$*-groups*. Proc. Amer. Math. Soc. **140**(2012), 779–793.

School of Mathematics, Tata Institute, Homi Bhabha Road, Mumbai 400005, India
e-mail: roushon@math.tifr.res.in
URL: http://www.math.tifr.res.in/~roushon

https://doi.org/10.4153/S0008414X2000036X Published online by Cambridge University Press