THE REVIEW OF SYMBOLIC LOGIC
Volume 2, Number 1, March 2009

MATRIX-BASED LOGIC FOR APPLICATION IN PHYSICS

PAUL WEINGARTNER
Department Of Philosophy, University Of Salzburg

Abstract. The paper offers a matrix-based logic (relevant matrix quantum physics) for propositions which seems suitable as an underlying logic for empirical sciences and especially for quantum physics. This logic is motivated by two criteria which serve to clean derivations of classical logic from superfluous redundancies and uninformative complexities. It distinguishes those valid derivations (inferences) of classical logic which contain superfluous redundancies and complexities and are in this sense "irrelevant" from those which are "relevant" or "nonredundant" in the sense of allowing only the most informative consequences in the derivations. The latter derivations are strictly valid in *RMQ*, whereas the former are only materially valid. *RMQ* is a decidable matrix calculus which possesses a semantics and has the finite model property. It is shown in the paper how *RMQ* by its strictly valid derivations can avoid the difficulties with commensurability, distributivity, and Bell's inequalities when it is applied to quantum physics.

§1. Introduction. According to the usual understanding, "quantum logics" are algebraic studies of the lattice of subspaces of a Hilbert space and its extensions. These kinds of logics have been constructed to solve a number of problems which emerge when classical logic (*CL*) is applied to modern physics, especially to quantum mechanics.[1] The respective proposals of the 70s however did not lead to a lattice-theoretic characterization of the class of all lattices of subspaces of a Hilbert space, but permitted an inflationary number of nonclassical Hilbert spaces. A solution for a unique restriction was then discovered by Solèr[2] which leads exactly to the three classical Hilbert spaces according to Piron. The respectively revised quantum logics (*QL*) offered a solution to several problems, in particular to the problem of commensurability and the problem of distributivity. On the other hand the usual systems of *QL* do not lead to a solution for the problems with Bell's inequalities: to use a system of *QL* as the underlying logic for the probability distributions of Bell's inequalities[3] does not lead to the classical probability distributions, since it is too weak. The systems of *QL* which were developed later concentrated on "unsharp systems" in the sense of considering unsharp properties and unsharp elementary propositions which are not value definite.[4] They are intended to handle unsharp observables which are understood to be objectively decided and thus to offer some solution of the problem of objectification. Although such unsharp systems of *QL* have several adequate applications, it turned out that they were not able to solve the quantum mechanical measurement problem so far.

Received: October 24, 2008

[1] For a survey see Dalla Chiara & Giuntini (2001).

[2] Solèr (1995). Compare Dalla Chiara *et al.* (2004, p. 72ff).

[3] Compare Mittelstaedt (1998, p. 74ff) for a detailed analysis.

[4] For a survey see Dalla Chiara *et al.* (2004).

© 2009 Association for Symbolic Logic

doi:10.1017/S1755020309090169

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

The logical system (*RMQ*) proposed here is not a quantum logic (*QL*) in the usual sense mentioned above. It is also not an unsharp system of *QL*, since in *RMQ* every proposition has a definite (truth)value and bivalence is generally valid (though this principle could be relaxed, cf. Section 4.4.2.). *RMQ* is however a kind of general logic which can serve as an underlying basis for the application in science and particularly quantum physics. It takes its logical laws from classical logic (*CL*) but purifies *CL* from redundancies and uninformative complexities. Moreover, it has a semantics represented by finite matrices.

The purpose of this paper is to show that this decidable matrix-based logic for propositions (called *RMQ*) can be used as an underlying logic for quantum physics which offers a way to avoid the difficulties with commensurability, distributivity, and Bell's inequalities. Moreover, this logic does not deviate essentially from classical logic since it rules out only redundancies and complexities in derivations of *CL*. However these redundancies and complexities contained in *CL* show that *CL* is too rich and makes too strong logical and ontological requirements compared with a logic (like *RMQ*) which is suitable for physics (especially for quantum physics). How it restricts *CL* is described in Sections 1.2. and 2. The logic *RMQ* is described in Sections 3 and 4. In Sections 4.3.1., 4.3.2., and 4.4.1., it is shown, together with the strictly valid theorems of Section 4.4., how *RMQ* avoids the problems of commensurability, distributivity, and Bell's inequalities by blocking their underlying logical assumptions (i.e., showing that they are too strong and not necessarily valid). Section 5 gives a comparison with other logical systems including systems of *QL*. Section 6 discusses the modal properties of *RMQ*. In Section 1.1. we start with a short description of the three problems mentioned above.

1.1. Problems.

(1) If (under conditions r) proposition p describes the physical state P (that the position of a particle has a certain precise value) and (under conditions r) the proposition q describes the physical state Q (that the momentum of that particle has a certain precise value), then it is not the case that (under conditions r) the conjunction $p \wedge q$ describes a measurable magnitude at all. But according to classical logic it should, since the domain of meaningful propositions $p, q, r, \ldots$ is truth functionally closed under the usual connectives and thus under the conjunction $\wedge$.[5] Thus *CL* presupposes that—under any meaningful condition—arbitrary pairs of states described by a pair of propositions are *commensurable.* The corresponding principle in the underlying classical propositional logic is this:

$$[(r \rightarrow p) \wedge (r \rightarrow q)] \Rightarrow [r \rightarrow (p \wedge q)]$$

(2) Assume a particle in a box possessing position (right R or left L) and momentum (even state E or odd state E'). Then the respective instance of the *law of distribution*

$$E \cap (L \cup R) = ((E \cap L) \cup (E \cap R))$$

is violated as has been shown already by Birkhoff and von Neumann.[6] Since $L \cup R$ gets value 1, the left side reduces to E; but $E \cap L$ and $E \cap R$ get value 0, such that

[5] This problem of universal logical closure is also stressed as an important problem by Dalla Chiara & Guintini (2001, p. 59) in connection with unsharp *QL*. Observe that this difficulty is not specific to physics: if p and q describe human actions, it does not follow that $p \wedge q$ describes a human action too. Similarly for states of animal behavior. See Weingartner (2004b, p. 234f).

[6] Birkhoff and von Neumann (1936, p. 831).

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

the right side reduces to 0. The corresponding principle in the underlying classical propositional calculus is this:

$$[p \wedge (q \vee r)] \Leftrightarrow [(p \wedge q) \vee (p \wedge r)]$$

(3) Consider one form of *Bell's inequalities*: Those pairs (of particles on which measurement is done) of which one member possesses property A and the other property B are included in the union of those pairs of which one member possesses A and the other C and of those pairs of which one member possesses $B$and the other non-C. The respective set theoretical principle (based on CL) is:

$$A \cap B \subseteq [(A \cap C) \cup (B \cap -C)]$$

It is well-known that this inequality (set theoretical theorem of inclusion, which underlies the classical probability distributions), is satisfied for all measurements where QM is not involved, but violated for some experiments in QM.[7] The corresponding principle in the underlying classical propositional logic is this:

$$(p \wedge r) \Rightarrow [(p \wedge q) \vee (r \wedge \neg q)]$$

1.2. Conjecture for a solution. Examples of that sort and similar difficulties coming up when CL is applied to other domains[8] lead to the conjecture CJ that there must be some properties of CL which are the common cause for these difficulties. More accurately, the conjecture accuses that subset of valid inferences $A \vdash \alpha$ or valid implications $A \Rightarrow \alpha$, which satisfy one or both of the following properties $P1$ and $P2$ of CL:

$P1$ Parts of α are replaceable by arbitrary parts of the same category *salva validitate* of the inference $A \vdash \alpha$ or $A \Rightarrow \alpha$; where "same category" means: propositional variables by (other) propositional variables, monadic predicates by monadic predicates, two-place predicates by two-place predicates ..., and so forth, and where "arbitrary part" means "arbitrary wff" (see below: RC).

$P2$ Parts of α are reducible to simpler (and usually shorter) parts or can be split up into conjuncts *salva validitate* of the inference $A \vdash \alpha$ or $A \Rightarrow \alpha$ and by preserving the logical content of (the conclusion) α.

Properties $P1$ and $P2$—represented more accurately as criteria RC and RD below—are possessed by valid inferences or implications which have *redundant* or *irrelevant* parts in the described sense. This can be grasped more easily from the following examples:

Examples of $P1$ are: $p \vdash (q \rightarrow p)$; $\neg p \vdash (p \rightarrow q)$; $p \vdash (p \vee q)$. Or the respective valid implicational formulas: $p \Rightarrow (q \rightarrow p)$, $\neg p \Rightarrow (p \rightarrow q)$, $p \Rightarrow (p \vee q)$, $(p \wedge q) \Rightarrow ((p \wedge r) \vee (q \wedge \neg r))$ (where $\Rightarrow$ stands for $\vdash (\ldots \rightarrow ___)$). Here q or $r$can be replaced at one or at both of its occurrences by any arbitrary propositional variable (also by its own negation) salva validitate and is therefore redundant or irrelevant in this sense.

Examples of $P2$ are: $\neg\neg p$; $p \wedge p$; $p \vee p$; $p \vee (q \wedge r)$; $(p \wedge q) \vee (p \wedge r)$.

Here we may reduce to: p; p; p; $(p \vee q) \wedge (p \vee r)$; $p \wedge (q \vee r)$.

[7] Compare Bell (1987, p. 147f).

[8] Such other domains are the theory of explanation and confirmation of scientific theories, of law statements, of approximation to truth, of Deontic logic, and so forth.

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

These properties concern both, the propositional logic of *CL* and the first- order predicate logic of *CL* (*PL*1), although the most important part are the inferences (derivations) of classical two-valued propositional logic, namely classical propositional calculus (abbreviated as *CPC*). This is in agreement with the experience of Dalla Chiara and Giuntini concerning quantum logics: "the most significant logical and metalogical pecularities of Quantum Logic arise at the sentential level."[9] The solution, that is, the confirmation of the mentioned conjecture *CJ* given in this paper, needs only *CPC*. The conjecture can now be formulated thus:

C J The above-mentioned difficulties coming up when *CPC* is applied to modern physics and to other domains in the empirical sciences can be avoided when *CPC* is restricted to *CPC**, which does not contain valid inferences or valid implications satisfying *P*1 or *P*2. In other words, the valid inferences $A \vdash \alpha$ and implications $A \Rightarrow \alpha$ of *CPC** do not contain replaceable or reducible parts in their consequence α.

Let us call that subset of theorems of *CPC*, which does not contain replaceable parts *P*1* and the respective one which does not contain reducible parts *P*2*.

*P*1* is however not a real restriction of *CPC*, since it can be proved that the set *S* of all consequences *s* of (an arbitrary premise or set of premises) *A* according to *CPC* is logically equivalent (according to *CPC*) to the set *S** of all (nonreplaceable) consequences *s** of *A* according to *P*1*.[10]

*P*2* is not a real restriction either w.r.t. redundant repetitions or double negations, but it is a kind of restriction w.r.t. the laws of distribution (which hold only in one direction) and in general w.r.t. the reduction of disjuncts (disjunctive parts) to conjuncts (conjunctive parts) wherever it is possible. But since the produced conjuncts are according to *CL* logically equivalent to the unreduced formula, the logical content is classically preserved. However, none of the important traditional rules (theorems) of inference (derivation) of *CPC* is affected by this restriction (see below Section 4.4., Theorems (1)–(13), (34), and (35)).

1.3. The system RMQ confirms conjecture CJ. In this paper the above conjecture *CJ* will be confirmed in detail as follows: First (Section 2), a more precise definition of the two criteria *RC* (replacement) and *RD* (reduction) will be given, which produce *P*1* and *P*2*. And it will be shown how *RC* and *RD* avoid the above discussed difficulties by ruling out as redundant or uninformative or irrelevant those principles of *CL* (*CPC*) which are responsible for them.

The system *CPC** (i.e., the system obtained by filtering *CPC* with *RC* and *RD*) has features which render it difficult to be applicable in a transparent way. This is so for the following reasons: *CPC** is not closed under substitution and in the application of *RC* and *RD* it is essential to distinguish a propositional variable from its concrete occurrences (in a formula). Consequently it is rather complicated to construct a semantics[11] for *CPC**

---

[9] Dalla Chiara & Giuntini (2001, p. 50).

[10] For the proof see Schurz & Weingartner (1987).

[11] A semantics for the system resulting from *CPC* when *RC* is applied (to *CPC*) was constructed by Schurz (1991b, section 8). This paper however contains a mistake in section 4, proposition 5: This result is correct if the criterion applied to *CPC* (no variable in the conclusion which is not in the premises) is applied to every $\rightarrow$ in the valid formula (as in Dunn's reconstruction of Parry's system of analytical implication). But it is not correct if the Aristotelian relevance criterion is applied only to the main connective $\rightarrow$ representing the inference relation as it appears in

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

(whereas *CPC* has a simple semantics with two-valued matrices, i.e., truth tables). Therefore, in Section 3 the system *RMQ* is presented, which approximates *RC* and *RD* (*CPC**) and which has the following properties:

(1) *RMQ* is a finite matrix system and so it contains its own semantics and is decidable like *CPC* or a Boolean algebra.
(2) *RMQ* satisfies *RC* with few exceptions, for example, that inferences (derivations) with contradictory premises behave like in *CPC*.
(3) *RMQ* satisfies *RD*, except that harmless equivalences like $p \Leftrightarrow \neg\neg p$, $p \Leftrightarrow (p \wedge p)$, and $p \Leftrightarrow (p \vee p)$ hold strictly and are reducible whenever needed.
(4) *RMQ* contains all theorems of *CPC*, but those which lead to difficulties because of containing redundancies or making too strong assumptions, hold only materially in *RMQ* (cf. Section 4.2. below), whereas the others hold strictly or with necessity in *RMQ* (cf. Section 4.4. below).
(5) *RMQ* is closed under transitivity of implication, under modus ponens and under equivalence substitution, whereas *RC* and *RD* do not satisfy these closure conditions (cf. Section 3.3.2.(d)–(f)).
(6) *RMQ* avoids the above-mentioned difficulties (cf. Section 1.1.) and thus conjecture *CJ* is confirmed by *RMQ*.
(7) *RMQ* has properties of a modal system which approximates the modal system *T* (of Feys).

In Section 4 the main theorems of *RMQ* are given. They will be separated into two parts: first into those which hold only materially in *RMQ* and of which many give rise to difficulties and paradoxes (cf. Section 4.2.); second into those which hold strictly (necessarily) and avoid the well-known difficulties and paradoxes when they are applied to empirical science (cf. Section 4.4.).

In Section 5 we shall make some comparison between *RMQ* and other systems of *QL*. In Section 6 the modal properties of *RMQ* together with some theorems are given.

§2. Replacement and reduction criterion. A very general criterion which avoids replaceable and reducible parts has been given elsewhere.[12] The gist of the criterion to avoid parts (in the conclusion) which are replaceable by arbitrary parts (replacement criterion, *RC*) is as follows:

RC α is a relevant (nonredundant) consequence of A iff $A \vdash \alpha$ or $A \Rightarrow \alpha$ (according to *CPC*) and it is not the case that a propositional variable is replaceable in α on some of its occurrences by any other propositional variable salva validate of $A \vdash \alpha$ or $A \Rightarrow \alpha$.[13]

Aristotle's syllogistics and as it is interpreted in Weingartner (1985). In this case, as shown in Weingartner (1985) the resulting logic is representable by finite matrices. This mistake however does not affect section 8 where Schurz constructs his semantics for *RC*.

12 Compare Weingartner (2000b) and Weingartner (2000a), chapter 9 and Weingartner (2004b). This criterion originates in Schurz & Weingartner (1987). Compare further Schurz (1991a). In these papers the replacement criterion is more generally formulated for *PL*1 (predicate logic of first order) including *CPC* (propositional variables are interpreted as zero-type predicates).

13 The *RC* defined in the papers referred to in Note 12 contains an additional clause to avoid redundant identity formulas. This is not needed here since the solution proposed here needs only propositional logic.

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

In order to avoid misunderstanding we might answer the following questions: (1) What does the expression "by any other propositional variable" mean: it means an arbitrary other variable which is a wff of the system of propositional logic. Since any propositional variable is an atomic wff we could also say: by an arbitrary wff. But *RC* could also be formulated by using just 'wff' instead of 'atomic wff' (or propositional variable). This has been proved in Schurz & Weingartner (1987).

(2) What does it mean "on some of its occurrences"? This presupposes a necessary distinction between "propositional variable (wff)" and "particular occurrence of a propositional variable (wff)": Thus in $p \Rightarrow (p \vee q)$ the variable 'q' has one occurrence whereas in $p \Rightarrow [(p \vee q) \wedge (q \vee \neg q)]$ it has three occurrences. And the variable 'q' can be replaced salva validitate by an arbitrary other variable in both valid formulas either on one of its occurrences or on all the three of its occurrences (uniformly, i.e., on all three occurrences by the same variable).

It is worth mentioning that already Aristotle's **syllogistics** contains a relevance criterion which is weaker than the replacement criterion (applied to predicates), but which is contained in it: Every correct syllogistic inference obeys the principle: there are no (new) predicates in the conclusion (consequence) which are not contained in the premises. Interpreted in propositional logic the Aristotelean criterion reads thus: In a correct inference there is no propositional variable in the conclusion which does not already occur in the premises. This criterion is contained in *RC*. As has been stated above (Section 1.2.) the result $P1*$ of the application of the replacement criterion *RC* to *CPC* (classical two-valued propositional calculus) is not a reduction in which one looses informative content in the consequence class. It has been proved that $P1*$ is logically equivalent (i.e., *CPC* equivalent) to the full consequence class of *CPC*. However $P1*$ is of course not *RC* equivalent with the full consequence class of *CPC*.

RD α is a reduced consequence of *A iff*

(a) α is a relevant (*RC*-) consequence of A.

(b) α is not logically equivalent to a conjunction $\alpha_1 \wedge \ldots \wedge \alpha_n$ $(n \geq 1)$ in which α_i is shorter than α.

The gist of the criterion to avoid reducible parts (reduction criterion) is as follows:

(1) It reduces repetitions.

(2) It reduces double negations.

(3) It splits complex *wff*s into smallest conjuncts.

(4) The process of splitting up is asymmetric in the sense that it leads from disjuncts to conjuncts (as the ultimate relevant consequence elements) but not back. However because *RD* splits up it does not allow building up arbitrarily conjunctions from separated parts.

2.1. RC rules out the following theorems which give rise to difficulties and paradoxes. (1) Ex falso quodlibet:[14] $\neg p \Rightarrow (p \rightarrow q)$; $\neg p \vdash (p \rightarrow q)$ where '$\Rightarrow$' stands for '$\vdash (\ldots \rightarrow ___)$'

Culprit for: Carnap's Paradox of Disposition Predicates, Paradox of Derived Obligation, Paradox of Commitment.

[14] These and other paradoxes are discussed in detail in Weingartner & Schurz (1986) and Weingartner (2001) and especially concerning versimilitude in Schurz & Weingartner (2008).

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

(2) Principles of addition: $p \Rightarrow (p \vee q)$

Culprit for: Hesse's Confirmation Paradox, Goodman's Paradox, Ross's Paradox, Paradox of Free Choice, Paradox of Versimilitude, Paradox of Irrelevant Explanandum Components.

(3) Adding premise:

$p \Rightarrow (q \rightarrow p)$

Culprit for: Assumption of Commensurability (cf. below), Paradox of Total Theoretical Explanation, Paradox of Irrelevant Law Specification.

(4) Assumption of commensurability:

$p \Leftrightarrow [(p \wedge q) \vee (p \wedge \neg q)]$. The direction $\Rightarrow$ is ruled out.

(5) Logical assumption of *CPC* underlying Bell's inequalities:

$(p \wedge r) \Rightarrow [(p \wedge q) \vee (r \wedge \neg q)]$ (cf. Problem (3) of Section 1.1.)

It is easily seen that according to the replacement criterion all the five *CPC* valid principles are irrelevant because 'q' can be replaced (on one or two occurrences) by any arbitrary variable (standing for any proposition, simple, or complex) salva validitate (veritate).

Similarly, in *RMQ* these principles are only materially valid and do not hold with necessity (L); that is, instead of $\Rightarrow$ only $\rightarrow$ holds.

The strict implication of the system *RMQ* behaves in a similar way as the quantum logical conditional introduced by Mittelstaedt and Finch[15] at least concerning several logical principles like the laws of distribution or the principles which presuppose commensurability. Moreover the following five principles listed as not valid for the quantum logical conditional by Dalla Chiara and Giuntini[16] are also strictly invalid in *RMQ*:

(1) $p \rightarrow (q \rightarrow p)$ compare Section 4.2.(9) below

(2) $p \rightarrow (q \rightarrow r) \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$

(3) $(p \rightarrow q) \rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow r)]$

(4) $[(p \wedge q) \rightarrow r] \rightarrow [p \rightarrow (q \rightarrow r)]$ compare Section 4.4.(32) and (33)

(5) $[p \rightarrow (q \rightarrow r)] \rightarrow [q \rightarrow (p \rightarrow r)]$

Moreover the strict implication $\Rightarrow$ of *RMQ* has a weak nonmonotonic behavior. Thus

$(p \Rightarrow q) \rightarrow [(p \wedge r) \Rightarrow q]$

holds only materially, but is strictly invalid. Compare also Principles (9), (10), and (11) of Section 4.2. which do not hold strictly either.

2.2. RD rules out the following theorems which give rise to redundancies, difficulties, and paradoxes. (1) Reduction of redundancies.

(1a) $p \Rightarrow p \wedge p \wedge \ldots$ (1b) $p \Rightarrow p \vee p \vee \ldots$

RD allows only the opposite arrows $\Leftarrow$.

(2) Reduction to conjuncts as most informative consequence elements. The following directions which lead from conjuncts to disjuncts or from separated parts to conjuncts are ruled out: Only the opposite directions $\Leftarrow$ (not listed here) are permitted by *RD*.

(2a) $[p \wedge (q \vee \neg q)] \Rightarrow [(p \wedge q) \vee (p \wedge \neg q)]$

(2b) $[p \wedge (q \vee r)] \Rightarrow [(p \wedge q) \vee (p \wedge r)]$

---

[15] Compare Mittelstaedt (1972), Mittelstaedt (1978, p. 37ff); Finch (1970).

[16] Dalla Chiara & Giuntini (2001, p. 25) and Dalla Chiara *et al.* (2004, p. 166f).

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

(2c) $[(p \vee q) \wedge (p \vee r)] \Rightarrow [p \vee (q \wedge r)]$

(2d) $[(r \rightarrow p) \wedge (r \rightarrow q)] \Rightarrow [r \rightarrow (p \wedge q)]$ (cf. Problem (1) of Section 1.1.)

Although the left parts like $p \wedge (q \vee r)$, $(p \vee q) \wedge (p \vee r)$ are not yet reduced consequences by *RD*, they are first steps which can be easily turned into reduced consequences by splitting up the conjunctions into their parts.

Or in other words: the right direction of the distribution laws is that direction which leads to a consequent (conclusion) which is a conjunction of reduced consequences.

Consequently the problematic distribution laws are ruled out, whereas the acceptable ones (leading to conjuncts) are permitted; they hold necessarily in *RMQ* (cf. Section 4.4. (22)–(26) below)

Also the system **R** of relevance logic allows the distribution laws in both directions and is therefore not suitable for the application to physics.

§3. The system *RMQ*. It was one of the purposes when constructing *RMQ*, to find a semantics with the help of finite matrices (truth tables), which approximates the effects of the relevance restriction of the replacement (*RC*) and reduction (*RD*) criteria in such a way that what is classically valid but redundant or irrelevant or leading to disjuncts by those criteria, is materially valid but strictly invalid in *RMQ*. This was indeed achieved by *RMQ* as one can see from the theorems listed in Section 4 below. Thus *RMQ* is a consistent model which on the one hand contains all the main principles of Aristotelean (classical two-valued) propositional logic as theorems which are valid necessarily (strictly) (cf. Section 4.4.), whereas it excludes from the necessarily valid theorems those principles which are violated in the applications to quantum physics (*QP*) and in other domains of empirical application

Deductive systems of propositional logic can be constructed in two ways. In the first way by the so-called *axiomatic method*: a usually finite set A (the axioms) are given and the set *Cn(A)*—the set of the logical consequences of $A-$ is formed. In the second way by the so-called *matrix method*: a matrix is given as an n-tupel consisting of two disjoint sets T and F of truth values (or more than two in many valued systems which contain some other values like *indefinite* between T and F) and usually at least two operations, like $\rightarrow$ and $\neg$. *RMQ* does not contain values between T and F, just more than one (three) for T and more than one (three) for F. Such a matrix is called *normal* (according to Tarski)[17] if the formulas $x \in T$ and $y \in F$ always imply $(x \rightarrow y) \in F$. It is easily seen from the matrix (truth table) for $\rightarrow$ below that any of the values for truth (1 or 2 or 3) implying any of the values for falsity (4 or 5 or 6) result in values for falsity (4 or 5 or 6). That means that the matrix system *RMQ* is *normal* in this sense.

3.1. Definition of the system RMQ. The system *RMQ* can be defined as the set of all formulas (*wff*s) which are satisfied by the matrix $M =< T, F, \neg, \vee, \wedge, \rightarrow, L >$, where $T = \{1, 2, 3\}$, $F = \{4, 5, 6\}$, and the operations $\neg, \vee, \wedge, \rightarrow, L$ are defined as follows:[18]

[17] Compare Tarski (1956, p. 40f).

[18] Where "a formula (set of formulas) is satisfied by the matrix M" is understood in Tarski's sense. Compare Tarski (1956, p. 41).

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

p	$\neg p$	$p \vee q$	1	2	3	4	5	6
1	6	1	1	1	1	1	1	1
2	5	2	1	2	2	2	1	2
3	4	3	1	2	3	1	3	3
4	3	4	1	2	1	4	4	5
5	2	5	1	1	3	4	5	5
6	1	6	1	2	3	5	5	6

$p \wedge q$	1	2	3	4	5	6	$p \rightarrow q$	1	2	3	4	5	6	Lp
1	1	2	3	4	5	6	1	1	2	3	5	5	6	1
2	2	2	3	4	6	6	2	1	1	3	5	5	5	3
3	3	3	3	6	5	6	3	1	2	1	4	5	5	6
4	4	4	6	4	5	6	4	1	2	3	1	3	3	6
5	5	6	5	5	5	6	5	1	2	2	2	1	2	6
6	6	6	6	6	6	6	6	1	1	1	1	1	1	6

$$p \leftrightarrow q = df\,[(p \rightarrow q) \wedge (q \rightarrow p)]$$

RMQ obeys the usual interdefinability between necessity (L) and possibility (M): $Mp = df \neg L \neg p$. Hence the matrices for all the seven positive modalities in *RMQ* are as follows:

p	$\neg p$	LLp	Lp	MLp	p	LMp	Mp	MMp
1	6	1	1	1	1	1	1	1
2	5	6	3	1	2	1	1	1
3	4	6	6	6	3	1	1	1
4	3	6	6	6	4	1	1	1
5	2	6	6	6	5	6	4	1
6	1	6	6	6	6	6	6	6

As it follows from the definition, every well-formed formula (*wff*) of *RMQ* is unambiguously determined by a particular matrix, according to the definition in Section 3.1., possessing either 6 or 36 or 216 ... etc (in general 6^n, $n = 1, 2, \ldots$) values. And any such particular matrix represents some well-formed formula (wff) of the system *RMQ*.[19]

For a comparison observe that the usual two-valued classical propositional calculus (*CPC*) can be analoguously defined as the set of all formulas (wffs) which are satisfied by the matrix $M2 =<\mathrm{T}, \mathrm{F}, \neg, \rightarrow>$ where T={1}, F={6}, where the operations $\neg$ and $\rightarrow$ are defined as follows:

[19] A similar system which differs slightly w.r.t. matrices of $\vee$ and $\rightarrow$, but agrees on $\neg$, $\wedge$ and modalities has been published as (Weingartner, 1968). This modal logic contains all theorems of *CL* in such a way that they hold necessarily (i.e., with $cv = 2$). Compare Section 6 below.

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

p	$\neg p$
1	6
6	1

$p \rightarrow q$	1	6
1	1	6
6	1	1

The other operations $\wedge$ and $\vee$ can be defined with the help of $\rightarrow$ and $\neg$ in such a way that the propositions in the definiendum and in the definiens have identical matrices; that is, $p \vee q$ has the same matrix as $\neg p \rightarrow q$. In *RMQ* however the operations $\wedge$ and $\vee$ are definable with the help of $\rightarrow$ and $\neg$ only via material equivalence, not via strict equivalence; that is, the respective propositions do not have completely identical matrices.

A further important point to observe is that both, the matrix $M2$ (of *CPC*) and the matrix M (of *RMQ*), define not only the well-formed formulas (wffs) of *CPC* or *RMQ* but also the validity of these wffs respectively. Because a formula of *CPC* is valid just if its matrix contains only values 1 and a formula of *RMQ* is materially or strictly valid just if its matrix contains values ≤ 3 or ≤ 2 respectively (see Section 3.2.).

Observe further that in a representation of *CPC* by matrices (truth tables), that is, by $M2$, $A \vdash B$ coincides with (valid) $A \rightarrow B$; and for the representation of *CPC* by *RMQ*, $A \vdash B$ coincides with (valid) $A \rightarrow B$ (material implication) since all valid formulas of *CPC* are materially valid in *RMQ* (see Section 3.3.). Further those formulas $A \vdash B$ of *CPC* which approximate *RC* and *RD* are represented in *RMQ* by (valid) $A \Rightarrow B$ (see Section 3.3.1. below).

3.2. Characteristic value of a matrix (representing a wff) and validity. The highest value which occurs in a matrix is called its characterstic value (*cv*). It defines the validity of a formula in the following ways: If the $cv = 3$, the respective (*wff*) is materially valid, if $cv = 2$, it is strictly (or necessarily) valid (i.e., valid with *L*), if $cv = 1$, it is strongly valid (i.e., valid with *LL*). If the $cv \geq 4$, and if the matrix contains at least one of the truth values 1, 2, or 3, then the respective formula (*wff*) is contingent; it is materially logically false if all values are ≥ 4, strictly logically false if all values are ≥ 5, strongly logically false if all values are 6.

Examples: $p \wedge \neg p$ is strongly logically false and $\neg(p \wedge \neg p)$ is strongly logically true (or strongly valid). Modus ponens is strictly valid (strictly logically true) and Frege's law $p \rightarrow (q \rightarrow p)$ is materially valid (materially logically true). The strong validity ($cv = 1$) is satisfied only by very simple tautologies like $p \rightarrow p$, $p \rightarrow \neg\neg p$ or $\neg(p \wedge \neg p)$ and by the laws of commutation. The calculus presented here to avoid difficulties in the application to empirical sciences is that which uses (at least) strict validity instead of material validity for its theorems.

3.3. RMQ contains CPC and approximates RC and RD. Proof: Reduce the six values to two by removing all values between 1 and 6. In a quadratic matrix take only the four

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

values in the corners. More complex matrices with more than two propositional variables involved can be built up from quadratic matrices (each having 36 values). They also have to be reduced respectively. By this reduction, one receives the usual truth tables of *CPC* (which define semantically *CPC*). To avoid misunderstanding: *RMQ* distinguishes between theorems which are only materially valid and others which are strictly (necessarily) valid. Only the materially valid theorems of *RMQ* contain the whole classical propositional calculus (*CPC*). The strictly valid theorems contain only a part of *CPC*. As will be seen below this part is suitable for the application to empirical sciences and to quantum physics.

Subsequently the system which contains all theorems which are at least materially valid in *RMQ* ($cv \leq 3$) will be denoted by RMQ^{-}. The system which contains all theorems which are at least strictly valid in *RMQ* ($cv \leq 2$) will be denoted by *RMQ**. It is *RMQ** which is proposed to solve the difficulties when propositional logic is applied to the sciences, especially to physics.

The following Sections 3.3.1. and 3.3.2. show how *RMQ* approximates *RC* and *RD*.

3.3.1. How RMQ agrees with RC and RD. (a) This can be seen from the theorems listed in Section 4.2. below: All theorems of *CPC* (1)–(11) and (13)–(21) are ruled out by *RC* or *RD* on the one hand and are strictly invalid (only materially valid) in *RMQ* on the other. For Theorems (12) and (22)–(24) see below Section 3.3.2.(a).

(b) How *RMQ* agrees with *RC* and *RD* can be also seen from the theorems listed in Section 4.4. below: All theorems of *CPC* (1)–(17), (22), (24), (26)–(28), (30), and (32) satisfy *RC* or *RD* on the one hand and are strictly valid in *RMQ* on the other.

3.3.2. How RMQ differs from RC and RD. (a) On the one hand there are theorems of *CPC* which are excluded from the strictly valid formulas by *RMQ* (i.e., which are only materially valid in *RMQ*) but which are not ruled out by *RC* or *RD* because *RC* and *RD* are not applicable. This is the case with theorems of *CPC* which do not have the structure of an implication or equivalence (i.e., where the implication or equivalence is the main connective, namely first connective in Polish notation). Examples are Theorems (22)–(24) of Section 4.2. which are also intuitionistically invalid. The consequent of Theorem (12) of Section 4.2. has to be split up into two implications $p \rightarrow q$ and $q \rightarrow p$ in order to apply *RC*.

(b) On the other hand there are some theorems of *CPC* which are ruled out by *RC* but which are strictly valid in *RMQ*. An example is the Principle (36) of Section 4.4. which says that from a contradiction an arbitrary sentence follows. For details and explanation see Section 4.4.2. below.

(c) The implication of *CPC* satisfying *RC* ($\rightarrow RC$) is not *RC* preserving; that is, if A is a premise satisfying *RC* and $A \rightarrow RC\ B$ then B does not generally satisfy *RC*.

Proof by counterexample:

Premise: $(p \wedge q) \rightarrow p$ satisfies *RC*
Implication: $((p \wedge q) \rightarrow p) \rightarrow (\neg p \rightarrow \neg(p \wedge q))$ satisfies *RC*
Conclusion: $\neg p \rightarrow \neg(p \wedge q)$ violates *RC*

On the other hand the strict implication $\Rightarrow$ of *RMQ* is preserving strict validity. Or in other words: If A is strictly valid and $A \rightarrow B$ is strictly valid then B is strictly valid; that is, *RMQ** preserves strict validity.

Proof: If any premise A is strictly valid, then the matrix of A has a $cv \leq 2$. If $A \rightarrow B$ is strictly valid then the matrix of $A \rightarrow B$ has also a $cv \leq 2$. It is easily seen from the matrix of the implication $(p \rightarrow q)$ in Section 3.1. that in this case $B(q)$ must also have a $cv = 2$: All the entries 1 or 2 of p result, via values 1 or 2 of the implication, in values 1 or 2 of q.

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

(d) *RC* is not closed under *transitivity*; that is, if $A \rightarrow RC\ B$ and $B \rightarrow RC\ C$ then not generally:

$A \rightarrow RC\ C$

Proof by counterexample:

$((p \vee q) \wedge r) \rightarrow (p \vee (q \wedge r))$ satisfies *RC*
$(p \vee (q \wedge r)) \rightarrow ((p \vee q) \wedge (p \vee r))$ satisfies *RC*
$((p \vee q) \wedge r) \rightarrow ((p \vee q) \wedge (p \vee r))$ violates *RC*

The last line violates *RC* because the last occurrence of the variable 'p' can be replaced by an arbitrary variable salva validitate.

On the other hand the strict implication $\Rightarrow$ of *RMQ* and consequently *RMQ** is closed under *transitivity.*

Proof: If $A \rightarrow B$ and $B \rightarrow C$ then the matrices of both implications have a $cv \leq 3$. In this case the matrix of the conjunction of both matrices cannot exceed $cv \leq 3$ either, as a look on the general matrix of conjunction in Section 3.1. shows. Now any matrix with $cv \leq 3$ leads via strict implication $\Rightarrow$ of *RMQ* (having a matrix with $cv \leq 2$) to the matrix of the conclusion $A \rightarrow C$ with $cv \leq 3$ which can be easily be checked by the respective matrices.

Remark: That the material implication $\rightarrow$ of *RMQ* is closed under transitivity is manifest from the fact that all classically valid theorems of *CPC* are materially valid in RMQ^{-} and that the implication of *CPC* is closed under transitivity of implication.

For closure w.r.t. transitivity where all implications are strictly valid, that is, for $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$ the proof is analogous where $cv \leq 3$ is replaced by $cv \leq 2$.

(e) *RC* is not closed under *modus ponens*; that is, if *A* satisfies *RC* and $A \rightarrow RC\ B$ then not generally: *B* satisfies *RC*.

Proof by counterexample:

$(p \wedge q) \rightarrow p$ satisfies *RC*
$[(p \wedge q) \rightarrow p] \rightarrow [\neg p \rightarrow \neg(p \wedge q)]$ satisfies *RC*
$\neg p \rightarrow \neg(p \wedge q)$ violates RC (q is replaceable)

On the other hand *RMQ** is closed under *modus ponens*. Proof: if $p \rightarrow q$ is true its matrix has $cv \leq 3$; the same holds for the truth of p. Then the matrix of the conjunction of both cannot have a higher *cv* than $cv \leq 3$ as is clear from the general matrix of conjunction (Section 3.1.). But a matrix with $cv \leq 3$ can lead via a strict implication $\Rightarrow$ of *RMQ* with $cv \leq 2$ only to a conclusion with a matrix not exceeding $cv \leq 3$.

Remark: That RMQ^{-} is closed under modus ponens is manifest by the same reason given above (d).

For the modus ponens where the premise is a strict implication the proof is analogous, where $cv \leq 3$ is replaced by $cv \leq 2$.

(f) *RC* and *RD* are not closed under equivalence substitution.

Proof by counterexample:

p is classically logically equivalent with $[(p \wedge q) \vee (p \wedge \neg q)]$, but $p \rightarrow [(p \wedge q) \vee (p \wedge \neg q)]$ violates *RC*.

$p \wedge q$ is classically logically equivalent with $p \wedge p \wedge q \wedge q$, but $(p \wedge q) \rightarrow (p \wedge p \wedge q \wedge q)$ violates *RD*.

RMQ^{-} is closed under equivalence substitution. This follows from the fact that RMQ^{-} includes the full classical two-valued propositional calculus *CPC* materially. *RMQ** is closed under strict equivalence substitution.

Proof: (1) Observe that the matrix for equivalence in *RMQ* is this:

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

$p \leftrightarrow q$	1	2	3	4	5	6
1	1	2	3	5	5	6
2	2	1	3	5	5	5
3	3	3	1	4	5	5
4	5	5	4	1	3	3
5	5	5	5	3	1	2
6	6	5	5	3	2	1

The strict equivalence $p \Leftrightarrow q$ is represented by a $cv \leq 2$ of that matrix, that is, by its values 1 and 2 (see frame in the matrix).

(2) The matrix of a strictly valid formula of *RMQ** has a $cv \leq 2$.

(2a) Assume it has the form $A \Rightarrow B (cv \leq 2)$ and we replace the propositional variable p in A by p'such that $p \Leftrightarrow p'$ and call the result A'. Then $A \Leftrightarrow A'$, since the change from p to p' satisfies the matrix of strict equivalence ($cv \leq 2$) and consequently the cv of the result A' cannot deviate from A more than what is permitted by strict equivalence:

(i) Values 1 and 2 can change only into values 1 or 2.

(ii) Values 5 and 6 can change only into values 5 or 6.

(iii) Values 3 can change only into 3 and 4 only into 4.

Since $A \Leftrightarrow A'$, the $cv \leq 2$ of $A \Rightarrow B$ is preserved if $A \Rightarrow B$ is replaced by $A' \Rightarrow B$ *as is clear from the matrix of* $\rightarrow$*RMQ*. The same holds if some variable q in B is replaced by q' such that $q \Leftrightarrow q'$.

(2b) Assume that it has the form $A \vee B (cv \leq 2)$ and we replace the propositional variable p in A by p' such that $p \Leftrightarrow p'$.Then $A \Leftrightarrow A'$ by the same reason as above and $cv \leq 2$ of $A \vee B$ is preserved if $A \vee B$ is replaced by $A' \vee B$ as is clear from the matrix of $\vee$ of *RMQ*.

(2c) Other strictly valid forms can only be conjunctions or disjunctions or implications of either of these two strictly valid forms which do not increase the $cv \leq 2$.

(g) *RC* and *RD* are not closed under uniform substitution.

Proof by counterexample:

$p \rightarrow p$ satisfies *RC* and *RD*

$(p \wedge \neg p) \rightarrow (p \wedge \neg p)$ violates *RC* (the third and fourth occurrence of p can be uniformly replaced by any wff) salva validitate

$(p \wedge p) \rightarrow (p \wedge p)$ violates *RD* (the consequent $\alpha (= p \wedge p)$ is equivalent to a conjunction of $\alpha_1 \wedge \alpha_2$ where α_i is shorter than α).

RMQ$^-$ is closed under uniform substitution. This follows from the fact that *RMQ*$^-$ includes the full classical two-valued propositional calculus *CPC* where all valid formulas of *CPC* are (at least) materially valid in *CPC* (cf. Section 3.3. above).

Conjecture: *RMQ** is also closed under uniform substitution.

3.3.3. Comparison to the structural rules of Gentzen. A part of the restrictive power of *RC* and *RD* and consequently *RMQ** can also be compared with the four types of structural rules of Gentzen's calculus of sequences.[20] The first type—exchange—concerning commutation is satisfied by *RC* and *RD* and also by *RMQ**. The second type—weakening—that is, adding conjuncts in the premises (antecedence) and disjuncts in the conclusion, is not

[20] Compare Takeuti (1987, §2).

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

generally permitted: neither by *RC* and *RD* nor by *RMQ**. This can already be grasped from the classical Principles (7), (9), (4), and (8) of Section 4.2. which are only materially valid in *RMQ*. Concerning a third type—contraction—*RD* and *RMQ** satisfy it. The fourth type, the cut rule is not satisfied by *RC*. *RC* is neither closed under modus ponens, nor closed under transitivity and consequently not closed under the cut rule. *RMQ** is closed under transitivity of implication and closed under modus ponens. But *RMQ** is not closed under the cut rule; the cut rule is only materially valid in *RMQ*. This can be proved by a counterinstance: If the cut rule is expressed as the following implication

$$[(r \to (p \vee q)) \wedge ((q \wedge f) \to g)] \to [(r \wedge f) \to (p \vee g)]$$

then for the value $v = 4$ of g this principle holds only materially (i.e., with $cv \leq 3$).

3.3.4. Comparison to substructural logics. Substructural logics are nonclassical logics which are weaker than classical logic, especially because of the absence of structural rules of Gentzen-type classical logic (cf. Section 3.3.3. above). We compare some important principles of substructural logics with that of *RMQ*:

Residuation or deduction theorem: As one can see of Principles (32) and (33) of Section 4.4. only one direction is strictly valid in *RMQ*. This is important, since the deduction theorem is frequently applied in both directions in substructural logics.

Forms of weakening: As is clear from the second type in Section 3.3.3. the general form of weakening by adding premises and by extending the conclusion with disjuncts is not satisfied in *RMQ**. Although certain specific forms of weakening are satisfied provided material implication is used in the antecedence and in the consequence: thus $(p \to q) \to ((p \wedge r) \to q)$ is satisfied in *RMQ*, but $(p \Rightarrow q) \to ((p \wedge r) \Rightarrow q)$ is not satisfied in *RMQ**.

Elimination rules: The following two principles which are used as elimination rules in substructural logics (translated into *RMQ*) are not satisfied in *RMQ**, that is, are only materially valid in *RMQ*:

$$[(r \to (p \to q)) \wedge (s \to p)] \to ((r \wedge s) \to q)$$

$$[(r \to p) \wedge (q \to s)] \to [((p \to q) \wedge r) \to s]$$

Other principles like $p \vee (p \to q)$ which hold in substructural logics are also not satisfied in *RMQ** (viz. are only materially valid in *RMQ*, cf. (22) of Section 4.2. below).

From these examples it seems clear that the kind of nonclassicality of substructural logics is different from that of *RMQ*. The main focus of *RMQ* is to avoid difficulties when propositional logic is applied to the empirical sciences, especially also to physics. And this means also that the implication has to have some important relevance properties.

3.3.5. More than two truth values. The reason for having more truth values than two is to be able to differentiate between firstly theorems of *CPC* which hold only materially in *RMQ* and secondly theorems of *CPC* which hold strictly in *RMQ*. Most of the former violate either criterion *RC* or criterion *RD* and cause difficulties in the application of logic to the sciences and to modern physics. The latter however do not cause such difficulties.

Such a differentiation is not possible with only two truth values. The six values are sufficient for such a differentiation and they lead moreover to an interesting modal system (see Section 6 below).

A four-valued matrix system allows already much more differentiation than the classical two-valued systems. We do not consider three values, because there should be no value

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

between (i.e., outside) truth or falsity and in case of three values for truth and falsity the respective matrices would be very asymmetric, at least w.r.t. negation.

A four matrix with values of p: 1, 2, 3, 4 and $\neg p$: 4, 3, 2, 1 (where 1 means strictly true and 2 means materially true) would not be suitable if $\neg p \vee q$ is strictly equivalent to $p \rightarrow q$ and to $\neg(p \wedge \neg q)$, that is, if the connectives were strictly interdefinable. Since then all the usual paradoxes of implication and the equivalence in the distribution laws would hold strictly. According to my investigation made so far, the most suitable four-valued matrix system, $M4$, would be the following:

p	$\neg p$	$\vee$	1	2	3	4	$\wedge$	1	2	3	4	$\rightarrow$	1	2	3	4
1	4	1	1	1	1	1	1	1	2	3	4	1	1	2	3	4
2	3	2	1	2	1	2	2	2	2	4	4	2	1	1	3	3
3	2	3	1	1	3	4	3	3	4	3	4	3	1	2	1	2
4	1	4	1	2	4	4	4	4	4	4	4	4	1	1	1	1

This system, $M4$, contains classical propositional calculus (*CPC*) in the sense that all its theorems hold at least materially (i.e., the highest value of the matrix of the *CPC* theorem is: 2). Some of the "paradoxes of implication" like $p \rightarrow (p \vee q)$ or $p \rightarrow (p \wedge (p \vee q))$ hold only materially. Also the distribution laws hold strictly in the "right direction" (from disjuncts to conjuncts), materially in the other direction. Also the traditional argument forms (implications) like modus ponens, modus tollens, hypothetical syllogism, and contraposition all hold strictly. However there are a number of drawbacks: Frege's law $p \rightarrow (q \rightarrow p)$ which presupposes the commensurability of the propositions p and q[21] is strictly valid in $M4$ instead of only materially. That means that commensurability is always satisfied on logical grounds (like in classical logic)—a situation which is not acceptable for applying logic to physics. Also a number of ex falso quodlibet principles, like (1), (2), and (3) of Section 4.2. and paradoxes of implication like (22), (23), and (24) of Section 4.2., are strictly valid, instead of materially. Moreover the modal logics obtained from $M4$ are either too simple or too complicated. Too simple if we define *Lp* by: 1 3 3 4 or by 1 4 4 4. Since then there are only three positive modalities: *Lp, p, Mp* (everything else collapses). Too complicated if we define *Lp* by: 1 3 4 4. Then there are eight positive modalities of which half of them are only materially connected (in contradistinction to Theorem (11) of Section 6.1. of the here proposed system *RMQ*).

Finally we might give an example how sophisticated *RMQ* is, due to its six values: The matrix of a principle with four variables has 6^4=1296 values. In case of the principle of substructural logic

$$[(r \rightarrow p) \wedge (q \rightarrow s)] \rightarrow [((p \rightarrow q) \wedge r) \rightarrow s]$$

its matrix satisfies strict validity, that is, has $cv \leq 2$ for 1294 values. Only 2 values out of 1296 have a $cv=3$ such that the formula is only materially valid but strictly invalid.

Because of these results a four-valued matrix does not seem to behave so well for a propositional logic which is suitable for the application to empirical sciences.

It may be asked how *RMQ* is axiomatisable; that is, how the properties of the above matrices can be formulated with the help of axioms. Such a task was solved for the matrix

[21] Compare Mittelstaedt & Weingartner (2005, p. 343).

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

system mentioned in Note 19 above by Czermak (1981) which can be represented by a set of 33 axioms. They may be reduced by independence proofs (and choosing a different basis for propositional logic) to 26. Since *RMQ* is more complicated than this system (since it distinguishes materially valid theorems of *CPC* from strictly valid ones) it is probable that *RMQ* needs more than 33 (26) axioms to represent its matrices. This only shows that the matrix-based logic *RMQ* is very specific.

3.4. Consistency of RMQ and RMQ*. A system is *negation consistent* iff for no *wff* p, both p and $\neg p$ are provable (are theorems). *RMQ* is *negation consistent.* Proof: in *RMQ* (*RMQ**) a *wff* is a theorem (strictly valid theorem), iff the *cv* of its matrix is $\leq 3 (\leq 2)$. But in this case the *cv* of its negation is ≥ 4 and thus not a theorem (not a strictly valid theorem).

A system is *trivially consistent* iff there is at least one *wff* which is not a theorem. *RMQ* (*RMQ**) is trivially consistent. Proof: The *wff*s $p \wedge \neg p$ and $[(p \rightarrow q) \wedge \neg p] \rightarrow \neg q$ are not theorems, since their *cv* is greater than 3.

A system is *consistent w.r.t. its atoms* iff no *wff* consisting of a single propositional variable is a theorem. *RMQ* (*RMQ**) is consistent w.r.t. its atoms. Proof: The matrix of every single propositional variable has a $cv \geq 4$ and is therefore not a theorem.

3.5. Decidablity of RMQ and RMQ*. Decidablity of a deductive system is established by a decision procedure. A decision procedure is a method which suffices to answer unambiguously with 'Yes' or 'No' the question whether or not any *wff* of the system (*RMQ*) is a theorem. *RMQ* (*RMQ**) are decidable. Proof: Such a decision procedure for *RMQ* (*RMQ**) consists in calculating the matrix of the *wff* in question for its *cv*: if its $cv \leq 3 (cv \leq 2)$, it is a theorem, if its $cv > 3$ (>2), it is not a theorem. In this case the *wff* may be either contingent (if its matrix contains both, at least one value between 1 and 3 and at least one value between 4 and 6) or logically false if all values are greater than 3.

3.6. RMQ has the finite model property. Proof: Every particular nontheorem of *RMQ* (*RMQ**) has its finite matrix with a $cv > 3$ $(cv > 2)$ which falsifies the nontheorem.

§4. Theorems of *RMQ*.

4.1. Conventions. If $\rightarrow$ is the main connective, then $\rightarrow$ means that the formula (*wff*) is only materially valid $(cv = 3)$; $\Rightarrow$ means that the formula is at least strictly valid (i.e., valid with L in front; $cv \leq 2$);

$(p \Rightarrow q) = df\ L(p \rightarrow q)$. Also *wff*s of which the main connective is $\vee$ (disjunction) may be only materially valid, that is, iff their $cv = 3$.

4.2. Theorems of CPC which are only materially valid (but strictly invalid) in RMQ.

(1) $\neg p \rightarrow (p \rightarrow q)$	Ex falso quodlibet
(2) $\neg p \rightarrow [p \rightarrow (q \wedge \neg q)]$	Ex falso quodlibet
(3) $(p \rightarrow \neg p) \rightarrow [p \rightarrow (q \wedge \neg q)]$	Ex falso quodlibet
(4) $p \rightarrow (p \vee q)$	Redundant element(s)
(5) $p \rightarrow [p \vee (q \wedge \neg q)]$	Redundant element(s)
(6) $p \rightarrow [(p \wedge q) \vee (p \wedge \neg q)]$	Redundant element(s)
(7) $(p \rightarrow q) \rightarrow [(p \wedge r) \rightarrow (q \wedge r)]$	Redundant element(s)
(8) $(p \rightarrow q) \rightarrow [(p \vee r) \rightarrow (q \vee r)]$	Redundant element(s)
(9) $p \rightarrow (q \rightarrow p)$	Adding premise
(10) $p \rightarrow [(q \vee \neg q) \rightarrow p]$	Adding premise

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

(11) $p \rightarrow [q \rightarrow (p \wedge q)]$	Adding premise
(12) $(p \wedge q) \rightarrow (p \leftrightarrow q)$	Conjunction and implication
(13) $(p \wedge q) \rightarrow (p \rightarrow q)$	Conjunction and implication
(14) $(p \wedge q) \rightarrow [(q \wedge \neg q) \vee (q \wedge p)]$	Conjunction and disjunction
(15) $(p \wedge q) \rightarrow [(p \wedge r) \vee (q \wedge \neg r)$	Conjunction and disjunction
(16) $[p \wedge (q \vee r)] \leftrightarrow [(p \wedge q) \vee (p \wedge r)]$	Distribution
(17) $[(p \vee q) \wedge (p \vee r)] \leftrightarrow [p \vee (q \wedge r)]$	Distribution
(18) $p \rightarrow [p \vee (p \wedge q)]$	Absorption
(19) $p \rightarrow [p \wedge (p \vee q)]$	Absorption
(20) $p \leftrightarrow [(p \wedge q) \vee (p \wedge \neg q)]$	Irrelevant disjuncts
(21) $p \leftrightarrow [p \wedge (q \vee \neg q)]$	Irrelevant disjuncts
(22) $p \vee (p \rightarrow q)$	Intuitionistically invalid $cv = 3$
(23) $(p \rightarrow q) \vee (p \rightarrow \neg q)$	Intuitionistically invalid $cv = 3$
(24) $(p \rightarrow q) \vee (\neg p \rightarrow q)$	Intuitionistically invalid $cv = 3$

With the exception of the last three (22)—(24), the principles (*CPC* theorems) (1)–(11), (13)–(15), and (18)–(21) are separated as classically valid but irrelevant by the replacement criterion *RC*: it is easily seen that at least one occurrence of the variable q (or p or r) in the consequent can be replaced by any arbitrary variable salva validitate (veritate) of the *CPC* theorem. The equivalence $p \leftrightarrow q$ in *CPC* theorem (12) has to be split into the two implications $(p \rightarrow q \wedge q \rightarrow p)$ in order to apply *RC*.

The direction $\rightarrow$ of (16) and (17) is forbidden by the reduction criterion *RD*. The last three (22)—(24) cannot be designated that way because the main connective is not an implication, but a disjunction.

However, for the system *RMQ* the form of the *wff* is not essential because the decision whether the *wff* is materially valid or strictly valid is determined by the *cv* of the matrix (cf. Section 3.2. above) which represents the *wff*. Thus it can easily be checked that all the above-listed Principles (1)–(24) of *CPC* are (only) materially valid, but not strictly valid in *RMQ*.

4.3. Requirements for a logic suitable for quantum physics.

4.3.1. The problem with distributivity and commensurability. It is an important feature of quantum logic that neither distributivity, nor commensurability is generally satisfied.[22] And if distributivity is not satisfied, then also commensurablity is not satisfied, that is,

$$\neg[p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)] \rightarrow \neg(p \sim q \wedge p \sim r)$$

(where '$p \sim q$' stands for 'p is commensurable with q'). That distributivity cannot generally be satisfied was already pointed out clearly by Birkhoff and von Neumann.[23] It can be shown as follows:

Assume a particle in a box. We are concerned with the complementary pair of position and momentum of the particle. For a rough estimate let the position be replaced by the determination of whether the particle is in the left part of the box (L) or in its right part ($R = L'$) and the momentum be replaced by the determination of whether the particle

[22] For principles of quantum logic in the usual understanding and their justification in a system of *QL* see Mittelstaedt (1978) and Dalla Chiara & Giuntini (2001). For a comparison of some systems of *QL* with *RMQ* see Section 5 below.

[23] Compare Birkhoff and von Neumann (1936, p. 831). Compare also Finkelstein (1979).

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

is in an even (symmetric) state (E) or odd state (E'). This is determined by the parity of a certain quantum number n (even or odd) proportional to the momentum which means that the wave function is either invariant or is changed. Then the respective instance of the distributive law is this:

$$E \cap (L \cup R) = [(E \cap L) \cup (E \cap R)].$$

Since $L \cup R$ means $L \cup L'$ and gets the value 1 in a Boolean algebra, the left side of the equation reduces to E. Translated into classical logic (*CPC*), $L \cup L'$ means that the particle is either on the left, or the particle is not on the left, which is a tautology. As to the right side of the equation, evenness (for momentum) and leftness or rightness (for position) are incommensurable properties such that the subspace of the wave functions of such a particle vanish, that is, they get the value 0. That is $E \cap L$ and also $E \cap R$ get the value 0 and so the whole right side gets the value 0. Translated into classical logic (*CPC*), this means that "particle s has evenness and particle s has rightness" is contradictory. Thus the left side of the distributive law gets the value E (i.e., a certain positive value) and the right side the value 0, which shows the violation.

The above instance of the set theoretical version of the law of distribution corresponds to the following instance in *CPC*:

$p \wedge (q \vee \neg q) \leftrightarrow [(p \wedge q) \vee (p \wedge \neg q)]$. It is derived from the general form of the law: $[p \wedge (q \vee r)] \leftrightarrow [(p \wedge q) \vee (p \wedge r)]$ (cf. Theorem (16) of Section 4.2. above). Now in *CPC* the right part and the left part of this instance are logically equivalent to p. That is in *CPC* the following are theorems:

$$p \leftrightarrow p \wedge (q \vee \neg q); \qquad p \leftrightarrow [(p \wedge q) \vee (p \wedge \neg q)]$$

But according to the replacement criterion, the implications $p \rightarrow p \wedge (q \vee \neg q)$ and $p \rightarrow [(p \wedge q) \vee (p \wedge \neg q)]$ cannot be permitted (as informative) because both occurrences of 'q' can be replaced (uniformly) by any other variable salva validitate (veritate). And in *RMQ* these directions holds only materially, but not strictly. Now these are exactly the directions which are forbidden in the application of logic to quantum theory. Since suppose p is the proposition that an electron has a certain position (is in a certain region) and q is the proposition that the momentum of this electron is equal to k. Then $p \wedge (q \vee \neg q) \rightarrow (p \wedge q \vee p \wedge \neg q)$ does not hold although the opposite direction holds, because both $p \wedge q$, $p \wedge \neg q$ are false. A similar consideration concerns the two-slit experiment: Let q or not-q stand for the particle going through one slit or the other, and let p represent the proposition that the particle has a certain momentum (as given by the plane wave). Then, since the derivation from $(p \wedge (q \vee \neg q))$ to $((p \wedge q) \vee (p \wedge \neg q))$ is blocked one cannot conclude that the particle goes through a definite slit and consequently one cannot derive the classical probability distribution (associated with no interference).

From these considerations it is clear that also the so-called postulate of weak objectification

$$B \leftrightarrow [(A \wedge B) \vee (\neg A \wedge B)]$$

where A and B are stating that the values a and b respectively belong to the system, is not strictly valid in *RMQ*, because the direction $\rightarrow$ is not strictly valid, only the opposite direction $\leftarrow$ is strictly valid.[24]

[24] Compare proposition (4.13) and the valid weaker implication (4.22) (which corresponds to the direction $\leftarrow$) in Mittelstaedt (1998, pp.74 and 77).

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

That is in a logic applicable to *QP* only the Principles (22) and (24) of Section 4.4. below are acceptable as distribution laws and just these are strictly valid in *RMQ*.

This is now connected with commensurability. Commensurability in *QL* is usually defined in one of the following ways[25] (where '$p \sim q$' stands for 'p is commensurable with q'):

(1) $p \sim q \leftrightarrow df\, p \leftrightarrow [(p \wedge q) \vee (p \wedge \neg q)]$
(2) $p \sim q \leftrightarrow df\, p \rightarrow (q \rightarrow p)$
(3) $p \sim q \leftrightarrow df\, q \rightarrow (p \rightarrow q)$

It is easily seen that in *CPC* all the right parts, that is, the definientia of (1), (2), and (3) are logically true (are theorems). Therefore, according to *CPC*, commensurability is always satisfied on logical grounds. This is untenable in *QL,* but also in any logic suitable for quantum physics (*QP*). Now observe that all right parts above (definentia) are irrelevant according to the replacement criterion (*RC*). The general distribution laws however are permitted according to the replacement criterion, but hold only in one direction (from disjuncts to conjuncts) by the reduction criterion *RD*.

RMQ manages to combine the virtues of both criteria: The general distribution laws hold as equivalences only materially (cf. Theorems (16) and (17) of Section 4.2.), but in the direction permitted by *QL* they hold necessarily. That is $[(p \wedge q) \vee (p \wedge \neg q)] \Rightarrow [p \wedge (q \vee \neg q)]$ holds necessarily in *RMQ* and the respective general law: $[(p \wedge q) \vee (p \wedge r)] \Rightarrow [p \wedge (q \vee r)]$ also holds necessarily.[26]

The same is the case with the definientia of commensurability above: they hold only materially. And of the first definition (1) only the following direction of its definiens holds necessarily: $[(p \wedge q) \vee (p \wedge \neg q)] \Rightarrow p$ (see the Theorem (27) in Section 4.4. below). The equivalence $p \leftrightarrow [(p \wedge q) \vee (p \wedge \neg q)]$ which is logically true in *CPC* expresses strong objectification in the sense that if the proposition p represents an observational result of the system S, then for an arbitrary proposition q, representing an observational result, q or non-q, is a result of S too; that is, what is too strong in *QP* is the implication $p \rightarrow [(p \wedge q) \vee (p \wedge \neg q)]$. The above equivalence leads (via Kolmogorov's axioms) to a postulate of probabilities which is also too strong for the application in *QP*. However, if one takes only that part of implication which is necessarily valid in *RMQ*, that is, $[(p \wedge q) \vee (p \wedge \neg q)] \Rightarrow p$ (cf. Theorem (27) of Section 4.4.), then the probability postulate obtained is weaker and in accordance with *QP*.[27]

4.3.2. The problem with Bell's inequalities. Proposition (15) (of Section 4.2.) is the propositional analogon to the following set theoretical theorem:

$$A \cap B \subseteq [(A \cap C) \cup (B \cap -C)]$$

It says w.r.t. the members (elements) of the respective sets: those members which have both property A and property B, are included in or equal to the union (sum) of those which have both, property A and property C, and those which have both property B and property non-C. This is one form of Bell's inequality.[28]

---

[25] See Mittelstaedt (1978, p. 30ff).
[26] For the complete list of the laws of distribution w.r.t. $\wedge$ and $\vee$ see Section 4.4.(22)–(26).
[27] For details see Mittelstaedt (1998, p. 74ff).
[28] Compare Bell (1987, p. 147f) and the elaboration in Zeilinger (1992, p. 108ff).

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

The statements (5)–(9) of inequality in Bell's book *Speakable and Unspeakable in Quantum Mechanics* (p. 147ff) have the same logical structure (form) as the one discussed by Zeilinger, that is, $(A \cap B) \subseteq [(A \cap C) \cup (B \cap -C)]$, which corresponds in propositional logic to Proposition (15) of Section 4.2., that is, to $(p \wedge q) \rightarrow [(p \wedge r) \vee (q \wedge \neg r)]$.

We may also use one of the formulations of Bell[29] to see how it represents Principle (15) of propositional logic (cf. Section 4.2. above).

The number that could pass at 0° (A) and the number that could not pass at 90° (B) is less than or equal to the number that could pass at 0 (A) and the number that could not pass at 45° (C) plus the number that could pass at 45° $(-C)$ and the number that could not pass at 90° (B).

Then the respective set theoretical principle is:

$(A \cap B) \subseteq [(A \cap C) \cup (B \cap -C)]$. If we convert this into propositional logic we have to replace $A \cap B$ by $x \in A \wedge x \in B$ and represent it by $p \wedge q$. Then we get the propositional analogue to this form of Bell's inequality as: $(p \wedge q) \rightarrow [(p \wedge r) \vee (q \wedge \neg r)]$ which is our Proposition (15) of Section 4.2. Observe that one can replace r (or C above) on both occurrences by any other variable salva validitate.

A lucid example for the above set theoretical principle underlying Bell's inequalities is due to D'Espagnat (1979, p. 27): The number of young (B) women (A) is less than or equal to the number of women (A) smokers (C) plus the number of young (B) nonsmokers $(-C)$. Observe that one can replace C on both occurrences by any arbitrary predicate (say by singers, scientists, teachers, ... etc.) salva validitate of the principle.

If the measurement has to be done on pairs of particles rather than singular particles—because incommensurable properties cannot be measured simultaneously—Bell's inequality reads thus: Those pairs of which one member possesses property A and the other property B, are included in or equal to the union (sum) of those pairs of which one member possesses property A and the other property C and of those pairs of which one member possesses property B and the other non-C (cf. Bell's examples (6) and (8)).

The main reason for the violation of Bell's inequalities for some experiments in *QP* seems to be similar to that concerning distributivity and commensurability: Some of the underlying laws of *CL*, of Boolean algebra or of set theory are too strong. This can be seen also from the probabilistic forms of Bell's inequalities. In order to obtain these probabilistic forms (see below) one needs certain classicality conditions plus the axioms of Kolmogorov. Now in the proof of the classicality conditions several assumptions of *CL* (of Boolean algebra or of set theory) have to be made like $\neg p \leftrightarrow ((\neg p \wedge q) \vee (\neg p \wedge \neg q))$ of Principle (20) of Section 4.2.[30] which are not strictly valid in *RMQ* and which violate *RC*. That means that the probabilistic forms of Bell's inequalities are relations of the classical probability theory. They deal with three observables or properties[31] a, b, c and with the corresponding probabilities $p(a), p(b), p(c), p(a, b), p(a, c), p(b, c)$ and can be formulated as follows[32]:

$$p(b, \neg c) \leq p(\neg a, b) + p(a, \neg c)$$

[29] Compare Bell (1987, p. 147(5)).

[30] Compare the proof of the first classicality condition in Mittelstaedt (1998, p. 130, appendix 6). The proof for the second classicality condition was given by Pitowsky (1989) and Beltrametti and Maczynski (1991).

[31] In the case of two observables the classical probability theory leads to inconsistencies with the wave-particle dualism.

[32] Compare Mittelstaedt (1998, p. 99).

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

$$p(b, c) \leq p(\neg a, b) + p(a, c)$$
$$p(\neg a, b) \leq p(\neg a, c) + p(b, \neg c)$$
$$p(\neg a, c) \leq p(\neg a, \neg b) + p(b, c)$$

These inequalities, like the ones in set theoretical form used by Bell are satisfied for all measurements where *QP* is not involved, but are violated for some experiments in *QP*. Observe now also here that in the right (consequent) part of the above probabilistic inequalities the variable *a* can be replaced on both occurrences by another arbitrary variable (standing for any other observable or property) salva validitate of the first two inequalities; the same can be done with the variables *c* and *b* respectively in the third and fourth inequality.

In the case of two observables or properties the classical probability theory leads to inconsistencies with the wave-particle dualism; however in the case of three observables the classical probability theory leads to inconsistencies with the locality of two (physical) systems. Here "locality" means that no additional information from observations made in space time regions spatially *separated* from a region *R* could modify our predictions concerning what will happen in *R*. In this sense Bell's inequalities may be said to test the separation or disentanglement of two systems: if the systems are in some way connected they are violated.

Now our conjecture is this: First one can see from a purely logical point of view that variables in that part which corresponds to the consequence part of laws of classical propositional logic of Boolean algebra (or of laws of set theory or equations[33] or inequalities) which can be replaced (on one or more than one occurrences uniformly) by arbitrary variables salva validitate are independent or separated or disentangled from the other part in the respective law or equation or inequality. And secondly as a consequence of that: if such variables represent states or events or systems or observables or properties they are (by the classical validity of the respective law or equation or inequality) described as independent or separated or disentangled from those states or events or systems or observables or properties represented by the other part of the respective law equation or inequality. And if one tries to describe nonseparated or entangled systems (events, observables, properties) by means of a model of separated or disentanglement systems (events, observables, properties), that is, by special principles of classical logic or probability theory like Bell's inequalities, then such principles are violated by experiment. Now logical or set theoretic principles should not dictate nature in the sense of prescribing which physical systems (or states, properties, observables) are separated or disentangled of others. But this is exactly what classical logic does; not only with the principles like (15) of Section 4.2. (and the respective set-theoretical principle) and with the probabilistic forms of Bell's inequalities above, but also with the distribution principles holding in both directions and with the principles underlying commensurability such that commensurability is classically satisfied universally on logical grounds.

That means that w.r.t. Bell's inequalities, *RMQ* shows that these inequalities should not be valid principles in a suitable logic for *QP*. Just like one direction of the laws of distribution (i.e., (23) and (25) of Section 4.4.) which hold in *CPC*, but are too strong for a logic, applied to *QP*, so Bell's inequalities in any of its forms are too strong: in their set theoretical form they hold in a Boolean Algebra (or their propositional analoga

[33] Equations (identities) are equivalences which can be replaced by a conjunction of two implications which each have a consequence part.

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

hold in *CPC*) and they underly their probabilistic versions; but they are too strong for a logic which can be suitably applied to *QP*. In *RMQ,* all these principles of classical logic are only materially valid, but as necessary laws they are invalid (i.e., they are not strictly valid). This means that *RMQ* is a consistent model for such a logic in which the principles which are violated in the application to *QP* are not valid necessarily, although all the main principles of Aristotelean logic, like the law of noncontradiction, modus ponens, modus tollens, hypothetical syllogism (cf. Section 4.4., Theorems (1)–(13) and (34) and (35)), are necessarily valid in *RMQ*. As has been pointed out already, the difficulties coming up when *CL* is applied to modern physics are not restricted to physics; they exist also in other domains, as mentioned in Sections 2.1. and 2.2., where conjecture *CJ* (with respect to these domains) has been confirmed already. Along these lines it is not astonishing that Bell's inequalities are also violated in phenomena on the macrolevel. Examples of experiments of that sort have been discussed by Aerts.[34]

4.4. Basic theorems of CPC which are strictly (or necessarily) valid (valid with L in front) in RMQ. The main theorems which are strictly valid, are recognisable by their main connective $\Rightarrow$. For comparison also some principles which are strictly invalid (i.e., only materially valid, with $\rightarrow$) in *RMQ*, are listed below. All the main theorems which hold strictly (with $\Rightarrow$), are valid in a logic suitable for *QP*. We shall denote that subsystem of *RMQ* which contains only theorems which are strictly valid in *RMQ* by *RMQ**.

(1) $(p \wedge q) \Leftrightarrow (q \wedge p)$	Commutation
(2) $(p \vee q) \Leftrightarrow (q \vee p)$	Commutation
(3) $[p \wedge (q \wedge r)] \Leftrightarrow [(p \wedge q) \wedge r)]$	Association
(4) $p \Rightarrow p$	
(5) $p \Leftrightarrow \neg\neg p$	Double negation
(6) $p \wedge q \Rightarrow p$	Simplification
(7) $p \wedge q \Rightarrow q$	Simplification
(8) $p \vee p \Rightarrow p$	Simplification
(9) $[(p \rightarrow q) \wedge p] \Rightarrow q$	Modus ponens
(10) $[(p \rightarrow q) \wedge \neg q] \Rightarrow \neg p$	Modus tollens
(11) $(p \rightarrow q) \Rightarrow (\neg q \rightarrow \neg p)$	Contraposition
(12) $[(p \rightarrow q) \wedge (q \rightarrow r)] \Rightarrow (p \rightarrow r)$	Hypothesis syllogism (transitivity of $\rightarrow$)
(13) $[(p \vee q) \wedge \neg p) \Rightarrow q$	Disjunctive syllogism
(14) $(p \wedge q) \Rightarrow \neg(\neg p \vee \neg q)$	De Morgan's law
(15) $(p \vee q) \Rightarrow \neg(\neg p \wedge \neg q)$	De Morgan's law
(16) $(\neg p \wedge \neg q) \Rightarrow \neg(p \vee q)$	De Morgan's law
(17) $(\neg p \vee \neg q) \Rightarrow \neg(p \wedge q)$	De Morgan's law
(18) $\neg(\neg p \vee \neg q) \rightarrow (p \wedge q)$	De Morgan's law
(19) $\neg(\neg p \wedge \neg q) \rightarrow (p \vee q)$	De Morgan's law
(20) $\neg(p \vee q) \rightarrow (\neg p \wedge \neg q)$	De Morgan's law
(21) $\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$	De Morgan's law

Looking at De Morgan's laws (14)–(21) shows that strict validity in *RMQ* interprets a negation before parenthesis of a compound *wff* as weaker than the negation which is

[34] Aerts (1985). Compare also Aerts & Aerts (2005), where they discuss several axioms and the role of disjunctions and conjunctions as presuppositions of such difficulties (sections 11.1–11.6).

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

applied immediately to the *wff*. This coincides partially with intuitionistic logic, such that Principles (18)–(21) hold only materially in *RMQ*.

(22) $[(p \wedge q) \vee (p \wedge r)] \Rightarrow [p \wedge (q \vee r)]$	Distribution
(23) $[p \wedge (q \vee r)] \rightarrow [(p \wedge q) \vee (p \wedge r)]$	Distribution
(24) $[p \vee (q \wedge r)] \Rightarrow [(p \vee q) \wedge (p \vee r)]$	Distribution
(25) $[(p \vee q) \wedge (p \vee r)] \rightarrow [p \vee (q \wedge r)]$	Distribution
(26) $[(p \wedge q) \vee (p \wedge \neg q)] \Rightarrow [p \wedge (q \vee \neg q)]$	Instance of distribution
(27) $[(p \wedge q) \vee (p \wedge \neg q)] \Rightarrow p$	
(28) $[p \rightarrow (q \wedge r)] \Rightarrow [(p \rightarrow q) \wedge (p \rightarrow r)]$	
(29) $[(p \rightarrow q) \wedge (p \rightarrow r)] \rightarrow [p \rightarrow (q \wedge r)]$	
(30) $[(p \rightarrow r) \vee (q \rightarrow r)] \Rightarrow [(p \wedge q) \rightarrow r]$	
(31) $[(p \wedge q) \rightarrow r] \rightarrow [(p \rightarrow r) \vee (q \rightarrow r)]$	
(32) $[r \rightarrow (p \rightarrow q)] \Rightarrow [(r \wedge p) \rightarrow q]$	
(33) $[(r \wedge p) \rightarrow q] \rightarrow [r \rightarrow (p \rightarrow q)]$	

4.4.1. Desiderata met by RMQ Those distribution laws which are strictly (or necessarily) valid (i.e., valid with L) are just the ones which must hold in a logic applicable to *QP* and which also hold in the usual systems of *QL*. Those which are only materially valid are valid in *CPC*, but not generally in a logic applicable to *QP* or in *QL*; they are however valid in *QL* only if commensurablity is satisfied.

Looking at the laws of distribution (22)–(26), one can see that those which hold necessarily in *RMQ* (and *QL*), represent a move from disjuncts (of conjuncts) to conjuncts (of single parts or of disjuncts)(cf. Theorems (22), (24), and (26) of Section 4.4.). This amounts to splitting up the consequence into maximally informative parts which is also a reduction of complexity. But the opposite way, that is, from maximally informative parts to complex disjuncts is not strictly valid.

The Principles (28) and (29) show a further important point: If, under a certain condition p, q represents and describes the physical state Q, and under the same condition p, r represents and describes the physical state R, does it then always follow that under condition p the conjunction $q \wedge r$ also represents a new physical state (which is some kind of fusion of states Q and R)? The answer is: No (recall Problem (1) of the Introduction)

Accordingly, Principle (29) above does not hold strictly in *RMQ*, but only materially. The strict validity holds only in the opposite direction (28). That this must be so is known from incommensurable magnitudes: If q represents the state of affairs that—under measurement condition p—the position has a certain sharp (enough) value and if r represents the state of affairs that—under measurement condition p—the momentum has a certain sharp (enough) value, then it is not the case that (under condition p)$q \wedge r$ represents a measurable magnitude at all. Physics is not alone w.r.t. such incommensurabilities. Analogous ones are known in animal behavior, in social help situations among animals, in human actions, and so forth.[35] A similar consideration can be made w.r.t. the Principles (30) and (31).

Concerning the further desiderata about commensurability and about Bell's inequalities it has been shown in Section 4.3. above that both: the definentia of commensurability

[35] Compare the examples discussed in Weingartner (2004b, p. 234f).

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

(valid in *CPC*) are not strictly valid in *RMQ* and the principles of *CPC* underlying Bell's inequalities are not strictly valid in *RMQ* either.

4.4.2. Noncontradiction, tertium non datur, and contradiction.

(34) $\neg(p \wedge \neg p)$ Principle of noncontradiction strongly valid
(35) $p \vee \neg p$ Principle of excluded middle or tertium non datur strongly valid

The strong validity ($cv = 1$) of the principle of noncontradiction means that no modal operator can change the value, since it is 1. Also: no modal operator can change the value 6 of the contradiction (see the Table in Section 3.1.). Since $\neg p$ is the mirror image of p, $p \wedge \neg p$ has the maximum value false, whereas $\neg(p \wedge \neg p)$ has the maximum value true.[36] The reason for it is the matrix of negation (mirror image) and of conjunction, which has values 6 in the diagonal from right top to left bottom. Similarly, the strong validity of the tertium non datur is caused by the matrix of negation and of disjunction, which has values 1 in the same diagonal.

One can easily see that one could change the values of the diagonal of the disjunction (from right top to left bottom) into 112211 or even 123321; in the latter case $p \vee \neg p$ (tertium non datur) would be only materially valid. And this would be so despite of the strong validity of the principle of noncontradiction. With such a weaker (materially valid) tertium non datur *RMQ* would still be a suitable logic for *QP*, since this would not affect the distribution laws (they would hold in the same direction strictly, compare (22) and (24) of Section 4.4.) and it would not affect purely implicational laws and also not modus ponens or modus tollens. There would though be an effect on some laws of modal logic (see below Section 6).

(36) $(p \wedge \neg p) \Rightarrow q$ Principle of explosion

The strong validity of the principle of explosion[37] shows that *RMQ* is not "paraconsistent" in the sense of Priest. It shows further that paraconsistency in this sense is independent of the other ex falso quodlibet principles which are only materially valid in *RMQ* (cf. (1)–(3) of Section 4.2. above). It further shows one can construct a model out of *RMQ* by changing the matrix for $\vee$ (value 3 of position (3,3) is changed into value 2) that the disjunctive syllogism (D. S.) can be only materially valid without any effect on the explosion principle; that is—contrary to opposite claims which interpret D. S. as a necessary condition for the 'principle of explosion'—D. S. is independent of the principle of explosion. Such a change in the matrix of $\vee$ would turn the strict validity of some theorems containing $\vee$ into material validity. But all theorems of *CPC* would still be materially valid in the new system. However, according to the replacement criterion, *RC*, the explosion principle is irrelevant because q can be replaced by any other variable including its negation.

As it was mentioned already in Section 3.3.1., *RMQ* satisfies the replacement criterion with respect to many theorems except for a few exceptions like inferences with contradictory premises. On the other hand *RMQ* differs from *RC* (cf. Section 3.3.2.) concerning

[36] As a consequence p is *Aristotle sharp* according to the terminology of algebraic notions of sharpness introduced in Dalla Chiara *et al.* (2004, p. 137). But (for p, q, r of *RMQ*)p is not *principal sharp* (sharp 1) according to this terminology. However, for any p of *RMQ*, p is *Boole sharp* (or sharp 5) because p is idempotent (p. 137).

[37] The principle was called so by Priest, compare Priest (2000). There he also defined "paraconsistent system" by "system in which the explosion principle is invalid."

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

closure conditions. However, it should be mentioned that there are other systems of paraconsistent logics describing weaker kinds of paraconsitency like the C-Logics of Newton da Costa and his coworkers; they distinguish well-behaved propositions for which classical logic holds from ill-behaved inconsistent ones for which a paraconsistent system as a supplement to classical logic holds.[38] *RMQ* however is not paraconsistent in this weaker sense either.

In general it should be mentioned that although the explosion principle is implausible (to permit to derive anything whatsoever from a contradiction) it seems that in the application of logic there is no known paradox which has this logical form. There are many known paradoxes which have the form of other ex falso quodlibet principles and many which have the form of the principle of addition or the one which permits to add premises, and so forth.[39] According to this consideration it seems better to accept the explosion principle, but render all harmful ex falso quodlibet principles (including others which lead to paradoxes) as only materially valid, but strictly invalid.

§5. Comparison of *RMQ* to other systems of *QL*. Subsequently *RMQ** represents that subsystem of *RMQ* which contains only theorems which are (at least) strictly valid ($cv \leq 2$) in *RMQ*.

5.1. The orthologic of Goldblatt. Goldblatt proposes an axiom system O called *Orthologic* consisting of nine axioms, which has the following properties.[40]

(1) O can be characterized by the class of ortholattices in such a way that B is in O derivable from A iff $v(A) \leq v(B)$ for all valuations v on all ortholattices.

(2) O has a semantics by the class of all orthoframes.

(3) O is decidable.

(4) O has the *finite model property.*

The system O is extended to the system Q (of quantum logic) by the addition of Axiom 10:

$$A \wedge (\neg A \vee (A \wedge B) \vdash B$$

Also for Q soundness and completeness w.r.t. quantum frames has been proved in (Goldblatt, 1974). Whether Q has the *finite model property* is an open question.

Comparison with *RMQ*:

Although O (Q) is a system of binary logic in the sense that it is a collection of ordered pairs $< A, B >$ such that B follows from A, which does not have an implication, it can easily be translated into a logic using implications for derivation as Goldblatt indicated on page 21 of his paper. Concerning *RMQ* recall from Section 3.1. that in a matrix representation of propositional logic $A \vdash B$ coincides with valid $A \rightarrow B$ (materially valid) or which $A \Rightarrow B$ (strictly valid). Classical (valid) implication (which is represented by materially valid implication in *RMQ*) leads to difficulties when applied in physics (in quantum logic) because it is not restricted according to criteria *RC* and *RD*. These restrictions or purifications are represented in *RMQ* by strict validity $A \Rightarrow B$. Goldblatt's logic meets all these restrictions (except Axiom 8 does not satisfy *RD*):

---

[38] Da Costa (1974).

[39] For a justification see the description of many different types of paradoxes in Weingartner & Schurz (1986).

[40] Goldblatt (1974).

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

Of the nine axioms of system O, all except Axiom 8 are strictly valid (valid with necessity) in *RMQ* (see Theorems (4)–(7), (36), (12), and (11) of Section 4.4.). Axiom 8 is Principle (29) of Section 4.4., which is only materially valid in *RMQ* (the reasons for this have been discussed in Section 4.4.1.).

Axiom 10, which extends system O to Q, is also strictly valid in *RMQ*. Therefore *RMQ** (i.e., the theorems of *RMQ* which are strictly or necessarily valid) is also a model for systems O and Q, although with the proviso that Axiom 8 is only materially (not strictly) valid. Moreover, *RMQ* and O are both decidable, possess a semantics and have the *finite model property.*

5.2. The orthocomplementary logic of Piron. In *axiomatique quantique*[41] Piron proposes an axiom system for quantum theory with four groups of axioms: O, T, C, and P. This axiom system is an orthocomplementary modular logic which has a specific relation of applicability to a finite projective geometry (theorem XVIII of appendix) and to closed vectorial subspaces of a Hilbert Space (theorem XXII).

Comparison with *RMQ*: Before we discuss the axioms of Piron's system we have to compare notations. Piron uses the ordering relation $\leq$ between two propositions a, b, that is, $a \leq b$ which he interprets as "b est vraie chaque fois que a est vraie" (b is true in all cases in which a is true). Since this is nothing but the definition of implication in propositional logic we translate $a \leq b$ as $a \rightarrow b$. In a similar way a' is translated as $\neg a$ and $a \cap b$ as $a \wedge b$. According to this translation the axioms of Piron's system are as follows:

Axioms of group O (Piron, 1964, p. 2):

$a \rightarrow a$ Theorem (4) of Section 4.4.
$(a \rightarrow b \wedge b \rightarrow c) \rightarrow a \rightarrow c$ Theorem (12) of Section 4.4.

Axiom T says that there is a greatest lower bound for all propositions. This implies immediately that there is the absurd proposition ("proposition absrude") $\emptyset$ such that it implies all propositions (Piron, 1964, p. 3):

$\emptyset \rightarrow (\forall p)\, p$

Since the absurd proposition has a matrix with exclusively values 6 and the conjunction of all propositions does have also a matrix of exclusively values 6 in *RMQ*, the propositional analogue of axiom T is strictly valid in *RMQ*.

Axioms of group C (Piron, 1964, p. 3):

$\neg\neg a \rightarrow a$	Theorem (5) of Section 4.4.
$(\neg a \wedge a) \rightarrow \emptyset$	Theorem (36) of Section 4.4.
$(\neg a \rightarrow \neg b) \rightarrow (b \rightarrow a)$	Theorem (11) of Section 4.4.

$\emptyset$ is the absurd proposition which follows from a contradiction. It can also be replaced by the arbitrary proposition as in Principle (36) of Section 4.4. or by the value 6, since the matrix of every contradiction contains only values of 6.

Axiom P: Two propositions a, b such that $a \leq b$ (in our notation: $a \rightarrow b$) are always compatible; where a is compatible with b iff $a \vee (\neg a \wedge b) \leftrightarrow b \vee (\neg b \wedge a)$.

This axiom holds strictly in *RMQ*, that is, two propositions a,b such that $a \rightarrow b$ are always compatible according to the above definition in *RMQ*. Since all the axioms O, T, C,

[41] Piron (1964).

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

and P hold strictly in *RMQ*, *RMQ**, is also a model of Piron's orthocomplementary logic and it seems to be an interesting further question whether *RMQ** has also a specific relation to a finite projective geometry and to closed vectorial subspaces of a Hilbert space in analogy to theorems XVIII and XXII of Piron.

5.3. Minimal quantum logic of Dalla Chiara. *MQL*[42] has 11 axioms and 3 rules of which Axiom (11) and Rule (iii) belong to *PL*1. Since our concern is only propositional logic, our comparison refers to Axioms (1)–(10) and Rules (i) and (ii). For *MQL* a soundness and completeness proof is given (the latter w.r.t. Kripkian and algebraic Semantics). With the help of a further axiom, *MQL* is extended to orthomodular logic.

Comparison with *RMQ*: Axioms (4)–(6) and (8)–(10) and Rule (i) (modus ponens) hold strictly in *RMQ* as Theorems (4), (6), (7), (36), (5), (11), and (9) respectively. Axiom (1), although it is a weaker form of Frege's axiom, is not strictly (i.e., only materially) valid in *RMQ*. This is also the case with Axioms (2), (3), and (7) and with Rule (ii) $\alpha \,/\, (\beta \rightarrow \alpha)$, which is used as a compensation of the weaker form of Axiom (1). Concerning Rule (ii), it is easily seen that β can be replaced by any other proposition (also by $\neg\beta$) salva validitate of the inference and is therefore irrelevant for the deduction. Such principles are ruled out by the *RC* criterion and in *RMQ* they are also ruled out from the strictly valid theorems except in the case, when the premises (the antecedents) are contradictory. Thus *RMQ** is only a partial model for the minimal quantum logic *MQL*. A further system of *QL*[43] seems not to be suitable, since even weak distributivity (our Theorem (22) of Section 4.4.) is not valid in it.

***5.4. The systems of QL of Mittelstaedt and Stachow*[44].** These systems have already been constructed in the 70s and have received a completeness proof w.r.t. a system of dialogical logic.[45] Stachow has shown also that that part of the law of distribution which leads from conjuncts to disjuncts (cf. Section 2.(4) and Section 4.4.1. above) is not satisfied by experimental setups. The axiom systems of both authors are very similar.[46] Axioms 1.1, 1.2, 2.1, 2.2, 4.1, 5.1, and 5.4 correspond in *RMQ* to Theorems (4), (12), (6), (7), (9), (36), and (35) respectively and are all strictly (necessarily) valid in *RMQ*. Moreover, Axioms 5.0, 5.2, and 5.3 are also strictly valid in *RMQ*. Axiom 2.3 is Principle (29) of Section 4.4. and is only materially valid in *RMQ* (the reasons have been discussed in Section 4.4.1. above). Only its opposite direction holds strictly. Axioms 3.1 and 3.2 correspond to (4) of Section 4.2. in which $q(B)$ is replaceable by any proposition whatsoever (including its own negation). Therefore they are not strictly valid in *RMQ*. Axiom 3.3 leads from conjuncts to disjuncts and so does not hold strictly in *RMQ*. The remaining Axioms 4.2, 4.3, and 4.4 also hold only materially (not strictly) in *RMQ*.

For the reasons stated above we have to say that *RMQ** is only a partial model of the system of *QL* by Mittelstaedt and Stachow.

§6. The modal logic contained in *RMQ*. The modal logic contained in *RMQ* has two kinds of necessity and two kinds of possibility, 14 different modalities altogether

---

42 Dalla Chiara (1977).

43 Cattaneo *et al.* (2004, p. 337).

44 Mittelstaedt (2004); Stachow (2004).

45 Stachow (1976) and Mittelstaedt very detailed in (1978).

46 We refer here to the one of Mittelstaedt (2004, p. 282). Compare Stachow (1978, pp. 363 and 372).

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

(7 positive and 7 negative ones). It derives from a similar six-valued matrix calculus.[47] Although both systems do not have strong necessitation (i.e., if $\vdash p$, then $\vdash Lp$, the system $SS1M$ contains all theorems of *CPC* as strictly (necessarily) valid, whereas *RMQ* allows as strictly valid only those theorems of *CPC* which (approximately) satisfy the criteria *RC* and *RD* (for avoiding the well-known difficulties), yet including all the important traditional principles of *CPC* (recall Sections 1 and 2 above). The modal system contained in *RMQ* is similar to the modal system *T* (of Feys or von Wright) concerning many theorems, except necessitation for all *CPC*-valid formulas. It also includes Brouwer's system B, though without necessitation w.r.t. all *CPC*-valid formulas.

6.1. Modal theorems of RMQ.

(1) $Lp \Leftrightarrow \neg M \neg p$ (5) $LLp \Leftrightarrow \neg\, MM\, \neg p$
(2) $Mp \Leftrightarrow \neg L \neg p$ (6) $MMp \Leftrightarrow \neg\, LL\, \neg p$
(3) $L \neg p \Leftrightarrow \neg Mp$ (7) $LL \neg p \Leftrightarrow \neg MMp$
(4) $\neg Lp \Leftrightarrow M \neg p$ (8) $\neg LLp \Leftrightarrow MM \neg p$
(9) $Lp \Rightarrow p$ (10) $p \Rightarrow Mp$
(11) $LLp \Rightarrow Lp \Rightarrow MLp \Rightarrow p \Rightarrow LMp \Rightarrow Mp \Rightarrow MMp$
(12) $L(p \rightarrow q) \Rightarrow (Lp \rightarrow Lq)$
(13) $M(p \rightarrow q) \rightarrow (Mp \rightarrow Mq)$
(14) $L(p \wedge q) \Leftrightarrow (Lp \wedge Lq)$
(15) $M(p \wedge q) \Rightarrow (Mp \wedge Mq)$
(16) $(Lp \vee Lq) \Rightarrow L(p \vee q)$
(17) $M(p \vee q) \rightarrow (Mp \vee Mq)$
(18) $(\neg Mp \wedge \neg Mq) \Rightarrow \neg M(p \vee q)$
(19) $L(p \rightarrow q) \Rightarrow (Mp \rightarrow Mq)$
(20) $L(\neg p \rightarrow p) \Leftrightarrow Lp$
(21) $[L(q \rightarrow p) \wedge L(\neg q \rightarrow p)] \Leftrightarrow Lp$
(22) $Lp \Rightarrow [Mq \rightarrow M(p \wedge q)]$
(23) $M(p \rightarrow q) \rightarrow (Lp \rightarrow Mq)$
(24) $L(p \vee q) \Rightarrow (Lp \vee Mq)$
(25) $[L(p \rightarrow q) \wedge p] \Rightarrow q$
(26) $[L(p \rightarrow q) \wedge \neg q] \Rightarrow \neg p$
(27) $[L(p \rightarrow q) \wedge L(q \rightarrow r)] \Rightarrow L(p \rightarrow r)$
(28) $L(p \rightarrow q) \Rightarrow L(\neg q \rightarrow \neg p)$

6.2. Comparison with other modal systems. *RMQ* is well comparable to the modal system *T*. Although *T* has a strong rule of necessitation which is not valid in *RMQ*, many of the main theorems of *T* are strictly valid in *RMQ*, too. Thus all Theorems (1)–(10) of Section 6.1. are theorems of *T*. Furthermore, Theorems (12), (14), (15), (17)–(24), and (27) are theorems of *T*. However, (17), (18), and (23) hold in *T* with a strict equivalence ($\Leftrightarrow$).

Concerning the modal system *S*4, *RMQ* does not have the reduction theorem $Lp \rightarrow LLp$ of *S*4, which is also valid in *S*5, because it is a task of *RMQ* to distinguish two kinds of

[47] Compare Weingartner (1968). This system was called there $SS1M$.

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

necessity, a stronger one and a weaker one; analogously *RMQ* distinguishes between two kinds of possibility, a stronger and a weaker one. Nevertheless, *RMQ* has the same number of different modalities as *S*4 (7 positive and 7 negative ones). But in *RMQ* they can be ordered into one line of strict implication (cf. Theorem (11) above), whereas in *S*4 this is not possible and the nonmodal proposition p is deductively connected only with Lp and Mp, but not with any other modality. Concerning modal system *S*5, the axiom which leads from *S*4 to *S*5, $Mp \rightarrow LMp$, is materially, but not strictly, valid in *RMQ*.

As has been said above, some theorems which are valid in system T with a strict equivalence, are valid in *RMQ* only with a strict or material implication. There are further cases in which a principle is valid in T with a strict equivalence containing connectives $\rightarrow$, but if $\rightarrow$ is replaced by $\Rightarrow$, they are only valid as a strict implication, whereas in *RMQ* they are only valid as a strict implication right from the beginning; an example is Theorem (30) of Section 4.4.

T and *RMQ* behave also similar concerning important invalid *wff*s. For example in both systems $(p \rightarrow q) \vee (q \rightarrow p)$ is valid, but $(p \Rightarrow q) \vee (q \Rightarrow p)$ is invalid. Moreover, $(p \wedge q) \rightarrow (p \rightarrow q)$ is valid in T and materially valid in *RMQ*, but $(p \wedge q) \rightarrow (p \Rightarrow q)$ is invalid in both (cf. Section 4.2.(13)).

6.3. Modal principles and situations in QP. There are several theorems of modal logic and of *RMQ* which can be helpful to interpret quantum mechanical situations.

As an example take Theorem (15) of Section 6.1. The opposite implication of (15) is invalid in *RMQ*: Assume that Mp is the proposition that a sharp measurement of position is possible on physical system S and that Mq is the proposition that a sharp measurement of momentum is possible on physical system S. From this it does not follow (i.e., it would be invalid to conclude) that both, a sharp measurement of position and a sharp measurement of momentum, are (jointly) possible.

As a second example we mention quasimodularity.[48] Its nonmodal version is only materially valid in *RMQ*:

$$[(q \rightarrow p) \wedge (r \rightarrow \neg p)] \rightarrow [(p \wedge (q \vee r)) \leftrightarrow ((p \wedge q) \vee (p \wedge r))]$$

But quasimodularity is strictly valid in *RMQ* if taken in that implicational form in which the distribution law is valid in *QL* independently of whether the antecedent implications are strengthened by necessity (see Section 4.4.(22)):

$$[(q \rightarrow p) \wedge (r \rightarrow \neg p)] \Rightarrow [((p \wedge q) \vee (p \wedge r)) \Rightarrow (p \wedge (q \vee r))]$$

$$[L(q \rightarrow p) \wedge L(r \rightarrow \neg p)] \Rightarrow [((p \wedge q) \vee (p \wedge r)) \Rightarrow (p \wedge (q \vee r))]$$

Weak modularity is only satisfied in the following weaker modal version:

$$L(q \rightarrow p) \Rightarrow [(p \wedge (q \vee \neg p)) \Rightarrow q].$$

§7. Conclusion. The application of classical logic (*CL*)—already in its form as classical propositional logic (*CPC*)—to modern physics leads to specific difficulties which come up in a similar way also in the application to other domains of empirical sciences. The conjecture *CJ* made at the beginning (Section 1) of the paper was that these difficulties (here: commensurability, distributivity, Bell's inequalities . . .) are caused by two Properties

[48] Compare Mittestaedt (1978) and Weingartner (2004b, p. 237).

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

$P1$ and $P2$ of *CL* (*CPC*); and that they can be avoided, when *CL* (*CPC*) is restricted by evading inferences with Properties $P1$ or $P2$. Property $P1$ says that *CL* (*CPC*) allows redundant or irrelevant elements in the consequence class of classically valid inferences in the sense that these elements may be replaceable by any other element (also its own negation) salva validitate of that inference. In Section 2 the criterion *RC* was defined, which avoids such inferences. Property $P2$ says that *CL* (*CPC*) allows to produce repetitions with $\wedge$ and $\vee$, and disjunctive parts out of conjunctive parts by classically valid inferences. Also in Section 2, the criterion *RD* was described, which avoids such inferences. The conjecture *CJ* is shown to be confirmed by a finite matrix calculus *RMQ* which is consistent and decidable and approximates *RC* and *RD*. The materially valid theorems of *RMQ* contain *CPC*. The strictly valid theorems of *RMQ* (*RMQ**) compare 4.4., satisfy *RC* (except for few cases like contradictory premises) and *RD* (except for the case of harmless logically equivalent repetitions). Those theorems of classical logic which lead to paradoxes when applied in different domains of empirical science and which are ruled out by *RC* or *RD* are also only materially valid in *RMQ*. In contradistinction to *RC* and *RD* however, *RMQ** is closed under modus ponens, under transitivity of implication and under equivalence substitution and is preserving the strict validity by applying strict implication. *RMQ** avoids the difficulties with commensurability, distributivity, and with Bell's inequalities (Section 4.3.). Moreover, *RMQ** is a model of Piron's orthocomplementary logic and also a model of the orthologic (with exception of Axiom 8) of Goldblatt. *RMQ* contains a modal logic which is close to the system *T* of Feys or von Wright.

§8. Acknowledgements. I would like to thank Prof. Peter Mittelstaedt and the two anonymous referees for valuable suggestions.

BIBLIOGRAPHY

Aerts, D. (1985). A possible explanation for the probabilities of quantum mechanics and example of a macroscopical system that violates Bell inequalities. In Mittelstaedt, P., and Stachow, E. W., editors. *Recent Developments in Quantum Logics*. Mannheim, Germany: B.I. Wissenschaftsverlag. pp. 235–249.

Aerts, D., & Aerts, S. (2005). Towards a general operational and realistic framework for quantum mechanics and relativity theory. In Elitzur, A., Dolev, S., and Kolenda, N., editors. *Quo Vadis Quantum Mechanics?* Heidelberg, Germany: Springer, pp. 153–207.

Bell, J. (1987). *Speakable and Unspeakable in Quantum Mechanics.* Cambridge, UK: Cambridge University Press.

Beltrametti, E. G., & Maczynski, M. J. (1991). On the characterization of classical and non-classical probabilities. *Journal of Mathematical Physics*, **32**, 1280–1286.

Birkhoff, G., & von Neumann, J. (1936). The logic of quantum mechanics. *Annals of Mathematics*, **37**, 823–843.

Cattaneo, G., Dalla Chiara, M. L., & Giuntini, R. (2004). An unsharp quantum logic from quantum computation. In Weingartner, P., editor. *Alternative Logics. Do Sciences Need Them?* Heidelberg, Germany: Springer. pp. 323–338.

Czermak, J. (1981). Eine endliche Axiomatisierung von SS1M. In Morscher, E., Neumaier, O., and Zecha, G., editors. *Philosophie als Wissenschaft*/ [*Essays in Scientific Philosophy*]. Comes, Bad Reichenhall. pp. 245–258.

Da Costa, N. (1974). On a theory of inconsistent formal systems. *Notre Dame Journal of Formal Logic*, **15**, 497–510.

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

Dalla Chiara, M. L. (1977). Quantum logic and physical modalities. *Journal of Philosohical Logic*, **6**, 391–404.
Dalla Chiara, M. L., & Guintini, R. (2001). *Quantum Logics.* arXiv: quant-ph/0101028 v2.
Dalla Chiara, M. L., Guintini, R., & Greechie, R. (2004). *Reasoning in Quantum Theory, Sharp and Unsharp Quantum Logics*. Dordrecht, The Netherlands: Kluwer.
D'Espagnat, B. (1979). *A la Recherche du Réel*. Paris, France: Gauthier-Villars.
Finch, P. D. (1970). Quantum logic as an implication algebra. *Bulletin of the Australian Mathematical Society*, **2**, 101–106.
Finkelstein, D. (1979). Matter, space and logic. In Hooker, C. A., editor. *The Logico-Algebraic Approach to Quantum Mechanics*. Berlin: Springer, Vol. II. pp. 123–139.
Goldblatt, R. I. (1974). Semantic analysis of orthologic. *Journal of Philosophical Logic*, **3**, 19–35.
Mittelstaedt, P. (1972). On the interpretation of the lattice of subspaces of Hilbert space as a propositional calculus. *Zeitschrift für Naturforschung*, **27a**, 1358–1362.
Mittelstaedt, P. (1978). *Quantum Logic*. Dordrecht, The Netherlands: Reidel.
Mittelstaedt, P. (1998). *The Interpretation of Quantum Mechanics and the Measurement Process.* Cambridge, UK: Cambridge University Press.
Mittelstaedt, P. (2004). Does quantum physics require a new logic? In Weingartner, P., editor. *Alternative Logics. Do Sciences Need Them?* Heidelberg, Germany: Springer. pp. 269–284.
Mittelstaedt, P., & Stachow, E. W., editors (1985). *Recent Developments in Quantum Logics*. Mannheim, Germany: B.I. Wissenschaftsverlag.
Mittelstaedt, P., & Weingartner, P. (2005). *Laws of Nature*. Heidelberg, Germany: Springer.
Piron, C. (1964). Axiomatique quantique. *Helvetica Physica Acta*, **37**, 439–468.
Pitowsky, I. (1989). *Quantum Probability-Quantum Logic*. Lecture Notes in Physics, Vol. 321. Berlin, Germany: Springer.
Priest, G. (2000). Motivation of paraconsistency: The slippery slope from classical logic to dialethism. In Batens, D., Morfensen, Ch., Priest, G., and van Bendegem, J. P., editors. *Frontiers in Paraconsistent Logics.* London: Research Studies Press, pp.223–233.
Schurz, G. (1991a). Relevant deduction. *Erkenntnis*, **35**, 391–437.
Schurz, G. (1991b). Relevant deductive inference: Criteria and logics. In Schurz, G., and Dorn, G., editors. *Advances in Scientific Philosophy. Essays in Honour of Paul Weingartner*. Amsterdam, The Netherlands: Rodopi, pp. 57–84.
Schurz, G., & Weingartner, P. (1987) Versimilitude defined by relevant consequence-elements. A new reconstruction of Popper's original idea. In Kuipers, T., editor. *What is Closer-to-the-Truth?* Amsterdam, The Netherland: Rodopi, pp. 47–77.
Schurz, G., & Weingartner, P. (2008). Zwart and Franssen's impossibility theorem holds for possible-world-accounts but not for consequence-accounts to versimilitude. *Synthese*. Forthcoming.
Solèr, M. P. (1995). Characterisation of Hilbert spaces by orthomudular lattices. *Communications in Algebra*, **23**(1), 219–243.
Stachow, E. W. (1976). Completeness of quantum logic. *Journal of Philosophical Logic*, **5**, 237–280.
Stachow, E. W. (1978). Quantum logical calculi and lattice structures. *Journal of Philosophical Logic*, **7**, 347–386.
Stachow, E. W. (2004). Experimental approach to quantum-logical connectives. In Weingartner, P., editor. *Alternative Logics. Do Sciences Need Them?* Heidelberg, Germany: Springer. pp. 285–298.
Takeuti, G. (1987). *Proof Theory*. Amsterdam: North Holland.

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press

Tarski, A. (1956). *Logic Semantics and Metamathematics*. Oxford, UK: Oxford University Press.

Weingartner, P. (1968). Modal logics with two kinds of necessity and possibility. *Notre Dame Journal of Formal Logic*, **9**(2), 97–159.

Weingartner, P. (1985). A simple relevance-criterion for natural language and its semantics. In Dorn, G., and Weingartner, P., editors. *Foundations of Logic and Linguistics: Problems and their Solutions*. New York: Plenum Press. pp. 563–575.

Weingartner, P. (2000a). *Basic Questions on Truth* (Series *Episteme* 24). Dordrecht, The Netherlands: Kluwer.

Weingartner, P. (2000b). Reasons for filtering classical logic. In Batens, D., Morfensen, Ch., Priest, G., and van Bendegem, J. P., editors. *Frontiers in Paraconsistent Logics*. London: Research Studies Press, pp. 315–327.

Weingarnter, P. (2001). Applications of logic outside logic and mathematics: Do such applications force us to deviate from classical logic?! In Stelzner, W., editor. *Zwischen traditioneller und moderner Logik*. Paderborn: Mentis, pp. 53–64.

Weingartner, P., editor (2004a). *Alternative Logics. Do Sciences Need Them?* Heidelberg, Germany: Springer.

Weingartner, P. (2004b). Reasons from science for limiting classical logic. In Weingartner, P., editor. *Alternative Logics. Do Sciences Need Them?* Heidelberg, Germany: Springer. pp. 233–248.

Weingartner, P., & Schurz, G. (1986). Paradoxes solved by simple relevance criteria. *Logique et Analyse*, **113**, 3–40.

Zeilinger, A. (1992). Physik und Wirklichkeit. Neuere Entwicklungen zum Einstein-Podolsk-Rosen Paradoxon. In Reichel, H. C., and Prat de la Riba, E., editors. *Naturwissenschaft und Weltbild.* Vienna, Austria: Hölder-Pichler-Temsky, pp. 99-121.

DEPARTMENT OF PHILOSOPHY
UNIVERSITY OF SALZBURG AND
INSTITUT FÜR WISSENSCHAFTSTHEORIE
SALZBURG, AUSTRIA, A-5020
E-mail: paul.weingartner@sbg.ac.at

https://doi.org/10.1017/S1755020309090169 Published online by Cambridge University Press