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Uniform approximation by universal series on arbitrary sets

BY VANGELIS STEFANOPOULOS

Department of Mathematics and Statistics, University of Cyprus,
1678 *Nicosia, Cyprus.*
e-mail: vstefan@ucy.ac.cy

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Abstract

By considering a tree-like decomposition of an arbitrary set we prove the existence of an associated series with the property that its partial sums approximate uniformly all elements in a relevant space of bounded functions. In a topological setting we show that these sums are dense in the space of continuous functions, hence in turn any Borel measurable function is the almost everywhere limit of an appropriate sequence of partial sums of the same series. The coefficients of the series may be chosen in c_0, or in a weighted ℓ^p with $1 < p < \infty$, but not in the corresponding weighted ℓ^1.

1. *Introduction*

Let X be a metrizable topological vector space over the real or complex numbers, let $(u_n)_{n=1}^{\infty}$ be a fixed sequence in X, and let $(a_n)_{n=1}^{\infty}$ be a sequence of scalars. By $\mathbb{N}$ we denote the set of positive integers.

Definition 1. A series $\sum_{k=1}^{\infty} a_k u_k$ is called a *universal series* in X if given any $x \in X$ there exists an increasing to infinity sequence $(\lambda_n)_{n=1}^{\infty}$ in $\mathbb{N}$ such that

$$\sum_{k=1}^{\lambda_n} a_k u_k \longrightarrow x, \quad \text{as } n \longrightarrow \infty,$$

in the topology of X, or equivalently we may say that the partial sums of the series are dense in X.

The notion of universality goes back, as far as we know, to 1914 where Fekete showed that there exists a power series $\sum_{n=1}^{\infty} a_n x^n$ on $[-1, 1]$ which approximates uniformly, along subsequences of partial sums, any continuous function f on $[-1, 1]$ satisfying $f(0) = 0$. In the course of all these years various types of universality were studied and several types of universal series were shown to exist. These include Menchoff's trigonometric series, Taylor series, Laurent series and so on. For the history of universal series we refer to the two excellent surveys [**2**] and [**4**].

Recently Nestoridis and Papadimitropoulos [**5**] presented an abstract theory on universal series in which a necessary and sufficient condition for the existence of such a series, in a quite general setting, is proved (c.f. Theorem 2). It turns out that in various cases of interest this condition is reasonably easily verified, thus allows the simplification of older proofs. It actually applies and does so in Fekete's result [**1**].

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Our point of departure is this abstract theory. One of the key ideas of universal series is that whatever can be approximated by finite linear combinations can be also approximated by partial sums of the series. This implies that we obtain the same approximation results either using the full series or its tail (c.f. Theorem 1). Following this principle we start with an arbitrary set E and we split it in subsets following a tree structure. Let $\phi_1, \phi_2, \ldots$ be an enumeration of the associated characteristic functions, then the uniform limits of finite linear combinations of the ϕ's, endowed with the supremum topology, form the topological vector space X. We establish the existence of a universal series

$$\sum_{k=1}^{\infty} a_k \phi_k(x),$$

with coefficients in an appropriate subspace A of real or complex sequences, that approximates uniformly, in the sense of the definition above, all elements in X.

Via now the abstract theory on universal series, mentioned above, a much stronger result is proved, namely that the set of sequences $(a_n)_{n=1}^{\infty}$ which generate universal series is a dense and G_δ set in A and contains all elements but the zero sequence of a dense vector subspace of A.

We prove the existence of such a series for various choices of A, namely for $A = c_0$, $A = \ell_w^p$, a weighted ℓ^p space relevant to the construction of X with $1 < p < \infty$, and for the whole space of sequences. Moreover we show that in the case where E is a compact space the partial sums of the universal series constructed are dense in the space of continuous functions on E. We also give an example in a familiar setting that exhibits the characteristics of our abstract construction.

This paper is organized as follows. In Section 2 we present, for self containment, the aspects of the abstract theory on universal series that are relevant to this work. In Section 3 the sequence of the ϕ_n's is constructed and the space X is defined. In Section 4 we show the existence of a universal series with coefficients in c_0. In Section 5 we define the ℓ_w^p space and we prove the existence of a series with this type of coefficients for $1 < p < \infty$, and the nonexistence for $p = 1$. In Section 6 and in a topological setting we show that the partial sums of any universal series constructed in Sections 4 and 5 with c_0 or ℓ_w^p coefficients approximate uniformly any continuous function or almost everywhere, with respect to any σ-finite Borel measure ν, any Borel measurable function. Finally in Section 7, by considering the dyadic decomposition of [0, 1], we give an example of a universal series that approximates along partial sums in the sense of uniform convergence, or almost everywhere convergence, or convergence in measure any function belonging to a relevant subclass of bounded functions on [0, 1].

2. *General theory*

Let (X, d) be a metric space, where $X = (X, +, \cdot)$ is a vector space over the field $\mathbb{F}$ of real or complex numbers, and d is a metric compatible with $+, \cdot$ and invariant under translations. Let $u = (u_n)_{n=1}^{\infty}$ be a fixed sequence of elements of X. By U_u we denote the class of sequences $a = (a_n)_{n=1}^{\infty}$ in $\mathbb{F}$ for which the partial sums $\sum_{k=1}^{n} a_k u_k$, $n = 1, 2, \ldots$ form a dense set in X. Thus each $a = (a_n)_{n=1}^{\infty} \in U_u$ generates an *unrestricted universal series*, provided that $U_u \neq \varnothing$.

The following existence result has been established in [**2**].

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THEOREM 1. *If X, u and U_u are as above the following statements are equivalent*:

(i) $U_u \neq \varnothing$;
(ii) *for every* $n = 1, 2, \ldots$ *the* $\operatorname{span}\{u_n, u_{n+1}, \ldots\}$ *is dense in* X;
(iii) U_u *is dense and* G_δ *in the space of sequences in* $\mathbb{F}$ *endowed with the Cartesian topology.*

Regarding now universal series with restrictions on the coefficients the following result is given in [**1**] and [**5**].

Let A be a vector subspace of the sequences in $\mathbb{F}$ and consider the metric space (A, d_A), where the metric d_A is compatible with $+, \cdot$ and invariant under translations. As it is usual we set $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots)$, ..., and assume that the following hold:

(A1) (A, d_A) is complete;
(A2) the projections $A \ni a \to a_n \in \mathbb{F}$ are continuous for all $n \in \mathbb{N}$;
(A3) $A \supset G = \{a = (a_n)_{n=1}^{\infty} : a_n \neq 0$ for only finitely many n's$\}$;
(A4) if $a = (a_n)_{n=1}^{\infty} \in A$ then $\sum_{k=1}^{n} a_k e_k \to a$, as $n \to \infty$.

THEOREM 2. *Let X, u, U_u and A be as above and let* 0 *be the zero sequence of A. The following statements are equivalent*:

(i) $U_u \cap A \neq \varnothing$;
(ii) *given* $x \in X$, $\epsilon > 0$, *and* $n \in \mathbb{N}$, *there exist* $N \geqslant n$ *and* $b_n, b_{n+1}, \ldots, b_N$ *in* $\mathbb{F}$ *such that*

$$d\left(\sum_{k=n}^{N} b_k u_k, x\right) < \epsilon, \qquad d_A\left(\sum_{k=n}^{N} b_k e_k, 0\right) < \epsilon;$$

(iii) *given* $x \in X$ *and* $\epsilon > 0$, *there exist* $N \geqslant 1$ *and* $b_1, b_2, \ldots, b_N$ *in* $\mathbb{F}$ *such that*

$$d\left(\sum_{k=1}^{N} b_k u_k, x\right) < \epsilon, \qquad d_A\left(\sum_{k=1}^{N} b_k e_k, 0\right) < \epsilon;$$

(iv) $U_u \cap A$ *is dense and* G_δ *in* A;
(v) $U_u \cap A$ *contains a dense vector subspace of* A *except* 0.

3. *The setting*

Let E be an arbitrary infinite set, and let $(m_n)_{n=1}^{\infty}$ be an arbitrary sequence of natural numbers. Split E into m_1 disjoint sets $E_1, E_2, \ldots, E_{m_1}$. Next split each E_l, $1 \leqslant l \leqslant m_1$, into m_2 disjoint sets $E_{l,1}, E_{l,2}, \ldots, E_{l,m_2}$. Continuing this way a tree of sets is formed with top E and roots extending to various levels. At this point we mention that some of these sets may be empty, therefore we allow the possibility that the number of descendants at some level of sets belonging to the same level is not necessarily the same. A typical set of the kth level may be denoted by $E_{l_1,\ldots,l_k}$ with $l_j \in \{1, 2, \ldots, m_j\}$ and $1 \leqslant j \leqslant k$. All these sets are countably many.

Definition 2. Let $(E_n)_{n=1}^{\infty}$ be an arbitrary but fixed enumeration of these sets and define $(\phi_n)_{n=1}^{\infty}$ to be the collection of the associated characteristic functions χ_{E_n}. Consider the space of bounded functions on E with values in $\mathbb{F}(= \mathbb{R}$ or $\mathbb{C})$ endowed with the supremum norm. We then define the vector space X by

$$X = \overline{\operatorname{span}\{\phi_1, \phi_2, \ldots\}},$$

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where the closure is taken in the supremum topology, and consider the metric space (X, d), where $d(f, g) := \|f - g\|_\infty = \sup_{x \in E} |f(x) - g(x)|$.

Each set E_n corresponds to an E_α, where α is a multi-index. By $[\alpha]$ we denote the length of α. Thus a set E_n belonging to the Nth level corresponds to some E_α with $\alpha = (l_1, \ldots, l_N)$, and $[\alpha] = N$. If $\alpha = (l_1, \ldots, l_k)$ is a multi-index by $\alpha l_{k+1} \cdots l_{k+p}$ we mean the index $(l_1, \ldots, l_{k+p})$. Observe that every set E_n can be expressed as a union of some or all of its descendants. More precisely if $E_n = E_\alpha$ and $[\alpha] = N$ we have the identity

$$E_\alpha = \bigcup_{l_1=1}^{m_{N+1}} E_{\alpha l_1} = \cdots = \bigcup_{l_1=1}^{m_{N+1}} \cdots \bigcup_{l_k=1}^{m_{N+k}} E_{\alpha l_1 \ldots l_k}. \tag{3·1}$$

If on the other hand E_n belongs to the Nth level and by $\phi_{k,j}^n$ we denote the characteristic function of a descendant of E_n in the $N + k$ level, we obtain

$$\phi_n = \sum_{j=1}^{m(N,k)} \phi_{k,j}^n, \qquad \text{with} \quad m(N, k) := \prod_{i=1}^{k} m_{N+i}. \tag{3·2}$$

LEMMA 1. *Given a set E_α with $[\alpha] = N$ and $k \in \mathbb{N}$ then*:

(i) *there exist finitely many disjoint sets $E_{\alpha\beta}$ with $[\beta] = k$ such that*

$$E_\alpha = \bigcup_{[\beta]=k} E_{\alpha\beta}; \tag{3·3}$$

(ii) *if ϕ_n is the characteristic function of E_α and M is a positive integer, then*

$$\phi_n = \frac{1}{M} \sum_{k=1}^{M} \sum_{j=1}^{m(N,k)} \phi_{k,j}^n, \tag{3·4}$$

where $m(N, k)$ is as in (3·2).

Proof. For proving (3·3) we take the union of all $E_{\alpha l_1 \ldots l_k}$ in (3·1), where $1 \leqslant l_i \leqslant m_{N+i}$, and $1 \leqslant i \leqslant k$. For (3·4) we consider the union of all $E_{\alpha l_1}, \ldots, E_{\alpha l_1 \ldots l_k}$ with $k = M$. Observing that in this case the set E_α is covered M times, the proof then follows from (3·2).

Remark 1. The proof of (3·4) is based on the identity

$$E_\alpha = \bigcup_{1 \leqslant [\beta] \leqslant k} E_{\alpha\beta},$$

i.e., the descendants of E_α belong to consecutive levels. It is true though that if S is a finite set of positive integers then

$$E_\alpha = \bigcup_{[\beta] \in S} E_{\alpha\beta},$$

hence (3·4) becomes

$$\phi_n = \frac{1}{|S|} \sum_{k \in S} \sum_{j=1}^{m(N,k)} \phi_{k,j}^n,$$

where $|S|$ denotes the cardinality of S.

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4. *The c_0 case*

We take here $A = c_0$ endowed with the supremum norm. We show that statement (3) of Theorem 2 is fulfilled. Thus in this setting we have the following:

LEMMA 2. *Let $f \in X$, and let $\epsilon > 0$, then there exist $M \in \mathbb{N}$ and scalars $b_1, b_2, \ldots, b_M$ such that*

$$\sup_{x \in E} \left| \sum_{j=1}^{M} b_j \phi_j(x) - f(x) \right| < \epsilon \tag{4·1}$$

$$\sup_{1 \leqslant j \leqslant M} |b_j| < \epsilon. \tag{4·2}$$

Notice that by the construction of X (4·1) is trivially satisfied. The main point of the lemma is that the coefficients can be chosen to be arbitrarily small.

Proof. As mentioned above we can always find a finite linear combination of ϕ_n's so that

$$\left| \sum_{i=1}^{n} c_i \phi_{l_i}(x) - f(x) \right| < \epsilon, \tag{4·3}$$

for all $x \in E$. Because of Lemma 1 we can take the sets E_{l_i}, $i = 1, 2, \ldots, n$ to be disjoint and belonging to the same level. By going to a higher level, if necessary, this level can be taken to be N so that $1/N < \epsilon$. Choose $M_0 \in \mathbb{N}$ so that $|c_i/M_0| < 1/N$, for each $i = 1, 2, \ldots, n$, and recall that

$$c_i \phi_{l_i} = \sum_{k=1}^{M_0} \sum_{r=1}^{m(N,k)} \frac{c_i}{M_0} \phi_{k,r}^{l_i},$$

via (3·4), where $m(N, k)$ is as in (3·2). Next consider the entire collection $\Phi = \{\phi_{k,r}^{l_i}\}$ where $1 \leqslant i \leqslant n$, $1 \leqslant k \leqslant M_0$, and $1 \leqslant r \leqslant m(N, k)$. Let M_1 be the higher index of ϕ in Φ so that $\Phi \subseteq \{\phi_1, \phi_2, \ldots, \phi_{M_1}\}$ and let $M = \max\{M_0, M_1\}$. For $j = 1, 2, \ldots, M$ now define

$$b_j = \begin{cases} 0, & \text{if } \phi_j \notin \Phi, \\ c_i/M, & \text{if } \phi_j = \phi_{k,r}^{l_i}. \end{cases} \tag{4·4}$$

Then clearly (4·1) is satisfied and from the definition of the b_j's and the choice of M we see

$$|b_j| \leqslant \frac{1}{N} < \epsilon,$$

by the choice of N. The proof is complete.

THEOREM 3. *There exists a sequence $(a_n)_{n=1}^{\infty}$ in c_0 with the following property: for every $f \in X$ there exists an increasing to infinity sequence $(\lambda_n)_{n=1}^{\infty}$ in $\mathbb{N}$ such that*

$$\sup_{x \in E} \left| \sum_{k=1}^{\lambda_n} a_k \phi_k(x) - f(x) \right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

Moreover the set of such sequences is G_δ and dense in c_0 and contains a dense vector subspace except the zero sequence.

Proof. The proof follows from Theorem 2 and Lemma 2.

A byproduct of Theorem 3, in the case $A = c_0$, in light of Theorem 2 is the following:

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COROLLARY 1. *The set of sequences $(a_n)_{n=1}^{\infty}$ for which the series $\sum a_n\phi_n$ is a universal series in X is a dense and G_δ set in the space of all sequences in $\mathbb{F}$, and contains a dense vector subspace except the zero sequence.*

5. *The weighted ℓ^p case*

Let E and E_n, $n=1,2,\ldots$, be as before and consider the measure space $(E,\mathfrak{M},\mu)$, where μ is an arbitrary positive measure so that $\mu(E)<\infty$. We require that $(E_n)_{n=1}^{\infty}\subset\mathfrak{M}$. For $1\leqslant p<\infty$ we take as A the space of all sequences $(a_k)_{k=1}^{\infty}$ for which

$$\sum_{k=1}^{\infty}|a_k|^p\mu(E_k)<\infty$$

with norm defined analogously. We denote this space by ℓ_w^p. We treat the cases $p>1$ and $p=1$ separately.

5·1. *The case $p>1$*

Again as in the c_0 case we prove the existence of a universal series by establishing that the assertion (3) of Theorem 2 is true.

LEMMA 3. *Let $f\in X$, and let $\epsilon>0$, then there exist $M\in\mathbb{N}$ and scalars $b_1,b_2,\ldots,b_M$ such that*

$$\sup_{x\in E}\left|\sum_{j=1}^{M}b_j\phi_j(x)-f(x)\right|<\epsilon \tag{5·1}$$

$$\left[\sum_{j=1}^{M}|b_j|^p\mu(E_j)\right]^{1/p}<\epsilon. \tag{5·2}$$

Proof. Choose N so that $1/N<\epsilon^p$ and proceed as in the proof of Lemma 2. Having established (4·3) we choose M_0 so that

$$\sum_{i=1}^{n}|c_i|^p\mu(E_{l_i})<\frac{M_0^{p-1}}{N} \tag{5·3}$$

and obtain finite sequences $(\phi_j)_{j=1}^{M}$ and $(b_j)_{j=1}^{M}$ with $M\geqslant M_0$ for which (5·1) holds, where the b_j's are as in (4·4). Then with the aid of Lemma 1 we compute

$$\begin{aligned}\sum_{j=1}^{M}|b_j|^p\mu(E_j)&=\sum_{j=1}^{M}\left|\frac{c_i}{M}\right|^p\sum_{k=1}^{M_0}\sum_{r=1}^{m(N,k)}\mu\big(E_{k,r}^{l_i}\big)\\&=\sum_{j=1}^{M}\left|\frac{c_i}{M}\right|^pM_0\mu\big(E_{l_i}\big)\\&\leqslant\sum_{j=1}^{M}\frac{|c_i|^p}{M_0^{p-1}}\mu\big(E_{l_i}\big),\end{aligned}$$

which via (5·3) and the choice of N proves (5·2) and completes the proof.

THEOREM 4. *Let $1<p<\infty$, then there exists a sequence $(a_n)_{n=1}^{\infty}$ in ℓ_w^p with the following property: for every $f\in X$ there exists an increasing to infinity sequence $(\lambda_n)_{n=1}^{\infty}$ in $\mathbb{N}$*

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such that

$$\sup_{x\in E}\left|\sum_{k=1}^{\lambda_n} a_k\phi_k(x) - f(x)\right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

Moreover the set of such sequences is G_δ and dense in ℓ^p_w and contains a dense vector subspace except the zero sequence.

Proof. It follows from Theorem 2 and Lemma 3.

5·2. *The case $p = 1$*

In this case we prove a negative result, namely:

PROPOSITION 1. *There exists no universal series*

$$\sum_{n=1}^{\infty} a_n\phi_n \tag{5·4}$$

with $(a_n)_{n=1}^{\infty} \in \ell^1_w$, i.e., satisfying

$$\sum_{n=1}^{\infty} |a_n|\mu(E_n) < \infty. \tag{5·5}$$

Proof. We argue by contradiction, so assume that there exists a sequence $(a_n)_{n=1}^{\infty}$ satisfying (5·5) for which (5·4) is a universal series. By the monotone convergence theorem we have

$$\int_E \sum_{n=1}^{\infty} |a_n|\phi_n(x)\, d\mu = \sum_{n=1}^{\infty} \int_E |a_n|\phi_n(x)\, d\mu = \sum_{n=1}^{\infty} |a_n|\mu(E_n)$$

which is finite by hypothesis. Then by Lebesgue's dominated convergence theorem one concludes that the series

$$\sum_{n=1}^{\infty} a_n\phi_n(x)$$

converges almost everywhere on E to an $L^1(\mu)$ function. If now x_0 is a point so that $\sum a_n\phi_n(x_0)$ converges, then the partial sums of this series form a Cauchy sequence. This contradicts though the fact that they are dense. This contradiction shows that no such series exists.

6. *Further approximations*

Let E and E_n, $n = 1, 2, \ldots$, be as before and consider a metric d_E on E so that the metric space (E, d_E) is compact. We require that diam $E_n \to 0$, as $n \to \infty$.

PROPOSITION 2. *There exists a sequence $(a_n)_{n=1}^{\infty}$ in c_0 with the following property: for every continuous function f on E there exists an increasing to infinity sequence $(\lambda_n)_{n=1}^{\infty}$ in $\mathbb{N}$ such that*

$$\sup_{x\in E}\left|\sum_{k=1}^{\lambda_n} a_k\phi_k(x) - f(x)\right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

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Proof. In light of Theorem 3 it is enough to show that the space X contains the continuous functions $C(E)$. Indeed let $f \in C(E)$ and let $\epsilon > 0$ be given. We can always write $E = \bigcup_{i=1}^{n} E_{l_i}$, where the E_{l_i}'s are on the same level, therefore are disjoint, and by uniform continuity and the assumption on the diameters of the E_n's chosen so that $|f(x) - f(y)| < \epsilon$, whenever $x, y \in E_{l_i}$. Let x_i be a fixed point in E_{l_i} and set $c_i = f(x_i)$. Then as in the proof of Theorem 3 we obtain

$$\left| \sum_{i=1}^{n} c_i \phi_{l_i}(x) - f(x) \right| < \epsilon \tag{6·1}$$

for all $x \in E$.

Remark 2. Let $1 < p < \infty$, then in Proposition 2 the sequence $(a_n)_{n=1}^{\infty}$ may be chosen to belong to ℓ_w^p, the space constructed in Section 5.

In addition to the above hypotheses on (E, d_E) consider now the measure space $(E, \mathfrak{B}, \nu)$, where $\mathfrak{B}$ is the Borel σ-algebra and ν is a σ-finite Borel measure.

Let $\sum_{n=1}^{\infty} a_n \phi_n$ be as in Proposition 2 then we have the following:

COROLLARY 2. *Let f be an ν-measurable function, then there exists an increasing to infinity sequence $(\lambda_n)_{n=1}^{\infty}$ in $\mathbb{N}$ such that*

$$\sum_{k=1}^{\lambda_n} a_k \phi_k \longrightarrow f, \quad \nu\text{-a.e.}, \quad \text{as } n \longrightarrow \infty. \tag{6·2}$$

Proof. By Lusin's theorem there exists a sequence of continuous functions $(g_n)_{n=1}^{\infty}$ that converges almost everywhere to f. Now via Proposition 6.1 for each n there exists a natural number λ_n such that

$$\sup_{x \in E} \left| \sum_{k=1}^{\lambda_n} a_k \phi_k(x) - g_n(x) \right| < \frac{1}{n}. \tag{6·3}$$

Then clearly $\sum_{k=1}^{\lambda_n} a_k \phi_k \to f$, a.e. in E, as $n \to \infty$.

7. *An example*

In this section we give a model example that illustrates several aspects of our previous abstract construction. The results, being transcriptions to this setting of the ones proved in previous sections, are presented without proofs.

We take $E = I = [0, 1]$ and consider all dyadic partitions of $[0, 1]$. Hence for each $n \in \mathbb{N}$ we define the intervals I_k^n with $k = 1, 2, \ldots, 2^n$ by

$$I_k^n = \begin{cases} [0,\ 1/2^n] & \text{if } k = 1, \\ ((k-1)/2^n,\ k/2^n] & \text{if } k = 2, 3, \ldots, 2^n. \end{cases} \tag{7·1}$$

Therefore for each $n = 1, 2, \ldots$ fixed the interval $[0, 1]$ is the union of the I_k^n's, with $1 \leqslant k \leqslant 2^n$. We say that the intervals I_k^n belong to the n – level.

Definition 3. Let $(I_n)_{n=1}^{\infty}$ be an enumeration of the collection of intervals defined in (7·1) and consider the sequence $(\phi_n)_{n=1}^{\infty}$, where $\phi_n = \chi_{I_n}$ is the characteristic function of I_n. By $|I_n|$ we denote the Lebesgue measure of I_n, i.e. the length of I_n. As X we take the closure in the uniform topology of $\operatorname{span}\{\phi_1, \phi_2, \ldots\}$ and endow it with the supremum norm.

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The space X is rich enough, contains in particular the continuous functions on I, as follows from Proposition 2, and each element of it is Lebesgue measurable.

Remark 3. Instead of the canonical dyadic partitions we used to construct the intervals I_n we could take other divisions. For example at each level we can take intervals of unequal length. In order though to have the continuous functions in X it is necessary that $|I_n| \to 0$ as $n \to \infty$. We could also take higher dimensional analogues, for example the unit cube of $\mathbb{R}^n$ and split it in dyadic cubes by taking cartesian products of the I_k^n's.

PROPOSITION 3. *Let $1 < p < \infty$, then there exist sequences $(a_n)_{n=1}^{\infty}$ in ℓ_w^p (respectively in c_0) with the following property: for every function f in X and in particular a continuous function on I there exists an increasing to infinity sequence $(\lambda_n)_{n=1}^{\infty}$ in $\mathbb{N}$ such that*

$$\sup_{x \in I} \left| \sum_{k=1}^{\lambda_n} a_k \phi_k(x) - f(x) \right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

COROLLARY 3. *Let f be a function in X and let $(a_n)_{n=1}^{\infty}$ be as in Proposition 3, then there exists an increasing to infinity sequence $(\lambda_n)_{n=1}^{\infty}$ in $\mathbb{N}$ such that*

$$\int_0^1 f(x)\,dx = \lim_{n \to \infty} \sum_{k=1}^{\lambda_n} a_k |I_k|. \tag{7·2}$$

Proof. The sequence S_{λ_n} of partial sums is a sequence of bounded measurable functions that converges uniformly to f on I. Since $|I| < \infty$, we have that

$$\int_0^1 f(x)\,dx = \lim_{n \to \infty} \int_0^1 S_{\lambda_n}(x)\,dx = \lim_{n \to \infty} \sum_{k=1}^{\lambda_n} a_k \int_0^1 \phi_k(x)\,dx,$$

from where (7·2) follows.

Let us now consider the measure space $(I, \mathfrak{B}, \mu)$, where $\mathfrak{B}$ is the Borel σ-algebra and μ is a finite Borel measure and let $\sum_{n=1}^{\infty} a_n \phi_n$ be as in Proposition 3, then we have:

COROLLARY 4. *Let f be an μ-measurable function and let $(a_n)_{n=1}^{\infty}$ be as in Proposition 3, then there exists an increasing to infinity sequence $(\lambda_n)_{n=1}^{\infty}$ in $\mathbb{N}$ such that*

$$\sum_{k=1}^{\lambda_n} a_k \phi_k \longrightarrow f, \quad \mu\text{-a.e., as } n \longrightarrow \infty.$$

Remark 4. Let $\mathcal{M}$ be the set of μ-measurable functions on I, where, as usual, two such functions are identified if they are μ-almost everywhere equal. On $\mathcal{M}$ define the metric

$$\rho(f, g) := \int_I \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}\,d\mu.$$

Then if $(f_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{M}$ it is known that $f_n \to f$ in measure on I if and only if $\rho(f_n, f) \to 0$, as $n \to \infty$. Since in a set of finite measure almost everywhere convergence implies convergence in measure a consequence of Corollary 4 is that if $f \in \mathcal{M}$, then there exists an increasing to infinity sequence $(\lambda_n)_{n=1}^{\infty}$ in $\mathbb{N}$ such that

$$\sum_{k=1}^{\lambda_n} a_k \phi_k \longrightarrow f \quad \text{in measure, as } n \longrightarrow \infty,$$

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or equivalently

$$\rho\left(\sum_{k=1}^{\lambda_n} a_k\phi_k,\, f\right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,$$

that is $\sum_{k=1}^{\infty} a_k\phi_k$ is a universal series in $(\mathcal{M}, \rho)$. On the other hand convergence in measure of a sequence implies almost everywhere convergence of a subsequence. Therefore the notion of universality in the sense of Corollary 4 coincides with that in $(\mathcal{M}, \rho)$.

8. *Concluding remarks*

Our result combined with those of [**1**] and [**5**] show that the partial sums $\sum_{j=1}^{N} a_j\phi_j(x)$ may be replaced by Cesàro means of them, see [**7**], and still have similar approximation results.

On the other hand combining also with a recent result of [**3**] we see that in the case where the coefficients are unrestricted, e.g. in $\mathbb{R}$ or $\mathbb{C}$ then the partial sums $\sum_{j=1}^{\lambda_n} a_j\phi_j(x)$ may be replaced by averaged sums of the type

$$\frac{1}{\lambda_n^{\tau}} \sum_{j=1}^{\lambda_n} a_j\phi_j(x),$$

where $\tau > 0$.

Finally our proofs can be slightly modified so to obtain similar results in the case where the coefficients are in $\mathbb{R}^N$. This gives the existence of N universal series where the approximations are realized simultaneously with the same approximative sequence $(\lambda_n)_{n=1}^{\infty}$, see also [**6**]. Furthermore, $\mathbb{R}^N$ may be replaced by the space of all real sequences, but there the simultaneous approximation will be valid only for arbitrary finite sets of coordinates [**6**].

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